LECTURE 5

LECTURE OUTLINE

• Directions of recession of convex functions
• Local and global minima
• Existence of optimal solutions

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DIRECTIONS OF RECESSION OF A FN

- We aim to characterize directions of monotonic decrease of convex functions.

- Some basic geometric observations:
  - The “horizontal directions” in the recession cone of the epigraph of a convex function $f$ are directions along which the level sets are unbounded.
  - Along these directions the level sets $\{x \mid f(x) \leq \gamma\}$ are unbounded and $f$ is monotonically nondecreasing.

- These are the directions of recession of $f$. 

![Diagram showing the recession cone, level sets, and epigraph of a convex function.](image-url)
RECESSION CONE OF LEVEL SETS

• **Proposition:** Let \( f : \mathbb{R}^n \mapsto (-\infty, \infty] \) be a closed proper convex function and consider the level sets \( V_\gamma = \{ x \mid f(x) \leq \gamma \} \), where \( \gamma \) is a scalar. Then:
  
  (a) All the nonempty level sets \( V_\gamma \) have the same recession cone:

  \[
  R_{V_\gamma} = \{ d \mid (d, 0) \in \text{Repi}(f) \}
  \]

  (b) If one nonempty level set \( V_\gamma \) is compact, then all level sets are compact.

**Proof:** (a) Just translate to math the fact that

\[ R_{V_\gamma} = \text{the “horizontal” directions of recession of epi}(f) \]

(b) Follows from (a).
**RECESSION CONE OF A CONVEX FUNCTION**

- For a closed proper convex function $f : \mathbb{R}^n \mapsto (-\infty, \infty]$, the (common) recession cone of the nonempty level sets $V_\gamma = \{ x \mid f(x) \leq \gamma \}$, $\gamma \in \mathbb{R}$, is the recession cone of $f$, and is denoted by $R_f$.

- **Terminology:**
  - $d \in R_f$: a direction of recession of $f$.
  - $L_f = R_f \cap (-R_f)$: the lineality space of $f$.
  - $d \in L_f$: a direction of constancy of $f$.

- **Example:** For the pos. semidefinite quadratic

  \[ f(x) = x'Qx + a'x + b, \]

  the recession cone and constancy space are

  \[ R_f = \{ d \mid Qd = 0, a'd \leq 0 \}, \quad L_f = \{ d \mid Qd = 0, a'd = 0 \} \]
RECESSION FUNCTION

• Function $r_f : \mathbb{R}^n \mapsto (-\infty, \infty]$ whose epigraph is $R_{\text{epi}(f)}$ is the recession function of $f$.

• Characterizes the recession cone:

$$R_f = \{ d \mid r_f(d) \leq 0 \}, \quad L_f = \{ d \mid r_f(d) = r_f(-d) = 0 \}$$

since $R_f = \{ (d, 0) \in R_{\text{epi}(f)} \}$.

• Can be shown that

$$r_f(d) = \sup_{\alpha > 0} f(x + \alpha d) - f(x) = \lim_{\alpha \to \infty} f(x + \alpha d) - f(x)$$

• Thus $r_f(d)$ is the “asymptotic slope” of $f$ in the direction $d$. In fact,

$$r_f(d) = \lim_{\alpha \to \infty} \nabla f(x + \alpha d)'d, \quad \forall \ x, d \in \mathbb{R}^n$$

if $f$ is differentiable.

• Calculus of recession functions:

$$r_{f_1 + \cdots + f_m}(d) = r_{f_1}(d) + \cdots + r_{f_m}(d),$$

$$r_{\sup_{i \in I} f_i}(d) = \sup_{i \in I} r_{f_i}(d)$$
• $y$ is a direction of recession in (a)-(d).
• This behavior is independent of the starting point $x$, as long as $x \in \text{dom}(f)$. 
LOCAL AND GLOBAL MINIMA

• Consider minimizing $f : \mathbb{R}^n \mapsto (-\infty, \infty]$ over a set $X \subset \mathbb{R}^n$

• $x$ is feasible if $x \in X \cap \text{dom}(f)$

• $x^*$ is a (global) minimum of $f$ over $X$ if $x^*$ is feasible and $f(x^*) = \inf_{x \in X} f(x)$

• $x^*$ is a local minimum of $f$ over $X$ if $x^*$ is a minimum of $f$ over a set $X \cap \{x \mid \|x - x^*\| \leq \epsilon\}$

**Proposition:** If $X$ is convex and $f$ is convex, then:

(a) A local minimum of $f$ over $X$ is also a global minimum of $f$ over $X$.

(b) If $f$ is strictly convex, then there exists at most one global minimum of $f$ over $X$. 

![Graph of convex function](image.png)
EXISTENCE OF OPTIMAL SOLUTIONS

- The set of minima of a proper $f : \mathbb{R}^n \mapsto (-\infty, \infty]$ is the intersection of its nonempty level sets.

- The set of minima of $f$ is nonempty and compact if the level sets of $f$ are compact.

- (An Extension of the) Weierstrass’ Theorem: The set of minima of $f$ over $X$ is nonempty and compact if $X$ is closed, $f$ is lower semicontinuous over $X$, and one of the following conditions holds:
  
  (1) $X$ is bounded.
  
  (2) Some set $\{x \in X \mid f(x) \leq \gamma\}$ is nonempty and bounded.
  
  (3) For every sequence $\{x_k\} \subset X$ s. t. $\|x_k\| \to \infty$, we have $\lim_{k \to \infty} f(x_k) = \infty$. (Coercivity property).

Proof: In all cases the level sets of $f \cap X$ are compact. Q.E.D.
Weierstrass’ Theorem specialized to convex functions: Let $X$ be a closed convex subset of $\mathbb{R}^n$, and let $f : \mathbb{R}^n \mapsto (-\infty, \infty]$ be closed convex with $X \cap \text{dom}(f) \neq \emptyset$. The set of minima of $f$ over $X$ is nonempty and compact if and only if $X$ and $f$ have no common nonzero direction of recession.

Proof: Let $f^* = \inf_{x \in X} f(x)$ and note that $f^* < \infty$ since $X \cap \text{dom}(f) \neq \emptyset$. Let $\{\gamma_k\}$ be a scalar sequence with $\gamma_k \downarrow f^*$, and consider the sets

$$V_k = \{x \mid f(x) \leq \gamma_k\}.$$

Then the set of minima of $f$ over $X$ is

$$X^* = \bigcap_{k=1}^{\infty} (X \cap V_k).$$

The sets $X \cap V_k$ are nonempty and have $R_X \cap R_f$ as their common recession cone, which is also the recession cone of $X^*$, when $X^* \neq \emptyset$. It follows $X^*$ is nonempty and compact if and only if $R_X \cap R_f = \{0\}$. Q.E.D.
EXISTENCE OF SOLUTION, SUM OF FNS

• Let \( f_i : \mathbb{R}^n \rightarrow (-\infty, \infty], i = 1, \ldots, m \), be closed proper convex functions such that the function

\[
f = f_1 + \cdots + f_m
\]

is proper. Assume that the recession function of a single function \( f_i \) satisfies \( r_{f_i}(d) = \infty \) for all \( d \neq 0 \). Then the set of minima of \( f \) is nonempty and compact.

• **Proof:** The set of minima of \( f \) is nonempty and compact if and only if \( R_f = \{0\} \), which is true if and only if \( r_f(d) > 0 \) for all \( d \neq 0 \). **Q.E.D.**

• **Example of application:** If one of the \( f_i \) is positive definite quadratic, the set of minima of the sum \( f \) is nonempty and compact.

• Also \( f \) has a unique minimum because the positive definite quadratic is strictly convex, which makes \( f \) strictly convex.
PROJECTION THEOREM

• Let $C$ be a nonempty closed convex set in $\mathbb{R}^n$.
  (a) For every $z \in \mathbb{R}^n$, there exists a unique minimum of
      $$f(x) = \|z - x\|^2$$
      over all $x \in C$ (called the projection of $z$ on $C$).
  (b) $x^*$ is the projection of $z$ if and only if
      $$(x - x^*)(z - x^*) \leq 0, \quad \forall x \in C$$

Proof: (a) $f$ is strictly convex and has compact level sets.
(b) This is just the necessary and sufficient optimality condition
      $$\nabla f(x^*)(x - x^*) \geq 0, \quad \forall x \in C.$$