LECTURE 4

LECTURE OUTLINE

• Algebra of relative interiors and closures
• Continuity of convex functions
• Closures of functions
• Recession cones and lineality space

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• The \( \text{ri}(C) \) and \( \text{cl}(C) \) of a convex set \( C \) “differ very little.”
  – Any set “between” \( \text{ri}(C) \) and \( \text{cl}(C) \) has the same relative interior and closure.
  – The relative interior of a convex set is equal to the relative interior of its closure.
  – The closure of the relative interior of a convex set is equal to its closure.

• Relative interior and closure commute with Cartesian product and inverse image under a linear transformation.

• Relative interior commutes with image under a linear transformation and vector sum, but closure does not.

• Neither relative interior nor closure commute with set intersection.
CLOSURE VS RELATIVE INTERIOR

- **Proposition:**
  
  (a) We have $\text{cl}(C') = \text{cl}(\text{ri}(C'))$ and $\text{ri}(C') = \text{ri}(\text{cl}(C'))$.
  
  (b) Let $C$ be another nonempty convex set. Then the following three conditions are equivalent:
  
  (i) $C$ and $C$ have the same rel. interior.
  
  (ii) $C$ and $C$ have the same closure.
  
  (iii) $\text{ri}(C) \subset C \subset \text{cl}(C)$.
  
  **Proof:** (a) Since $\text{ri}(C) \subset C$, we have $\text{cl}(\text{ri}(C)) \subset \text{cl}(C)$. Conversely, let $x \in \text{cl}(C)$. Let $x \in \text{ri}(C)$. By the Line Segment Principle, we have
  
  $$\alpha x + (1 - \alpha)x \in \text{ri}(C'), \quad \forall \, \alpha \in (0, 1].$$
  
  Thus, $x$ is the limit of a sequence that lies in $\text{ri}(C')$, so $x \in \text{cl}(\text{ri}(C'))$.

  ![Diagram](image)

  The proof of $\text{ri}(C') = \text{ri}(\text{cl}(C'))$ is similar.
LINEAR TRANSFORMATIONS

- Let $C$ be a nonempty convex subset of $\mathbb{R}^n$ and let $A$ be an $m \times n$ matrix.

  (a) We have $A \cdot \text{ri}(C) = \text{ri}(A \cdot C)$.

  (b) We have $A \cdot \text{cl}(C) \subset \text{cl}(A \cdot C)$. Furthermore, if $C$ is bounded, then $A \cdot \text{cl}(C) = \text{cl}(A \cdot C)$.

**Proof:** (a) Intuition: Spheres within $C$ are mapped onto spheres within $A \cdot C$ (relative to the affine hull).

(b) We have $A \cdot \text{cl}(C) \subset \text{cl}(A \cdot C)$, since if a sequence $\{x_k\} \subset C$ converges to some $x \in \text{cl}(C)$ then the sequence $\{Ax_k\}$, which belongs to $A \cdot C$, converges to $Ax$, implying that $Ax \in \text{cl}(A \cdot C)$.

To show the converse, assuming that $C$ is bounded, choose any $z \in \text{cl}(A \cdot C)$. Then, there exists $\{x_k\} \subset C$ such that $Ax_k \to z$. Since $C$ is bounded, $\{x_k\}$ has a subsequence that converges to some $x \in \text{cl}(C)$, and we must have $Ax = z$. It follows that $z \in A \cdot \text{cl}(C)$. **Q.E.D.**

Note that in general, we may have

$$A \cdot \text{int}(C) \neq \text{int}(A \cdot C), \quad A \cdot \text{cl}(C) \neq \text{cl}(A \cdot C)$$
INTERSECTIONS AND VECTOR SUMS

• Let $C_1$ and $C_2$ be nonempty convex sets.

(a) We have

\[ \text{ri}(C_1 + C_2) = \text{ri}(C_1) + \text{ri}(C_2), \]
\[ \text{cl}(C_1) + \text{cl}(C_2) \subset \text{cl}(C_1 + C_2) \]

If one of $C_1$ and $C_2$ is bounded, then

\[ \text{cl}(C_1) + \text{cl}(C_2) = \text{cl}(C_1 + C_2) \]

(b) If $\text{ri}(C_1) \cap \text{ri}(C_2) \neq \emptyset$, then

\[ \text{ri}(C_1 \cap C_2) = \text{ri}(C_1) \cap \text{ri}(C_2), \]
\[ \text{cl}(C_1 \cap C_2) = \text{cl}(C_1) \cap \text{cl}(C_2) \]

Proof of (a): $C_1 + C_2$ is the result of the linear transformation $(x_1, x_2) \mapsto x_1 + x_2$.

• Counterexample for (b):

\[ C_1 = \{ x \mid x \leq 0 \}, \quad C_2 = \{ x \mid x \geq 0 \} \]
CARTESIAN PRODUCT - GENERALIZATION

- Let $C$ be convex set in $\mathbb{R}^{n+m}$. For $x \in \mathbb{R}^n$, let

$$C_x = \{ y \mid (x, y) \in C \},$$

and let

$$D = \{ x \mid C_x \neq \emptyset \}.$$

Then

$$\text{ri}(C) = \{ (x, y) \mid x \in \text{ri}(D), y \in \text{ri}(C_x) \}.$$

**Proof:** Since $D$ is projection of $C$ on $x$-axis,

$$\text{ri}(D) = \{ x \mid \text{there exists } y \in \mathbb{R}^m \text{ with } (x, y) \in \text{ri}(C) \},$$

so that

$$\text{ri}(C) = \bigcup_{x \in \text{ri}(D)} \left( M_x \cap \text{ri}(C) \right),$$

where $M_x = \{ (x, y) \mid y \in \mathbb{R}^m \}$. For every $x \in \text{ri}(D)$, we have

$$M_x \cap \text{ri}(C) = \text{ri}(M_x \cap C) = \{ (x, y) \mid y \in \text{ri}(C_x) \}.$$

Combine the preceding two equations. **Q.E.D.**
CONTINUITY OF CONVEX FUNCTIONS

- If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex, then it is continuous.

**Proof:** We will show that $f$ is continuous at 0. By convexity, $f$ is bounded within the unit cube by the max value of $f$ over the corners of the cube.

Consider sequence $x_k \rightarrow 0$ and the sequences $y_k = x_k / \|x_k\|_{\infty}$, $z_k = -x_k / \|x_k\|_{\infty}$. Then

$$f(x_k) \leq (1 - \|x_k\|_{\infty}) f(0) + \|x_k\|_{\infty} f(y_k)$$

$$f(0) \leq \frac{\|x_k\|_{\infty}}{\|x_k\|_{\infty} + 1} f(z_k) + \frac{1}{\|x_k\|_{\infty} + 1} f(x_k)$$

Take limit as $k \rightarrow \infty$. Since $\|x_k\|_{\infty} \rightarrow 0$, we have

$$\limsup_{k \rightarrow \infty} \|x_k\|_{\infty} f(y_k) \leq 0, \quad \limsup_{k \rightarrow \infty} \|x_k\|_{\infty} + 1 f(z_k) \leq 0$$

so $f(x_k) \rightarrow f(0)$. **Q.E.D.**

- Extension to continuity over $\text{ri} (\text{dom}(f))$. 
CLOSURES OF FUNCTIONS

• The closure of a function \( f : X \mapsto \mathbb{R}^n \) is the function \( \text{cl} f : \mathbb{R}^n \mapsto \mathbb{R} \) with

\[
\text{epi}(\text{cl} f) = \text{cl}(\text{epi}(f))
\]

• The convex closure of \( f \) is the function \( \tilde{\text{cl}} f \) with

\[
\text{epi}(\tilde{\text{cl}} f) = \text{cl}(\text{conv}(\text{epi}(f)))
\]

• \textbf{Proposition}: For any \( f : X \mapsto \mathbb{R}^n \)

\[
\inf_{x \in X} f(x) = \inf_{x \in \mathbb{R}^n} (\text{cl} f)(x) = \inf_{x \in \mathbb{R}^n} (\tilde{\text{cl}} f)(x).
\]

Also, any vector that attains the infimum of \( f \) over \( X \) also attains the infimum of \( \text{cl} f \) and \( \tilde{\text{cl}} f \).

• \textbf{Proposition}: For any \( f : X \mapsto \mathbb{R}^n \):

(a) \( \text{cl} f \) (or \( \tilde{\text{cl}} f \)) is the greatest closed (or closed convex, resp.) function majorized by \( f \).

(b) If \( f \) is convex, then \( \text{cl} f \) is convex, and it is proper if and only if \( f \) is proper. Also,

\[
(\text{cl} f)(x) = f(x), \quad \forall x \in \text{ri}(\text{dom}(f)),
\]

and if \( x \in \text{ri}(\text{dom}(f)) \) and \( y \in \text{dom}(\text{cl} f) \),

\[
(\text{cl} f)(y) = \lim_{\alpha \downarrow 0} f(y + \alpha(x - y)).
\]
RECESSION CONE OF A CONVEX SET

- Given a nonempty convex set $C$, a vector $d$ is a *direction of recession* if starting at *any* $x$ in $C$ and going indefinitely along $d$, we never cross the relative boundary of $C$ to points outside $C$:

$$x + \alpha d \in C, \quad \forall \ x \in C, \ \forall \ \alpha \geq 0$$

- *Recession cone* of $C$ (denoted by $R_C$): The set of all directions of recession.
- $R_C$ is a cone containing the origin.
RECESSION CONE THEOREM

• Let $C$ be a nonempty closed convex set.

(a) The recession cone $R_C$ is a closed convex cone.

(b) A vector $d$ belongs to $R_C$ if and only if there exists some vector $x \in C$ such that $x + \alpha d \in C$ for all $\alpha \geq 0$.

(c) $R_C$ contains a nonzero direction if and only if $C$ is unbounded.

(d) The recession cones of $C$ and $\text{ri}(C)$ are equal.

(e) If $D$ is another closed convex set such that $C \cap D \neq \emptyset$, we have

$$R_{C \cap D} = R_C \cap R_D$$

More generally, for any collection of closed convex sets $C_i, i \in I$, where $I$ is an arbitrary index set and $\bigcap_{i \in I} C_i$ is nonempty, we have

$$R_{\bigcap_{i \in I} C_i} = \bigcap_{i \in I} R_{C_i}$$
Let $d \neq 0$ be such that there exists a vector $x \in C$ with $x + \alpha d \in C$ for all $\alpha \geq 0$. We fix $x \in C$ and $\alpha > 0$, and we show that $x + \alpha d \in C$. By scaling $d$, it is enough to show that $x + d \in C$.

For $k = 1, 2, \ldots$, let

$$z_k = x + kd, \quad d_k = \frac{(z_k - x)}{\|z_k - x\|} \|d\|$$

We have

$$d_k = \frac{\|z_k - x\|}{\|z_k - x\| \|d\|} d + \frac{x - x}{\|z_k - x\|}, \quad \|z_k - x\| \to 1, \quad \|z_k - x\| \to 0,$$

so $d_k \to d$ and $x + d_k \to x + d$. Use the convexity and closedness of $C$ to conclude that $x + d \in C$. 

LINEALITY SPACE

• The lineality space of a convex set $C$, denoted by $L_C$, is the subspace of vectors $d$ such that $d \in R_C$ and $-d \in R_C$:

$$L_C = R_C \cap (-R_C)$$

• If $d \in L_C$, the entire line defined by $d$ is contained in $C$, starting at any point of $C$.

• Decomposition of a Convex Set: Let $C$ be a nonempty convex subset of $\mathbb{R}^n$. Then,

$$C = L_C + (C \cap L_C^\perp).$$

• Allows us to prove properties of $C$ on $C \cap L_C^\perp$ and extend them to $C$.

• True also if $L_C$ is replaced by a subspace $S \subset L_C$. 

[Diagram of a convex set $C$ and its lineality space $L_C$]