INVERSION OF GEOPHYSICAL LOGS TO DETERMINE LITHOLOGY

by

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ABSTRACT

We develop a linear programming inversion scheme to estimate lithology from well log data. The inversion described severely limits the dependence on core retrieval, and makes accurate mapping of new parameters possible. There are no limitations on the number of input well logs or output lithologic fractions. Because the method uses an L-1 norm error measure, it should be more robust than otherwise similar techniques. Other advantages of the technique include the straightforward imposition of inequality constraints, and strictly positive volume fraction results. As specific examples we apply the method in a variety of geologic situations, and compare our results to those achieved via coring. We also test the new method against conventional least squares inversion algorithms. While both methods perform well, the linear programming inversion yields improved results when formation petrophysical properties are not known accurately.

INTRODUCTION

In this paper we investigate further the solution of the well log lithology inverse problem. Although many log analysts continue to use crossplot techniques, the numerical study of the lithology problem was first undertaken by Burke et al. (1969). Widespread availability of computer power now makes these techniques useful in even the remotest locations. The driving force behind lithology inversion is that it obviates or severely lessens the need for core or outcrop rock samples. Since inversion provides a continuous record, it may be possible to obtain results spatially dense enough for mapping in two or three dimensions. These data have important implications for the study of paleoenvironments and areas of potential hydrocarbon accumulation. Several applications of full waveform acoustic logging (e.g., mechanical properties for sanding or hydrofracturing) also require independent lithology information.
A variety of solutions for the lithology inverse problem have already been proposed. These include least squares methods (see for example Doveton, 1986; Gilchrist et al., 1982), and a priori methods (McCammon, 1970). Statistical techniques, including principal components and clustering, are discussed in Wolff and Pelissier-Combescure, (1982). The most frequently encountered difficulties shared by most of these methods are: (1) a lack of robustness manifested by lithologic fractions greater than unity or less than zero; (2) missing or insufficient estimates of the solution error; and (3) a strong dependence on formation end member properties.

In the remaining sections we develop an inversion scheme which uses linear programming. As with other methods, there are no special requirements on the number of input logs or on the number of lithologic components. We only require that a linear model relates the observables and the lithologic fractions. We test the new method in two widely disparate geologic situations, and in each case we compare our results to those of the least squares techniques and available core data. We show that while both methods yield useful results, the linear programming inverse is less affected when formation properties are not accurately known.

MATHEMATICAL FORMULATION

As an example consider the case in which four observable quantities (e.g., density, neutron, sonic, and gamma ray) are used to study a lithology consisting of three end-members. Using $x_i$ to represent the lithologic fractions, we can write the following mathematical model:

$$\begin{bmatrix}
1 & 1 & 1 \\
\rho_1 & \rho_2 & \rho_3 \\
\phi_1 & \phi_2 & \phi_3 \\
\Delta t_1 & \Delta t_2 & \Delta t_3 \\
\gamma_1 & \gamma_2 & \gamma_3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
\rho_{meas} \\
\phi_{meas} \\
\Delta t_{meas} \\
\gamma_{meas}
\end{bmatrix}
$$

or the equivalent matrix equation $Ax = b$. The first equation in (1) expresses conservation of volume. The remaining equations are simple linear forms of log response equations. We emphasize that this lithology model is complete, i.e., it is assumed that the formation is composed entirely of mineral fractions $x_1 + x_2 + x_3$. The formulation is general because the dimensions of $x$ and $b$ (and hence $A$) can change to accommodate different measurements and different lithologic models. In practice it is usually possible to have $m > n$ (row and column dimensions of $A$; i.e., more measurements than unknowns) and this is desirable because a priori information about lithologic variation is often difficult to obtain.

Equations (1) express the forward or direct problem, in which we can calculate the
observable log responses given relevant rock properties and a model of the formation. In contrast we are generally faced with solving the inverse problem of estimating a formation model based on a set of observable log data. The least squares operator for such a problem is well known (e.g. Aki and Richards, 1980):

\[ L = (A^T W A)^{-1} A^T W. \]

\( W \) is a diagonal weighting matrix used to equalize the ranges of the various log data. Then the estimate of the model component volumes is

\[ x_{LS} = Lb \]

Errors in the input logs \( \sigma_d \) are mapped into the least squares solutions according to

\[ \text{cov}_{LS} a = (\Delta x_{LS} \Delta x_{LS}^T) = \sigma_d^2 LL^T \]

because least squares uses an L-2 norm error measure.

Linear Programming Formulation

In contrast to least squares, linear programs minimize (or maximize) an objective, or cost, function. The objective function is any linear function of the unknowns. Since in well logging one component is of particular interest (e.g., porosity, oil fraction, or ore component) we use this component as the objective function. By in turn minimizing, then maximizing, the objective we find upper and lower bounds for that component's estimate. Because the objective function is linear, we are minimizing an L-1 norm measure of error. Therefore the linear programming method should be more robust because it is less sensitive to outlier data. Three other points about linear programming are advantages in so far as they relate to the well log lithology problem. First, linear programming solution vectors can only contain non-negative entries, i.e., the search space contains no negative \( x_i \). Second, it is straightforward to impose inequality constraints such as \( x_i \leq 1 \) (again, volume fractions > 1 are non-physical) or \( x_1 > x_3 \) (the volume of matrix is greater than the volume of water, for example.) Third, volume conservation is exactly maintained.

The essence of a linear program has two parts: the first is an objective function, a linear function of the model parameters, to be minimized (or maximized). The second part is a set of constraints which confine the possible solution space. The simplex procedure (due to Dantzig, 1963) is the grandfather of all solution techniques for linear programs. The algorithm begins by finding any solution vector in the subspace defined by the constraint equations, and then moving to an edge of the subspace (the "simplex") such that the objective function is minimized. Since the objective function is linear, the optimal solution must lie at a corner of the simplex. The process continues
until no move can further decrease the value of the objective function. The reader can find interesting details in Strang (1986) or Lunenberg (1973).

We construct the linear programming problem by rewriting each of the log response equations (Eqs. 1) as a set of two inequality constraints of the form

\[ \sum_{j=1}^{n} A_{ij} x_j \geq b_i - \sigma_i \quad \sum_{j=1}^{n} A_{ij} x_j \leq b_i + \sigma_i \quad \] (5)

\( \sigma_i \), the standard deviation of the \( i \)th observable, is uniformly taken as \( 0.1 \times b_i \). In solution space, each pair of constraints defines a tabular region sandwiched between planes defined by the two halfspaces in (5). We could further confine the solution space twice more into permissible halfspaces with inequality constraints such as

\[ x_1 > x_3 \quad \text{and} \quad x_1 > x_2 \quad \] (6)

Finally we conserve volume by insisting the optimal solution lies on the plane defined by \( x_1 + x_2 + x_3 = 1 \). On the resulting plane region, we first minimize, then in turn maximize, the objective, \( f(x) = x_i \), to provide upper and lower bounds on the estimates at each successive depth.

It is easy to imagine that the intersection of all the constraint planes does not form a convex set. If a solution exists which satisfies all the constraints, then the set of solutions is a convex set, i.e., one in which all points on a line joining two candidate solutions are themselves candidates. Conversely if the log response constraints do not form a simplex, we can increase the separation between conjugate planes by increasing the standard deviations (Equation 5). We begin with uniform 10% data standard deviations, increasing in 10% steps, until all the observed log data could be fit. While we occasionally used values as high as 40%, most data could be fit with 10% or 20% standard deviation.

Error analysis for the linear programming inverse proceeds differently than for the weighted least squares solution. We address the question of solution variance in linear programming by defining an average model

\[ \hat{x} = \frac{1}{2} (x_{up} + x_{low}) \quad \] (7)

The variance in the average model \( \hat{x} \) is \( \Delta \hat{x} \) (Cheng, 1978):

\[ \Delta \hat{x} = \frac{1}{2} (x_{up} - x_{low}) \quad \] (8)

This is obviously justified since the bounds correspond to minimizing and maximizing the objective function. Variances for the other phases may be calculated by suitably changing the objective function.
DISCUSSION OF INVERSION RESULTS

A key issue surrounding any inverse problem is the question "how meaningful is the answer?" When two different techniques are investigated, we naturally try to choose between them. Each theory provides a different estimate of solution error and resolution. Further, we often have core-measured data. These data sometimes come from small samples, and are not error-free themselves. In this section we discuss a variety of means by which to judge two different inversion results. Our results indicate that linear programming performs better when formation end member properties are not accurately known.

We will discuss in turn two different applications of the linear programming lithology inversion. In each case we compare our results to those of a weighted least squares algorithm. The first application comes from a well drilled by the Ocean Drilling Project in the Mediterranean Sea (ODP Site 651). In this case it was important to estimate the fraction of volcanic sediment in the section. Since core retrieval was difficult in this well, lithology inversion proved especially useful. Whenever possible, inversion results are compared with direct measurement. The second example is taken from four wells in the British sector of the North Sea and one well in California. In these wells it was of interest to determine the fraction of organic material in a source rock shale. The results of the inversion algorithms are also compared to geochemical measurements of organic content made in the laboratory.

ODP Site 651

This hole was drilled near the center of a young back arc basin which subsided very quickly. The basin now lies in 5000 meter of water, yet the oldest sediments (1.3 Ma) were deposited in shallow water. Laboratory analyses made aboard the drilling vessel revealed the sediment had three distinct end members:

1. pelagic carbonate ooze
2. terrigenous reworked sediments
3. volcanic ash

The sporadic addition of ash to the total sedimentary volume is important in modeling the rapid subsidence of the basin.

We now use linear programming to invert for each of the lithologic volume fractions in the above model. We will estimate upper and lower bounds on the volcanic fraction by making it the cost function. We used three logs (uranium, potassium, and Δt) and
the end member properties shown in Table 1. We chose these logs because of their large variability over the logged section, and their insensitivity to borehole wall rugosity. An interesting discussion of the abundance of radioactive elements in the volcanic sediments in Italian land sections (Locardi et al., 1976) was useful in estimating the contribution of these rocks in the measured radioactivity logs.

Table 1: Petrophysical end-member properties

<table>
<thead>
<tr>
<th></th>
<th>Pelagic</th>
<th>Terrigenous</th>
<th>Volcanigenic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uranium, ppm</td>
<td>5</td>
<td>12</td>
<td>18</td>
</tr>
<tr>
<td>Potassium, %</td>
<td>0.1</td>
<td>0.8</td>
<td>0.5</td>
</tr>
<tr>
<td>$\Delta t, \mu\text{sec/ft}$</td>
<td>185</td>
<td>130</td>
<td>165</td>
</tr>
</tbody>
</table>

The results of the linear programming inversion are plotted in Figure 1. We calculate an inversion result approximately twice per meter from 120 meters below sea floor to 260 m.b.s.f., the depth at which spectral gamma ray data ends. Figure 1 shows two lithologic cross-sections, corresponding to minimizing the volcanic fraction (right) and maximizing the volcanic fraction (left). In each cross-section, the shaded region represents the volcanic fraction, the central region represents the pelagic fraction, and the leftmost region depicts the terrigenous fraction.

As a first check on the inversion results we compare our predictions of the pelagic fraction with shipboard measurements of CaCO$_3$ on recovered material from the same depth. This is shown, plotted versus depth, in Figure 2. In general we see qualitative agreement between the core measurements and the inversion results. Agreement is best towards the bottom of the logged section, with increasing scatter towards the top. Even when absolute agreement is poor, trends in the core data are mimicked by the linear programming inversion results. The average r.m.s. error relative to the core measurements is 20.4%. Linear programming variance in the volcanic faction estimate is equal to 10.5% over the logged interval (from Eq. 8). Variances for the other phases may be calculated by suitably changing the objective function.

Figure 3 shows the results of least squares inversion plotted in the same way as for Figure 1. We used the same model rock properties as in the linear programming inverse. Note that in this case many of the volume fraction results are negative and or greater than unity. This makes it impossible to calculate the r.m.s. core error without rescaling the least squares results, and severely degrades the usefulness of the results.

Another important characteristic of the inversion is its ability to fit the observed log data. We obtain the fit to the data by first inverting for the model, and then substituting this result back into the forward problem. We can then compare errors in the same units as the log data. The original and reconstructed logs from linear programming are shown in Figure 4. The reconstructed logs from both minimum and
maximum volcanic fraction estimates are plotted as points, while the actual log data is plotted as a curve. Ideally the points should surround measured values. When this is not the case we say that the inversion result has been degraded, perhaps due to excessive noise in the data or an unrepresentative model.

The results of the linear programming inversion can be summarized over the logged interval as follows. Table 2 gives the percent of the total thickness (140 m) made up of each of the three lithologies. The first column corresponds to minimizing the volcanic contribution, while the second column corresponds to maximizing the same.

<table>
<thead>
<tr>
<th>Lithology</th>
<th>Percent Thickness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Terrigenous</td>
<td>min. volc.</td>
</tr>
<tr>
<td></td>
<td>30.9</td>
</tr>
<tr>
<td>Pelagic</td>
<td>44.7</td>
</tr>
<tr>
<td>Volcanigenic</td>
<td>24.5</td>
</tr>
</tbody>
</table>

The most striking result of this inversion is that the fraction of volcanigenic sediment lies roughly between 25% and 45%. This is a good deal higher than is indicated by the study of recovered material (average < 10%). We suggest that in fact there is a built-in bias in studying the core material, because some lithologies are readily retrieved while others are not. To support this argument, we redraw in Figure 5 the percentage of recovered core material as a function of depth. Comparison of Figure 5 with Figures 1 and 2 shows that carbonate-rich sections were generally recovered, and at the same time, volcanic-rich sections were preferentially lost.

**North Sea and California**

In this section we show a significant reduction of r.m.s. core error is achieved by linear programming over least squares when the end member log properties are not accurately known. The shale sections of four wells in the North Sea and one well in California were selected for source rock analysis. We invert for total organic content, an important basin modeling parameter in potential hydrocarbon areas. We also invert for a shale matrix and a pore water fraction. An inversion scheme which is relatively less sensitive to the choice of model rock properties will be called robust. The use of a robust method allows maps of parameters like organic content with greater accuracy than is possible by other methods.

We used the following four input logs: density, neutron, sonic Δt, and total gamma ray. We tabulated the log responses from depths where geochemical measurements of
organic content were available. These measurements come from sidewall cores, and in the California well, from drill core. A total of 110 samples were measured. Each of the logs was modeled as inequality constraints of the type in (5). We also used constraints of the type (6) such that the matrix fraction was greater than the pore or organic fractions. When comparing lab and inversion estimates of organic content, we convert the lab values from weight to volume fractions using the density log.

The model rock properties were determined individually for each data set by either \textit{a priori} information (in the case of matrix and porosity components) or extrapolation of crossplot results (in the case of the organic fraction). Unfortunately end member constituent properties of the organic phase are not well studied, and extrapolation may introduce errors into the model. Additional sources of error come from the assumption that the properties do not change within a given borehole data set, and from linearizations such as in the time average equation.

The results of the linear programming inversion are plotted for each well in Figure 6. All three components were calculated, but only the organic fraction upper and lower bounds are shown. Laboratory measurements are also shown; these are plotted as crosses. The estimated linear programming bounds are well correlated with the observed organic content data. In almost all cases the bounds enclose the core data. An exception is North Sea Well A, where the discrepancy may be due to poor modeling of some data points.

For comparison, Figure 7 shows, for each well, crossplots of core measured organic fraction versus the least squares inversion result. Perfect agreement would lie along the 45° diagonal. Error bounds are drawn at ±2% volume, which corresponds roughly to the least squares variance on organic fraction predicted by (4) when \( \sigma_d = 10\% \). The variances predicted by the two different theories are given in Table 3.

<table>
<thead>
<tr>
<th>Well</th>
<th>Linear Programming</th>
<th>Least Squares</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2.60</td>
<td>1.33</td>
</tr>
<tr>
<td>B</td>
<td>2.29</td>
<td>6.02</td>
</tr>
<tr>
<td>C</td>
<td>2.12</td>
<td>2.99</td>
</tr>
<tr>
<td>D</td>
<td>4.33</td>
<td>14.42</td>
</tr>
<tr>
<td>California</td>
<td>2.83</td>
<td>1.77</td>
</tr>
</tbody>
</table>

Note that while there is no clear winner in this category, linear programming variances are smaller three out of five times.

A different estimate of the error in the inversion is the difference between predicted and core-measured organic fractions. The lab values are themselves quite accurate (less than 1% absolute error) but are taken from sample volumes of only 5 cm\(^3\) of rock. Rock volumes "seen" by the well logs are on the order of 1 m\(^3\), so a potential problem
exists when comparing these observations. Nonetheless we do so, giving the r.m.s. error in units of volume percent, for both the linear programming and least squares solutions. These errors vary between 1.58% and 5.05% volume, with both methods giving similar errors in all five data sets.

Table 4: Errors in predicted data relative to core data. Units are volume percent.

<table>
<thead>
<tr>
<th></th>
<th>Well A</th>
<th>Well B</th>
<th>Well C</th>
<th>Well D</th>
<th>California</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear Programming</td>
<td>4.11</td>
<td>1.88</td>
<td>4.38</td>
<td>3.50</td>
<td>2.74</td>
</tr>
<tr>
<td>Least Squares</td>
<td>4.17</td>
<td>1.58</td>
<td>5.05</td>
<td>3.32</td>
<td>2.66</td>
</tr>
</tbody>
</table>

We show now that the linear programming inverse is more reliable when the model properties are poorly known. To do this we construct an “average” set of rock properties and attempt to model all five data sets simultaneously. We know from the preceding sections that this is a bad assumption, because the end-member petrophysical properties are not the same for each data set. When we invert all of the data (110 points) with the average properties we get the following results. Both variances and r.m.s. organic fraction errors are smaller for the linear programming inverse (see Table 4). In both inversions, however, the results are degraded compared to those achieved with the petrophysical properties determined by modeling. Thus, while it is clear that the linear programming inverse is less dependent on an accurate knowledge of rock properties, both methods perform better when these properties are known. Even when rock properties are not well known, the linear programming upper and lower bounds provide smaller r.m.s. core errors and significant reductions in solution variance.

Table 5: Inversion of all data points with a single model; r.m.s. errors are from core observations.

<table>
<thead>
<tr>
<th></th>
<th>r.m.s. error</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Least Squares</td>
<td>7.40</td>
<td>5.17</td>
</tr>
<tr>
<td>Linear Programming</td>
<td>6.82</td>
<td>3.74</td>
</tr>
</tbody>
</table>

CONCLUSIONS

We develop and demonstrate a new method for well log lithology inversion problems. The method is presently restricted to linear log responses, but in this form is applicable to a broad spectrum of problems. To derive best and worst case results, we use linear programming, an L-1 norm optimization. Our method has three main advantages
when compared to other techniques such as least squares. First, it is straightforward to impose inequality constraints on the solution. Second, solutions are always positive, and come from the plane $\Sigma x_i = 1$. Third, the method should be more robust (i.e., less sensitive to outlier data) because it minimizes a linear error measure. A consequence of this is to lessen the dependence on an accurate knowledge of the model petrophysical properties.

In studying ODP Site 651, we found that the fraction of volcanics in the uppermost formations was severely underestimated by shipboard core recovery. We saw evidence that a bias in the coring technique instead favored carbonaceous material. Well log inversion indicates an upper bound on the volcanic fraction of 45% and a lower bound of 24%.

The section on organic rich source rocks reveals that if end member petrophysical properties are well known (e.g., by forward modeling) both linear programming and least squares provide reliable lithology results. However, when an accurate knowledge of the end member properties is not obtainable, linear programming results showed smaller variance and decreased error relative to lab measurement.

**ACKNOWLEDGEMENTS**

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REFERENCES


Figure 1: Inversion results by linear programming, ODP Site 651. Cross-section on left shows maximum volcanic fraction, right shows minimum volcanic fraction. In each cross-section, leftmost region represents terrigenous fraction, central region represents pelagic fraction, and shaded region represents volcanic fraction.
Figure 2: Comparison of shipboard CaCO₃ measurements (points) and inversion estimate of pelagic fraction (solid curve). Inversion result is the average solution defined in Eq. 7.
Figure 3: Inversion results by weighted least squares, ODP Site 651. Leftmost region represents terrigenous fraction, central region represents pelagic fraction, and shaded region represents volcanic fraction. Fractions less than zero or greater than unity are explained in the text.
Figure 4: Measured (solid curve) and reconstructed log data. Also at each depth, two points are plotted corresponding to solutions which minimize and maximize the volcanic fraction.
Figure 5: Recovered fraction, cores 14R-28R, Site 651.
Figure 6: Results of the weighted least squares inverse. Axes are the same for all wells: Y-axis is inversion result, X-axis is the laboratory measured value. Units are volume percent. Bands are plotted at ±2%, which corresponds approximately to the average standard deviation for all the wells.
Figure 7: Results of the weighted least squares inverse. Axes are the same for all wells: X-axis is inversion result, Y-axis is the laboratory measured value. Units are volume percent. Bands are plotted at ±2%, which corresponds approximately to the average standard deviation for all the wells.