TRANSVERSELY ISOTROPIC SATURATED POROUS FORMATIONS
II. WAVE PROPAGATION AND APPLICATION TO MULTIPOLe LOGGING

by

D.P. Schmitt*

Earth Resources Laboratory
Department of Earth, Atmospheric, and Planetary Sciences
Massachusetts Institute of Technology
Cambridge, MA 02139

ABSTRACT

The wavefields generated by monopole and dipole sources in a fluid filled borehole embedded in multilayered transversely isotropic saturated porous formations are studied. The layers are modeled following Biot theory modified in accordance with homogenization theory. It allows to take into account a transversely isotropic skeleton and/or a transversely isotropic complex permeability tensor. Their axes of symmetry are assumed to coincide, parallel to the vertical axis of the borehole. A general formulation, valid for any order of multipole source and based on the Thomson Haskell method, allows to take into account any combination of elastic and saturated porous layers, either isotropic or transversely isotropic. The presence of an external fluid layer is also possible. The study focuses on the modes behavior. It is achieved through the computation of dispersion and attenuation curves, sensitivity coefficients with respect to the stiffness constants of the skeleton(s), and full waveform synthetic microseismograms using the discrete wavenumber method.

In the simple hole model with an impermeable borehole wall, whatever the type of the formation (fast or slow), the behavior of the modes is analogous to that in the presence of simple elastic formations with body wave attenuations added. The phase velocity of the Stoneley wave generated by a monopole source is sensitive to the horizontally propagating \( SH \)-wave velocity. Such a coupling decreases with increasing frequency and stiffness of the formation. The low frequency part of the zero-th order (i.e., flexural) mode generated by a multi(di)pole source measures the vertically propagating \( SV \)-wave velocity. The shear wave anisotropy may then be evaluated. With a fast formation, the vertically propagating \( SV \)-wave velocity can also be obtained.

*Now at Etudes et Productions Schlumberger, 26 Rue de la Cavée, 92142 Clamart, France.
from the low frequency (high velocity) part of the pseudo-Rayleigh mode generated by the monopole source. The anisotropy of the complex permeability tensor cannot be detected. Moreover, only the attenuation of the vertically propagating $P$ wave is sensitive to the only vertical permeability. Any anelastic (anisotropic) attenuation will supersede the latter.

When the borehole wall is permeable, the fluid flow which takes place at the interface refers to the horizontal mobility (i.e., horizontal permeability/saturant fluid viscosity). Assuming greater horizontal velocities, the decrease of the Stoneley wave phase velocity and the increase of its low frequency attenuation are enhanced. The shear wave transverse isotropy cannot be anymore detected and any estimation of the horizontal permeability based on Stoneley wave characteristics may become questionable with a high anisotropy degree of the skeleton. However, detection of permeability variation may still be reasonably performed.

In the presence of an invaded zone, whatever the boundary conditions at the borehole wall, Stoneley wave integrates the properties of the inner layer in the entire frequency range. This coupling phenomenon increases with increasing thickness and decreasing body wave velocities of the inner layer. As a result, both estimations of the shear wave transverse anisotropy and the permeability of the virgin formation from the Stoneley wave characteristics are ill posed. Of course, such a result hold true in a cased borehole, whatever the quality of the bonding. In any of the multilayered configuration, the low frequency part of both the flexural mode and the pseudo-Rayleigh mode, when it exists, measures the characteristics of the vertically propagating $SV$ wave of the virgin formation. Such an interesting information may be however difficult to extract.

**INTRODUCTION**

Since the pioneering work of Rosenbaum (1974), most of the following studies also consider the simple model composed of a fluid filled borehole embedded in a radially semi-infinite isotropic saturated porous formation. The goal of such an amount of work has been to investigate the feasibility of an indirect determination of the in situ permeability using essentially the Stoneley wave velocity and attenuation. If the understanding of the physical phenomenon may be considered as acquired, the reliability of the method is still the subject of numerous debates due to more or less inconsistent discrepancies between measured or expected values and results of the inversion. Because the reservoirs are known to be anisotropic, we choose to investigate the effects of such a situation, limiting the study to the simple case of the transverse isotropy. The porous media are then modeled following Biot theory modified in accordance with homogenization theory as described in Part I.
The drilling process can modify the physical properties of the formations close to the borehole wall, leading to the presence of several coaxial shells. Such a configuration has been the subject of various studies, mostly with a monopole source. Considering isotropic elastic formations, Baker (1984), Tubman (1984), and Schmitt and Bouchon (1985) presented numerical studies involving an invaded or flushed zone. Stephen et al. (1985), using the finite difference method, analyzed the case of a velocity gradient. Burns (1986) studied the partition coefficients distribution. Tubman (1984) and Burns (1986) also extensively studied the cased hole configuration. After White and Tongtaow (1981), Chan and Tsang (1983) presented synthetic microseismograms when the layers are transversely isotropic elastic focusing on the quasi body wave behaviors. Schmitt (1987b) investigated the effects of a radial variation of the permeability and/or the porosity, with or without modification of the saturant fluid, within isotropic saturated porous formations. The multilayered formation situation has been less extensively studied when the source is a multipole. Baker and Winbow (1985) studied the depth of penetration of both P and S waves generated by dipole and quadrupole sources in the presence of isotropic elastic formations while Schmitt (1987c) focused on the modes behavior. Schmitt (1987a) considered transversely isotropic formations, and Schmitt and Cheng (1987) included isotropic saturated porous formations. Everhart and Chang (1985) and Schmitt (1987c) investigated the cased hole configuration.

In the first part, we present a general formulation based on the Thomson Haskell method which allows to take into account several coaxial shells, either fluid or elastic or saturated porous, isotropic or transversely isotropic. It is valid for any order of the multipole source. The components of the displacement-stress vectors needed to propagate the wavefields are given in the Appendices for all four types of layers.

The second part is devoted to a numerical analysis focused on the guided modes behavior generated by both a monopole and a dipole source. When possible, both situations of an impermeable and a permeable fluid-porous formation interface are considered. In the simple hole model, the effects of the only transversely isotropic permeability, the borehole radius and of the transversely isotropic mass coupling coefficient are investigated. Also discussed is the presence of an elastic tool at the center of the borehole. Studies of the wavefields behavior in the presence of an invaded zone and in the cased hole configuration, either well bonded or not, are finally performed.
THEORETICAL DEVELOPMENT

Configuration

We define a cylindrical coordinates system as \((r, \theta, z)\). The model consists of a fluid-filled borehole, extending to infinity in the \(z\) direction, embedded in a radially layered formation. The radii \(R_{j-1}\) and \(R_j\) bound the \(j\)th layer. The borefluid is denoted by the subscript \(I\) so that the inner borehole radius is \(R_I\). Each layer of the formation can be elastic or saturated porous. It can be either transversely isotropic or simply isotropic. In the former case, the vertical axis of the borehole is assumed to coincide with the axis of symmetry of the formation which is common to both the complex permeability tensor and the skeleton when the shell is saturated porous (see Part I). One of the layers can be a fluid layer, allowing the study of not well bonded cased hole, for example.

A fluid layer is characterized by 2 parameters: the \(P\)-wave velocity \(c_{pj}\), and the density \(\rho_j\).

An isotropic elastic layer is characterized by 3 parameters: the \(P\)-wave velocity \(c_{pj}\), the \(S\)-wave velocity \(c_{sj}\), and the density \(\rho_j\).

A transversely isotropic elastic layer is characterized by 6 parameters: five elastic components of the stiffness tensor (i.e. \(c_{11,j}, c_{13,j}, c_{33,j}, c_{44,j}\), and \(c_{66,j}\) or, alternatively, \(c_{11,j}, c_{12,j}, c_{13,j}, c_{33,j},\) and \(c_{44,j}\), using the abbreviated notation as defined by Auld (1973)), and the density \(\rho_j\).

An isotropic saturated porous layer is characterized by 10 parameters: the saturant fluid properties which are its \(P\)-wave velocity \(c_{pj}\), its density \(\rho_{j}\) and its dynamic viscosity \(\eta_j\); the bulk modulus \(K_{sj}\) and the density \(\rho_{sj}\) of the constitutive grains; the porosity \(\varphi_j\); the compressional \((c_{pj})\) and shear \((c_{sj})\) dry rock velocities from which the bulk \((K_{sj})\) and shear \((N_{j})\) moduli can be evaluated as

\[
K_{sj} = (1 - \varphi_j) \rho_{sj} \left( c_{pj}^2 - 4 c_{sj}^2 / 3 \right),
\]

\[
N_{j} = (1 - \varphi_j) \rho_{sj} c_{sj}^2;
\]

the tortuosity \((\tilde{\tau}_j)\) (assuming that the pores are modeled as unidirectional cylindrical ducts (see Part I)), and the intrinsic permeability \(k_j\).

A transversely isotropic saturated porous layer is characterized by 15 parameters: the saturant fluid properties which are its \(P\)-wave velocity \(c_{pj}\), its density \(\rho_{j}\) and its dynamic viscosity \(\eta_j\); the bulk modulus \(K_{sj}\) and the density \(\rho_{sj}\) of the constitutive grains; five elastic components of the stiffness tensor of the skeleton (i.e. \(c_{11,j}, c_{13,j}, c_{33,j}, c_{44,j}\), and \(c_{66,j}\) or, alternatively, \(c_{11,j}, c_{12,j}, c_{13,j}, c_{33,j},\) and \(c_{44,j}\) ); the porosity \(\varphi_j\), the horizontal and vertical tortuosities \((\tilde{\tau}_{j}^H, \tilde{\tau}_{j}^V)\) (again, assuming that the
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pores are modeled as unidirectional cylindrical ducts); and the horizontal and vertical intrinsic permeabilities ($k_f^H$ and $k_f^V$).

Anelastic attenuation may be introduced in any fluid, elastic, or skeleton of a porous layer. It can be achieved by considering complex velocities or stiffness constants. As a result, the number of parameters increases by 1, 2 or 5 depending on the type of the layer.

Source Formulation and Solution for the Bore Fluid

We consider multipole point sources as defined by Kurkjian and Chang (1986) and Schmitt and Cheng (1987). The total wavefield in the bore fluid corresponds to the sum of the source contribution and the reflected wavefield. In the frequency axial wavenumber domain, the displacement potential of a multipole source of the nth order is then given by:

$$\Phi^{(n)}_1 = \frac{\varepsilon_n}{2^n n!} (k p_1 R_0)^n K_n(k p_1 R_0) \cos[n(\theta - \theta_0)] e^{i k z} e^{i \omega t},$$  \hspace{1cm} (1)

where $\varepsilon_n = 2 - \delta_{n0}$ is the Neumann coefficient, $k$ is the axial wavenumber, $\omega$ is the angular frequency, and $k p_1^2 = k^2 - \omega^2 / c_p^2$ is the radial wavenumber. $K_n$ denotes the nth order modified Bessel function of the second kind. $R_0$ is the multipole separation (i.e., the radius of the circle over which the n monopoles are distributed), while $\theta_0$ is an arbitrary angle of reference.

With the radiation condition prescribing that the displacement potential of the bore fluid remains finite at the axis of the borehole, its expression with respect to an nth order multipole point source is, in the frequency axial wavenumber,

$$\Phi^{(n)}_1 = \frac{1}{2^n n!} (k p_1 R_0)^n \tilde{A}^{(n)}_1 I_n(k p_1 r) \cos[n(\theta - \theta_0)] e^{i k z} e^{i \omega t},$$  \hspace{1cm} (2)

where $\tilde{A}^{(n)}_1$ is function of $k$ and $\omega$, and $I_n$ denotes the nth order modified Bessel function of the first kind.

The associated radial displacement $U^{(n)}_1$ and stress $\sigma^{(n)}_1$ are:

$$U^{(n)}_1 = \Phi^{(n)}_1 r,$$
$$\sigma^{(n)}_1 = -\omega^2 \rho_1 \Phi^{(n)}_1.$$

Solution for the Layers of the Formation

The displacement vector $u_j^{(n)} = (u_j^{(n)}, v_j^{(n)}, w_j^{(n)})^T$ in the jth elastic layer or in the solid phase of the jth saturated porous layer, and the displacement vector $U_j^{(n)} =$
In the liquid phase of the $j$th saturated porous layer, are separated into their compressional, and vertically, and horizontally polarized shear components:

\[
\begin{align*}
\psi_s^{(n)} &= \nabla \cdot \Phi_s^{(n)} + \nabla \times \left( \zeta_s^{(n)} \right) + \nabla \times \left( \Gamma_s^{(n)} \right), \\
\psi_f^{(n)} &= \nabla \cdot \Phi_f^{(n)} + \nabla \times \left( \zeta_f^{(n)} \right) + \nabla \times \left( \Gamma_f^{(n)} \right),
\end{align*}
\]

where the supplementary subscripts $s$ and $f$ refer to the solid and liquid phase of the saturated porous layer, respectively. For an elastic layer, the subscript $s$ is implicitly mute. $\Phi_s^{(n)}$ and $\Phi_f^{(n)}$ are compressional potentials. $\zeta$ is the unit vector in the $z$ direction, $\zeta_s^{(n)}$, $\zeta_f^{(n)}$ are horizontally polarized shear potentials ($SH$-wave), and $\Gamma_s^{(n)}$, $\Gamma_f^{(n)}$ are vertically polarized shear potentials ($SV$-wave).

In the following, we present the general solutions of these three kinds of potentials for each type of layer, along with the displacements and stresses formulae.

**Transversely isotropic saturated porous layer**

In a transversely isotropic saturated porous layer, there are four body waves (see Part I): a quasi compressional wave of the first kind, $P_1$, which displays high velocity, a quasi compressional wave of the second kind, $P_2$, associated with low velocity and near viscous characteristics, a quasi $SV$-wave, and a $SH$-wave. In addition to the coupling between the solid and liquid phases which leads to the existence of the two compressional waves, additional “coupling” phenomena occur between both compressional waves and the $SV$-wave. Each of those three has then a compressional part and a rotational part. They are called quasi waves since they approach the isotropic $P_1$-wave, $P_2$-wave and $SV$-wave as the degree of anisotropy of both the complex permeability tensor and the skeleton vanishes. The general solutions for the solid and liquid phases displacement potentials are then:

\[
\begin{align*}
\Phi_s^{(n)} &= \Phi_j^{(n)P_1} + \Phi_j^{(n)P_2} + \psi_j^{(n)SV} \Gamma_j^{(n)SV}, \\
\Gamma_s^{(n)} &= \psi_j^{(n)P_1} \Phi_j^{(n)P_1} + \psi_j^{(n)P_2} \Phi_j^{(n)P_2} + \psi_j^{(n)SV} \Gamma_j^{(n)SV}, \\
\Phi_f^{(n)} &= \psi_j^{(n)P_1} \Phi_j^{(n)P_1} + \psi_j^{(n)P_2} \Phi_j^{(n)P_2} + \psi_j^{(n)SV} \Gamma_j^{(n)SV}, \\
\Gamma_f^{(n)} &= \psi_j^{(n)P_1} \Phi_j^{(n)P_1} + \psi_j^{(n)P_2} \Phi_j^{(n)P_2} + \psi_j^{(n)SV} \Gamma_j^{(n)SV}, \\
\psi_s^{(n)} &= \chi_j^{SH} \psi_j^{(n)SH}, \\
\psi_f^{(n)} &= \zeta_f^{(n)SH},
\end{align*}
\]

where $\psi_j^{P_1}$, $\psi_j^{P_2}$, $\psi_j^{SV}$, $\psi_j^{P_1}$, $\psi_j^{SV}$, $\psi_j^{P_2}$, $\psi_j^{SV}$, $\psi_j^{P_2}$, $\psi_j^{SV}$, and $\chi_j^{SH}$ denote the complex frequency dependent factors which express the coupling between the body waves and/or the two phases of the saturated porous layer (see Part I). The $SH$-wave only excites the
horizontal permeability, indicated by the superscript $^H$. The fluid does not support any shear displacement, but it affects the (quasi) shear waves through inertial effects.

Analogous to the bore fluid (equ.(2)), the displacement potentials relative to the (quasi) body waves which propagate in a bounded $j$th transversely isotropic saturated porous layer with respect to a $n$th order multipole point source can be written as:

$$
\Phi_j^{(n)P1} = \left\{ \tilde{A}_{1j}^{(n)} \left(k_{P1j} r \right) + \tilde{B}_{1j}^{(n)} K_0 \left(k_{P1j} r \right) \right\} \frac{1}{n!} \left( \frac{k_{P1} R_0}{2} \right)^n e^{\iota \omega t} e^{i k z} \cos \left( n[\theta - \theta_0] \right),
$$

$$
\Phi_j^{(n)P2} = \left\{ \tilde{A}_{2j}^{(n)} \left(k_{P2j} r \right) + \tilde{B}_{2j}^{(n)} K_0 \left(k_{P2j} r \right) \right\} \frac{1}{n!} \left( \frac{k_{P1} R_0}{2} \right)^n e^{\iota \omega t} e^{i k z} \cos \left( n[\theta - \theta_0] \right),
$$

$$
\Gamma_j^{(n)SV} = \left\{ \tilde{E}_j^{(n)} \left(k_{SVj} r \right) + \tilde{E}_j^{(n)} K_0 \left(k_{SVj} r \right) \right\} \frac{1}{n!} \left( \frac{k_{P1} R_0}{2} \right)^n e^{\iota \omega t} e^{i k z} \cos \left( n[\theta - \theta_0] \right),
$$

$$
\Psi_j^{(n)SH} = \left\{ \tilde{C}_j^{(n)} \left(k_{SHj} r \right) + \tilde{D}_j^{(n)} K_0 \left(k_{SHj} r \right) \right\} \frac{1}{n!} \left( \frac{k_{P1} R_0}{2} \right)^n e^{\iota \omega t} e^{i k z} \sin \left( n[\theta - \theta_0] \right),
$$

where $k_{P1j}^2 = k^2 - \omega^2/c_{P1j}^2$, $k_{P2j}^2 = k^2 - \omega^2/c_{P2j}^2$, and $k_{SVj}^2 = k^2 - \omega^2/c_{SVj}^2$ are the radial wavenumbers of the quasi body waves, solutions of a third degree equation (see Part I), and $k_{SHj}^2 = k^2 - \omega^2/c_{SHj}^2$ is the radial wavenumber of the $SH$-wave. $\tilde{A}_{1j}^{(n)}$, $\tilde{A}_{2j}^{(n)}$, $\tilde{C}_j^{(n)}$, and $\tilde{E}_j^{(n)}$ are weighting coefficients, function of the axial wavenumber $k$ and the angular frequency $\omega$, which can be interpreted as the transmission coefficients of the outcoming (quasi) body waves, while $\tilde{B}_{1j}^{(n)}$, $\tilde{B}_{2j}^{(n)}$, $\tilde{D}_j^{(n)}$, and $\tilde{F}_j^{(n)}$ stand for the incoming (quasi) body waves.

The radiation conditions implying that the potentials have to remain finite when $r$ tends to infinity, we have for a semi-infinite transversely isotropic saturated porous layer of index $N$,

$$
\tilde{A}_{1N}^{(n)} = \tilde{A}_{2N}^{(n)} = \tilde{B}_{1N}^{(n)} = \tilde{B}_{2N}^{(n)} = \tilde{C}_N^{(n)} = \tilde{E}_N^{(n)} = 0.
$$

In terms of the potentials, the components of the displacement in the solid phase are given by:

$$
u_j^{(n)} = \frac{1}{r} \Phi_j^{(n)} + \frac{1}{r} \Psi_j^{(n)} + \Gamma_j^{(n)},
$$

$$
u_j^{(n)} = \frac{1}{r} \Phi_j^{(n)} - \frac{1}{r} \Psi_j^{(n)} + \Gamma_j^{(n)},
$$

$$
u_j^{(n)} = \Phi_j^{(n)} - \nabla^2 \Phi_j^{(n)} + \Gamma_j^{(n)}.
$$

The expressions of the components of the displacement in the liquid phase ($U_j^{(n)}$, $V_j^{(n)}$, and $W_j^{(n)}$) are identical but as functions of $\Phi_j^{(n)}$, $\Psi_j^{(n)}$, and $\Gamma_j^{(n)}$. 
The stresses in the solid phase are:

\[
\sigma_{\partial z}^{(n)} = (d_{11j} - d_{13j}) \left\{ -\Phi_{s_j}^{(n)} + \left( \nabla^2 \Gamma_{s_j}^{(n)} - \Gamma_{s_j}^{(n)} \right) \right\} + d_{11j} \nabla^2 \Phi_{s_j}^{(n)}
\]

\[
+ \frac{2}{r} d_{6\partial z} \left\{ \Phi_{s_j}^{(n)} + \frac{1}{r} \Phi_{s_j}^{(n)} \right\} + \Gamma_{s_j}^{(n)} + \frac{1}{r} \Gamma_{s_j}^{(n)} + \frac{1}{r} \Phi_{s_j}^{(n)} + \frac{1}{r} \Psi_{s_j}^{(n)} + \frac{1}{r} \Psi_{s_j}^{(n)} \right\} + Q_{1ij} \nabla^2 \Phi_{s_j}^{(n)}
\]

\[
\sigma_{\partial z}^{(n)} = (d_{11j} - d_{13j}) \left\{ -\Phi_{s_j}^{(n)} + \left( \nabla^2 \Gamma_{s_j}^{(n)} - \Gamma_{s_j}^{(n)} \right) \right\} + d_{11j} \nabla^2 \Phi_{s_j}^{(n)}
\]

\[
+ 2 d_{6\partial z} \left\{ \Phi_{s_j}^{(n)} + \frac{1}{r} \Psi_{s_j}^{(n)} \right\} + \Gamma_{s_j}^{(n)} + \frac{1}{r} \Phi_{s_j}^{(n)} + \frac{1}{r} \Psi_{s_j}^{(n)} + \frac{1}{r} \Psi_{s_j}^{(n)} + \frac{1}{r} \Psi_{s_j}^{(n)} \right\} + Q_{1ij} \nabla^2 \Phi_{s_j}^{(n)}
\]

\[
\sigma_{\partial z}^{(n)} = (d_{11j} - d_{13j}) \left\{ -\Phi_{s_j}^{(n)} + \left( \nabla^2 \Gamma_{s_j}^{(n)} - \Gamma_{s_j}^{(n)} \right) \right\} + d_{11j} \nabla^2 \Phi_{s_j}^{(n)}
\]

\[
+ Q_{3ij} \nabla^2 \Phi_{s_j}^{(n)}
\]

\[
\sigma_{\partial z}^{(n)} = (d_{11j} - d_{13j}) \left\{ -\Phi_{s_j}^{(n)} + \left( \nabla^2 \Gamma_{s_j}^{(n)} - \Gamma_{s_j}^{(n)} \right) \right\} + d_{11j} \nabla^2 \Phi_{s_j}^{(n)}
\]

\[
+ Q_{5ij} \nabla^2 \Phi_{s_j}^{(n)} + Q_{7ij} \nabla^2 \Phi_{s_j}^{(n)}
\]

\[
\sigma_{\partial z}^{(n)} = (d_{11j} - d_{13j}) \left\{ -\Phi_{s_j}^{(n)} + \left( \nabla^2 \Gamma_{s_j}^{(n)} - \Gamma_{s_j}^{(n)} \right) \right\} + d_{11j} \nabla^2 \Phi_{s_j}^{(n)}
\]

\[
+ Q_{9ij} \nabla^2 \Phi_{s_j}^{(n)} + Q_{11ij} \nabla^2 \Phi_{s_j}^{(n)}
\]

where \(d_{klj}\) are the stiffness constants of the saturated porous layer, and \(Q_{kj}\) are elastic coefficients (see Part I).

For the liquid phase, the stress \(s_j^{(n)}\) which is related to the pore fluid pressure by \(s_j^{(n)} = -\Phi_j^{(n)} p_j^{(n)}\) is,

\[
s_j^{(n)} = \left( Q_{1ij} - Q_{3ij} \right) \left\{ -\Phi_{s_j}^{(n)} + \left( \nabla^2 \Gamma_{s_j}^{(n)} - \Gamma_{s_j}^{(n)} \right) \right\} + Q_{1ij} \nabla^2 \Phi_{s_j}^{(n)} + Q_{3ij} \nabla^2 \Phi_{s_j}^{(n)}
\]

where \(\kappa_j\) is an elastic coefficient of the saturated porous layer.

Transversely isotropic elastic layer

A transversely isotropic elastic layer is a zero porosity porous layer. Hence, only three body waves propagate in such a layer: a quasi P-wave, a quasi SV-wave and a SH wave. Similarly to the previous derivation, the general solutions for the displacement potentials can be expressed as:

\[
\Phi_j^{(n)} = \Phi_j^{(n)P} + \gamma_j^{SV} \Gamma_j^{(n)SV} \]

\[
\Gamma_j^{(n)} = \gamma_j^{FP} \Phi_j^{(n)P} + \Gamma_j^{(n)SV} \]

\[
\Psi_j^{(n)} = \Psi_j^{(n)SH} \]
where $Y_{f}^{SV}$ and $Y_{f}^{P}$ are complex frequency dependent factors which express the coupling between the quasi body waves.

The expressions of the displacement potentials relative to each body wave can be deduced from the equations (6), muting the subscript which refers to the compressional waves. The radial wavenumbers $k_{PF}^{2} = k^2 - \omega^2/c_{PF}^2$ and $k_{SF}^{2} = k^2 - \omega^2/c_{SF}^2$ are now solutions of a biquadratic equation (see Part I and Tongtaow, 1982).

In terms of the displacement potentials of the layer, the components of the displacement vector are given by equations (8), dropping the subscript $s$. Similarly, the stresses are given by equations (9), setting $d_{kj} = c_{kj}$ and $Q_{k} = 0$.

Isotropic saturated porous layer

When both the permeability tensor and the skeleton of the saturated porous layer are isotropic, only the coupling between the solid and liquid phases remains. There are then only three body waves: a compressional wave of the first kind, $P_1$, a compressional wave of the second kind, $P_2$, and a shear wave. The general solutions for the solid and liquid phases displacement potentials are then:

$$
\Phi^{(n)}_{s} = \Phi^{(n)P1}_{j} + \Phi^{(n)P2}_{j}, \\
\Gamma^{(n)}_{s} = \Gamma^{(n)S}_{j}, \\
\Phi^{(n)}_{f} = \xi_{1j} \Phi^{(n)P1}_{j} + \xi_{2j} \Phi^{(n)P2}_{j}, \\
\Gamma^{(n)}_{f} = \chi_{j} \Gamma^{(n)S}_{j}, \\
\Psi^{(n)}_{s} = \Psi^{(n)S}_{j}, \\
\Psi^{(n)}_{f} = \chi_{j} \Psi^{(n)S}_{j}
$$

(12)

where $\xi_{1j}$, $\xi_{2j}$, and $\chi_{j}$ are complex frequency dependent factors which characterize the only influence of one phase on the other one (Part I, Schmitt, 1986a, b).

The expressions of the displacement potentials relative to each body wave are given by equations (6). Both radial wavenumbers $k_{PF}^{2} = k^2 - \omega^2/c_{PF}^2$ and $k_{SF}^{2} = k^2 - \omega^2/c_{SF}^2$ are now solutions of a biquadratic equation (see Part I). There is only one shear radial wavenumber $k_{S}^{2} = k^2 - \omega^2/c_{S}^2$. In terms of the displacement potentials of the layer, the components of the displacement vectors in both phases are given by equations (8) (using the appropriate phase potentials) Finally, the stresses in the solid and in the liquid phase are given by equations (9) and (10), respectively, using the known elastic coefficients of an isotropic saturated porous formation, i.e., $d_{11j} = d_{33j} = A_{j} + 2N_{j}$, $d_{13j} = A_{j}$, $d_{44j} = d_{66j} = N_{j}$, $Q_{kj} = Q_{j}$, and $\tilde{R}_{j} = \tilde{R}_{j}$ (e.g., Part I).
Isotropic elastic layer

For such a well known case, there are only two uncoupled body waves. The needed equations can be deduced either by setting the porosity to zero (so that $Q_j = R_j = 0$) and replacing the elastic coefficients $A_j$, $N_j$ by the Lamé's coefficients $\lambda_j$, and $\mu_j$, respectively, in the just above situation; or by replacing the stiffness constants of a transversely isotropic elastic layer by the appropriate (combination of) Lamé's coefficient(s) ($C_{11j} = C_{33j} = \lambda_j + 2\mu_j = c_{13j} + 2c_{44j}$ ($= c_{66j}$)). In any case, the supplementary subscripts (1, 2, or $\varepsilon$) are implicitly mute.

Wave Propagation

The computation of any displacement or stress in the bore fluid requires the knowledge of the continuity equations at the borehole wall as well as those at the various interfaces located within the formation. In the following, the boundary conditions at several kinds of interface are first briefly reviewed. They allow the definition of suitable displacement-stress vectors needed to propagate the wavefield through the different layers using the Thomson-Haskell method as described in the second part.

Boundary conditions

The complexity of the situation that might be encountered, leads to the presence of several types of interface: porous-porous, elastic-elastic, fluid-elastic and, last, fluid-porous. The order just outlined is thereafter followed as the continuity equations can be easily deduced from the previous case for the next one. As already mentioned, the cylindrical interface is located at $r = R_j$ so that it lies between the $j$th and $(j + 1)$th layers.

Porous-porous interface: Few authors discuss the conditions at the interface between two porous media. Using the working rate of the forces acting on a saturated porous medium, Deresiewick and Skalak (1963) established sufficient conditions which ensure the unicity of the wavefield. They obtained a theorem similar to that of Neumann for the elastic media. Berryman and Thigpen (1985), considering partially saturated media, obtained similar equations using a formulation based on the variational principle. Bonnet (1985) and Schmitt (1985, 1986a) used mixtures theory, the porous media being considered as continu. Eight equations are obtained. The first three state the continuity of the displacement vectors of the skeletons, emphasizing the no slip condition (i.e., both skeletons are supposed to remain in welded contact). The fourth equation ensures the balance of the flux, The next three express the continuity
of the total radial, axial and azimuthal stresses. Finally, the continuity of the saturant fluid pressures is stated, which implicitly assumes that both media are fully connected. Symbolically, they can be written

\[
\begin{align*}
  u_j^{(n)}(R_j) &= u_j^{(n)}(R_j) + \varphi_j \left[ U_j^{(n)}(R_j) - u_j^{(n)}(R_j) \right] \\
  v_j^{(n)}(R_j) &= v_j^{(n)}(R_j) \\
  w_j^{(n)}(R_j) &= w_j^{(n)}(R_j) \\
  \sigma_{rj}^{(n)}(R_j) + s_j^{(n)}(R_j) &= \sigma_{rj+1}^{(n)}(R_j) + s_j^{(n)}(R_j) \\
  \sigma_{\theta j}^{(n)}(R_j) &= \sigma_{\theta j+1}^{(n)}(R_j) \\
  \sigma_{zj}^{(n)}(R_j) &= \sigma_{zj+1}^{(n)}(R_j) \\
  p_j^{(n)}(R_j) &= p_j^{(n)}(R_j)
\end{align*}
\]

This set of equations is valid whatever the nature of the saturant fluid on both sides, and whatever the degree of anisotropy of each media. A suitable displacement-stress vector is then,

\[
G_{ij}^{P(n)}(R_j) = T_{ij}^{P(n)}(R_j) X_{ij}^{P(n)};
\]

where \( G_{ij}^{P(n)} = (u_j^{(n)}, v_j^{(n)}, w_j^{(n)}, \varphi_j \left[ U_j^{(n)} - u_j^{(n)} \right], \sigma_{rj}^{(n)} + s_j^{(n)}, \sigma_{\theta j}^{(n)}, \sigma_{zj}^{(n)}, p_j^{(n)})^T \), and \( X_{ij}^{P(n)} = (\lambda_{ij}^{(n)}, \lambda_{2j}^{(n)}, B_{ij}^{(n)}, B_{2j}^{(n)}, C_j^{(n)}, D_j^{(n)}, E_j^{(n)}, F_j^{(n)})^T \), whatever the degree of anisotropy. \( T_{ij}^{P(n)} \) and \( T_{ij}^{P(n)} \) are 8x8 complex matrices whose elements are given in Appendix A and B, respectively, dropping the unnecessary \( e^{i\omega t} e^{i\kappa z} \frac{1}{n!} \left( \frac{kp_1 R_0}{2} \right)^n \) terms, and the angular dependence.

**Elastic-porous interface:** A solid elastic medium can be considered as a zero porosity porous medium. Only seven continuity equation are then necessary. They express the continuity of the solid displacement vectors, the total radial, axial and azimuthal stresses, and the fact that that there is no relative motion between the two phases of the saturated porous layer at the interface. Again, they are valid whatever the degree
of anisotropy of any of the layers. Assuming that the elastic layer is the \( j \)th one yields:

\[
\begin{align*}
    u_j^{(n)}(R_j) &= u_{j+1}^{(n)}(R_j), \\
    v_j^{(n)}(R_j) &= v_{j+1}^{(n)}(R_j), \\
    w_j^{(n)}(R_j) &= w_{j+1}^{(n)}(R_j), \\
    0 &= \tilde{\varphi}_{j+1}^{(n)} \left[ U_{j+1}^{(n)}(R_j) - u_{j+1}^{(n)}(R_j) \right], \\
    \sigma_{rr_j}^{(n)}(R_j) &= \sigma_{rr_{j+1}}^{(n)}(R_j) + s_j^{(n)}(R_j), \\
    \sigma_{rz_j}^{(n)}(R_j) &= \sigma_{rz_{j+1}}^{(n)}(R_j), \\
    \sigma_{r\theta_j}^{(n)}(R_j) &= \sigma_{r\theta_{j+1}}^{(n)}(R_j).
\end{align*}
\]  

(15)

**Elastic-elastic interface:** For such a well-known situation, six boundary conditions are used. They state the continuity of the displacement vectors, the radial, axial, and azimuthal stress, i.e.,

\[
\begin{align*}
    u_j^{(n)}(R_j) &= u_{j+1}^{(n)}(R_j), \\
    v_j^{(n)}(R_j) &= v_{j+1}^{(n)}(R_j), \\
    w_j^{(n)}(R_j) &= w_{j+1}^{(n)}(R_j), \\
    \sigma_{rr_j}^{(n)}(R_j) &= \sigma_{rr_{j+1}}^{(n)}(R_j), \\
    \sigma_{rz_j}^{(n)}(R_j) &= \sigma_{rz_{j+1}}^{(n)}(R_j), \\
    \sigma_{r\theta_j}^{(n)}(R_j) &= \sigma_{r\theta_{j+1}}^{(n)}(R_j).
\end{align*}
\]  

(16)

A suitable propagator matrix is then,

\[
\Gamma_j^{E(n)}(R_j) = \mathcal{T}_j^{E(n)}(R_j) X_j^{E(n)},
\]  

(17)

where \( \Gamma_j^{E(n)} = (u_j^{(n)}, v_j^{(n)}, w_j^{(n)}, \sigma_{rr_j}^{(n)}, \sigma_{rz_j}^{(n)}, \sigma_{r\theta_j}^{(n)}) \) and \( X_j^{E(n)} = (\tilde{A}_j^{(n)}, \tilde{B}_j^{(n)}, \tilde{C}_j^{(n)}, \tilde{D}_j^{(n)}, \tilde{E}_j^{(n)}, \tilde{F}_j^{(n)}) \), whatever the degree of anisotropy. \( \mathcal{T}_j^{E(n)} \) and \( X_j^{E(n)} \) are 6x6 complex matrices whose elements are given in Appendix C and D, respectively, dropping the unnecessary \( e^{iwt}e^{ikz} \frac{1}{n!} \left( \frac{kp_1R_0}{2} \right)^n \) terms, and the angular dependence.

**Fluid-elastic interface:** When the \( j \)th layer is a perfect fluid, only the continuity of the radial displacement and stress remain. The axial and azimuthal stresses vanish.
One then obtains:

\[
\begin{align*}
U_j^{(n)} (R_j) &= u_{j+1}^{(n)} (R_j), \\
\sigma^{(n)}_{rr_j} (R_j) &= \sigma_{rr_{j+1}}^{(n)} (R_j), \\
0 &= \sigma_{r\theta_{j+1}}^{(n)} (R_j), \\
0 &= \sigma_{\theta\theta_{j+1}}^{(n)} (R_j).
\end{align*}
\]  \tag{18}

**Permeable fluid-porous interface:** A fluid medium can be considered as a unit porosity porous medium. Taking the limit of equations (13) and assuming that the fluid layer is the \( j \)th one yields:

\[
\begin{align*}
U_j^{(n)} (R_j) &= u_{j+1}^{(n)} (R_j) + \tilde{\varphi}_{j+1} \left[ U_{j+1}^{(n)} (R_j) - u_{j+1}^{(n)} (R_j) \right], \\
\sigma^{(n)}_{rr_j} (R_j) &= \sigma_{rr_{j+1}}^{(n)} (R_j) + \sigma_{\theta\theta_{j+1}}^{(n)} (R_j), \\
0 &= \sigma_{r\theta_{j+1}}^{(n)} (R_j), \\
0 &= \sigma_{\theta\theta_{j+1}}^{(n)} (R_j), \\
p_j^{(n)} (R_j) &= p_{j+1}^{(n)} (R_j). \\
\end{align*}
\]  \tag{19}

The first equation now ensures the balance of the fluid volume. Such a set of continuity equations corresponds to the free fluid flow situation.

**Impermeable fluid-porous interface:** Writing the continuity equations relative to a fluid-elastic interface (18) and to an elastic-porous interface (15) and making the elastic layer thickness tends to zero, one obtains with a \( j \)th fluid layer:

\[
\begin{align*}
U_j^{(n)} (R_j) &= u_{j+1}^{(n)} (R_j), \\
\sigma^{(n)}_{rr_j} (R_j) &= \sigma_{rr_{j+1}}^{(n)} (R_j) + \sigma_{\theta\theta_{j+1}}^{(n)} (R_j), \\
0 &= \sigma_{r\theta_{j+1}}^{(n)} (R_j), \\
0 &= \sigma_{\theta\theta_{j+1}}^{(n)} (R_j), \\
p_j^{(n)} (R_j) &= U_{j+1}^{(n)} (R_j) - u_{j+1}^{(n)} (R_j).
\end{align*}
\]  \tag{20}

This set of equations corresponds to the sealed pores situation for which there is no relative motion between the two phases of the porous medium at the interface. Thus, the fluid pressures are no longer continuous. It *does not mimic* the effects related to the presence of an elastic layer whatever its properties are. It is only a limit case for a zero thickness elastic layer presence whose study allows to separate the effects upon the wavefields of the porous body wave attenuations from the fluid flow.
Thomson-Haskell Method

With the above defined displacement-stress vectors, it is easy to propagate the wavefield through the different layers. Let us consider \( M \) saturated porous layers ranging from \( j = m + 1 \) to \( j = M + m \). Each layer is bonded by the radii \( r = R_{j-1} \) and \( r = R_j \), the \( M \) th one being eventually radially semi-infinite. For \( j = m + 1 \) to \( j = M + m - 1 \), the displacement-stress vectors are

at \( r = R_{j-1} \)

\[
G_{j}^{P(n)}(R_{j-1}) = T_{j}^{P(n)}(R_{j-1}) X_{j}^{P(n)} ,
\]

(21.a)

and at \( r = R_j \)

\[
G_{j}^{P(n)}(R_j) = T_{j}^{P(n)}(R_j) X_{j}^{P(n)} ,
\]

(21.b)

where \( \cdot P(n) \) implies that the degree of anisotropy of each layer is free. As a result,

\[
G_{j}^{P(n)}(R_{j-1}) = T_{j}^{P(n)}(R_{j-1}) \left[ T_{j}^{P(n)}(R_j) \right]^{-1} G_{j}^{P(n)}(R_j) .
\]

(22)

The boundary conditions at a porous-porous interface (equ. (13)) state that:

\[
G_{j}^{P(n)}(R_j) = G_{j+1}^{P(n)}(R_j) .
\]

(23)

The combination of the last two equations yields

\[
G_{m+1}^{P(n)}(R_m) = \left\{ \prod_{j=m+1}^{M+m-1} T_{j}^{P(n)}(R_{j-1}) \left[ T_{j}^{P(n)}(R_j) \right]^{-1} \right\} G_{M+m}^{P(n)}(R_{M+m-1}) ,
\]

(24)

where \( \overline{H}^{P(n)} \) is a 8x8 complex matrix. The same operation can be repeated for a unit composed of only \((K - 1)\) elastic layers, ranging from \( j = k + 1 \) to \( j = K + k - 1 \) leading to

\[
G_{k+1}^{E(n)}(R_k) = \left\{ \prod_{j=k+1}^{K+k-2} T_{j}^{E(n)}(R_{j-1}) \left[ T_{j}^{E(n)}(R_j) \right]^{-1} \right\} G_{K+k-1}^{E(n)}(R_{K+k-2}) ,
\]

(25)

where \( \overline{H}^{E(n)} \) is a 6x6 complex matrix.

Any combination of the previous equations allows the computation of the dispersion and attenuation characteristics of any mode as well as of the synthetic microseismograms as it is shown in the following.
Period Equations

Five different configurations, leading to eight models, are thereafter analyzed. They correspond to the more usual situations (more or less well modeled ...) that might be encountered. We let the reader define other combinations. The dispersion relations are obtained by finding values of the (angular) frequency and the phase velocity (or axial wavenumber) for which the period equations hold true.

Well bonded saturated porous and/or elastic layers

The surrounding formation can be a single semi-infinite layer or radially layered. Such a model allows the analysis of the wavefields in the presence of a damaged or flushed zone, or when the borehole is well cased.

When the formation is composed of only \( M \) saturated porous layers, the borehole wall can be either permeable (Equ. (19)), allowing fluid flow effects, or impermeable (Equ. (20)). For both situations, the displacement-stress vector of the formation at the borehole wall (i.e., at \( r = R_1 \)) is given by equation (24) with \( m = 1 \). The period equations are then, with a permeable borehole wall,

\[
\mathbf{M}_1^{(n)} \mathbf{A}_1^{(n)} = \mathbf{P}_1^{(n)},
\]

and with an impermeable borehole wall,

\[
\mathbf{M}_2^{(n)} \mathbf{A}_2^{(n)} = \mathbf{P}_2^{(n)},
\]

where \( \mathbf{M}_1^{(n)} \) and \( \mathbf{M}_2^{(n)} \) are 5x5 complex matrices,

\[
\mathbf{A}_1^{(n)} = \left( \mathbf{A}_1^{(n)} , \mathbf{B}_1^{(n)} , \mathbf{D}_1^{(n)} , \mathbf{F}_1^{(n)} \right)^T, \quad \text{and} \quad \mathbf{P}_1^{(n)} = \mathbf{P}_2^{(n)} = \left( 0, 0, 0, 0 \right)^T.
\]

If the formation is composed of only \( (K - 1) \) elastic layers, the period equation is obtained by setting equation (18) at \( r = R_1 \), using equation (25) with \( k = 1 \) for the displacement-stress vector of the surrounding formation. One then obtains a matrix equation which can be written as:

\[
\mathbf{M}_3^{(n)} \mathbf{A}_3^{(n)} = \mathbf{P}_3^{(n)},
\]

where \( \mathbf{M}_3^{(n)} \) is a 4x4 complex matrix, \( \mathbf{A}_3^{(n)} = \left( \mathbf{A}_1^{(n)} , \mathbf{B}_K^{(n)} , \mathbf{D}_K^{(n)} , \mathbf{F}_K^{(n)} \right)^T, \) and \( \mathbf{P}_3^{(n)} = \left( 0, 0, 0, 0 \right)^T. \)

It is easy to combine these two situations. In the following, only investigated is the case for which the saturated porous layers make the outermost part of the formation.
Assume that \((K - 1)\) elastic layers (steel and cement or mud cake, for example) are located between the bore fluid and \(M\) saturated porous layers. The period equation is obtained by setting equations (18) at \(r = R_1\) using equation (25) with \(k = 1\) as previously, and equations (15) at \(r = R_K\) using equation (24) with \(m = K\). The period equation is then of the form:

\[
\mathbf{M}^{(n)}_{\Delta} \Delta^{(n)}_\mathbf{\Delta} = \mathbf{P}^{(n)}_\mathbf{\Delta},
\]

where \(\mathbf{M}^{(n)}_{\Delta}\) is a 11x11 complex matrix,

\[
\Delta^{(n)}_{\mathbf{\Delta}} = (A_{1}^{(n)}, A_{K}^{(n)}, \bar{B}_{K}^{(n)}, \bar{C}_{K}^{(n)}, \bar{D}_{K}^{(n)}, \bar{E}_{K}^{(n)}, \bar{F}_{K}^{(n)}, \bar{B}_{1K+M}^{(n)}, \bar{B}_{2K+M}^{(n)}, \bar{D}_{K+M}^{(n)}, \bar{F}_{K+M}^{(n)})^T
\]

and

\[
\mathbf{P}^{(n)}_{\mathbf{\Delta}} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T.
\]

Unbonded elastic and saturated porous layers

Casing may be not a successful operation. As a result, the cement layer may be not well bonded either to the steel casing or to the formation. Such situations are modeled through the introduction of a (micro annulus) external fluid layer (Tubman, 1984; Schmitt and Bouchon, 1985; Tubman et al., 1986; after Chang and Everhart, 1983). In order to not lose any generality, let us consider an external fluid layer of index \(L\) within \((K - 1)\) elastic layers, assuming that the outermost part of the formation is composed of \(M\) saturated porous layers.

Let us first consider the case where the external fluid layer is not in contact with a saturated porous layer (i.e., \(2 < L < K\)). The period equations is obtained by setting the boundary equations (18) successively at \(r = R_1, R_{L-1}\) and \(R_L\) respectively with

\[
G_2^{E(n)}(R_1) = \left\{ \prod_{j=2}^{L-2} T_{E_{(j-1)}}(R_{j-1}) \left[ T_{E_{(j)}}(R_{j}) \right]^{-1} \right\} T_{E_{(L-1)}}(R_{L-2}) X_{L-1}^{E(n)}, \quad (30)
\]

and

\[
G_{L+1}^{E(n)}(R_L) = \left\{ \prod_{j=L}^{K-1} T_{E_{(j-1)}}(R_{j-1}) \left[ T_{E_{(j)}}(R_{j}) \right]^{-1} \right\} T_{E_{(K-1)}}(R_{K-1}) X_{L}^{E(n)}. \quad (31)
\]

At \(r = R_K\), one sets the boundary equations (15) using equation (24) with \(m = K\). The final result is a matrix equation of the 19th order:

\[
\mathbf{M}^{(n)}_{\Delta} \Delta^{(n)}_\mathbf{\Delta} = \mathbf{P}^{(n)}_\mathbf{\Delta},
\]

where \(\mathbf{M}^{(n)}_{\Delta}\) is a 19x19 complex matrix,

\[
\Delta^{(n)}_{\mathbf{\Delta}} = (A_{1}^{(n)}, A_{L-1}^{(n)}, \bar{B}_{L-1}^{(n)}, \bar{C}_{L-1}^{(n)}, \bar{D}_{L-1}^{(n)}, \bar{E}_{L-1}^{(n)}, \bar{F}_{L-1}^{(n)}, \bar{A}_{L}^{(n)}, \bar{B}_{L}^{(n)}, \bar{A}_{K}^{(n)}, \bar{B}_{K}^{(n)}, \bar{C}_{K}^{(n)}, \bar{D}_{K}^{(n)}),
\]

and

\[
\mathbf{P}^{(n)}_{\mathbf{\Delta}} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T.
\]
Transversely Isotropic Saturated Porous Formations II

\[ \tilde{p}_K^{(n)}, \tilde{f}_K^{(n)}, \tilde{b}_K^{(n)}, \tilde{d}_K^{(n)}, \tilde{f}_K^{(n)} \] and
\[ \mathcal{P}_a^{(n)} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T. \]

If no saturated porous layer is present, the period equation is given by neglecting the boundary conditions at \( r = R_K \), so that the above matrix equation reduces to the 12th order, i.e.,
\[ \mathcal{M}_6^{(n)} \mathcal{N}_6^{(n)} = \mathcal{P}_6^{(n)}, \]
(33)

where \( \mathcal{M}_6^{(n)} \) is a 12x12 complex matrix,
\[ \mathcal{N}_6^{(n)} = (\tilde{A}_1^{(n)}, \tilde{A}_L^{(n)}, \tilde{B}_L^{(n)}, \tilde{D}_L^{(n)}, \tilde{E}_L^{(n)}, \tilde{F}_L^{(n)}, \tilde{A}_L, \tilde{B}_L, \tilde{D}_K, \tilde{F}_K)^T \]
and \( \mathcal{P}_6^{(n)} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T. \)

When the \( M \) saturated porous layers are present, the external fluid layer can be in contact with the inner one (i.e., for \( L = K \)). With a permeable interface located at \( r = R_K = R_L \), the period equation is given by equation (30) and by setting the boundary conditions (19) at \( r = R_L \), using equation (24) with \( m = L \). One then obtains:
\[ \mathcal{M}_7^{(n)} \mathcal{N}_7^{(n)} = \mathcal{P}_7^{(n)}, \]
(34)

where \( \mathcal{M}_7^{(n)} \) is a 13x13 complex matrix,
\[ \mathcal{N}_7^{(n)} = (\tilde{A}_1^{(n)}, \tilde{A}_L^{(n)}, \tilde{B}_L^{(n)}, \tilde{D}_L^{(n)}, \tilde{E}_L^{(n)}, \tilde{F}_L^{(n)}, \tilde{A}_L, \tilde{B}_L, \tilde{B}_{1L+M}, \tilde{D}_{1L+M}, \tilde{F}_{1L+M})^T \]
and \( \mathcal{P}_7^{(n)} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T. \)

When the fluid-porous interface is sealed, the continuity equations at \( r = R_L \) have to be replaced by the equations (20), leading to the new period equation
\[ \mathcal{M}_8^{(n)} \mathcal{N}_8^{(n)} = \mathcal{P}_8^{(n)}, \]
(35)

where \( \mathcal{M}_8^{(n)} \) is a 13x13 complex matrix, \( \mathcal{N}_8^{(n)} = \mathcal{N}_7^{(n)} \) and \( \mathcal{P}_8^{(n)} = \mathcal{P}_7^{(n)}. \)

Synthetic Microseismograms

The calculation of full wave synthetic microseismograms in the borehole with respect to a \( n \)th order multipole point source requires the determination of the weighting coefficient \( \tilde{A}_1^{(n)} \). This is achieved by solving the systems deduced from the period equations with non-zero second members \( \mathcal{P}_7^{(n)}. \) These last are obtained through the introduction of the radial displacement and stress \( (U_s^{(n)} \) and \( \sigma_s^{(n)} \) radiated by the source.
Schmitt

at the borehole wall (evaluated from the displacement potential given by equation (1) using equations (3)). For the examples previously described, this yields:

\[ P_1^{(n)} = (-U_z^{(n)}, -\sigma_z^{(n)}, 0, 0, \sigma_z^{(n)} e^{-\omega t})^T, \]  
\[ P_2^{(n)} = (-U_z^{(n)}, -\sigma_z^{(n)}, 0, 0, 0)^T, \]  
\[ P_3^{(n)} = (-U_z^{(n)}, -\sigma_z^{(n)}, 0, 0)^T, \]  
\[ P_4^{(n)} = (-U_z^{(n)}, -\sigma_z^{(n)}, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T, \]  
\[ P_5^{(n)} = (-U_z^{(n)}, -\sigma_z^{(n)}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T, \]  
\[ P_6^{(n)} = (-U_z^{(n)}, -\sigma_z^{(n)}, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T, \]  
\[ P_7^{(n)} = P_8^{(n)} = (-U_z^{(n)}, -\sigma_z^{(n)}, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T. \]  

With a multipole point source, the analysis is limited to the study of the wavefields on the axis of the borehole. It is assumed that pressure sensors are used if the source is a monopole, that displacement sensors are used if the source is a dipole, and that gradients of the displacement are sensed for multipole source of higher order. The solution is then

with a monopole source,

\[ P_1^{(0)}(z, t) = \int_{-\infty}^{+\infty} S(\omega)D^{(0)}(\omega) e^{i\omega t} d\omega + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tilde{A}_1^{(0)}(\omega)S(\omega)e^{ikz e^{-\omega t}} d\omega d\omega, \]  

with a nth order multipole source:

\[ Re_1^{(n)}(z, t) = \int_{-\infty}^{+\infty} S(\omega)D^{(n)}(\omega) e^{i\omega t} d\omega + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{n!} \left( \frac{kp_1 R_0}{2} \right)^n \tilde{A}_1^{(n)}(\omega)S(\omega)e^{ikz e^{i\omega t}} d\omega d\omega, \]  

where \( S(\omega) \) denotes the source spectrum and \( D^{(n)}(\omega) \) is the source contribution, i.e.,

\[ D^{(n)}(\omega) = \frac{\pi \varepsilon R_0^n}{n! 2^{m+1}} \sum_{m=0}^{n} nC^n_m \frac{2^m}{2^{m+1}} (-i\omega)^m \left( \frac{2n-m}{\omega z} \right) e^{-\omega z} \frac{e^{-\omega z}}{cp_1}; \]  

\( C^n_m \) being the binomial factor.

Whatever the nature of the inner layer, the boundary conditions, and the order of the source, the reflected wavefield has a contribution from a branch point related to the bore fluid velocity. This fluid branch point contribution precisely cancels the direct arrival from the source.
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The full integration can be performed using the discrete wavenumber method (Bouchon and Aki, 1977). When a saturated porous layer is present, the (quasi) $P_2$ wave is associated with a low phase velocity and is inhomogeneous in most of the practical situations. It cannot be critically refracted at the borehole wall and thus cannot be recorded.

The full waveforms are composed of several wave types which interfere due to overlapping. “Mode by mode” analysis can also be performed. The body waves are associated with branch points. Their contribution to the wavetrains can be evaluated using a contour integral along a branch cut in the complex axial wavenumber plane (Peterson, 1974; Tsang and Rader, 1979; Kurkjian, 1985; Paillet and Cheng, 1986). The modes correspond to poles whose contributions can be evaluated using the residue theory (Kurkjian, 1985; Burns, 1986). However, rippling artefacts may be obtained for the modes that do have a low cut-off frequency or coalesce with a branch point.

Sensitivity Coefficients

Guided modes propagation is complex. In order to gain a better understanding of the effects of the isotropic elastic layers parameters, Cheng et al. (1982) and Burns (1986) used the variational principle to analytically derive the normalized partial derivative of phase velocity with respect to body wave velocity (i.e., the partition coefficient) for any guided mode generated by a monopole point source. For small values of temporal $Q^{-1}$, the attenuation of a guided wave can then be represented by the sum of the layer attenuation values weighted by their respective partition coefficients (Anderson and Archarumbeau, 1964; Aki and Richards, 1980). Burns and Cheng (1987) used such a definition to derive an attenuation inversion process, the partition coefficients being also a measure of the fractional strain energy of a guided mode.

In the present paper, we just define the frequency dependent sensitivity coefficients as:

$$S_x(c) = \frac{x}{c} \frac{\partial c}{\partial x}, \quad (45)$$

where $c$ is the guided mode phase velocity, and $x$ is a stiffness constant of either the skeleton of a transversely isotropic saturated porous layer or of a transversely isotropic elastic layer. For an isotropic layer, $x$ stands for $\mu_b$ or $\lambda_b + 2\mu_b$, $\mu_b$ and $\lambda_b$ being the Lamé coefficients. Their variations are then analogous to those of the partition coefficients, weighted by the density effects. The calculation is performed numerically although analytical expressions can be derived for a transversely isotropic elastic layer (see Ellefsen and Cheng, this issue).
NUMERICAL EXAMPLES

In this section, we analyze the wavefields generated by monopole and dipole sources in various configurations including transversely isotropic saturated porous virgin formations. The study focuses on the effects of the transversely isotropic permeability and the anisotropy of the skeleton on the interface and guided waves when the borehole wall is either impermeable or permeable. It is achieved through the computation of dispersion and attenuation curves of the modes in presence, sensitivity coefficients with respect to the stiffness constants of the skeleton(s), and full waveform synthetic microseismograms. Both cases of a fast and a slow formation are investigated for the simple hole model.

Any pole (i.e., the Stoneley wave, the flexural wave, the pseudo-Rayleigh modes and the trapped modes) corresponds at a given frequency to a complex wavenumber root \( k = \text{Re}(k) + i\text{Im}(k) \) of the period equations. Its phase velocity \( c \) is equal to \( \omega/\text{Re}(k) \) while its attenuation \( Q^{-1} \) is equal to \( 2\text{Im}(k)/\text{Re}(k) \). The so defined attenuation then corresponds to the spatial \( Q^{-1} \) (Aki and Richards, 1980). In every dispersion figure, the phase and group velocities are normalized with respect to the bore fluid velocity which is water.

As already mentioned, the partial derivatives of the phase velocity with respect to the stiffness constants of a skeleton are numerically evaluated. Although not normalized, the obtained values are consistent from one case to another.

All synthetic microseismograms have been evaluated in the frequency range \([0, 21]\) kHz and have been computed at 256 points in time. The source waveform is a non-zero phase Ricker wavelet. On each figure, the scaling coefficient at the upper left of a wavetrain gives the relative value of the peak amplitude compared to the maximum of the whole series denoted by 1.00. The indicated arrival times have been computed using ray theory. When impossible, the indicated times correspond to the offset divided by the wave velocity. Only (large) iso-offset comparisons are displayed. It is of course understood that the larger the number of receivers and the offset range, the better.

Table 1 gives the physical parameters of the saturated porous formations used in this study. Table 2 gives the characteristics of the (quasi) body waves in the low frequency range for horizontal and vertical propagation. They correspond to those of equivalent elastic formations whose density are \( \rho_j = (1-\varphi)\rho_{sj} + \varphi\rho_f \). The parameters of the simple isotropic elastic layers are given in Table 3. The pore shape considered in each principal direction is the unidirectional cylindrical duct model (see Part I), so that the critical frequency along the \( i \)th one is given by:

\[
f_{ci} = \frac{3\eta_i \varphi}{8\pi \kappa_i \tau_{ij} \rho_f}.
\]
Unless specified otherwise, both horizontal and vertical tortuosities are equal to 1 and the saturant fluid is water. Although the saturant fluid mobility (i.e., the permeability to the saturant fluid viscosity) is the key factor, we will refer to the only permeability. No anelastic attenuation is taken into account. Whatever the example, the borehole radius is 10 cm while the multipole separation (i.e., $R_0$) is equal to 1 cm.

We recall that the SH-wave is affected by the only horizontal permeability, whatever its angle of propagation. When only the permeability tensor is transversely isotropic, the quasi $P_1$-wave and quasi $SV$-wave behaviors in the low and high frequency range of the saturated porous formation can be approximated by considering an effective permeability defined as a function of the angle of propagation with respect to the vertical ($\psi$) (see Part I):

$$\tilde{k}(P_1, \psi) = \tilde{k}_v \cos^2(\psi) + \tilde{k}_h \sin^2(\psi),$$  

and

$$\tilde{k}(SV, \psi) = \tilde{k}_v \sin^2(\psi) + \tilde{k}_h \cos^2(\psi).$$

On the other hand, whatever the frequency, the quasi $P_1$-wave, quasi $SV$-wave and SH-wave velocities are primarily governed by the anisotropy of the skeleton.

**Simple Open Hole Model**

**Fast formation. Monopole source**

In the presence of a fast formation, a usual monopole (axisymmetric) source excites two types of guided waves: the Stoneley wave and the pseudo-Rayleigh wave. The last is composed of an infinite number of modes, each having a low cut-off frequency that increases in frequency with each higher mode. Only the first one is considered as it is the principal contributor in most of the actual recorded wavetrains.

The formation considered is a water saturated sandstone whose porosity is 15%. The anisotropy of the skeleton has been introduced by increasing the horizontal $P$-wave velocity by $\pm 5\%$ and the horizontal $S$-wave velocity by $\pm 12\%$ (see Table 1). The horizontal and vertical permeabilities are respectively equal to 1 darcy and 100 mdarcies so that the associated critical frequencies are equal to 18.142 kHz and 181.42 kHz, respectively.

When displayed, the iso-offset ($z = 5\text{m}$) comparison is performed between the waveforms computed in the presence of a saturated porous formation with (1) an isotropic skeleton and isotropic 100 mdarcies permeability; (2) an isotropic skeleton and isotropic 1 darcy permeability; (3) an isotropic skeleton and transversely isotropic
permeability (1 darcy in the horizontal plane, 100 mdarcies in the vertical direction); and (4) a skeleton and a permeability tensor both transversely isotropic.

Stoneley wave. Impermeable borehole wall.— In the presence of complete isotropic saturated porous formations, the phase and group velocities of the Stoneley wave are practically unaffected by any permeability variation when the pores are sealed at the borehole wall. Only its attenuation varies, correlative with the formation shear wave attenuation. A simple transversely isotropic permeability mainly leads to an anisotropy of the quasi body wave attenuations (see Part I). It thus only affects the attenuation of the interface wave, analogously to an isotropic permeability variation from which it cannot be differentiated. Figure 1 illustrates this behavior. The calculations have been performed with isotropic permeabilities equal to 1 darcy and 100 mdarcies, and when these last refer either to the horizontal plane and the vertical direction. The high frequency attenuations associated with the transversely isotropic permeabilities vary as the quasi $SV$-wave attenuation, defined for a small angle of propagation with respect to the vertical.

In the presence of the completely anisotropic saturated porous formation, compared to the complete isotropic case with a 1 darcy permeability, Stoneley wave reverse dispersion is shifted toward higher values, with a difference that decreases with increasing frequency (Figure 2). Such a behavior is analogous to that in the presence of simple tranversely isotropic elastic formations (White and Tongtaow, 1981; Tongtaow, 1982) and characterizes the low frequency coupling of the interface wave with the horizontally propagating $SH$-wave. The zero frequency limit of both phase and group velocities is:

$$\lim_{\omega \to 0} c_{ST}(\text{Imp}) = c_{p1} (1 + \rho_1 c_{p1}^2 / d_{662})^{-1/2} \left(1 + \rho_1 c_{p1}^2 / c_{662}\right)^{-1/2},$$

which is independent from any permeability as well as of the saturant fluid, similarly to the isotropic case.

The attenuation is representative of the quasi $SV$-wave "effective" permeability, slightly less than that with an isotropic skeleton (Figure 1). It is also much less than the corresponding shear wave attenuation.

The variations as a function of frequency of the sensitivity coefficients allow a quantification of the previous observations (Figure 3). Over 80% of the sensitivity refers to the bore fluid in the entire frequency range, emphasizing a poor coupling of the interface wave with such a fast formation. The shear motion in the formation only corresponds to $\sim 15\%$, maximum in the quasi-static range where it is associated with the only $c_{66}$ (i.e, the horizontally propagating $SH$-wave). With increasing frequency, the sensitivity with respect to $c_{66}$ decreases, contrary to that with respect to $c_{44}$ which
thus dominates above \( \simeq 8 \text{ kHz} \). Starting at zero in the quasi-static range, the amount of compressional motion in the formation gradually increases with frequency, but never exceeds 5%, distributed among \( c_{11}, c_{33} \) and \( c_{13} \) by decreasing order of magnitude. Similar results are obtained with a transversely isotropic elastic formation. Because of the boundary condition which imposes no relative motion between the two phases of the saturated porous formation, the sensitivity coefficient with respect to the saturant fluid properties is practically zero over the entire frequency range. In accordance with the increase in the horizontal stiffnesses, the sensitivity with respect to the bore fluid is increased compared to the isotropic situation (Figure 4): more of the interface wave energy travels into the bore fluid. At the same time, the overall sensitivities with respect to the shear and compressional stiffnesses are decreased.

Figure 5 shows the iso-offset comparison \( (z = 5 \text{m}) \) of the synthetic microseismograms computed with a 1 kHz source center frequency (the arrows indicate the arrival time as computed from equ. (49)). The variations of the amplitude maxima are representative of the (quasi) body wave attenuations, slightly weighted by the stiffnesses variations. They also indicate an increase of the Stoneley wave low frequency excitation. However, because of the poor coupling with the formation, such small amplitude (and attenuations) variations will be superseded by any anelastic attenuation in the bore fluid and/or in the skeleton. Hence, only the anisotropy of the skeleton may be a reliable information in such a situation.

**Stoneley wave. Permeable borehole wall.**— When the pores are open, a fluid flow takes place at the interface, resulting from the pressure continuity between the bore and saturant fluids, and the balance of the fluid volume (equ. (19)). In the presence of completely isotropic formations, most of the effects are known to be concentrated in the low frequencies where the interface wave energy is maximum and Darcy’s law is the dominant phenomenon: the formation behavior is governed by a homogeneous diffusion equation in pore pressure (Chandler, 1981; Chandler and Johnson, 1981). In such a low frequency range (i.e., up to \( \simeq 5 \text{ kHz} \)), the flow results in a decrease of the phase (and group) velocity of the Stoneley wave, and in an important increase of its attenuation. Correlatively, its energy decreases and its useful part is relegated in a narrower low frequency band (Schmitt, 1986b). For given isotropic formation and saturant fluid, the effects increase with the permeability. These effects also increase with increasing porosity, all other parameters being kept constant. This is related to an increase of the fluid volume, so that more of the energy is transmitted to the saturant fluid in the low frequency range.

The analysis of the sensitivity coefficients variations bring insights in the physics of Stoneley wave propagation with a permeable borehole wall. In the quasi-static range (Figure 6, up to 1 kHz), the sensitivity with respect to \( c_{66} \) tends to zero, contrary to that with respect to the saturant fluid, \( c_{11} \), and, in a lesser extent, \( c_{13} \) and \( c_{33} \), which
increases (see Figure 3). At the same time, the sensitivity with respect to the bore fluid slightly decreases (both in absolute and relative values). Stoneley wave is then representative of a radial compressional motion. Such a behavior is associated with a diffusion process.

Considering only radial (i.e., horizontal) propagation, the quasi-static limit of Stoneley wave phase velocity can be easily evaluated. In such a direction, the zero frequency limit of the diffusive compressional wave of the second kind is (see Part I):

\[ \lim_{\omega \to 0} c_{PH}^2 = 2\omega \frac{\frac{k^H}{\eta \rho}}{d_{11} + \frac{\mathcal{R}}{Q_1} + 2Q_1} = 2\omega C_H^H, \]

where the subscript 2 referring to the formation has been omitted for clarity. \( C_H^H \) is a diffusivity coefficient, less than the diffusivity within a rigid body \( C_H^r = \frac{k^H}{\eta \rho}, \rho, c_p^2 \).

Following Chandler (1981) and Chandler and Johnson (1981), the homogeneous diffusion equation which governs the porous formation in the quasi-static range (i.e., neglecting the inertial terms) may then be written in the radial direction:

\[ C_H^H \nabla^2 p = \frac{\partial p}{\partial t}, \]

where \( \nabla^2 \) is the Laplacian in the cylindrical coordinates system and \( p \) is the pore fluid pressure. Then, following the developments of White (1983), as suggested by Norris (1986) in the isotropic case, the quasi-static limit of the Stoneley wave phase velocity is:

\[ \lim_{\omega \to 0} c_{ST(Per)} = \frac{1}{c_{66}} \left[ 1 + \frac{\rho_1 c_p^2}{c_{66}} + \frac{2\rho_1 c_p^2 k^H}{i\omega R_1} \frac{K_1}{\eta_2} \sqrt{\frac{i\omega}{C_{D2}}} \frac{R_1}{K_0} \left( \sqrt{\frac{i\omega}{C_H^r}} \frac{R_1}{K_0} \right) \right]^{-1/2}, \]

where \( K_n \) denote nth order modified Bessel functions of the second kind. The supplementary term compared to the limit when the pores are sealed (equ. (49)), is of the form \( \frac{2\rho_1 c_p^2}{i\omega R_1} \frac{1}{Z} \), where \( Z \) is the borehole wall impedance.

Similarly to the complete isotropic case (Norris, 1986), this limit is exact only up to \( \simeq 1 \) kHz, i.e., as long as the inertial forces do not play any role. In a higher frequency range (Figure 7), the overall variations and distribution of the sensitivity coefficients are very similar to those obtained when the pores are sealed. Little differences can be noticed, mostly in absolute value (see Figures 26a-h for more details). Also, the crossover between the sensitivities with respect to \( c_{66} \) and \( c_{44} \) occurs at a lower frequency.
The quasi-static limit of Stoneley wave phase velocity only involves the horizontal permeability which also controls the attenuation. This is illustrated by the calculations performed up to 5 kHz with an isotropic skeleton (Figure 8). Both velocities and attenuations relative to transversely isotropic permeabilities are little differentiated (only over 2 kHz) from those obtained with an isotropic permeability equal to the horizontal one. The frequency range of validity of the only horizontal permeability effects increases with decreasing horizontal permeability, i.e., with increasing horizontal critical frequency (equ. (48)), so that the horizontal inertial forces are negligible over a wider frequency range. Such effects hold in the same way for any saturant fluid, the pore fluid mobility (i.e., $\frac{k^H}{\mu}$) being the key factor. In addition, the more compressible the saturant fluid, in absolute value and compared to the bore fluid, the greater the effects of the driving phenomenon over a wider frequency range (Schmitt, 1986b). Such an effect can be characterized by the diffusivities ratio $\frac{C^B}{C^H}$ which qualitatively expresses the amount of energy transmitted to the saturant fluid in the very low frequency range (the greater, the larger the relative motion between the two phases). Higher in frequency (Figure 9), both velocities and attenuations vary as a function of the “effective” permeability defined for the quasi $SV$-wave for a small angle of propagation. With high horizontal permeability, the phase velocity may even be greater than when the pores are sealed.

Figure 10 shows the dispersion and attenuation of the interface wave in the presence of the complete transversely isotropic formation compared to the isotropic case with a 1 darcy permeability. The differences between the phase velocities are representative of the sensitivity with respect to $c_{66}$ while the low frequency attenuation is representative of the horizontal permeability. However, the relative decrease of the phase velocity is slightly more important when the formation is transversely isotropic (see Figure 2). This is in accordance with a stiffer skeleton in the radial direction: the Stoneley wave is less coupled to the formation (Figure 11) so that it is more affected by any fluid motion. Correlatively, its attenuation is then little enhanced.

In the time domain (Figure 12a) where the decrease of the interface wave energy is indicated by that of the maximum of the whole series compared to the sealed pores situation (Figure 5), the relative variations of amplitude also emphasize the predominant role of the horizontal permeability in the low frequency range (the source center frequency is 1 kHz). Any permeability estimation based on the Stoneley wave characteristics (e.g., Burns et al., 1987) will then refer to this parameter. Although weak in the present example, the effects of the transverse anisotropy of the skeleton are still detectable, even with a slightly higher source center frequency (3 kHz, Figure 12b). The reliability of the horizontal permeability (relative) determination then needs the knowledge of the transverse isotropy of the skeleton, especially if it is based on the phase velocity variations. The use of the attenuation variations requires information about any (anisotropic) anelastic attenuation.
Pseudo-Rayleigh mode.— The pseudo-Rayleigh modes can be identified as a hybridization of surface waves and multiple reflected waves (associated with proper resonances of the borehole), the cut-off frequency corresponding to the case of critical refraction (Paillet and White, 1982; Schmitt and Bouchon, 1985). The shear wave involved at the cut-off frequency is thus the vertically propagating, horizontally polarized $SV$-wave ($SV_H$), which only excites the horizontal permeability (see Part I).

Figure 13 shows the dispersion and attenuation of the first pseudo-Rayleigh mode compared to those obtained with a complete isotropic 1 darcy formation when the borehole wall is impermeable (a) and permeable (b). The increase in the horizontal $P$-wave and $S$-wave of the skeleton leads to a shift toward higher frequencies of the low cut-off frequency and to a decrease of the Airy phase (similarly to the isotropic case, the borehole wall is stiffer). The overall dispersion is little affected and practically independent from the boundary conditions. The phase (and group) velocities start at the same formation shear velocity and tend toward that of the bore fluid at higher frequencies. The attenuation at the cut-off frequency is slightly greater than for the isotropic case, following the increase with frequency of the $SV_H$-wave in the low frequency range of the formation behavior. Such a distribution is a function of the ratio of the critical frequency of the formation in the horizontal direction to the cut-off frequency of the mode. Contrary to the velocity, the attenuation is sensitive to the borehole wall conditions. When the pores are open, the attenuation maxima are greater and located at the Airy phase of the mode which corresponds to the maximum of excitation. This high frequency behavior emphasizes that the bore fluid penetrates into the formation, leading to a greater attenuation of the multiple reflected waves. Similar to the Stoneley wave, the maximum of attenuation, greater in the presence of the transversely isotropic skeleton which leads to a stiffer borehole wall, is representative of the horizontal permeability. As a function of this last parameter, the distribution of the attenuation maxima may then depends on the ratio of the horizontal critical frequency of the formation to the low cut-off frequency of the mode with a saturant fluid such as water. Any anelastic attenuation will result in a linear scale effect of the described attenuation phenomena.

The sensitivity with respect to the stiffness constants are displayed on Figure 14a. In accordance with the dispersion curves, the sensitivity close to the cut-off frequency is relative to shear motion associated with $c_{44}$, while it refers to the bore fluid at high frequencies. The sensitivities with respect to the other stiffnesses little vary and never exceeds few percents. The boundary conditions has no significant effects except on the very small saturant fluid sensitivity (Figure 14b) following the attenuation variations. A more compressible and/or less viscous saturant fluid will lead to greater effects (and values). Figure 15 shows the results obtained with the complete isotropic 1 darcy formation.

The attenuation and coupling phenomena described in the frequency domain are
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easily recognizable in the time domain with a 7.5 kHz source center frequency (Figures 16a, b). Such calculations now exhibit the $P$ wavetrain. The critically refracted $P$-wave travels vertically along the borehole wall. It is then the only wave which is affected by the only vertical permeability (see Part I). However, such an interesting information will be superseded by any anelastic attenuation. The $P$-wave leaky modes have similar behavior than the pseudo-Rayleigh modes, as their phase velocity starts at the vertically propagating $P$-wave velocity of the formation and tends toward that of the bore fluid (see Paillet and Cheng, 1986).

Despite the fact little energy of the mode propagates close to the cut-off frequency, the most reliable informations are the $SVH$-wave velocity and attenuation brought by the low frequency (high velocity) part, regardless of the boundary conditions. Successful determinations have been obtained in the presence of isotropic elastic formations (Burns, 1986). Assuming that a reliable estimation of the horizontally propagating $SH$ wave can be obtained from the low frequency part of the Stoneley wave, the transverse isotropy of the skeleton can be detected when the borehole wall is impermeable. When the borehole wall is permeable, only the attenuation of the shear wave, equal to that of the horizontally propagating $SH$ wave, is a truly useful information.

**Mass coupling coefficient effects.**— Increasing the tortuosity in one of the principal direction, the associated mass coupling coefficient ($p_{22}$) increases and the critical frequency ($f_q$) decreases by the same amount. All other parameters being held constant, all three body waves velocities and attenuations also decrease in the high frequency range because the saturated porous medium is less biphase. Their characteristics remain unchanged in the low frequency range because of Darcy's law. In the presence of the transversely isotropic formation, we hereafter consider tortuosity values equal to 1, 2, and 3 in both directions. Effects of the transverse isotropy of the tortuosity tensor is also investigated with a horizontal value equal to unity and a vertical value equal to 2 and 3.

Whatever the tortuosity distribution, the static and low frequency values of the viscous coupling coefficient ($h(\omega)$) remain unchanged. Thus, when the borehole wall is permeable, the tortuosity has few practical effects on Stoneley wave characteristics in the low frequency range of interest (Figure 17a). It is only higher in frequency (Figure 17b, up to 25 kHz) that both the attenuation and the velocity are representative of the less biphase character of the formation in the horizontal direction. These calculations also emphasize the very little effects of the vertical properties of the complex permeability tensor. Similar results are obtained with an impermeable borehole wall. The pseudo-Rayleigh mode is more affected because of its higher frequency excitation (Figure 18, the pores are open). With increasing tortuosity in the horizontal direction, the maximum of attenuation associated with the Airy phase is seen to decrease both in absolute value and compared to the attenuation at the low cut-off frequency.
which follows the attenuation of the vertically propagating $SV$-wave. Such a behavior corresponds, in the horizontal direction, to equal viscous forces which compete with greater friction forces occurring lower in frequency. Also involved is the decrease of the ratio between the horizontal critical frequency of the formation and the low cut-off frequency of the mode. Again, the vertical tortuosity does not play any significant role, emphasizing the coupling with the quasi-vertically propagating $SV$-wave. In the time domain, for given formation and source center frequency, the pseudo-Rayleigh wave-train will be more developed compared to the Stoneley wave pulse with increasing horizontal tortuosity.

Radius effects.—— The borehole radius is known to play an important role upon the guided wave propagation in the presence of an isotropic formation, either elastic (Schoenberg et al., 1981; Cheng and Toksöz, 1981, for example) or saturated porous (Schmitt, 1986b). In the following we analyzed the results obtained with a 6 cm borehole radius in the presence of the transversely isotropic formation.

With decreasing borehole radius, the adimensional frequency defined by $f_a = f R_t / c p_t$ remains constant if the frequency is increased by the same amount. Hence, when the borehole wall is impermeable, the reverse dispersion of the Stoneley wave decreases, the zero frequency limit of both the phase and group velocities remaining constant (Figure 19a). The sensitivity coefficients (Figure 19b) are affected by the same "geometric" phenomenon. Correlatively to the borehole radius decrease, both the excitation and the energy of the Stoneley wave increase (Tubman et al., 1984; Schmitt and Bouchon, 1985). Thus, when the borehole wall is permeable, the phase velocity more decreases and the attenuation more increases in the low frequencies (Figures 20a, b, c). Also involved is the increase curvature of the borehole wall. Compared to the isotropic situations, such effects, which decrease with increasing frequency, are reinforced by the horizontal stiffnesses increase and enhance the predominant role of the horizontal permeability.

The behavior of the pseudo-Rayleigh mode is analogous (Figure 21). The low cut-off frequency is shifted toward higher frequencies with decreasing borehole radius. In the present example, the attenuation also increases, whatever the boundary conditions. Various situations may occur, depending on the relative values, compared to the horizontal critical frequency of the formation, of the new cut-off frequency of the mode and of the new frequency location of its Airy phase when the fluid flow is free.

Any variation of the borehole radius, even smooth and large wavelength, must then be taken into account in order to avoid any misinterpretation or misdetermination of the parameters previously studied (see also, Bouchon and Schmitt, 1988).
Fast formation. Dipole source

In the presence of a fast formation, dipole sources excite two types of waves, similar to the usual monopole (axisymmetric) source. The zero-th order mode, associated with a pure bending of the hole and referred to as the flexural mode, is the counterpart of the Stoneley wave. The higher trapped modes behave very closely like the pseudo-Rayleigh modes. In the following, we focus on the properties of the flexural mode as it is the main contributor to the wavetrains at typical frequencies ($\leq$3 kHz) excited by the shear wave acoustic logging tool (Zemanek et al., 1984).

Flexural mode.— Figure 22 shows the dispersion and attenuation of the flexural mode in the presence of the completely transversely isotropic formation compared to those obtained when the formation is totally isotropic with a 1 darcy permeability, computed with an impermeable (a) and permeable (b) borehole wall.

The phase velocity starts at the vertically propagating, horizontally polarized $SV$-wave (i.e., $SVH$) velocity of the formation and tends towards that of a Scholte wave (i.e., a Stoneley wave propagating at a plane fluid-formation interface). The mode has no theoretical cut-off frequency. The absence of low frequency continuity on the figure is due to a coalescence of the pole related to the mode with the $SVH$-wave branch point. The excitation curve (e.g., Schmitt et al., 1987) corroborates this observation as it sharply drops off on two sides of a peak associated with the Airy phase of the mode giving in practice a sharp low cut-off frequency. In the remainder of this paper, we will speak of useful starting energy. The high frequency limit is identical to that of the Stoneley wave. With such characteristics, it is understandable that the increase in the horizontal stiffnesses leads to a shift toward higher frequencies and higher values of the phase velocity compared to the isotropic situation. Whatever the boundary conditions, the dispersion is little affected.

In the low frequencies the attenuation follows that of the $SVH$ wave of the formation, related to the only horizontal permeability. Higher in frequency and when the pores are sealed, it is less for the transversely isotropic formation because of a weaker coupling of the mode with this last. When the pores are open, the attenuation maximum greatly increases (nearly by a factor of 8), associated with the Airy phase which corresponds to the maximum of excitation. It is slightly greater in the presence of the transversely isotropic formation owing to the stiffnesses variations, similar to the Stoneley wave. Of course, it is representative of the horizontal permeability.

The sensitivity coefficients (Figure 23a) emphasize the dominant low frequency dependence with regards to $c_{44}$ and the high frequency one with regards to the bore fluid which is more important than in the presence of the isotropic formation (Figure 23b). The other stiffnesses plays little role, mostly around the Airy phase in correlation
with the beginning of the interface character of the mode. The boundary conditions have little effects, except with regards to the saturant fluid sensitivity (see Figure 26b), which is small in absolute value.

With a 1 kHz source center frequency, the synthetic microseismograms are composed of a simple low energy $SVH$-wave pulse (Figures 24a, b). Its amplitude, which is independent from the boundary conditions, decreases with increasing shear wave attenuation and mostly with the shift toward higher frequencies of the useful starting energy of the flexural mode. Hence, the greater the degree of transverse isotropy, the lower the amplitude of the shear pulse. Reliable detection then requires high energy low frequency logging tools. As a function of the offset, the "geometrical spreading" is known to be proportional to $1/z$ provided that the shear wavelength is greater than 10 times the borehole radius and that the offset is at least on the order of the shear wavelength (e.g., Kurkjian, 1986). With a 3 kHz source center frequency (Figures 25a, b), the waveforms are dominated by the Airy phase of the mode. The variations of the amplitude maxima are then mainly representative of the $SV$-wave attenuation and the boundary conditions, characterizing the only horizontal permeability. Any anelastic attenuation in the skeleton and/or in the bore fluid will be superimposed.

The type and usefulness of the informations brought by the low frequency part of the flexural mode are similar to those brought by the pseudo-Rayleigh mode as analyzed above. In the case of an important frequency dependence of the horizontal shear attenuation, the determination based on the flexural mode measurements can be considered as more valuable.

The low frequency complementarity and high frequency similarity between the Stoneley wave and the flexural mode are emphasized in Figures 26a to 26g which display, for both boundary conditions, their sensitivity coefficients with respect to the bore and saturant fluid Lamé coefficients, to $c_{44}$, $c_{66}$, $c_{11}$, $c_{13}$, and $c_{33}$, successively. The transition between the pure flexural and interface behaviors of the flexural mode is located around its Airy phase. This allows to infer a negligible effect of the only transversely isotropic permeability. A decrease in the borehole radius will lead to shift toward higher frequencies of the useful starting energy while an increase of the horizontal mass coupling coefficient will decrease the attenuation effects related to the two phase character of the formation.

**Trapped mode.**— As shown in Figures 27 and 28 with a permeable borehole wall, the behavior of the trapped mode is analogous to that of the pseudo-Rayleigh mode (see Figures 13 to 15).
Slow formation

In the presence of a slow formation, only the fundamental modes are excited. The porosity of the formation considered is 20%. The transverse isotropy in the skeleton has been introduced by increasing the horizontal $P$-wave velocity by $\pm 7\%$ and the horizontal $S_H$-wave velocity by $\pm 15\%$ (see Tables 1 and 2). The horizontal and vertical permeabilities are respectively equal to 1 darcy and 100 mdarcies so that the associated critical frequencies are equal to 21.412 kHz and 214.12 kHz.

In the following, we analyze the behavior of the modes compared to those obtained in the presence of an isotropic formation whose permeability is 1 darcy. The effects of the only transversely isotropic permeability are similar to those described for the fast formation and need not be illustrated. The iso-offset ($z = 4\text{m}$) comparisons of the synthetic microseismograms refer to the same cases as for the fast formation.

Monopole source.— With such a type of formation, the dispersion of the Stoneley wave is direct with an impermeable borehole wall (Figure 29). Similar to the fast formation, the increase of the horizontal stiffnesses increases the phase and group velocities in the entire frequency range (the effect is even relatively more pronounced, taking into account the difference in the stiffness increase). The zero frequency limit, still given by equation (49), is now greater than the vertically propagating, horizontally polarized, $SV$-wave velocity. The interface wave is thus leaky in the very low frequency range, up to $\sim 1.8\text{ kHz}$. This is emphasized by a slightly greater attenuation below this frequency for the transversely isotropic formation, while the distribution is reverse above.

The sensitivity coefficient with respect to $c_{44}$ is now dominant in the high frequencies emphasizing a better coupling of the interface wave with the formation (Figure 30a). The frequency location of the crossing point with the sensitivity coefficient with respect to $c_{66}$ is also lower than in the presence of a fast formation (around 2 kHz, the extrapolation of the low frequency values is due to numerical instabilities related to the leaky character of the Stoneley wave), correlatively with a narrower low frequency excitation. At the same time, the bore fluid sensitivity coefficient, dominant only in the very low frequencies, decreases with increasing frequency, being less than those with respect to both shear stiffnesses (Figure 30b illustrates the behavior when the formation is isotropic). The high frequency attenuation of the Stoneley wave is then greater than those of the shear waves, the difference decreasing with increasing anisotropy degree of the surrounding medium, and contrary to the fast formation situation.

When the borehole wall is permeable, the dispersion is more complex (Figure 31). Because of the fluid flow, which refers to the horizontal permeability, it is reverse in the low frequencies. In relative values, the phase velocity decreases more in the presence
of the transversely isotropic formation (so that the interface wave is no more leaky) and its dispersion becomes direct again at a higher frequency, emphasizing a greater sensitivity toward the bore fluid motion. Such an enhancement is also characterized by a greater low frequency attenuation and by the frequency variations of the sensitivity coefficients (Figures 32a, c). Of course, the effects are much less pronounced and last for a smaller frequency range than in the presence of the fast formation (Figure 32b compared with Figure 6. See also Figure 8). The high frequency phase velocities are slightly higher and are associated with a lower attenuation than when the pores are sealed. Such an attenuation behavior is typical of a water saturation situation. With a more compressible and less viscous saturant fluid, the attenuation with open pores at the borehole wall will be greater in the entire frequency range (Schmitt et al., 1987).

Figures 33 and 34 illustrate in the time domain, with respective 1 kHz and 5 kHz source center frequencies, the strong sensitivity of the interface wave with respect to the horizontal stiffness. However, with only monopole measurements, no detection of the anisotropy degree of the formation can be performed.

**Dipole source.**—The dispersion and attenuation of the flexural mode, compared to those obtained in the presence of the 1 darcy isotropic formation are given in Figures 35a, b when the borehole wall is impermeable and permeable, respectively. The general trends are similar to those described in the presence of the fast formation.

The phase (and group) velocity starts at the vertically propagating, horizontally polarized $SV$-wave velocity and decrease with frequency to reach that of a Sholte wave (Figures 36a, b illustrate the high frequency convergence between the Stoneley wave and the flexural mode). In accordance with the stiffness variations, the useful starting energy is located higher in frequency when the formation is transversely isotropic, but at a lower frequency than when the formation is fast. This reinforces again the need of high energy in the low frequencies for the logging tools. The attenuation is also representative of the stiffness increase, being lower in the transversely isotropic situation. Close to the useful starting energy, it follows that of the $SVH$ wave. It then increases, being slightly more pronounced when the pores are open. Higher in frequency, it is greater with sealed pores at the borehole wall, similar to the Stoneley wave. The frequency location of the crossing point between both boundary conditions attenuations decreases in the isotropic situation because of the better coupling with the surrounding medium. It will also decrease with increasing horizontal permeability (e.g., Schmitt et al., 1987). The synthetic microseismograms exhibit no significant variations of amplitude as a function of the boundary conditions, whatever the source center frequency (Figure 37 (1 kHz), and Figure 38 (3 kHz)). The maxima of amplitude are lower in the transversely isotropic case in accordance with the shift toward higher frequencies of both the useful starting energy and the maximum of excitation, associated with the Airy phase. With a less viscous and/or more compressible saturant fluid.
fluid, the attenuation of the flexural mode as a function of the boundary conditions will be analogous to that observed with a fast formation (Schmitt et al., 1987), similar to the Stoneley wave.

The sensitivity coefficient with respect to $c_{44}$ has a dominant role over the entire frequency range (Figure 39a), similar to that with respect to the shear modulus of an isotropic formation (Figure 39b). The other stiffnesses have little effects, as well as the boundary conditions. Such a behavior leads to similar conclusions than for the fast formation. They are discussed in the next section.

General discussion

When the borehole is impermeable, similar to the elastic situation (e.g., Schmitt, 1987a; Rice, 1987), the transverse isotropy of the skeleton of a formation, fast or slow, may be detected and evaluated. Such a process needs two steps. The vertically propagating $SV$-wave velocity is measured from the low frequency (high velocity) part of the flexural mode generated by a dipole source (or that of any fundamental mode generated by multipole source of higher order). In the presence of a fast formation, the same information can be obtained from the low frequency (high velocity) part of the pseudo-Rayleigh mode generated by the monopole source. The horizontally propagating $SH$-wave velocity is indirectly determined from the very low frequency limit of the Stoneley wave phase velocity. Such calculations require useful low frequency energy logging tool, whatever its multipole order and knowledge about the bore fluid characteristics. Of course, the larger the number of receivers and the offset range, the better, so that elaborated signal processing techniques such as those based on Prony method (e.g., Lang et al., 1987; Ellefsen et al., 1987) can be performed.

The most difficult step is the inversion of Stoneley wave velocity whose reliability is strongly dependent on the adequacy of the model used. One has to keep in mind that the logging tool geometry may significantly affect the zero frequency limit of the Stoneley wave phase velocity (at least in the order of the sensitivity with respect to $c_{66}$ of the formation) and reinforces the role of the borehole radius (e.g., Cheng and Toksöz, 1981; Schmitt, 1985). As a reminder, when the tool is rigid, the equation (49) becomes:

$$\lim_{\omega \to \infty} c_{ST}(Imp) = c_p \left\{ \left( 1 - \frac{R_t^2}{R_t^2} \right)^{1/2} \left( 1 - \frac{R_t^2}{R_t^2} + \frac{\rho_1 c_p^2}{c_{66}} \right)^{-1/2} \right\},$$

(53)

where $R_t$ denotes the tool radius. When the tool is elastic, the zero frequency limit of
the Stoneley wave phase velocity is the lowest root the following biquadratic equation

\[ c_{ST}^4 \left[ \rho_t (c_p^2 - c_s^2) \left\{ \rho_1 c_p^2 + c_{662} \left( 1 - \frac{R_1^2}{R_2^2} \right) \right\} + \rho_1 \frac{R_2^2}{R_1^2} c_p^2 c_{662} \right] 
\]

\[ + c_{ST}^2 \left[ -c_p^2 c_{662} \left( 1 - \frac{R_2^2}{R_1^2} \right) \rho_t (c_p^2 - c_s^2) - \rho_1 c_p^2 c_p^2 c_{662} \frac{R_2^2}{R_1^2} \right] 
\]

\[ + \rho_t (4c_s^4 - 3c_p^2 c_s^2) \left\{ \rho_1 c_p^2 + c_{662} \left( 1 - \frac{R_2^2}{R_1^2} \right) \right\} \]

\[ - \rho_t (4c_s^4 - 3c_p^2 c_s^2) c_p^2 c_{662} \left( 1 - \frac{R_2^2}{R_1^2} \right) \], \hspace{1cm} (54)

where \( c_p, c_s, \) and \( \rho_t \) denote respectively the compressional and shear velocity and the density of the tool. The second root is close to the extensional velocity of the tool given by:

\[ c_E = \left\{ \frac{(3c_p^2 c_s^2 - 4c_s^4)}{(c_p^2 - c_s^2)} \right\}^{1/2} \]. \hspace{1cm} (55)

The Stoneley wave phase velocity evaluated with a steel tool is only slightly lower than that obtained with a simple rigid tool. Moreover, such a geometry may decrease or even cancel the low frequency leaky character of the Stoneley wave associated with a slow formation. With a rigid tool whose radius is 4 cm, the zero frequency limit is decreased down to 1171.7 m/s (compared to 1210.1 m/s with the point source model). With a rubber tool (i.e., \( c_p = 3350 \) m/s, \( c_s = 1980 \) m/s, and \( \rho_t = 1200 \) kg/m³) of the same radius, the phase velocity is equal to 1151 m/s, practically equal to the vertically propagating \( SV \)-wave velocity (1146.2 m/s).

The small attenuation effects induced by the transversely isotropic permeability cannot be differentiated from simple variations of those induced by an isotropic permeability and will be superseded by any anelastic (anisotropic) attenuation.

When the borehole wall is permeable, the transverse isotropy of the formation cannot be reliably evaluated. The increase in the horizontal stiffnesses increases the fluid flow effects on Stoneley wave phase velocity and attenuation which last over a wider frequency range and primarily refer to the only horizontal permeability. The only truly useful information brought by the low frequency part of the flexural mode is the attenuation of the vertically propagating quasi \( SV \) wave, equal to that of the horizontally propagating \( SH \) wave and also related to the only horizontal permeability. In the case of a strongly frequency dependent horizontal shear attenuation, such a data is more valuable than that brought either by the low frequency part of the pseudo-Rayleigh mode or of any fundamental mode generated by higher order multipole source which are located higher in frequency (see, for example, the mass coupling coefficients effects). Any more or less quantitative estimation of the horizontal permeability based on Stoneley wave velocity and attenuation variation may be much more questionable than in the simple isotropic case, especially when the anisotropy degree of the skeleton
is important. On the other hand, detection of permeability variation may still be reasonably performed. It is understood that any borehole radius changes has to be taken into account.

Invaded Zone

In the following, we investigate the presence of an invaded (damaged) zone when the formation is fast. Although it may be more complicated than a single layer, only the configuration including two coaxial transversely isotropic saturated porous shells is studied. Following equation (13), the saturant fluid pressures are assumed to be continuous at the invaded zone-virgin formation interface. The porosity of the supplementary layer and the vertical permeability are equal to those of the fast formation. The horizontal permeability is decreased down to 500 mdarcies. The saturant fluid is still water and both horizontal vertical toruosities are equal to unity. The horizontal and vertical critical frequencies are then respectively equal to 36.284 kHz and 181.42 kHz. With the chosen lower stiffnesses of the skeleton (see Table 1), the low frequency vertically propagating $P$ and $S$-wave velocities correspond to a decrease of 9% and 11% of those of the fast formation while the horizontally ones correspond to a decrease of 11% and 14%, respectively (see Table 2). In the presence of a damaged zone, the inner borehole radius remains constant (i.e., 10 cm). We consider two different thicknesses of the invaded layer: 3cm and 15 cm. Both boundary conditions are also studied. As a result, although little compressional energy is carried by the modes, we will display the sensitivity coefficient with respect to $C_{11}$. The dispersion and attenuations are displayed along with those associated with radially semi-infinite virgin formation and invaded zone. For the iso-offset comparisons ($z = 5m$), only the 3 cm thick invaded zone case is shown along with the two last mentioned.

Monopole source

Stoneley wave.— The dispersion and attenuation of the Stoneley wave when the borehole is impermeable are displayed in Figure 40. Figures 41 and 42 show the sensitivity coefficients for the 3 cm and 15 cm thick invaded zone, respectively. Because it is generated at the borehole wall, and propagates along, the interface wave integrates the properties of the inner layer even in the very low frequency range, so that the phase velocity decreases. The sensitivity coefficient with respect to the bore fluid also decreases, emphasizing a better coupling with the slower inner layer (see Figure 3). Such effects increase with increasing frequency and increasing thickness of the invaded zone. Correlatively, the decreasing influence of the virgin formation is relegated to the very low frequency range. As a result, most of the energy which does not propagates into the bore fluid refers to the horizontally propagating $SH$-wave (i.e., $c_{66}$) of the
inner layer in the low frequencies and to $c_{44}$ of the same shell higher in frequency. One can notice that the overall variations of the sensitivity coefficients with respect to these last two parameters tends to be similar to those in the presence of a slow formation (see Figure 30). For given thickness and frequency, the described effects will also increase with decreasing stiffnesses of the inner layer.

When the pores are open, both the lower horizontal permeability of the inner layer and the stronger coupling of the interface wave with this shell lead to less efficient fluid flow effects. As a result, the relative decrease of the phase velocity and the increase of the attenuation in the low frequencies (Figure 43, $[0, 5]$ kHz) are mostly representative of the inner layer porous properties. The diffusion process, characterized by an increase of the sensitivity coefficients with respect to the saturant fluid and $c_{11}$ and by a decrease of those with respect to $c_{66}$ and the bore fluid, is seen to refer to the virgin formation only below 200 Hz with a $3$ cm thick invaded zone (Figure 44), less efficiently than in the simple hole model (see Figure 8). With a greater thickness (Figure 45), the inner layer controls the diffusion process. The sensitivity coefficient with respect to the horizontal shear stiffness of the formation is slightly increased. It is only around $30$ Hz that the permeability of the virgin formation plays a weak role. In a higher frequency range (up to $25$ kHz, Figure 46), the phase velocity variations are mostly representative of the stiffnesses variations, while the attenuations cannot be differentiated from the one of the invaded zone. The small variations of the sensitivity coefficients (Figure 47 ($3$ cm case), and Figure 48 ($15$ cm case)) are analogous to those described in the simple hole section (Figures 9, 26, 32). The synthetic microseismograms computed with a $1$ kHz source center frequency (Figure 49) characterize in the time domain the predominant effects of the invaded zone.

**Pseudo-Rayleigh mode.**— Whatever the boundary conditions, the low cut-off frequency of the first pseudo-Rayleigh mode is shifted toward lower frequencies with increasing thickness of the invaded zone (Figure 50a, b). The phase velocity still starts at the vertically propagating $SV$-wave velocity of the formation. The attenuation at the low cut-off frequency then decreases, following the frequency variation of the quasi body wave in such a frequency range. When the pores are open (Figure 50b), the attenuation at the Airy phase of the mode is characteristic of the horizontal stiffness and permeability of the inner layer, similar to the Stoneley wave. As in the open hole situation (Figure 14), the sensitivity coefficient with respect to $c_{44}$ of the formation dominates close to the cut-off frequency while most of the energy travels into the bore fluid at high frequencies (Figure 51 ($3$ cm case), and Figure 52 ($15$ cm case)). The effects of the invaded zone, mainly characterized by the variations relative to $c_{44}$, increase and are shifted toward intermediate frequencies with increasing thickness of this inner layer leading to a less predominant effect of the bore fluid in this frequency range. In the time domain (Figure 53, the source center frequency is $7.5$ kHz), the amplitude variations are also representative of the coupling with the inner layer, whatever the
boundary conditions. Because of the large offset and the small thickness of the invaded zone, the first body wave arrivals are relevant to the virgin formation. One can notice an increase of the absolute amplitude of the \( P \) wavetrain compared to the simple virgin formation model. Similar to the isotropic elastic case, this related to a focus of the energy, associated with a curvature, without rupture, of the quasi \( P \) wavefront.

Dipole source

The low frequency part of the flexural mode starts at the virgin formation vertically propagating \( SV \)-wave characteristics (velocity and attenuation) with a shift toward lower frequencies of its useful starting energy that increases with increasing thickness of the invaded zone, whatever the boundary conditions (Figures 54a, b). Higher in frequency, the properties of the mode follow those of the Stoneley wave (see Figure 40). The variations of the sensitivity coefficients (Figure 55 (3 cm case) and Figure 56 (15 cm case)) also emphasize such a behavior with a decrease of the high frequency energy traveling into the borefluid correlated with an increase of the invasion zone effects located around the Airy phase (see Figure 23a). Figures 57 and 58 show the iso-offset comparison with a 1 kHz and 3 kHz source center frequency, respectively. They emphasize the need of useful low frequency energy.

Discussion

These calculations show that the determination of both the horizontally propagating \( SH \)-wave velocity and the in situ horizontal permeability of the virgin formation is ill posed when based on the only Stoneley wave characteristics. Whatever the boundary conditions, the most reliable informations are the vertically propagating \( SV \)-wave velocity and attenuation of the virgin formation brought by the low frequency (high velocity) part of both the pseudo-Rayleigh mode and the flexural mode. This analysis also emphasizes that the presence of a simple invaded zone, even resulting only from a permeability contrast, can be sufficient to explain the discrepancy between the measured permeabilities and the ones derived from the interface wave using a simple hole model. It is not necessary to consider any questionable pressure boundary conditions such as the one stated by Deresiewick and Skalak (1963), nor to call only for (unknown) mud cake effects. One has to keep in mind that the deposit at the borehole wall of the particles in suspension in the bore fluid may result from, or simply follows, the invasion process.
Cased Hole

When the borehole is cased, the formation body wave velocities can still be determined with more or less difficulty, depending on the quality of the bonding (Tubman, 1984; Block et al., 1987). With transversely isotropic formations, these results of course hold true, referring to the velocities of the vertically propagating quasi $P$ and quasi $SV$ waves. In the well bonded cased hole situation, when all the interfaces are welded contacts, the Stoneley and guided wave behaviors are also analogous to those observed in the presence of simple isotropic elastic formations (e.g., Schoenberg et al., 1981; Tubman, 1984; Baker, 1984; Schmitt and Bouchon, 1985; Burns, 1986; and Everhart and Chang, 1985; Schmitt, 1987c, for the multipole sources). The permeable character of a saturated porous formation can induce some specific effects only in the unbonded configuration corresponding to the presence of an external fluid annulus located between the cement and the formation (Schmitt, 1985). In the following, we then investigate the so called free pipe situation in the extreme case where the cement is totally replaced by a fluid, considering both boundary conditions at the fluid-formation interface (see equ. (34) and (35)). As a reference, we first present the well bonded configuration. The physical properties of the casing and the cement are given in Table 3. Their respective thicknesses are equal to 1 cm and 3 cm so that the inner borehole radius is decreased down to 6 cm. The external fluid layer is water. Only the fast transversely isotropic water saturated porous formation case is considered.

Monopole source

Stoneley wave.— In a well bonded cased hole, Stoneley wave phase velocity dispersion is little pronounced (Figure 59). As shown in Figure 60, there is almost no coupling of the interface wave with the solid layers at very low frequencies as the sensitivity with respect to the bore fluid is increased compared to the simple hole model (Figure 3). The effects of the formation are very small over the entire frequency range, while the sensitivity coefficients related to the shear moduli of both the casing and the cement increase with increasing frequency. At the same time, the sensitivity with respect to the bore fluid drops out.

Figure 61 shows the dispersion and attenuation of the interface wave in the free pipe situation for both boundary conditions. The associated sensitivity coefficients are displayed in Figure 62. When the fluid annulus-formation interface is impermeable, the phase velocity less varies with frequency than in the previous example. The sensitivity with respect to the bore fluid, slightly smaller in the low frequencies, more sharply decreases with increasing frequency to be less than that with respect to the fluid annulus which dominates at high frequencies. More of the small remaining energy is coupled to the formation both in the low frequencies (associated with $c_{66}$) and the
Transversely Isotropic Saturated Porous Formations II

high frequencies (associated with $c_{44}$). Correlatively, the sensitivity coefficients with respect to the shear and compressional properties of the casing are smaller and little vary with frequency.

When the interface is permeable, the behavior is quite different. The phase velocity direct dispersion is more pronounced starting at lower values in the low frequencies. It is associated with a much more important attenuation, greater than in the open hole model with an impermeable borehole wall (Figure 3), which increases with frequency. Such a behavior is related to the fluid flow induced at the annulus-formation interface by the ringing of the casing. In the very low frequencies, the diffusion process leads to a complete decoupling from the formation. As a result, the sensitivity coefficients with respect to the casing increase by more than a factor of 2, while those with respect to the others layers are smaller. The associated lower phase velocity is only apparently inconsistent: it simply tends toward the phase velocity of an interface wave propagating inside a steel pipe embedded in an infinite fluid. With increasing frequency, the behavior tends to be analogous than when the interface is impermeable, with however more high frequency energy relative to the fluid annulus.

In addition to the usual Stoneley wave, a second Stoneley wave is generated in the unbonded configuration (e.g., Tubman, 1984). Figure 63 shows the dispersion and attenuation of this supplementary annulus mode. As a function of the boundary conditions, their variations are similar to those of an interface wave in the simple hole model with a fast formation (see Figures 2 and 11), although the velocity is much less. The sensitivity coefficients (Figure 64) confirm this observation and clearly indicate that the mode primarily propagates in the fluid annulus. Compared to the central Stoneley wave, it is more sensitive to the formation properties. At low frequencies, when the interface is impermeable, most of the sensitivity refers to the fluid annulus, while the remaining energy is distributed among $c_{66}$ of the formation, shear motion in the casing, the bore fluid, and compressional motion in the casing, by decreasing order of magnitude. At high frequencies, the mode uncouples from the formation and is primarily governs by the casing and both fluid layers. When the fluid annulus-formation is permeable, the variations of the sensitivity coefficients are again similar to those in the simple hole model (see Figure 10 compare to Figure 4). Above 10 kHz, the boundary conditions have no practical effects. Such properties

Figure 65 shows the iso-offset ($z = 5$ m) comparisons of the synthetic microseismograms computed in the free pipe situation for both boundary conditions with a 3 kHz (a) and a 5 kHz (b) source center frequency. The attenuation effects due to the induced fluid flow when the annulus-formation interface is permeable are clearly detectable. As a result, the second (annulus) Stoneley wave, whose energy is much less that of the central one and is located in a much narrower low frequency band (e.g., Burns, 1986), is present only when the interface is impermeable. In practice, however, only the attenuation of the usual Stoneley wave is a reliable information.
Pseudo-Rayleigh mode.— Because of the borehole radius decrease, the low cut-off frequency of the pseudo-Rayleigh mode is shifted toward higher frequencies when the cased hole is well bonded. (Figure 66, compared to Figure 13a, and see Figure 21). Figure 67 shows the variations of the sensitivity coefficients. Similar to the invaded zone case, the casing and cement layers play a more important role at intermediate frequencies, leading to a significant decrease of the sensitivity with respect to the bore fluid (see Figure 14a). This last increases again at higher frequencies where the sensitivity with respect to compressional motion in the casing dominates those related to the formation. Close to the cut-off frequency, most of the wave’s energy refer to $c_{44}$ of the formation, as usual.

In the free pipe situation (Figure 68), the low cut-off frequency of the mode is shifted toward lower frequencies compared to the previous situation, and the Airy phase is lower. The boundary conditions have no significant effects on the dispersion. Similar to the simple hole case (Figure 13) but in a lesser extent, they lead to an increase of the maximum of attenuation associated with the Airy phase when the annulus-formation interface is permeable. The sensitivity coefficients are given in Figure 69. Compared to the previous example, the ringing of the casing leads to a slightly greater sensitivity towards the shear modulus of this layer at intermediate frequencies. More energy also travels into the bore fluid in the same frequency range, while at high frequencies it decreases owing to the part associated with the fluid annulus.

Dipole source

As in the previous situations, only the flexural mode is studied.

In the well bonded configuration, the borehole radius decrease leads to a shift toward higher frequency of the useful starting energy of the flexural mode whose phase velocity is still equal to that of the vertically propagating $SV$ wave of the formation (Figure 70, compared to Figure 23a). The effects of the non attenuating steel casing and cement layers at intermediate and high frequencies are emphasized by a sharp decrease with frequency of the attenuation. The sensitivity coefficients (Figure 71) emphasize the high frequency interface behavior of the mode through a decrease of the sensitivity coefficient with respect to the bore fluid (see Figure 60). The maximum effects of the bore fluid and of the the shear modulus of both the casing and the cement layer are located around the Airy phase of the mode. At lower frequencies, the sensitivity with respect to $c_{44}$ of the formation is dominant. In the time domain, the frequency shift will result in much less energy contained in the quasi $SV$-wave pulse.

Replacing the cement elastic layer by a fluid layer (Figure 72), the useful starting energy is shifted toward lower frequencies, reaching a frequency close to the one in the simple hole model (Figure 23a), and the dispersion is less sharp. The boundary
conditions at the fluid annulus-formation interface essentially affect the attenuation which greatly increases when the fluid flow is free, similar to the Stoneley wave (Figure 61), although it is more important in the frequency range considered. The high frequency repartition of the sensitivity coefficients (Figure 73a) is also similar to that of the Stoneley wave (Figure 62) in the high frequencies where most of the energy travels into the fluid annulus. In the intermediate frequencies (i.e., [5, 10] kHz), the shear modulus of the casing exhibits the dominant effects. Although the boundary conditions play little role, one can notice that the differences occur predominantly in this intermediate frequency range, emphasizing the transition between the flexural and interface character of the mode correlated with its maximum of excitation. Similar to the simple hole model, the largest effects of the boundary conditions refer to the small sensitivity coefficient with respect to the saturant fluid of the formation (Figure 73a, compared to Figure 26b). In the time domain, the modes associated with the “ringing” of the casing may lead to a difficult detection of the quasi SV-wave signal.

**Discussion**

Any effects described in the free pipe situation with a free fluid flow refer to the horizontal permeability. They will increase with decreasing viscosity and/or increasing compressibility of the saturant fluid. On the other hand, they will be reduced by the presence of cement well bonded to the casing in relation with the correlated decrease of the fluid annulus thickness and the damping of the ringing of the casing. We also remind that the second Stoneley wave phase velocity and excitation decrease with decreasing thickness of the fluid annulus.

There is little, not to say none, possibility to extract lithologic information from the Stoneley wave. Again, only the low frequency (high velocity) part of the pseudo-Rayleigh and flexural mode contain reliable informations. Contrary, the attenuation behavior of the three modes, and especially that of the interface wave, may be of practical interest. Case studies seem to confirm such an assertion.
CONCLUSIONS

We have presented a general formulation, based on the Thomson Haskell method, which allows the study of the wavefield generated by any order of multipole source located in a fluid filled borehole embedded in a multilayered formation. Each layer can be fluid, elastic or saturated porous, modeled following Biot theory modified in accordance with homogenization theory. Each shell can be either transversely isotropic or simply isotropic. In the former case, the vertical axis of the borehole is assumed to coincide with the axis of symmetry of the formation, common to both the complex permeability tensor and the skeleton when the layer is saturated porous.

In the simple hole model, when the borehole wall is impermeable, the transverse isotropy of the skeleton of a fast or slow saturated porous formation may be detected and evaluated, similarly to the elastic situation. The vertically propagating $SV$-wave velocity is measured from the low frequency (high velocity) part of the flexural mode generated by a dipole source (or that of any fundamental mode generated by a multipole source of higher order). In the presence of a fast formation, the same information can be obtained from the low frequency part of the pseudo-Rayleigh mode generated by the monopole source. The horizontally propagating $SH$-velocity is indirectly determined from the very low frequency limit of the Stoneley wave phase velocity. Such calculations require useful low frequency energy logging tools, whatever their multipole order and knowledge about the density of the formation and the bore fluid properties. The logging tool geometry may also be taken into account. The transverse isotropy of both the permeability and the mass coupling coefficient cannot be detected. The induced attenuation effects cannot be differentiated from those associated by variations of simple isotropic parameters. In addition, any anelastic (anisotropic) attenuation will be superimposed and generally dominant. Finally, the critically refracted $P$ wave is the only wave affected by the vertical permeability.

With a permeable borehole wall, the fluid flow effects primarily refer to the only horizontal permeability. The transverse isotropy of the skeleton cannot be anymore reliably evaluated. An increase in the horizontal stiffness constants increases the fluid flow effects on the Stoneley wave velocity and attenuation through a decoupling of the interface wave from the formation. They also last over a wider frequency range. The only truly useful information brought by the low frequency part of the flexural mode is the attenuation of the vertically propagating $SV$ wave, equal to that of the horizontally propagating $SH$ wave and related to the only horizontal components of the permeability tensor. In the case of a strongly frequency dependent shear attenuation, such a data is more valuable than that brought by any fundamental mode generated by multipole source of higher order or by the pseudo-Rayleigh mode because of their higher frequency location. For example, the value given by the latter can be strongly affected by the mass coupling coefficient. Any more or less quantitative estimation
of the horizontal permeability based on Stoneley wave velocity and attenuation may then be much more questionable than in the simple isotropic case, especially when the degree of anisotropy is important. However, detection of permeability variation may still be reasonably performed. It is understood that any change in the borehole radius must be taken into account.

In the presence of an invaded zone, the determination of both the horizontally propagating $SH$-wave velocity and the in situ horizontal permeability of the virgin formation is ill posed when based on the only Stoneley wave characteristics. This is related to a stronger coupling of the interface with the inner slower layer, whatever the boundary conditions. Such effects increase with increasing thickness and decreasing body wave velocities of the invasion zone. A simple permeability contrast can be also sufficient to explain the discrepancy between the measured permeabilities and those derived from the interface wave using a simple hole model. It is only in the free pipe situation with a permeable fluid annulus-formation interface that the modes, especially the Stoneley wave and the flexural mode, may exhibit significant attenuation characteristic of the biphasic character of the formation. In any of the multilayered configuration, the low frequency part of both the flexural mode and the pseudo-Rayleigh mode measures the characteristics of the vertically propagating $SV$ wave of the virgin formation. Such an interesting information may be however difficult to extract.

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REFERENCES


<table>
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<th>Parameter</th>
<th>Fast formation</th>
<th>Slow formation</th>
<th>Invaded zone</th>
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<td>1500</td>
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Table 1. Transversely isotropic saturated porous formations parameters.

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<tr>
<th>Formation</th>
<th>Direction of propagation</th>
<th>$c_{P1} \ (m/s)$</th>
<th>$c_{SV} \ (m/s)$</th>
<th>$c_{SH} \ (m/s)$</th>
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</thead>
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<td>Vertical</td>
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<td>Horizontal</td>
<td>3950.5</td>
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<td>Horizontal</td>
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Table 2. Phase velocities of the quasi body waves in the low frequency range
Table 8. Elastic layers parameters.

<table>
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<th>cs (m/s)</th>
<th>ρ (kg/m³)</th>
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<td>1000</td>
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<td>Casing</td>
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</table>
Elements of the $T_{TIP}^{(n)}$ Matrix

Before giving the elements of the propagator matrix, a convenient system of notations for some lengthy expressions of the modified Bessel functions is first outlined. In the following, $q_j$ denotes any radial wavenumber.

For the modified Bessel functions of the first kind:

$$A_1(q_j r) = I_n(q_j r)_r = q_j I_{n-1}(q_j r) - \frac{n}{r} I_n(q_j r),$$
$$A_2(q_j r) = A_1(q_j r) - \frac{1}{r} I_n(q_j r),$$
$$A_3(q_j r) = A_1(q_j r) - \frac{n^2}{r} I_n(q_j r).$$  \hfill (A - 1)

For the modified Bessel functions of the second kind:

$$B_1(q_j r) = K_n(q_j r)_r = -q_j K_{n-1}(q_j r) - \frac{n}{r} K_n(q_j r),$$
$$B_2(q_j r) = B_1(q_j r) - \frac{1}{r} K_n(q_j r),$$
$$B_3(q_j r) = B_1(q_j r) - \frac{n^2}{r} K_n(q_j r).$$  \hfill (A - 2)

The use of the derivatives of the $n$th order as function of the $(n - 1)$th and $n$th ones presents the advantage of speed and accuracy from the numerical point of view. The relations hold whatever the value of $n$ because

$$I_{-n}(q_j r) = I_n(q_j r),$$
$$K_{-n}(q_j r) = K_n(q_j r);$$

(Abramovitz and Stegun, 1965; Gradshteyn and Rhyzik, 1965).
Components of $u_j^{(n)}$:

\[
\begin{align*}
T_{11}^{TP(n)} &= A_1(kpu_r) \left\{ 1 + ik\gamma_{ij}^{P1} \right\}, \\
T_{12}^{TP(n)} &= A_1(kpu_r) \left\{ 1 + ik\gamma_{ij}^{P2} \right\}, \\
T_{13}^{TP(n)} &= B_1(kpu_r) \left\{ 1 + ik\gamma_{ij}^{P1} \right\}, \\
T_{14}^{TP(n)} &= B_1(kpu_r) \left\{ 1 + ik\gamma_{ij}^{P2} \right\}, \\
T_{15}^{TP(n)} &= \frac{n}{r} I_n(ks_H r), \\
T_{16}^{TP(n)} &= \frac{n}{r} K_n(ks_H r), \\
T_{17}^{TP(n)} &= A_1(ksV_r) \left\{ ik + \gamma_{ij}^{SV} \right\}, \\
T_{18}^{TP(n)} &= B_1(ksV_r) \left\{ ik + \gamma_{ij}^{SV} \right\}.
\end{align*}
\]

Components of $v_j^{(n)}$:

\[
\begin{align*}
T_{21}^{TP(n)} &= -\frac{n}{r} I_n(kpv_r) \left\{ 1 + ik\gamma_{ij}^{P1} \right\}, \\
T_{22}^{TP(n)} &= -\frac{n}{r} I_n(kpv_r) \left\{ 1 + ik\gamma_{ij}^{P2} \right\}, \\
T_{23}^{TP(n)} &= -\frac{n}{r} K_n(kpv_r) \left\{ 1 + ik\gamma_{ij}^{P1} \right\}, \\
T_{24}^{TP(n)} &= -\frac{n}{r} K_n(kpv_r) \left\{ 1 + ik\gamma_{ij}^{P2} \right\}, \\
T_{25}^{TP(n)} &= -A_1(ks_H r), \\
T_{26}^{TP(n)} &= -B_1(ks_H r), \\
T_{27}^{TP(n)} &= -\frac{n}{r} I_n(ksV_r) \left\{ ik + \gamma_{ij}^{SV} \right\}, \\
T_{28}^{TP(n)} &= -\frac{n}{r} K_n(ksV_r) \left\{ ik + \gamma_{ij}^{SV} \right\}.
\end{align*}
\]
Components of $w_j^{(n)}$:

\[
T_{31j}^{TIP^{(n)}} = I_n(kp_{11j} r) \left\{ ik - kp_{11j}^2 \gamma_{1j}^{P1} \right\} ,
\]
\[
T_{32j}^{TIP^{(n)}} = I_n(kp_{12j} r) \left\{ ik - kp_{12j}^2 \gamma_{1j}^{P2} \right\} ,
\]
\[
T_{33j}^{TIP^{(n)}} = K_n(kp_{11j} r) \left\{ ik - kp_{11j}^2 \gamma_{1j}^{P1} \right\} ,
\]
\[
T_{34j}^{TIP^{(n)}} = K_n(kp_{12j} r) \left\{ ik - kp_{12j}^2 \gamma_{1j}^{P2} \right\} ,
\]
\[
T_{35j}^{TIP^{(n)}} = 0 ,
\]
\[
T_{36j}^{TIP^{(n)}} = 0 ,
\]
\[
T_{37j}^{TIP^{(n)}} = I_n(ks_{Vj} r) \left\{ -ks_{Vj} \gamma_{1j}^{SV} + ik\gamma_{1j}^{SV} \right\} ,
\]
\[
T_{38j}^{TIP^{(n)}} = K_n(ks_{Vj} r) \left\{ -ks_{Vj} \gamma_{1j}^{SV} + ik\gamma_{1j}^{SV} \right\} .
\]

Components of $\tilde{\phi}_j \left( U_j^{(n)} - u_j^{(n)} \right)$:

\[
T_{41j}^{TIP^{(n)}} = \tilde{\phi}_j A_1(kp_{11j} r) \left\{ P_{2j}^{P1} - 1 + ik \left( \gamma_{3j}^{P1} - \gamma_{1j}^{P1} \right) \right\} ,
\]
\[
T_{42j}^{TIP^{(n)}} = \tilde{\phi}_j A_1(kp_{12j} r) \left\{ P_{2j}^{P2} - 1 + ik \left( \gamma_{3j}^{P2} - \gamma_{1j}^{P2} \right) \right\} ,
\]
\[
T_{43j}^{TIP^{(n)}} = \tilde{\phi}_j B_1(kp_{11j} r) \left\{ P_{2j}^{P1} - 1 + ik \left( \gamma_{3j}^{P1} - \gamma_{1j}^{P1} \right) \right\} ,
\]
\[
T_{44j}^{TIP^{(n)}} = \tilde{\phi}_j B_1(kp_{12j} r) \left\{ P_{2j}^{P2} - 1 + ik \left( \gamma_{3j}^{P2} - \gamma_{1j}^{P2} \right) \right\} ,
\]
\[
T_{45j}^{TIP^{(n)}} = \tilde{\phi}_j \left( X_j^H - 1 \right) T_{13j}^{TIP} ,
\]
\[
T_{46j}^{TIP^{(n)}} = \tilde{\phi}_j \left( X_j^H - 1 \right) T_{15j}^{TIP} ,
\]
\[
T_{47j}^{TIP^{(n)}} = \tilde{\phi}_j A_2(ks_{Vj} r) \left\{ ik \left( \gamma_{3j}^{SV} - 1 \right) + \gamma_{2j}^{SV} - \gamma_{1j}^{SV} \right\} ,
\]
\[
T_{48j}^{TIP^{(n)}} = \tilde{\phi}_j B_2(ks_{Vj} r) \left\{ ik \left( \gamma_{3j}^{SV} - 1 \right) + \gamma_{2j}^{SV} - \gamma_{1j}^{SV} \right\} .
\]
Components of $\sigma^{(n)}_{TIP} + s^{(n)}_j$:

\begin{align*}
T^{TIP}_{51j} &= \left\{ (d_{11j} + Q_{1j}) \left( 1 + ikY_{1j}^{P1} \right) k_{p11j}^2 + ik \left( d_{13j} + Q_{3j} \right) \left( ik - k_{p11j}Y_{1j}^{P1} \right) \\
&\quad + \left( Q_{1j} + \bar{\kappa}_j \right) \left( k_{p11j}^2 - k^2 \right) Y_{1j}^{P1} \right\} I_n(kp_{11j} r) - \frac{2}{r} d_{66j} \left\{ 1 + ikY_{1j}^{P1} \right\} A_3(kp_{11j} r), \\
T^{TIP}_{52j} &= \left\{ \left( d_{11j} + Q_{1j} \right) \left( 1 + ikY_{1j}^{P2} \right) k_{p12j}^2 + ik \left( d_{12j} + Q_{3j} \right) \left( ik - k_{p12j}Y_{1j}^{P2} \right) \\
&\quad + \left( Q_{1j} + \bar{\kappa}_j \right) \left( k_{p12j}^2 - k^2 \right) Y_{1j}^{P2} \right\} I_n(kp_{12j} r) - \frac{2}{r} d_{66j} \left\{ 1 + ikY_{1j}^{P2} \right\} A_3(kp_{12j} r), \\
T^{TIP}_{53j} &= \left\{ \left( d_{11j} + Q_{1j} \right) \left( 1 + ikY_{1j}^{P1} \right) k_{p11j}^2 + ik \left( d_{13j} + Q_{3j} \right) \left( ik - k_{p11j}Y_{1j}^{P1} \right) \\
&\quad + \left( Q_{1j} + \bar{\kappa}_j \right) \left( k_{p11j}^2 - k^2 \right) Y_{1j}^{P1} \right\} K_n(kp_{11j} r) - \frac{2}{r} d_{66j} \left\{ 1 + ikY_{1j}^{P1} \right\} B_3(kp_{11j} r), \\
T^{TIP}_{54j} &= \left\{ \left( d_{11j} + Q_{1j} \right) \left( 1 + ikY_{1j}^{P2} \right) k_{p12j}^2 + ik \left( d_{12j} + Q_{3j} \right) \left( ik - k_{p12j}Y_{1j}^{P2} \right) \\
&\quad + \left( Q_{1j} + \bar{\kappa}_j \right) \left( k_{p12j}^2 - k^2 \right) Y_{1j}^{P2} \right\} K_n(kp_{12j} r) - \frac{2}{r} d_{66j} \left\{ 1 + ikY_{1j}^{P2} \right\} B_3(kp_{12j} r), \\
T^{TIP}_{55j} &= 2d_{66j} \frac{n}{r} A_2(ksh_j r), \quad T^{TIP}_{56j} = 2d_{66j} \frac{n}{r} B_2(ksh_j r), \\
T^{TIP}_{57j} &= \left\{ \left( d_{11j} + Q_{1j} \right) \left( ik + Y_{1j}^{SV} \right) k_{s1j}^2 + ik \left( d_{13j} + Q_{3j} \right) \left( -k_{s1j}^2 + ikY_{1j}^{SV} \right) \\
&\quad + \left( Q_{1j} + \bar{\kappa}_j \right) \left( k_{s1j}^2 - k^2 \right) Y_{1j}^{SV} \right\} I_n(ksv_j r) - \frac{2}{r} d_{66j} \left\{ ik + Y_{1j}^{SV} \right\} A_3(ksv_j r), \\
T^{TIP}_{58j} &= \left\{ \left( d_{11j} + Q_{1j} \right) \left( ik + Y_{1j}^{SV} \right) k_{s1j}^2 + ik \left( d_{13j} + Q_{3j} \right) \left( -k_{s1j}^2 + ikY_{1j}^{SV} \right) \\
&\quad + \left( Q_{1j} + \bar{\kappa}_j \right) \left( k_{s1j}^2 - k^2 \right) Y_{1j}^{SV} \right\} K_n(ksv_j r) - \frac{2}{r} d_{66j} \left\{ ik + Y_{1j}^{SV} \right\} B_3(ksv_j r). \\
\end{align*}

Components of $\sigma^{(n)}_{TIP}$:

\begin{align*}
T^{TIP}_{61j} &= -2d_{66j} \frac{n}{r} \left\{ 1 + ikY_{1j}^{P1} \right\} A_2(kp_{11j} r), \\
T^{TIP}_{62j} &= -2d_{66j} \frac{n}{r} \left\{ 1 + ikY_{1j}^{P2} \right\} A_2(kp_{12j} r), \\
T^{TIP}_{63j} &= -2d_{66j} \frac{n}{r} \left\{ 1 + ikY_{1j}^{P1} \right\} B_2(kp_{11j} r), \\
T^{TIP}_{64j} &= -2d_{66j} \frac{n}{r} \left\{ 1 + ikY_{1j}^{P2} \right\} B_2(kp_{12j} r), \\
T^{TIP}_{65j} &= -d_{66j} \left\{ k_{s1j}^2 I_n(ksh_j r) - \frac{2}{r} A_3(ksh_j r) \right\}, \\
T^{TIP}_{66j} &= -d_{66j} \left\{ k_{s1j}^2 K_n(ksh_j r) - \frac{2}{r} B_3(ksh_j r) \right\}, \\
T^{TIP}_{67j} &= -2d_{66j} \left\{ ik + Y_{1j}^{SV} \right\} A_2(ksv_j r), \\
T^{TIP}_{68j} &= -2d_{66j} \left\{ ik + Y_{1j}^{SV} \right\} B_2(ksv_j r),
\end{align*}

(A-7)
Components of $\sigma_{ij}^{(n)}$:

\[
T_{IP(\hat{n})_{ij}}^{(n)} = d_{44j} \left\{ 2ik - \left( kp_{11j}^2 + k^2 \right) Y_{1j}^{P1} \right\} A_1(kp_{11j}r), \\
T_{IP(\hat{n})_{ij}}^{(n)} = d_{44j} \left\{ 2ik - \left( kp_{12j}^2 + k^2 \right) Y_{1j}^{P2} \right\} A_1(kp_{12j}r), \\
T_{IP(\hat{n})_{ij}}^{(n)} = d_{44j} \left\{ 2ik - \left( kp_{11j}^2 + k^2 \right) Y_{1j}^{P1} \right\} B_1(kp_{11j}r), \\
T_{IP(\hat{n})_{ij}}^{(n)} = d_{44j} \left\{ 2ik - \left( kp_{12j}^2 + k^2 \right) Y_{1j}^{P2} \right\} B_1(kp_{12j}r), \\
T_{IP(\hat{n})_{ij}}^{(n)} = ik_n d_{44j} I_n(ksH_j r), \\
T_{IP(\hat{n})_{ij}}^{(n)} = ik_n d_{44j} K_n(ksH_j r), \\
T_{IP(\hat{n})_{ij}}^{(n)} = d_{44j} \left\{ -ks\hat{V}_j - k^2 + 2ikY_{ij}^{SV} \right\} A_1(ksV_j r), \\
T_{IP(\hat{n})_{ij}}^{(n)} = d_{44j} \left\{ -ks\hat{V}_j - k^2 + 2ikY_{ij}^{SV} \right\} B_1(ksV_j r).
\]

(A - 9)

Components of $p_j^{(n)}$:

\[
T_{IP(\hat{n})_{ij}}^{(n)} = -\frac{1}{\phi_j} \left\{ Q_{1j} \left( 1 + ikY_{ij}^{P1} \right) kp_{11j}^2 + ikQ_{3j} \left( ik - kp_{11j}^2 Y_{ij}^{P1} \right) \right. \\
+ K_j \left( kp_{11j}^2 - k^2 \right) Y_{ij}^{P1} \left\} I_n(kp_{11j}r), \\
T_{IP(\hat{n})_{ij}}^{(n)} = -\frac{1}{\phi_j} \left\{ Q_{1j} \left( 1 + ikY_{ij}^{P2} \right) kp_{12j}^2 + ikQ_{3j} \left( ik - kp_{12j}^2 Y_{ij}^{P2} \right) \right. \\
+ K_j \left(kp_{12j}^2 - k^2 \right) Y_{ij}^{P2} \left\} I_n(kp_{12j}r), \\
T_{IP(\hat{n})_{ij}}^{(n)} = -\frac{1}{\phi_j} \left\{ Q_{1j} \left( 1 + ikY_{ij}^{P1} \right) kp_{11j}^2 + ikQ_{3j} \left( ik - kp_{11j}^2 Y_{ij}^{P1} \right) \right. \\
+ K_j \left( kp_{11j}^2 - k^2 \right) Y_{ij}^{P1} \left\} K_n(kp_{11j}r), \\
T_{IP(\hat{n})_{ij}}^{(n)} = -\frac{1}{\phi_j} \left\{ Q_{1j} \left( 1 + ikY_{ij}^{P2} \right) kp_{12j}^2 + ikQ_{3j} \left( ik - kp_{12j}^2 Y_{ij}^{P2} \right) \right. \\
+ K_j \left(kp_{12j}^2 - k^2 \right) Y_{ij}^{P2} \left\} K_n(kp_{12j}r), \\
T_{IP(\hat{n})_{ij}}^{(n)} = 0 , \\
T_{IP(\hat{n})_{ij}}^{(n)} = 0 , \\
T_{IP(\hat{n})_{ij}}^{(n)} = -\frac{1}{\phi_j} \left\{ Q_{1j} \left( ik + \gamma_{ij}^{SV} \right) ks\hat{V}_j + Q_{3j} \left( -ks\hat{V}_j + ikY_{ij}^{SV} \right) \right. \\
+ K_j \left( ks\hat{V}_j - k^2 \right) Y_{ij}^{SV} \left\} I_n(ksV_j r), \\
T_{IP(\hat{n})_{ij}}^{(n)} = -\frac{1}{\phi_j} \left\{ Q_{1j} \left( ik + \gamma_{ij}^{SV} \right) ks\hat{V}_j + Q_{3j} \left( -ks\hat{V}_j + ikY_{ij}^{SV} \right) \right. \\
+ K_j \left( ks\hat{V}_j - k^2 \right) Y_{ij}^{SV} \left\} K_n(ksV_j r).
\]

(A - 10)
APPENDIX B

Elements of the $TIP^{(n)}$ Matrix

The functions $A_1, A_2, A_3, B_1, B_2, \text{ and } B_3$ are defined in Appendix A.

Components of $u_j^{(n)}$:

\[
\begin{align*}
T_{11}^{IP(n)} &= A_1(kp_{1j} r), \\
T_{12}^{IP(n)} &= A_1(kp_{2j} r), \\
T_{13}^{IP(n)} &= B_1(kp_{1j} r), \\
T_{14}^{IP(n)} &= B_1(kp_{2j} r), \\
T_{15}^{IP(n)} &= \frac{n}{r} I_n(ks_j r), \\
T_{16}^{IP(n)} &= \frac{n}{r} K_n(ks_j r), \\
T_{17}^{IP(n)} &= i k A_1(ks_j r), \\
T_{18}^{IP(n)} &= i k B_1(ks_j r).
\end{align*}
\]

Components of $v_j^{(n)}$:

\[
\begin{align*}
T_{21}^{IP(n)} &= -\frac{n}{r} I_n(kp_{1j} r), \\
T_{22}^{IP(n)} &= -\frac{n}{r} I_n(kp_{2j} r), \\
T_{23}^{IP(n)} &= -\frac{n}{r} K_n(kp_{1j} r), \\
T_{24}^{IP(n)} &= -\frac{n}{r} K_n(kp_{2j} r), \\
T_{25}^{IP(n)} &= -A_1(ks_j r), \\
T_{26}^{IP(n)} &= -B_1(ks_j r), \\
T_{27}^{IP(n)} &= -i k \frac{n}{r} I_n(ks_j r), \\
T_{28}^{IP(n)} &= -i k \frac{n}{r} K_n(ks_j r).
\end{align*}
\]
Components of $u_j^{(n)}$:

\[
T_{31j}^{(n)} = ik I_n(kp_1 r), \\
T_{32j}^{(n)} = ik I_n(kp_2 r), \\
T_{33j}^{(n)} = ik K_n(kp_1 r), \\
T_{34j}^{(n)} = ik K_n(kp_2 r), \\
T_{35j}^{(n)} = 0, \\
T_{36j}^{(n)} = 0, \\
T_{37j}^{(n)} = -ks_j^2 I_n(k s_j r), \\
T_{38j}^{(n)} = -ks_j^2 K_n(k s_j r).
\]

(B - 3)

Components of $\tilde{\phi}_j (U_j^{(n)} - u_j^{(n)})$:

\[
T_{41j}^{(n)} = \tilde{\phi}_j (\xi_{1j} - 1) T_{11j}^{(n)}, \\
T_{42j}^{(n)} = \tilde{\phi}_j (\xi_{2j} - 1) T_{12j}^{(n)}, \\
T_{43j}^{(n)} = \tilde{\phi}_j (\xi_{3j} - 1) T_{13j}^{(n)}, \\
T_{44j}^{(n)} = \tilde{\phi}_j (\xi_{4j} - 1) T_{14j}^{(n)}, \\
T_{45j}^{(n)} = \tilde{\phi}_j (\xi_{5j} - 1) T_{15j}^{(n)}, \\
T_{46j}^{(n)} = \tilde{\phi}_j (\xi_{6j} - 1) T_{16j}^{(n)}, \\
T_{47j}^{(n)} = \tilde{\phi}_j (\xi_{7j} - 1) T_{17j}^{(n)}, \\
T_{48j}^{(n)} = \tilde{\phi}_j (\xi_{8j} - 1) T_{18j}^{(n)}.
\]

(B - 4)
Components of $\sigma^{(n)}_{rj} + s^{(n)}_j$:

\[
\begin{align*}
T_{51}^{IP(n)} &= \left\{ 2N_j k p_{j1}^2 + \left( k p_{j2}^2 - k^2 \right) \left\{ A_j + Q_j + (Q_j + \ddot{R}_j) \xi_{j1} \right\} \right\} I_n(k p_{1j} r) \\
&\quad - \frac{2}{r} N_j A_3(k p_{1j} r), \\
T_{52}^{IP(n)} &= \left\{ 2N_j k p_{j2}^2 + \left( k p_{j1}^2 - k^2 \right) \left\{ A_j + Q_j + (Q_j + \ddot{R}_j) \xi_{j2} \right\} \right\} I_n(k p_{2j} r) \\
&\quad - \frac{2}{r} N_j A_3(k p_{2j} r), \\
T_{53}^{IP(n)} &= \left\{ 2N_j k p_{j1}^2 + \left( k p_{j2}^2 - k^2 \right) \left\{ A_j + Q_j + (Q_j + \ddot{R}_j) \xi_{j3} \right\} \right\} K_n(k p_{1j} r) \\
&\quad - \frac{2}{r} N_j B_3(k p_{1j} r), \\
T_{54}^{IP(n)} &= \left\{ 2N_j k p_{j2}^2 + \left( k p_{j1}^2 - k^2 \right) \left\{ A_j + Q_j + (Q_j + \ddot{R}_j) \xi_{j4} \right\} \right\} K_n(k p_{2j} r) \\
&\quad - \frac{2}{r} N_j B_3(k p_{2j} r), \\
T_{55}^{IP(n)} &= 2N_j \frac{n}{r} A_2(k s_j r), \\
T_{56}^{IP(n)} &= 2N_j \frac{n}{r} B_2(k s_j r), \\
T_{57}^{IP(n)} &= 2ik N_j \left\{ k s_j^2 I_n(k s_j r) - \frac{1}{r} A_3(k s_j r) \right\}, \\
T_{58}^{IP(n)} &= 2ik N_j \left\{ k s_j^2 K_n(k s_j r) - \frac{1}{r} B_3(k s_j r) \right\}. 
\end{align*}
\]  

(B - 5)

Components of $\sigma^{(n)}_{rj}$:

\[
\begin{align*}
T_{61}^{IP(n)} &= -2N_j \frac{n}{r} A_2(k p_{1j} r), \\
T_{62}^{IP(n)} &= -2N_j \frac{n}{r} A_2(k p_{2j} r), \\
T_{63}^{IP(n)} &= -2N_j \frac{n}{r} B_2(k p_{1j} r), \\
T_{64}^{IP(n)} &= -2N_j \frac{n}{r} B_2(k p_{2j} r), \\
T_{65}^{IP(n)} &= -N_j \left\{ k s_j^2 I_n(k s_j r) - \frac{2}{r} A_3(k s_j r) \right\}, \\
T_{66}^{IP(n)} &= -N_j \left\{ k s_j^2 K_n(k s_j r) - \frac{2}{r} B_3(k s_j r) \right\}, \\
T_{67}^{IP(n)} &= -2ik \frac{n}{r} N_j A_2(k s_j r), \\
T_{68}^{IP(n)} &= -2ik \frac{n}{r} N_j B_2(k s_j r). 
\end{align*}
\]  

(B - 6)
Components of $\sigma_{ij}^{(n)}$:  

\[
T_{71}^{IP(n)} = 2ikN_jA_1(kp_{1j} r),
T_{72}^{IP(n)} = 2ikN_jA_1(kp_{2j} r),
T_{73}^{IP(n)} = 2ikN_jB_1(kp_{1j} r),
T_{74}^{IP(n)} = 2ikN_jB_1(kp_{2j} r),
T_{75}^{IP(n)} = ikN_j^n r J n(ks_j r),
T_{76}^{IP(n)} = ikN_j^n r J n(K_n(ks_j r)),
T_{77}^{IP(n)} = -N_j \left\{ks_j^2 + k^2\right\} A_1(ks_j r),
T_{78}^{IP(n)} = -N_j \left\{ks_j^2 + k^2\right\} B_1(ks_j r).
\]

(B - 7)

Components of $p_j^{(n)}$:  

\[
T_{81}^{IP(n)} = -\frac{1}{\phi_j} \left\{kp_{1j}^2 - k^2\right\} \left\{Q_j + \bar{R}_j t_{1j}\right\} I_n(kp_{1j} r),
T_{82}^{IP(n)} = -\frac{1}{\phi_j} \left\{kp_{2j}^2 - k^2\right\} \left\{Q_j + \bar{R}_j t_{2j}\right\} I_n(kp_{2j} r),
T_{83}^{IP(n)} = -\frac{1}{\phi_j} \left\{kp_{1j}^2 - k^2\right\} \left\{Q_j + \bar{R}_j t_{1j}\right\} K_n(kp_{1j} r),
T_{84}^{IP(n)} = -\frac{1}{\phi_j} \left\{kp_{2j}^2 - k^2\right\} \left\{Q_j + \bar{R}_j t_{2j}\right\} K_n(kp_{2j} r),
T_{85}^{IP(n)} = 0,
T_{86}^{IP(n)} = 0,
T_{87}^{IP(n)} = 0,
T_{88}^{IP(n)} = 0.
\]

(B - 8)
APPENDIX C

Elements of the $T_{ij}^{\text{TIE}}(n)$ Matrix

The functions $A_1, A_2, A_3, B_1, B_2$, and $B_3$ are defined in Appendix A.

Components of $u_j^{(n)}$:

\[
T_{11j}^{\text{TIE}(n)} = A_1(kpt_j r) \left\{ 1 + ik Y_j^P \right\},
\]
\[
T_{12j}^{\text{TIE}(n)} = B_1(kp_t j r) \left\{ 1 + ik Y_j^P \right\},
\]
\[
T_{13j}^{\text{TIE}(n)} = \frac{n}{r} I_n(k s_{Hj} r),
\]
\[
T_{14j}^{\text{TIE}(n)} = \frac{n}{r} K_n(k s_{Hj} r),
\]
\[
T_{15j}^{\text{TIE}(n)} = A_1(k s v_j r) \left\{ ik + Y_j^{SV} \right\},
\]
\[
T_{16j}^{\text{TIE}(n)} = B_1(k s v_j r) \left\{ ik + Y_j^{SV} \right\}.
\]

Components of $v_j^{(n)}$:

\[
T_{21j}^{\text{TIE}(n)} = -\frac{n}{r} I_n(k p_t j r) \left\{ 1 + ik Y_j^P \right\},
\]
\[
T_{22j}^{\text{TIE}(n)} = -\frac{n}{r} K_n(k p_t j r) \left\{ 1 + ik Y_j^P \right\},
\]
\[
T_{23j}^{\text{TIE}(n)} = -A_1(k s_{Hj} r),
\]
\[
T_{24j}^{\text{TIE}(n)} = -B_1(k s_{Hj} r),
\]
\[
T_{25j}^{\text{TIE}(n)} = -\frac{n}{r} I_n(k s v_j r) \left\{ ik + Y_j^{SV} \right\},
\]
\[
T_{26j}^{\text{TIE}(n)} = -\frac{n}{r} K_n(k s v_j r) \left\{ ik + Y_j^{SV} \right\}.
\]

Components of $w_j^{(n)}$:

\[
T_{31j}^{\text{TIE}(n)} = I_n(k p_t j r) \left\{ ik - k p^2_{ji} Y_j^P \right\},
\]
\[
T_{32j}^{\text{TIE}(n)} = K_n(k p_t j r) \left\{ ik - k p^2_{ji} Y_j^P \right\},
\]
\[
T_{33j}^{\text{TIE}(n)} = 0,
\]
\[
T_{34j}^{\text{TIE}(n)} = 0,
\]
\[
T_{35j}^{\text{TIE}(n)} = I_n(k s v_j r) \left\{ ik Y_j^{SV} - k s_{Vj} \right\},
\]
\[
T_{36j}^{\text{TIE}(n)} = K_n(k s v_j r) \left\{ ik Y_j^{SV} - k s_{Vj} \right\}.
\]
Components of $\sigma_{rr}^{(n)}$:

\[
T_{41j}^{\text{TIE}} = \left\{ c_{11j} k_p^2 \left( 1 + ik \gamma_j^P \right) + i c_{13j} \left( ik - k_p^2 \gamma_j^P \right) \right\} I_n(k_p \gamma_j) \\
- \frac{2}{r} c_{66j} A_3(k_p \gamma_j) \left\{ 1 + ik \gamma_j^P \right\},
\]

\[
T_{42j}^{\text{TIE}} = \left\{ c_{11j} k_p^2 \left( 1 + ik \gamma_j^P \right) + i c_{13j} \left( ik - k_p^2 \gamma_j^P \right) \right\} K_n(k_p \gamma_j) \\
- \frac{2}{r} c_{66j} B_3(k_p \gamma_j) \left\{ 1 + ik \gamma_j^P \right\},
\]

\[
T_{43j}^{\text{TIE}} = 2 c_{66j} \frac{n}{r} A_2(k s_H \gamma_j),
\]

\[
T_{44j}^{\text{TIE}} = 2 c_{66j} \frac{n}{r} B_2(k s_H \gamma_j),
\]

\[
T_{45j}^{\text{TIE}} = \left\{ c_{11j} k s^2 \gamma_j \left( ik + \gamma_j^S \right) + i c_{13j} \left( ik \gamma_j^S - k s^2 \gamma_j \right) \right\} I_n(k s \gamma_j) \\
- \frac{2}{r} c_{66j} A_3(k s \gamma_j) \left\{ ik + \gamma_j^S \right\},
\]

\[
T_{46j}^{\text{TIE}} = \left\{ c_{11j} k s^2 \gamma_j \left( ik + \gamma_j^S \right) + i c_{13j} \left( ik \gamma_j^S - k s^2 \gamma_j \right) \right\} K_n(k s \gamma_j) \\
- \frac{2}{r} c_{66j} B_3(k s \gamma_j) \left\{ ik + \gamma_j^S \right\}.
\]  

\[\text{(C - 4)}\]

Components of $\sigma_{r\phi}^{(n)}$:

\[
T_{51j}^{\text{TIE}} = -2 \frac{n}{r} c_{66j} A_2(k_p \gamma_j) \left\{ 1 + ik \gamma_j^P \right\},
\]

\[
T_{52j}^{\text{TIE}} = -2 \frac{n}{r} c_{66j} B_2(k_p \gamma_j) \left\{ 1 + ik \gamma_j^P \right\},
\]

\[
T_{53j}^{\text{TIE}} = -c_{66j} \left\{ k s^2 \gamma_j I_n(k s \gamma_j) - \frac{2}{r} A_3(k s \gamma_j) \right\},
\]

\[
T_{54j}^{\text{TIE}} = -c_{66j} \left\{ k s^2 \gamma_j K_n(k s \gamma_j) - \frac{2}{r} B_3(k s \gamma_j) \right\},
\]

\[
T_{55j}^{\text{TIE}} = -2 \frac{n}{r} c_{66j} A_2(k s \gamma_j) \left\{ ik + \gamma_j^S \right\},
\]

\[
T_{56j}^{\text{TIE}} = -2 \frac{n}{r} c_{66j} B_2(k s \gamma_j) \left\{ ik + \gamma_j^S \right\}.
\]  

\[\text{(C - 5)}\]

Components of $\sigma_{\phi \phi}^{(n)}$:

\[
T_{61j}^{\text{TIE}} = c_{44j} A_1(k_p \gamma_j) \left\{ 2 i k \gamma_j^P \left\{ k^2 + k_p^2 \right\} \right\},
\]

\[
T_{62j}^{\text{TIE}} = c_{44j} B_1(k_p \gamma_j) \left\{ 2 i k \gamma_j^P \left\{ k^2 + k_p^2 \right\} \right\},
\]

\[
T_{63j}^{\text{TIE}} = i k c_{44j} \frac{n}{r} J_n(k s \gamma_j),
\]

\[
T_{64j}^{\text{TIE}} = i k c_{44j} \frac{n}{r} K_n(k s \gamma_j),
\]

\[
T_{65j}^{\text{TIE}} = c_{44j} \left\{ 2 i k \gamma_j^S - k^2 - k s^2 \gamma_j \right\} A_1(k s \gamma_j),
\]

\[
T_{66j}^{\text{TIE}} = c_{44j} \left\{ 2 i k \gamma_j^S - k^2 - k s^2 \gamma_j \right\} B_1(k s \gamma_j).
\]  

\[\text{(C - 6)}\]
APPENDIX D

Elements of the $T_{ij}^{(n)}$ Matrix

The functions $A_1, A_2, A_3, B_1, B_2$, and $B_3$ are defined in Appendix A.

Components of $u_j^{(n)}$:

$$
T_{11j}^{(n)} = A_1(kp_j r),
T_{12j}^{(n)} = B_1(kp_j r),
T_{13j}^{(n)} = \frac{n}{r} I_n(ks_j r),
T_{14j}^{(n)} = \frac{n}{r} K_n(ks_j r),
T_{15j}^{(n)} = ik A_1(ks_j r),
T_{16j}^{(n)} = ik B_1(ks_j r).
$$  \hspace{1cm} (D - 1)

Components of $v_j^{(n)}$:

$$
T_{21j}^{(n)} = -\frac{n}{r} I_n(kp_j r),
T_{22j}^{(n)} = -\frac{n}{r} K_n(kp_j r),
T_{23j}^{(n)} = -A_1(ks_j r),
T_{24j}^{(n)} = -B_1(ks_j r),
T_{25j}^{(n)} = -ik \frac{n}{r} I_n(ks_j r),
T_{26j}^{(n)} = -ik \frac{n}{r} K_n(ks_j r).
$$  \hspace{1cm} (D - 2)

Components of $w_j^{(n)}$:

$$
T_{31j}^{(n)} = ik I_n(kp_j r),
T_{32j}^{(n)} = ik K_n(kp_j r),
T_{33j}^{(n)} = 0,
T_{34j}^{(n)} = 0,
T_{35j}^{(n)} = -ks_j^2 I_n(ks_j r),
T_{36j}^{(n)} = -ks_j^2 K_n(ks_j r).
$$  \hspace{1cm} (D - 3)
Components of $\sigma^{(n)}_{rj}$:

\[
T_{41}^{IE(n)} = \mu_j \left\{ \left( k^2 + k s_j^2 \right) I_n(k p_j r) - \frac{2}{r} \alpha_3(k p_j r) \right\},
\]

\[
T_{42}^{IE(n)} = \mu_j \left\{ \left( k^2 + k s_j^2 \right) K_n(k p_j r) - \frac{2}{r} \beta_3(k p_j r) \right\},
\]

\[
T_{43}^{IE(n)} = 2 \frac{n}{r} \mu_j A_2(k s_j r),
\]

\[
T_{44}^{IE(n)} = 2 \frac{n}{r} \mu_j B_2(k s_j r),
\]

\[
T_{45}^{IE(n)} = 2 i k \mu_j \left\{ k s_j^2 I_n(k s_j r) - \frac{1}{r} \alpha_3(k s_j r) \right\},
\]

\[
T_{46}^{IE(n)} = 2 i k \mu_j \left\{ k s_j^2 K_n(k s_j r) - \frac{1}{r} \beta_3(k s_j r) \right\}.
\]

Components of $\sigma^{(n)}_{r'j}$:

\[
T_{51}^{IE(n)} = -2 \frac{n}{r} \mu_j A_2(k p_j r),
\]

\[
T_{52}^{IE(n)} = -2 \frac{n}{r} \mu_j B_2(k p_j r),
\]

\[
T_{53}^{IE(n)} = -\mu_j \left\{ k s_j^2 I_n(k s_j r) - \frac{2}{r} \alpha_3(k s_j r) \right\},
\]

\[
T_{54}^{IE(n)} = -\mu_j \left\{ k s_j^2 K_n(k s_j r) - \frac{2}{r} \beta_3(k s_j r) \right\},
\]

\[
T_{55}^{IE(n)} = -2 i k \frac{n}{r} \mu_j A_2(k s_j r),
\]

\[
T_{56}^{IE(n)} = -2 i k \frac{n}{r} \mu_j B_2(k s_j r).
\]

Components of $\sigma^{(n)}_{r''j}$:

\[
T_{61}^{IE(n)} = 2 i k \mu_j A_1(k p_j r),
\]

\[
T_{62}^{IE(n)} = 2 i k \mu_j B_1(k p_j r),
\]

\[
T_{63}^{IE(n)} = i k \mu_j \frac{n}{r} I_n(k s_j r),
\]

\[
T_{64}^{IE(n)} = i k \mu_j \frac{n}{r} K_n(k s_j r),
\]

\[
T_{65}^{IE(n)} = -\mu_j \left( k s_j^2 + k^2 \right) A_1(k s_j r),
\]

\[
T_{66}^{IE(n)} = -\mu_j \left( k s_j^2 + k^2 \right) B_1(k s_j r).
\]
Figure 1: Effects of the only transversely isotropic permeability with an impermeable borehole wall. Dispersion and attenuation of the Stoneley wave in the presence of the water saturated porous fast formation with an isotropic skeleton. The numbers indicate the permeability values in darcies. The superscript H and V denote the horizontal plane and the vertical direction, respectively. The velocities are normalized with respect to the bore fluid velocity.
Figure 2: Impermeable borehole wall. Dispersion and attenuation of the Stoneley wave in the presence of the transversely isotropic water saturated porous fast formation (TIP) and of the isotropic fast formation with a 1 darcy isotropic permeability (ISO). The velocities are normalized with respect to the bore fluid velocity.
Figure 3: Impermeable borehole wall. Sensitivity coefficients for the Stoneley wave in the presence of the transversely isotropic saturated porous fast formation. L1 and LF refer to the bore and saturant fluid Lamé coefficients, respectively. C11, C13, C33, C44, and C66 refer to the stiffness constant of the skeleton. These notations will be used in the subsequent figures as well.
Figure 4: Impermeable borehole wall. Sensitivity coefficients for the Stoneley wave in the presence of the isotropic saturated porous fast formation with a 1 darcy isotropic permeability. M2 and P2 refer to the shear modulus of the skeleton ($\mu_b$), and to $\lambda_b + 2\mu_b$, respectively. The number 2 indicates the index of the layer. Compare with Figure 3.
Figure 5: Impermeable borehole wall. Monopole source. Iso-offset comparison ($z = 5$ m) of the synthetic microseismograms obtained at the center of the borehole with a 1 kHz source center frequency in the presence of the water saturated porous fast formation with an isotropic skeleton and isotropic 100 mdarcies permeability (Iso. 100.md), an isotropic skeleton and isotropic 1 darcy permeability (Iso. 1.d), an isotropic skeleton and transversely isotropic permeability (1 darcy in the horizontal plane and 100 mdarcies in the vertical direction, i.e., I. 1dH-100.mdV). Each wavetrain is normalized with respect to the maximum of the whole series indicated by max in arbitrary units. The scaling coefficient at the upper left gives the relative value of the peak amplitude compared to the latter. The arrows indicate the arrival time of the Stoneley wave computed using the zero frequency limit of the group velocity (equ. (49)).
Figure 6: Permeable borehole wall. Sensitivity coefficients for the Stoneley wave in the presence of the transversely isotropic water saturated porous fast formation in the low frequencies.
Figure 7: Permeable borehole wall. Sensitivity coefficients for the Stoneley wave in the presence of the transversely isotropic water saturated porous fast formation up to 25 kHz. Compare with Figure 3.
Figure 8: Permeable borehole wall. Sensitivity coefficients for the Stoneley wave in the presence of the isotropic water saturated porous fast formation with a 1 darcy isotropic permeability. Compare with Figures 4 and 7.
Figure 9: Effects of the only transversely isotropic permeability in the low frequency range with a permeable borehole wall. Dispersion and attenuation of the Stoneley wave in the presence of the water saturated porous formation with an isotropic skeleton. The numbers indicate the permeability values in darcies. The superscript H and V denote the horizontal plane and the vertical direction, respectively. The velocities are normalized with respect to the bore fluid velocity. Compare with Figure 1.
Figure 10: Same as Figure 9, up to 25 kHz. Compare with Figure 1.
Figure 11: Permeable borehole wall. Dispersion and attenuation of the Stoneley wave in the presence of the transversely isotropic water saturated porous fast formation (TIP) and of the isotropic fast formation with a 1 darcy isotropic permeability (ISO). The velocities are normalized with respect to the bore fluid velocity. Compare with Figure 2.
Figure 12: Permeable borehole wall. Monopole source. Iso-offset comparison ($z = 5$ m) of the synthetic microseismograms obtained at the center of the borehole with a $1$ kHz (a) and $3$ kHz (b) source center frequency. Compare with Figure 5.
Permeable, $f=3.0\text{ kHz}$

\begin{align*}
1.000 & \quad \text{I}\text{so=100 md} \\
0.232 & \quad \text{I=1 d} \\
0.231 & \quad \text{I=1 d+100 mdV} \\
0.225 & \quad \text{TIP}
\end{align*}

(b) TIME (ms)
Figure 13: Dispersion and attenuation of the first pseudo-Rayleigh mode in the presence of the transversely isotropic water saturated porous fast formation (TIP) and the isotropic formation with a 1 darcy isotropic permeability (ISO) when the borehole wall is impermeable (a) and permeable (b). The velocities are normalized with respect to the bore fluid velocity.
Figure 14: Sensitivity coefficients for the first pseudo-Rayleigh mode in the presence of the transversely isotropic water saturated porous fast formation for both boundary conditions.
Figure 15: Sensitivity coefficients for the first pseudo-Rayleigh mode in the presence of the isotropic water saturated porous fast formation with a 1 darcy isotropic permeability.
Figure 16: Monopole source. Iso-offset (z = 5 m) comparison of the synthetic microseismograms obtained at the center of the borehole with a 7.5 kHz source center frequency when the borehole wall is impermeable (a) and permeable (b).
Permeable, $f=7.5$ kHz

max = 33.19

Is = 100 m$\text{d}$

Is = 1 d

I = 1 dH - 100 m$\text{dV}$

TIP

TIME (ms)
Figure 17: Mass coupling coefficient effects. Dispersion and attenuation of the Stoneley wave with a permeable borehole wall in the presence of the transversely isotropic water saturated porous fast formation. The numbers refer to the tortuosity values (i.e., infinite frequency limits of the adimensional mass coupling coefficients $\rho_{22}(\omega)/\bar{\phi}p_f$). The superscripts H and V indicate the horizontal plane and the vertical direction, respectively. The velocities are normalized with respect to the bore fluid velocity.
Figure 18: Mass coupling coefficients effects. Dispersion and attenuation of the first pseudo-Rayleigh mode with a permeable borehole wall in the presence of the transversely isotropic water saturated porous fast formation. The numbers refer to the tortuosity values (i.e., infinite frequency limits of the adimensional mass coupling coefficients $\rho_{22}(\omega)/\phi \rho_f$). The superscripts H and V indicate the horizontal plane and the vertical direction, respectively. The velocities are normalized with respect to the bore fluid velocity.
Figure 19: Borehole radius effects with an impermeable borehole wall. Dispersion (a) and sensitivity coefficients (b) of the Stoneley wave in the presence of the transversely isotropic water saturated porous fast formation. The velocities are normalized with respect to the bore fluid velocity.
Figure 20: Borehole radius effects with a permeable borehole wall. Dispersion (a), attenuation (b), and sensitivity coefficients (c) of the Stoneley wave in the presence of the transversely isotropic water saturated porous fast formation. The velocities are normalized with respect to the bore fluid velocity.
Figure 21: Borehole radius effects. Dispersion and attenuation of the first pseudo-Rayleigh mode in the presence of the transversely isotropic water saturated porous fast formation with both boundary conditions. The velocities are normalized with respect to the bore fluid velocity. Compare with Figure 13.
Figure 22: Dispersion and attenuation of the flexural mode in the presence of the transversely isotropic water saturated porous fast formation (TIP) and the isotropic formation with a 1 darcy isotropic permeability (ISO) when the borehole wall is impermeable (a) and permeable (b). The velocity are normalized with respect to the bore fluid velocity.
(b)
Figure 23: Sensitivity coefficients of the flexural mode in the presence of the transversely isotropic water saturated porous fast formation for both boundary conditions (a), and in the presence of the isotropic water saturated porous fast formation with a 1 darcy isotropic permeability when the borehole is permeable (b).
Figure 24: Dipole source. Iso-offset ($z = 5$ m) comparison of the synthetic microseismograms obtained with a 1 kHz source center frequency in the presence of the fast water saturated porous formation with an isotropic skeleton and isotropic 100 mdarcies permeability (Iso. 100.md), an isotropic skeleton and isotropic 1 darcy permeability (Iso. 1.d), an isotropic skeleton and transversely isotropic permeability (1 darcy in the horizontal plane and 100 mdarcies in the vertical direction, i.e., 1.1dH-100.mdV). The borehole wall is impermeable (a) and permeable (b). Each wavetrain is normalized with respect to the maximum of the whole series indicated by max in arbitrary units. The scaling coefficient at the upper left gives the relative value of the peak amplitude compared to the latter. The arrows indicate the vertically propagating $SV$-wave arrival time computed using ray theory.
Permeable, \( f = 1.0 \text{ kHz} \)

\[
\begin{array}{c}
\text{TIP} \\
0.942 \\
0.942 \\
0.703 \\
0.000
\end{array}
\]

\( \max = 5.02 \times 10^{-3} \)

TIME (ms)
Figure 25: Same as Figure 24 with a 3 kHz source center frequency.
Permeable, $f = 3.0 \text{ kHz}$

![Graph showing waveforms and time measurements.](image-url)
Figure 26: Comparison of the sensitivity coefficients of the Stoneley wave and the flexural mode for both boundary conditions. (a): bore fluid Lamé coefficient, (b): saturant fluid Lamé coefficient, (c): $c_{44}$, (d): $c_{66}$, (e): $c_{11}$, (f): $c_{13}$, (g): $c_{33}$. 
The graph shows the sensitivity for different types of wave propagation in a material.

- **C33**
  - **Flexural**: Sensitivity values peak around mid-frequency, with a sharp decline on either side.
  - **Stoneley**: Sensitivity values remain relatively flat across the frequency range.

Two curves represent the sensitivity for permeable and impermeable materials, indicating different behaviors at various frequencies.
Figure 27: Dispersion and attenuation of the first trapped mode in the presence of the transversely isotropic water saturated porous fast formation when the borehole wall is permeable. The velocities are normalized with respect to the bore fluid velocity. Compare with Figure 13b.
TRAPPED MODE
Permeable borehole wall

Figure 28: Sensitivity coefficients for the first trapped mode in the presence of the transversely isotropic water saturated porous fast formation when the borehole wall is permeable. Compare with Figure 14a.
Figure 29: Impermeable borehole wall. Dispersion and attenuation of the Stoneley wave in the presence of the transversely isotropic water saturated porous slow formation (TIP) and the isotropic formation with a 1 darcy isotropic permeability (ISO). The velocities are normalized with respect to the bore fluid velocity.
Figure 30: Impermeable borehole wall. Sensitivity coefficients for the Stoneley wave in the presence of the transversely isotropic water saturated slow formation (a) and the isotropic formation with a 1 darcy isotropic permeability (b). In the case (a), the low frequency values have been extrapolated due to numerical instability related to the leaking character of the interface wave (see Figure 29).
Figure 31: Same as Figure 29 with a permeable borehole wall.
Figure 32: Permeable borehole wall. Sensitivity coefficients for the Stoneley wave in the presence of the transversely isotropic water saturated slow formation (a, b) and the isotropic formation with a 1 darcy isotropic permeability (c). Compare with Figure 30.
STONELEY WAVE
Isotropic formation
Permeable borehole wall

SENSITIVITY

FREQUENCY (kHz)
Figure 33: Monopole source. Iso-offset (z = 4 m) comparison of the synthetic microseismograms obtained with a 1 kHz source center frequency in the presence of the slow water saturated porous formation with an isotropic skeleton and isotropic 100 mdarcies permeability (Iso. 100.md), an isotropic skeleton and isotropic 1 darcy permeability (Iso. 1.d), an isotropic skeleton and transversely isotropic permeability (1 darcy in the horizontal plane and 100 mdarcies in the vertical direction, i.e., 1.1dH-100.mdV). The borehole wall is impermeable (a) and permeable (b). Each wavetrain is normalized with respect to the maximum of the whole series indicated by max in arbitrary units. The scaling coefficient at the upper left gives the relative value of the peak amplitude compared to the latter. The arrows indicate the arrival time of the interface wave computed using the zero frequency limit of the group velocity (equ. (49)).
Permeable $f=1.0 \text{ kHz}$

Time (ms)

- ISO $1d$
- ISO $1dH \rightarrow 1dV$
- TIP

Max = 57.13
Figure 34: Same as Figure 33 with a 5 kHz source center frequency.
Permeable, $f = 5.0 \text{ kHz}$

(b) TIME (ms)

- ISO. 1d
- TIP
Figure 35: Dispersion and attenuation of the flexural mode, when the borehole wall is impermeable (a) and permeable (b), in the presence of the transversely isotropic water saturated porous slow formation (TIP) and the isotropic formation with a 1 darcy isotropic permeability (ISO). The velocities are normalized with respect to the bore fluid velocity.
Transversely Isotropic Saturated Porous Formations II

(b) FREQUENCY (kHz)

Normalized Velocity

- phase
- group

Frequency (kHz)

Attenuation (\(1/Q\) x 10\(^{-1}\))

Iso.

TIP

Iso.

TIP
Figure 36: Comparison of the dispersion and attenuation of the Stoneley wave (St.) and the flexural mode (Flex.) in the presence of the transversely isotropic water saturated porous slow formation when the borehole wall is impermeable (a) and permeable (b).
Transversely Isotropic Saturated Porous Formations II

![Diagram showing normalized velocity and attenuation as functions of frequency.](image)

**Normalized Velocity:**
- **Phase:**
- **Group:**

**Attenuation (1/Q) x 10^6:**
- **St.**
- **Flex.**

**Frequency (kHz):**
- 0
- 5
- 10
- 15

**Normalized Velocity:**
- 0.820
- 0.763
- 0.707
- 0.650

**Attenuation (1/Q) x 10^6:**
- 0.400
- 0.267
- 0.133
- 0.000
Figure 37: Dipole source. Iso-offset ($z = 4$ m) comparison of the synthetic microseismograms obtained with a $1$ kHz source center frequency in the presence of the slow water saturated porous formation with an isotropic skeleton and isotropic $100$ mdarcies permeability (Iso. 100.md), an isotropic skeleton and isotropic $1$ darcy permeability (Iso. 1.d), an isotropic skeleton and transversely isotropic permeability ($1$ darcy in the horizontal plane and $100$ mdarcies in the vertical direction, i.e., 1.dH-100.mdV). The borehole wall is impermeable (a) and permeable (b). Each wavetrain is normalized with respect to the maximum of the whole series indicated by max in arbitrary units. The scaling coefficient at the upper left gives the relative value of the peak amplitude compared to the latter. The arrows are at a time equal to the offset divided by the velocity of the vertically propagating $SV$ wave of the formation.
Permeable, $f = 1.0\, \text{kHz}$

$0.000$

$I_{0.879}^1$

$0.879$

$I_{0.879}^1$

$0.521$

$T_{I_{0.521}}$

TIME (ms)
Figure 38: Same as Figure 37 with a 3 kHz source center frequency.
Transversely Isotropic Saturated Porous Formations II

Permeable, \( f = 3.0 \text{ kHz} \)

\[
\begin{align*}
1.000 & \quad \text{max} = 0.49E+00 \\
0.447 & \quad \text{ISO. 1 d} \\
0.457 & \quad \text{ISO. 1 dH → 1 dV} \\
0.317 & \quad \text{TIP}
\end{align*}
\]

TIME (ms)
Figure 39: Sensitivity coefficients of the flexural mode in the presence of the transversely isotropic water saturated porous slow formation for both boundary conditions (a), and in the presence of the isotropic formation with a 1 darcy isotropic permeability when the borehole wall is permeable.
Figure 40: Invaded zone effects with an impermeable borehole wall. Dispersion and attenuation of the Stoneley wave in the presence of the transversely isotropic water saturated porous fast formation alone (Virg.), the invaded zone alone (Inv.), and when the thickness of this last is equal to 3 cm and 15 cm. The velocities are normalized with respect to the bore fluid velocity. The parameters are given in Table 1.
Figure 41: Invaded zone effects with an impermeable borehole wall. Sensitivity coefficients of the Stoneley wave in the presence of a 3 cm thick invaded zone. The virgin formation is the transversely isotropic water saturated porous fast formation (see Table 1). Compare with Figure 3.
Figure 42: Same as Figure 41 with a 15 cm thick invaded zone.
Figure 43: Invaded zone effects with a permeable borehole wall. Dispersion and attenuation of the Stoneley wave up to 5 kHz in the presence of the transversely isotropic water saturated sandstone fast formation alone (Virg.), the invaded zone alone (Inv.), and when the thickness of this last is equal to 3 cm and 15 cm. The velocities are normalized with respect to the bore fluid velocities.
Figure 44: Invaded zone effects with a permeable borehole wall. Sensitivity coefficients in the low frequencies for the Stoneley wave in the presence of a 3 cm thick invaded zone. The virgin formation is the transversely isotropic water saturated fast formation (see Table 1). Compare with Figure 6.
Figure 45: Same as Figure 44 with a 15 cm thick invaded zone.
Figure 46: Same as Figure 43 up to 25 kHz.
Figure 47: Same as Figure 44 up to 25 kHz. Compare with Figure 7.
Figure 48: Same as Figure 45 up to 25 kHz.
Figure 49: Monopole source. Invaded zone effects. Iso-offset ($z = 5$ m) comparison of the synthetic microseismograms obtained at the center of the borehole with a 1 kHz source center frequency in the presence of the invaded zone alone, when it is 3 cm thick (composite), and when the virgin formation is alone. The borehole wall is impermeable (a), and permeable (b).
Figure 50: Invaded zone effects. Dispersion and attenuation of the first pseudo-Rayleigh mode in the presence of the transversely isotropic water saturated porous fast formation alone (Virg.), the invaded zone alone (Inv.), and when the thickness of this last is equal to 3 cm and 15 cm. The borehole wall is impermeable (a) and permeable (b). The parameters are given in Table 1.
(b)
Figure 51: Invaded zone effects with a permeable borehole wall. Sensitivity coefficients of the first pseudo-Rayleigh mode in the presence of a 3 cm thick invaded zone. The virgin formation is the transversely isotropic water saturated fast formation (see Table 1). Similar results are obtained when the borehole wall is impermeable. Compare with Figure 14a.
Figure 52: Same as Figure 51 for a 15 cm thick invaded zone.
Figure 53: Same as Figure 49 with a 7.5 kHz source center frequency.
Figure 54: Invaded zone effects. Dispersion and attenuation of the flexural mode in the presence of the transversely isotropic water saturated porous fast formation alone (Virg.), the invaded zone alone (Inv.), and when the thickness of this last is equal to 3 cm and 15 cm. The borehole wall is impermeable (a) and permeable (b). The parameters are given in Table 1.
Figure 55: Invaded zone effects with a permeable borehole wall. Sensitivity coefficients of the flexural mode in the presence of a 3 cm thick invaded zone. The virgin formation is the transversely isotropic water saturated porous fast formation (see Table 1). Similar results are obtained with an impermeable borehole wall. Compare with Figure 23a.
Figure 56: Invaded zone effects with a permeable borehole wall. Sensitivity coefficients of the flexural mode in the presence of 15 cm thick invaded zone. The virgin formation is the transversely isotropic water saturated porous fast formation (see Table 1). Similar results are obtained with an impermeable borehole wall. Compare with Figures 54 and 23a.
Figure 57: Dipole source. Invaded zone effects. Iso-offset \( (z = 5 \text{ m}) \) comparison of the synthetic microseismograms obtained at the center of the borehole with a 1 kHz source center frequency in the presence of the invaded zone alone, when it is 3 cm thick (composite), and when the virgin formation is alone. The borehole wall is impermeable (a), and permeable (b). The arrows indicate the vertically propagating \( SV \)-wave arrival times computed using ray theory.
Figure 58: Dipole source. Invaded zone effects. Iso-offset ($z = 5$ m) comparison of the synthetic microseismograms obtained at the center of the borehole with a 3 kHz source center frequency in the presence of the invaded zone alone, when it is 3 cm thick (composite), and when the virgin formation is alone. The borehole wall is impermeable (a), and permeable (b). The arrows indicate the vertically propagating $SV$-wave arrival times computed using ray theory.
Figure 59: Well bonded cased hole. Dispersion and attenuation of the Stoneley wave in the presence of the transversely isotropic water saturated porous fast formation. The layers parameters are given in Tables 1 and 3. Compare with Figure 2.
Figure 60: Well bonded cased hole. Sensitivity coefficients of the Stoneley wave in the presence of the transversely isotropic saturated porous fast formation. The parameters are given in Tables 1 and 3. Compare with Figure 3. MX and PX refer to the shear modulus of the layer of index X $\mu_b$, and to $\lambda_b + 2\mu_b$, respectively. 2= Casing; 3= Cement.
Figure 61: Free pipe model. Dispersion and attenuation of the central Stoneley wave in the presence of the transversely isotropic water saturated porous fast formation. The parameters are given in Tables 1 and 3. Compare with Figure 59.
Figure 62: Free pipe model. Sensitivity coefficients of the central Stoneley wave in the presence of the transversely isotropic water saturated porous formation for both boundary conditions at the fluid annulus-formation interface. L3 indicate the fluid annulus Lamé coefficient.
Figure 63: Free pipe model. Dispersion and attenuation of the annulus Stoneley wave in the presence of the transversely isotropic water saturated porous formation for both boundary conditions at the fluid annulus-formation interface.
Figure 64: Free pipe model. Sensitivity coefficients for the annulus Stoneley wave in the presence of the transversely isotropic water saturated porous fast formation for both boundary conditions at the fluid annulus-formation interface.
Figure 65: Monopole source. Free pipe model. Iso-offset ($z = 5 \text{ m}$) comparison of the synthetic microseismograms relative to both boundary conditions obtained with a 3 kHz (a) and 5 kHz (b) source center frequency at the center of the borehole.
Figure 66: Well bonded cased hole. Dispersion and attenuation of the first pseudo-Rayleigh mode in the presence of the transversely isotropic water saturated fast formation. The parameters are given in Tables 1 and 3. Compare with Figures 13 and 21.
Figure 67: Well bonded configuration. Sensitivity coefficients for the first pseudo-Rayleigh mode in the presence of the transversely isotropic water saturated porous fast formation. Compare with Figure 14.
Figure 68: Free pipe model. Dispersion and attenuation of the first pseudo-Rayleigh mode in the presence of the transversely isotropic water saturated porous fast formation for both boundary conditions at the fluid annulus-formation interface. Compare with Figure 66.
Figure 69: Free pipe model. Sensitivity coefficients for the first pseudo-Rayleigh mode in the presence of the transversely isotropic water saturated porous fast formation for both boundary conditions at the fluid annulus-formation interface. Compare with Figure 67.
Figure 70: Well bonded cased hole. Dispersion and attenuation of the flexural mode in the presence of the transversely isotropic water saturated porous fast formation. Compare with Figure 22.
Figure 71: Well bonded cased hole. Sensitivity coefficients for the flexural mode in the presence of the transversely isotropic water saturated porous fast formation.
Figure 72: Free pipe model. Dispersion and attenuation of the flexural mode in the presence of the transversely isotropic water saturated porous fast formation for both boundary conditions at the fluid annulus-formation. Compare with Figure 70.
Figure 73: Free pipe model. (a) Sensitivity coefficients for the flexural mode in the presence of the transversely isotropic water saturated porous fast formation for both boundary conditions at the fluid annulus-formation interface. (b) Focus on the sensitivity coefficient with respect to the saturant fluid Lamé coefficient.