6.253: Convex Analysis and Optimization
Homework 4

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Problem 1

Let $f : \mathbf{R}^n \mapsto \mathbf{R}$ be the function

$$f(x) = \frac{1}{p} \sum_{i=1}^{n} |x_i|^p$$

where $1 < p$. Show that the conjugate is

$$f^\star(y) = \frac{1}{q} \sum_{i=1}^{n} |y_i|^q,$$

where q is defined by the relation

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Problem 2

(a) Show that if $f_1 : \mathbf{R}^n \mapsto (-\infty, \infty]$ and $f_2 : \mathbf{R}^n \mapsto (-\infty, \infty]$ are closed proper convex functions, with conjugates denoted by ${f_1}^\star$ and ${f_2}^\star$, respectively, we have

$$f_1(x) \leq f_2(x), \qquad \forall\, x \in \mathbf{R}^n,$$

if and only if

$${f_1}^\star(y) \geq {f_2}^\star(y), \qquad \forall\, y \in \mathbf{R}^n.$$

(b) Show that if C_1 and C_2 are nonempty closed convex sets, we have

$$C_1 \subset C_2,$$

if and only if

$$\sigma_{C_1}(y) \leq \sigma_{C_2}(y), \qquad \forall\, y \in \mathbf{R}^n.$$

Construct an example showing that closedness of C_1 and C_2 is a necessary assumption.

Problem 3

Let $X_1, \ldots, X_r$, be nonempty subsets of $\mathbf{R}^n$. Derive formulas for the support functions for $X_1 + \cdots + X_r$, $conv(X_1) + \cdots + conv(X_r)$, $\cup_{j=1}^r X_j$, and $conv\left(\cup_{j=1}^r X_j\right)$.

Problem 4

Consider a function ϕ of two real variables x and z taking values in compact intervals X and Z, respectively. Assume that for each $z \in Z$, the function $\phi(\cdot, z)$ is minimized over X at a unique point denoted $\hat{x}(z)$. Similarly, assume that for each $x \in X$, the function $\phi(x, \cdot)$ is maximized over Z at a unique point denoted $\hat{z}(x)$. Assume further that the functions $\hat{x}(z)$ and $\hat{z}(x)$ are continuous over Z and X, respectively. Show that ϕ has a saddle point (x^*, z^*). Use this to investigate the existence of saddle points of $\phi(x, z) = x^2 + z^2$ over $X = [0, 1]$ and $Z = [0, 1]$.

Problem 5

In the context of Section 4.2.2, let $F(x, u) = f_1(x) + f_2(Ax + u)$, where A is an $m \times n$ matrix, and $f_1 : \mathbf{R}^n \mapsto (-\infty, \infty]$ and $f_2 : \mathbf{R}^m \mapsto (-\infty, \infty]$ are closed convex functions. Show that the dual function is

$$q(\mu) = -f_1^\star(A'\mu) - f_2^\star(-\mu),$$

where $f_1^\star$ and $f_2^\star$ are the conjugate functions of f_1 and f_2, respectively. *Note:* This is the Fenchel duality framework discussed in Section 5.3.5.