6.01: Introduction to EECS I

Solving Circuits
Circuit Abstractions

Week 8 *October 27, 2009*

Analyzing Simple Circuits

Simple circuits (of the type that we have been building in lab) can usually be analyzed by

- recognizing equivalent representations (that are even simpler) e.g., series and parallel combinations
- recognizing common patterns e.g., voltage and current dividers
- serendipitous formulation of circuit equations

Analyzing complicated circuits requires a more algorithmic approach.

Analyzing Complicated Circuits

All circuits can be analyzed by systematically applying

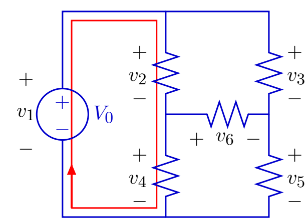
- Kirchoff's voltage law (KVL),
- Kirchoff's current law (KCL), and
- current-voltage laws (constituitive relations) for the components

and then solving the resulting equations.

Developing a systematic approach is especially important for automated simulation tools (such as CMax).

Step 1: KVL

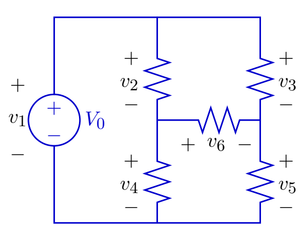
The sum of the voltages around any closed path is zero.



Example: $-v_1 + v_2 + v_4 = 0$

Check Yourself

How many KVL equations can be written for this circuit?

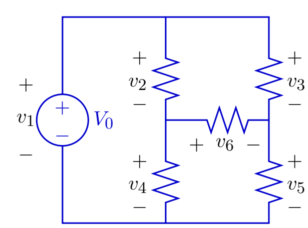


KVL Equation Solver

To solve circuits algorithmically using KVL, we must

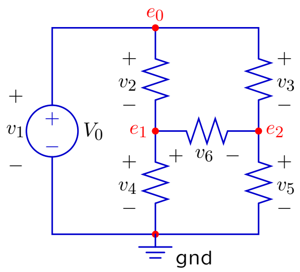
- enumerate a complete set of linearly independent KVL equations
- eliminate those that are linearly dependent on others.

This task is not trivial, even for just moderately complicated circuits.



Alternative Representation: Node Voltages

Node voltages represent the voltage between each node in a circuit and an arbitrarily selected ground.

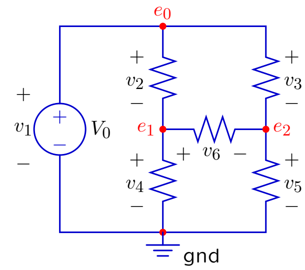


Node voltages and component voltages are different but equivalent **representations** of voltage.

- **component voltages** represent the voltages across components.
- **node voltages** represent the voltages in a circuit.

Node Voltages

Node voltages are linear combinations of component voltages. Component voltages are differences between node voltages.

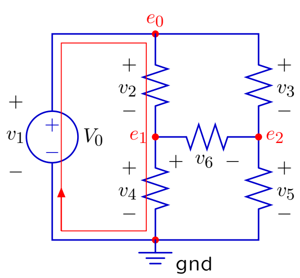


Examples: $e_0 = v_2 + v_4$

$v_2 = e_0 - e_1$

Node Voltages

Node voltages automatically satisfy KVL.

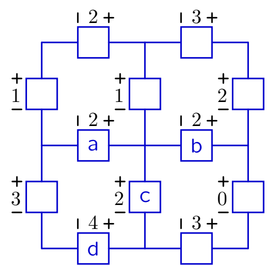


using component voltages: $-v_1 + v_2 + v_4 = 0$

using node voltages $-e_0 + (e_0 - e_1) + (e_1) \equiv 0$

Check Yourself

The following voltages are not consistent with KVL but can be made consistent by changing just one. Which one?



1. a
2. b
3. c
4. d
5. none of the above

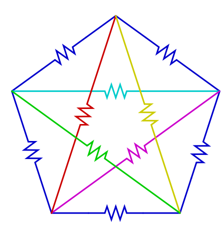
Node Voltages

Node voltages eliminate the need to enumerate **any** KVL equations.

This is especially helpful when the KVL equations are difficult to enumerate!

Check Yourself

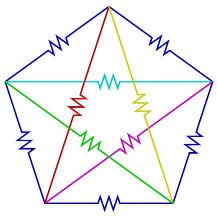
Many KVL equations can be written for this circuit.
How many contain exactly three component voltages?



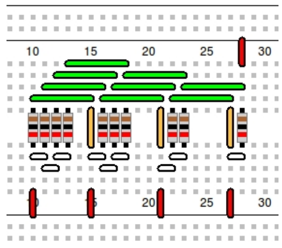
1. 4 2. 5 3. 10 4. 16 5. none of these

Analyzing Complicated Circuits

Using node voltages is much easier than formulating KVL equations for complicated circuits.



Pentagonal circuit



CMax representation

CMax may have to solve this circuit regardless of whether it is useful.

Node Voltages

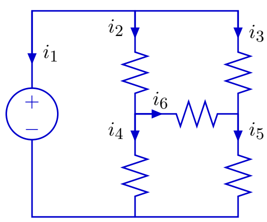
Node voltages

- eliminate the need to enumerate **any** KVL equations
- slightly complicate component (constitutive) relations:
  e.g., Ohm's law: $V_1 = I_1 R_1 \rightarrow e_6 - e_7 = I_1 R_1$

Eliminating all KVL equations can be well-worth the added complication to Ohm's law, especially when the KVL equations are difficult to find.

Step 2: KCL

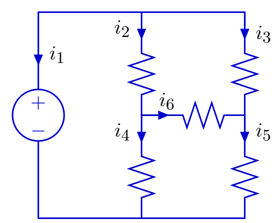
The net current into (or net current out of) any node is zero.



Example: $i_1 + i_2 + i_3 = 0$

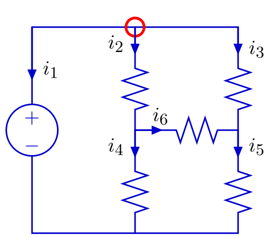
Check Yourself

How many distinct KCL relations can be written for this circuit?



Analyzing Circuits: KCL

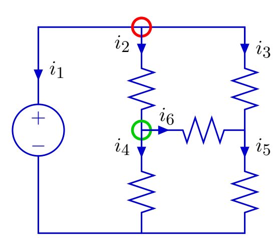
The net current out of any closed surface (which can contain multiple nodes) is zero.



node 1: $i_1 + i_2 + i_3 = 0$

Analyzing Circuits: KCL

The net current out of any closed surface (which can contain multiple nodes) is zero.

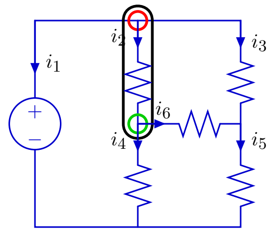


node 1: $i_1 + i_2 + i_3 = 0$

node 2: $-i_2 + i_4 + i_6 = 0$

Analyzing Circuits: KCL

The net current out of any closed surface (which can contain multiple nodes) is zero.



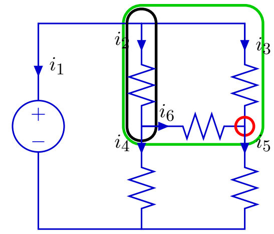
node 1: $i_1 + i_2 + i_3 = 0$

node 2: $-i_2 + i_4 + i_6 = 0$

nodes 1+2: $i_1 + i_3 + i_4 + i_6 = 0$

Analyzing Circuits: KCL

The net current out of any closed surface (which can contain multiple nodes) is zero.



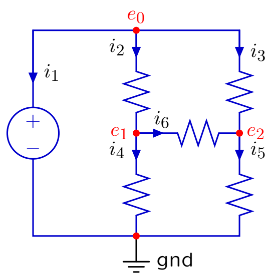
nodes 1+2: $i_1 + i_3 + i_4 + i_6 = 0$

node 3: $-i_3 - i_6 + i_5 = 0$

nodes 1+2+3: $i_1 + i_4 + i_5 = 0$

KCL with Node Voltages

The sum of the currents entering "ground" is equal to the sum of the currents exiting all of the other nodes.



We need only write KCL equations at e_0, e_1, and e_2.

KCL: Summary

The sum of the currents out of any node is zero.

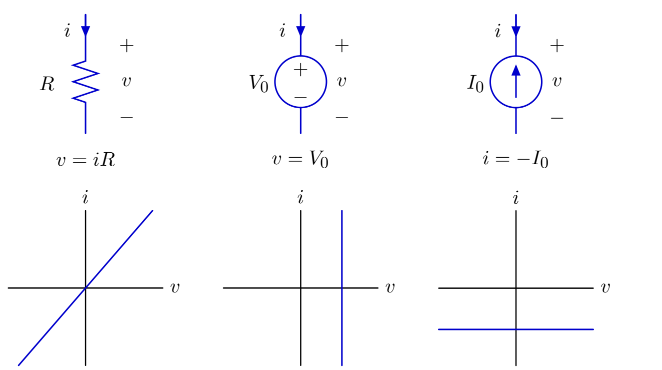
One KCL equation can be written for every closed surface (which contain one or more nodes) in a circuit.

Sets of KCL equations are not necessarily linearly independent.

KCL equations for every primitive node except one (ground) are linearly independent.

Step 3: Component (constitutive) Equations

One equation is needed to characterize each **linear** component.



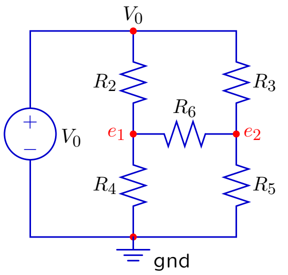
Node-Voltage-and-Component-Current (NVCC) Method

Combining steps 1, 2, and 3 leads to the NVCC method for solving circuits:

- Assign **node voltage variables** to every node except ground (whose voltage is arbitrarily taken as zero). $\to n-1$ variables
- Assign **component current variables** to every component in the circuit. $\to m$ variables
- Write one **constitutive relation** for each component in terms of the component current variable and the component voltage, which is the difference between the node voltages at its terminals. $\to m$ equations
- Express **KCL** at each node except ground in terms of the component currents. $\to n-1$ equations
- **Solve** the resulting $m+n-1$ equations in $m+n-1$ unknowns.

Node Method

The "node method" is a variant of NVCC in which component currents are not represented by variables but are calculated as needed, using the node voltage variables. Also, nodes connected to voltage sources are represented by constants rather than by variables.



KCL at e_1:

$$\frac{e_1 - V_0}{R_2} + \frac{e_1 - e_2}{R_6} + \frac{e_1}{R_4} = 0$$

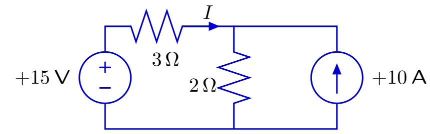
KCL at e_2:

$$\frac{e_2 - V_0}{R_3} + \frac{e_2 - e_1}{R_6} + \frac{e_2}{R_5} = 0$$

- solve (here just 2 equations and 2 unknowns)

Check Yourself

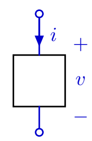
Determine the current I in the circuit below.



1. $1\,\text{A}$
2. $\frac{5}{3}\,\text{A}$
3. $-1\,\text{A}$
4. $-5\,\text{A}$
5. none of the above

Circuit Abstractions: One-ports

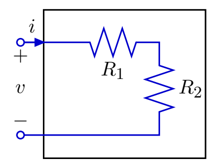
A "one-port" is a circuit that can be represented as a single element.

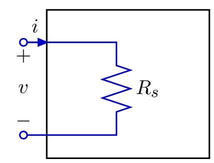


A one-port has two terminals. Current enters one terminal (+) and exits the other (−), producing a voltage (v) across the terminals.

Series Combinations

The series combination of two resistors is equivalent to a single resistor whose resistance is the sum of the two original resistances.





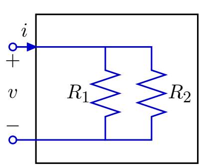
$$v = R_1 i + R_2 i \qquad v = R_s i$$

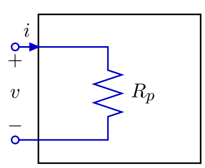
$$R_s = R_1 + R_2$$

The resistance of a series combination is always larger than either of the original resistances.

Parallel Combinations

The parallel combination of two resistors is equivalent to a single resistor whose conductance (1/resistance) is the sum of the two original conductances.





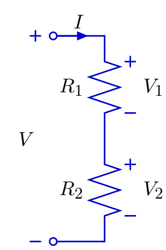
$$i = \frac{v}{R_1} + \frac{v}{R_2} \qquad i = \frac{v}{R_p}$$

$$\frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_1 + R_2}{R_1 R_2} \quad \rightarrow \quad R_p = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{R_1 R_2}{R_1 + R_2} \equiv R_1 || R_2$$

The resistance of a parallel combination is always smaller than either of the original resistances.

Voltage Divider

Resistors in series act as voltage dividers.



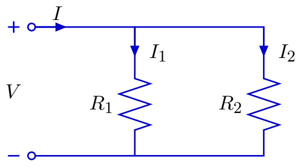
$$I = \frac{V}{R_1 + R_2}$$

$$V_1 = R_1 I = \frac{R_1}{R_1 + R_2} V$$

$$V_2 = R_2 I = \frac{R_2}{R_1 + R_2} V$$

Current Divider

Resistors in parallel act as current dividers.



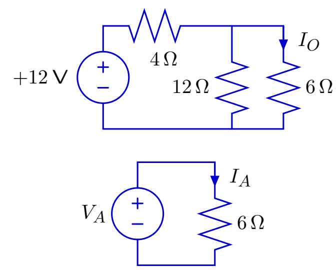
$$V = (R_1||R_2)\, I$$

$$I_1 = \frac{V}{R_1} = \frac{R_1||R_2}{R_1} I = \frac{1}{R_1}\frac{R_1R_2}{R_1+R_2} I = \frac{R_2}{R_1+R_2} I$$

$$I_2 = \frac{V}{R_2} = \frac{R_1||R_2}{R_2} I = \frac{1}{R_2}\frac{R_1R_2}{R_1+R_2} I = \frac{R_1}{R_1+R_2} I$$

Check Yourself

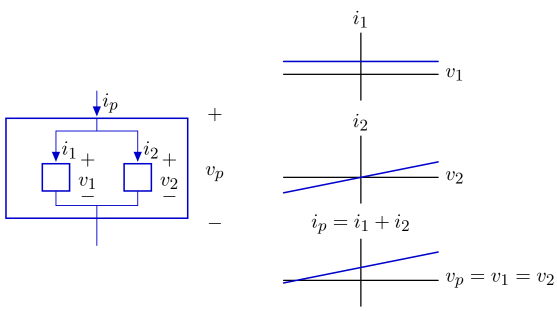
Find V_A so that $I_A = I_O$



Linear Circuits

If a one-port contains just resistors, voltage sources, and current sources, then the relation between its terminal voltage and current is a straight line.

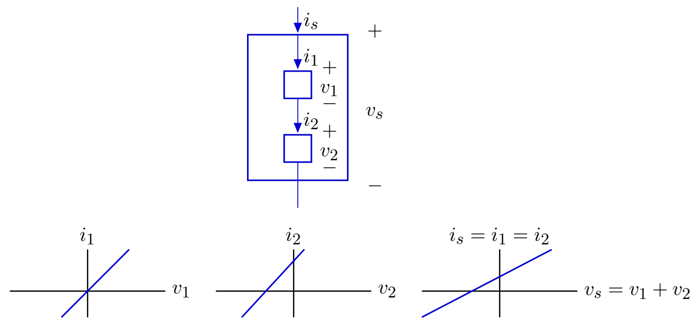
Example: parallel combination



Linear Circuits

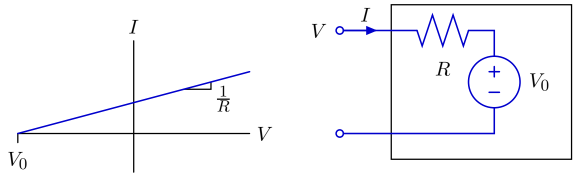
If a one-port contains just resistors, voltage sources, and current sources, then the relation between its terminal voltage and current is a straight line.

Example: series combination



Thevenin Equivalents

If the relation between terminal voltage and current can be represented by a straight line, then the terminal behavior of the one-port can be represented by a voltage source in series with a resistor.

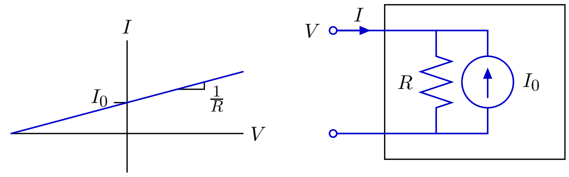


The voltage V_0 is equal to the voltage where $I = 0$.

The resistance R is the reciprocal of the slope.

Norton Equivalents

If the relation between terminal voltage and current can be represented by a straight line, then the terminal behavior of the one-port can be represented by a current source in parallel with a resistor.



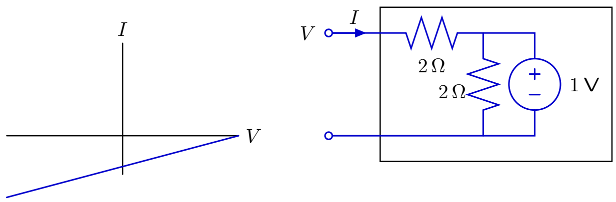
The current I_0 is equal to the current where $V = 0$.

The resistance R is the reciprocal of the slope.

Linear One-Ports

If a one-port contains just resistors and current and voltage sources, then its terminal behavior can be characterized by determining just two points on its v-i curve.

Example: open circuit voltage and short circuit current.

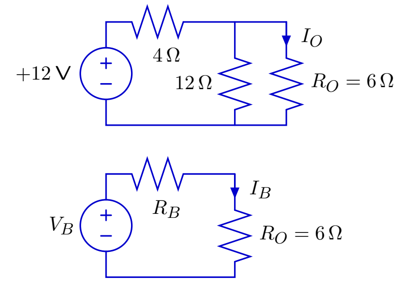


V_0 is the voltage V when $I = 0$, i.e., $V_0 = 1\,\text{V}$.

I_0 is the current I when $V = 0$, i.e., $I_0 = -1/2\,\text{A}$.

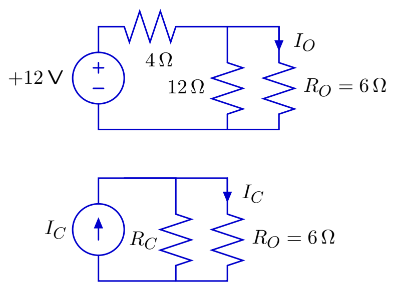
Check Yourself

Find V_B and R_B so that $I_B = I_O$.
Choose values so that $I_B = I_O$ even if $R_O \neq 6\,\Omega$.



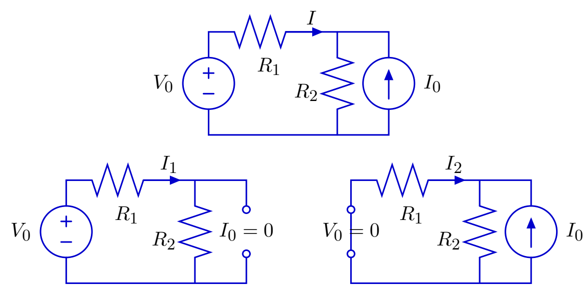
Check Yourself

Find I_C and R_C so that $I_C = I_O$.
Choose values so that $I_C = I_O$ even if $R_O \neq 6\,\Omega$.



Superposition

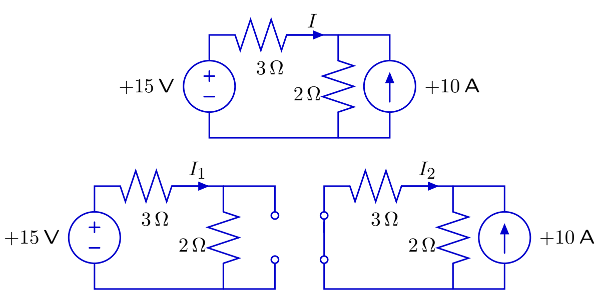
If a circuit contains only linear parts (resistors, current and voltage sources), then any voltage (or current) can be computed as the sum of those that result when each source is turned on one-at-a-time.



$$I = I_1 + I_2 = \frac{V_0}{R_1 + R_2} - \frac{R_2}{R_1 + R_2} I_0$$

Superposition

For many circuits, superposition is even easier to apply than the node or the loop methods.



$$I = I_1 + I_2 = \frac{15}{2+3} - \frac{2}{2+3} 10 = 3 - 4 = -1\,\text{A}$$

Summary

Circuits represent systems as connections of elements

- through which currents (through variables) flow and
- across which voltages (across variables) develop.

We have seen three (of many) methods for **analyzing** circuits. Each one is based on a different set of variables:

- currents and voltages for each component
- node voltages and component currents
- node voltages

There are circuit abstraction methods that facilitate **design**:

- series and parallel combinations
- voltage and current dividers
- Thevenin and Norton equivalents
- Superposition

PCAP

To design and analyze complex systems, we have to find organizing structures that are *compositional*:

- primitives
- means of composition
- means of abstraction
- abstract entities can do anything a primitive can

6.01 Introduction to Electrical Engineering and Computer Science I
Fall 2009