LECTURE 6

LECTURE OUTLINE

- Nonemptiness of closed set intersections
- Existence of optimal solutions
- Linear and quadratic programming
- Preservation of closure under linear transformation

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• **A fundamental question:** Given a sequence of nonempty closed sets \( \{C_k\} \) in \( \mathbb{R}^n \) with \( C_{k+1} \subseteq C_k \) for all \( k \), when is \( \bigcap_{k=0}^{\infty} C_k \) nonempty?

• Set intersection theorems are significant in at least three major contexts, which we will discuss in what follows:

1. Does a function \( f : \mathbb{R}^n \mapsto (-\infty, \infty] \) attain a minimum over a set \( X \)? This is true if and only if

   \[
   \text{Intersection of nonempty } \{ x \in X \mid f(x) \leq \gamma_k \}
   \]

   is nonempty.

![Level Sets of \( f \)](image-url)
**ROLE OF CLOSED SET INTERSECTIONS II**

2. If $C$ is closed and $A$ is a matrix, is $AC$ closed?  
   Special case:  
   - If $C_1$ and $C_2$ are closed, is $C_1 + C_2$ closed?

3. If $F(x, z)$ is closed, is $f(x) = \inf_z F(x, z)$ closed? (Critical question in duality theory.) Can be addressed by using the relation

$$P(\text{epi}(F)) \subset \text{epi}(f) \subset \text{cl}(P(\text{epi}(F)))$$

where $P(\cdot)$ is projection on the space of $(x, w)$. 
ASYMPTOTIC SEQUENCES

- Given nested sequence \( \{C_k\} \) of closed convex sets, \( \{x_k\} \) is an asymptotic sequence if

\[
x_k \in C_k, \quad x_k \neq 0, \quad k = 0, 1, \ldots
\]

\[
\|x_k\| \to \infty, \quad \frac{x_k}{\|x_k\|} \to \frac{d}{\|d\|}
\]

where \( d \) is a nonzero common direction of recession of the sets \( C_k \).

- As a special case we define asymptotic sequence of a closed convex set \( C \) (use \( C_k \equiv C \)).

- Every unbounded \( \{x_k\} \) with \( x_k \in C_k \) has an asymptotic subsequence.

- \( \{x_k\} \) is called retractive if for some \( k \), we have

\[
x_k - d \in C_k, \quad \forall k \geq k.
\]
RETRACTIVE SEQUENCES

- A nested sequence \( \{C_k\} \) of closed convex sets is *retractive* if all its asymptotic sequences are retractive.

- A closed halfspace (viewed as a sequence with identical components) is retractive.

- Intersections and Cartesian products of retractive set sequences are retractive.

- A polyhedral set is retractive. Also the vector sum of a convex compact set and a retractive convex set is retractive.

- Nonpolyhedral cones and level sets of quadratic functions need not be retractive.
SET INTERSECTION THEOREM I

Proposition: If \( \{C_k\} \) is retractive, then \( \cap_{k=0}^{\infty} C_k \) is nonempty.

- Key proof ideas:
  
  (a) The intersection \( \cap_{k=0}^{\infty} C_k \) is empty iff the sequence \( \{x_k\} \) of minimum norm vectors of \( C_k \) is unbounded (so a subsequence is asymptotic).

  (b) An asymptotic sequence \( \{x_k\} \) of minimum norm vectors cannot be retractive, because such a sequence eventually gets closer to 0 when shifted opposite to the asymptotic direction.
SET INTERSECTION THEOREM II

**Proposition:** Let \( \{C_k\} \) be a nested sequence of nonempty closed convex sets, and \( X \) be a retractive set such that all the sets \( C_k = X \cap C_k \) are nonempty. Assume that

\[
R_X \cap R \subset L,
\]

where

\[
R = \bigcap_{k=0}^{\infty} R_{C_k}, \quad L = \bigcap_{k=0}^{\infty} L_{C_k}
\]

Then \( \{C_k\} \) is retractive and \( \bigcap_{k=0}^{\infty} C_k \) is nonempty.

- **Special cases:**
  - \( X = \mathbb{R}^n, \ R = L \) ("cylindrical" sets \( C_k \))
  - \( R_X \cap R = \{0\} \) (no nonzero common recession direction of \( X \) and \( \bigcap_{k} C_k \))

**Proof:** The set of common directions of recession of \( C_k \) is \( R_X \cap R \). For any asymptotic sequence \( \{x_k\} \) corresponding to \( d \in R_X \cap R \):

1. \( x_k - d \in C_k \) (because \( d \in L \))
2. \( x_k - d \in X \) (because \( X \) is retractive)

So \( \{C_k\} \) is retractive.
NEED TO ASSUME THAT $X$ IS RETRACTIVE

Consider $\cap_{k=0}^{\infty} C_k$, with $C_k = X \cap C_k$

- The condition $R_X \cap R \subset L$ holds
- In the figure on the left, $X$ is polyhedral.
- In the figure on the right, $X$ is nonpolyhedral and nonretractivie, and

$$\cap_{k=0}^{\infty} C_k = \emptyset$$
LINEAR AND QUADRATIC PROGRAMMING

- **Theorem:** Let
  \[ f(x) = x'Qx + c'x, \quad X = \{x \mid a'_j x + b_j \leq 0, \ j = 1, \ldots, r\} \]
  where \( Q \) is symmetric positive semidefinite. If the minimal value of \( f \) over \( X \) is finite, there exists a minimum of \( f \) over \( X \).

**Proof:** (Outline) Write

Set of Minima = \( \cap_{k=0}^{\infty} (X \cap \{x \mid x'Qx+c'x \leq \gamma_k\}) \)

with

\[ \gamma_k \downarrow f^* = \inf_{x \in X} f(x). \]

Verify the condition \( R_X \cap R \subset L \) of the preceding set intersection theorem, where \( R \) and \( L \) are the sets of common recession and lineality directions of the sets

\[ \{x \mid x'Qx + c'x \leq \gamma_k\} \]

**Q.E.D.**
CLOSURE UNDER LINEAR TRANSFORMATION

- Let $C$ be a nonempty closed convex, and let $A$ be a matrix with nullspace $N(A)$.

(a) $AC$ is closed if $RC \cap N(A) \subset LC$.

(b) $A(X \cap C)$ is closed if $X$ is a retractive set and

$$RX \cap RC \cap N(A) \subset LC,$$

Proof: (Outline) Let $\{y_k\} \subset AC$ with $y_k \to y$. We prove $\cap_{k=0}^{\infty} C_k \neq \emptyset$, where $C_k = C \cap N_k$, and

$$N_k = \{x \mid Ax \in W_k\}, \quad W_k = \{z \mid \|z-y\| \leq \|y_k-y\|\}$$

- Special Case: $AX$ is closed if $X$ is polyhedral.
NEED TO ASSUME THAT $X$ IS RETRACTIVE

Consider closedness of $A(X \cap C)$

- In both examples the condition

  $$R_X \cap R_C \cap N(A) \subset L_C$$

  is satisfied.

- However, in the example on the right, $X$ is not retractive, and the set $A(X \cap C)$ is not closed.
CLOSEDNESS OF VECTOR SUMS

- Let $C_1, \ldots, C_m$ be nonempty closed convex subsets of $\mathbb{R}^n$ such that the equality $d_1 + \cdots + d_m = 0$ for some vectors $d_i \in R_{C_i}$ implies that $d_i = 0$ for all $i = 1, \ldots, m$. Then $C_1 + \cdots + C_m$ is a closed set.

- **Special Case:** If $C_1$ and $-C_2$ are closed convex sets, then $C_1 - C_2$ is closed if $R_{C_1} \cap R_{C_2} = \{0\}$.

**Proof:** The Cartesian product $C = C_1 \times \cdots \times C_m$ is closed convex, and its recession cone is $R_C = R_{C_1} \times \cdots \times R_{C_m}$. Let $A$ be defined by

$$A(x_1, \ldots, x_m) = x_1 + \cdots + x_m$$

Then

$$AC = C_1 + \cdots + C_m,$$

and

$$N(A) = \{(d_1, \ldots, d_m) \mid d_1 + \cdots + d_m = 0\}$$

$$R_C \cap N(A) = \{(d_1, \ldots, d_m) \mid d_1 + \cdots + d_m = 0, \ d_i \in R_{C_i}, \forall i\}$$

By the given condition, $R_C \cap N(A) = \{0\}$, so $AC$ is closed. **Q.E.D.**