LECTURE 7

LECTURE OUTLINE

- Partial Minimization
- Hyperplane separation
- Proper separation
- Nonvertical hyperplanes

Reading: Sections 3.3, 1.5

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PARTIAL MINIMIZATION

• Let $F : \mathbb{R}^{n+m} \mapsto (-\infty, \infty]$ be a closed proper convex function, and consider

$$f(x) = \inf_{z \in \mathbb{R}^m} F(x, z)$$

• 1st fact: If $F$ is convex, then $f$ is also convex.

• 2nd fact:

$$P(\text{epi}(F)) \subset \text{epi}(f) \subset \text{cl}\left( P(\text{epi}(F)) \right),$$

where $P(\cdot)$ denotes projection on the space of $(x, w)$, i.e., for any subset $S$ of $\mathbb{R}^{n+m+1}$, $P(S) = \{(x, w) \mid (x, z, w) \in S\}$.

• Thus, if $F$ is closed and there is structure guaranteeing that the projection preserves closedness, then $f$ is closed.

• ... but convexity and closedness of $F$ does not guarantee closedness of $f$. 
PARTIAL MINIMIZATION: VISUALIZATION

- Connection of preservation of closedness under partial minimization and attainment of infimum over $z$ for fixed $x$.

- **Counterexample:** Let

$$ F(x, z) = \begin{cases} 
    e^{-\sqrt{xz}} & \text{if } x \geq 0, \ z \geq 0, \\
    \infty & \text{otherwise.}
\end{cases} $$

- $F$ convex and closed, but

$$ f(x) = \inf_{z \in \mathbb{R}} F(x, z) = \begin{cases} 
    0 & \text{if } x > 0, \\
    1 & \text{if } x = 0, \\
    \infty & \text{if } x < 0,
\end{cases} $$

is not closed.
PARTIAL MINIMIZATION THEOREM

- Let $F : \mathbb{R}^{n+m} \mapsto (-\infty, \infty]$ be a closed proper convex function, and consider $f(x) = \inf_{z \in \mathbb{R}^m} F(x, z)$.
- Every set intersection theorem yields a closed-ness result. The simplest case is the following:

- **Preservation of Closedness Under Compactness:** If there exist $\bar{x} \in \mathbb{R}^n$, $\bar{\gamma} \in \mathbb{R}$ such that the set

  \[ \{ z \mid F(\bar{x}, z) \leq \bar{\gamma} \} \]

  is nonempty and compact, then $f$ is convex, closed, and proper. Also, for each $x \in \text{dom}(f)$, the set of minima of $F(x, \cdot)$ is nonempty and compact.
HYPERPLANES

- A hyperplane is a set of the form \( \{ x \mid a'x = b \} \), where \( a \) is nonzero vector in \( \mathbb{R}^n \) and \( b \) is a scalar.

- We say that two sets \( C_1 \) and \( C_2 \) are separated by a hyperplane \( H = \{ x \mid a'x = b \} \) if each lies in a different closed halfspace associated with \( H \), i.e.,

  \[
  \begin{align*}
  \text{either} & \quad a'x_1 \leq b \leq a'x_2, \quad \forall x_1 \in C_1, \forall x_2 \in C_2, \\
  \text{or} & \quad a'x_2 \leq b \leq a'x_1, \quad \forall x_1 \in C_1, \forall x_2 \in C_2
  \end{align*}
  \]

- If \( x \) belongs to the closure of a set \( C \), a hyperplane that separates \( C \) and the singleton set \( \{ x \} \) is said be supporting \( C \) at \( x \).
• Separating and supporting hyperplanes:

![Visualization](image)

• A separating \( \{ x \mid a'x = b \} \) that is disjoint from \( C_1 \) and \( C_2 \) is called \textit{strictly} separating:

\[
a'x_1 < b < a'x_2, \quad \forall \ x_1 \in C_1, \forall \ x_2 \in C_2
\]
SUPPORTING HYPERPLANE THEOREM

- Let $C$ be convex and let $x$ be a vector that is not an interior point of $C$. Then, there exists a hyperplane that passes through $x$ and contains $C$ in one of its closed halfspaces.

\begin{figure}
\includegraphics[width=\textwidth]{hyperplane_theorem_diagram.png}
\end{figure}

**Proof:** Take a sequence $\{x_k\}$ that does not belong to $\text{cl}(C)$ and converges to $x$. Let $\hat{x}_k$ be the projection of $x_k$ on $\text{cl}(C)$. We have for all $x \in \text{cl}(C)$

$$a'_k x \geq a'_k x_k, \quad \forall \ x \in \text{cl}(C), \ \forall \ k = 0, 1, \ldots,$$

where $a_k = (\hat{x}_k - x_k)/\|\hat{x}_k - x_k\|$. Let $a$ be a limit point of $\{a_k\}$, and take limit as $k \to \infty$. **Q.E.D.**
SEPARATING HYPERPLANE THEOREM

Let $C_1$ and $C_2$ be two nonempty convex subsets of $\mathbb{R}^n$. If $C_1$ and $C_2$ are disjoint, there exists a hyperplane that separates them, i.e., there exists a vector $a \neq 0$ such that

$$a'x_1 \leq a'x_2, \quad \forall x_1 \in C_1, \forall x_2 \in C_2.$$

**Proof:** Consider the convex set

$$C_1 - C_2 = \{x_2 - x_1 \mid x_1 \in C_1, x_2 \in C_2\}$$

Since $C_1$ and $C_2$ are disjoint, the origin does not belong to $C_1 - C_2$, so by the Supporting Hyperplane Theorem, there exists a vector $a \neq 0$ such that

$$0 \leq a'x, \quad \forall x \in C_1 - C_2,$$

which is equivalent to the desired relation. \hspace{1cm} Q.E.D.
**STRICT SEPARATION THEOREM**

- **Strict Separation Theorem:** Let $C_1$ and $C_2$ be two disjoint nonempty convex sets. If $C_1$ is closed, and $C_2$ is compact, there exists a hyperplane that strictly separates them.

![Diagram](image)

**Proof:** (Outline) Consider the set $C_1 - C_2$. Since $C_1$ is closed and $C_2$ is compact, $C_1 - C_2$ is closed. Since $C_1 \cap C_2 = \emptyset$, $0 \notin C_1 - C_2$. Let $x_1 - x_2$ be the projection of 0 onto $C_1 - C_2$. The strictly separating hyperplane is constructed as in (b).

- **Note:** Any conditions that guarantee closedness of $C_1 - C_2$ guarantee existence of a strictly separating hyperplane. However, there may exist a strictly separating hyperplane without $C_1 - C_2$ being closed.
ADDITIONAL THEOREMS

• **Fundamental Characterization:** The closure of the convex hull of a set $C \subset \mathbb{R}^n$ is the intersection of the closed halfspaces that contain $C$. (Proof uses the strict separation theorem.)

• We say that a hyperplane *properly separates* $C_1$ and $C_2$ if it separates $C_1$ and $C_2$ and does not fully contain both $C_1$ and $C_2$.

• **Proper Separation Theorem:** Let $C_1$ and $C_2$ be two nonempty convex subsets of $\mathbb{R}^n$. There exists a hyperplane that properly separates $C_1$ and $C_2$ if and only if

\[
\text{ri}(C_1) \cap \text{ri}(C_2) = \emptyset
\]
PROPER POLYHEDRAL SEPARATION

- Recall that two convex sets $C$ and $P$ such that
  \[
  \text{ri}(C) \cap \text{ri}(P) = \emptyset
  \]
can be properly separated, i.e., by a hyperplane that does not contain both $C$ and $P$.
- If $P$ is polyhedral and the slightly stronger condition
  \[
  \text{ri}(C) \cap P = \emptyset
  \]
holds, then the properly separating hyperplane can be chosen so that it does not contain the non-polyhedral set $C$ while it may contain $P$.

On the left, the separating hyperplane can be chosen so that it does not contain $C$. On the right where $P$ is not polyhedral, this is not possible.
NONVERTICAL HYPERPLANES

- A hyperplane in \( \mathbb{R}^{n+1} \) with normal \((\mu, \beta)\) is nonvertical if \(\beta \neq 0\).
- It intersects the \((n+1)\)st axis at \(\xi = (\mu/\beta)'u + w\), where \((u, w)\) is any vector on the hyperplane.

- A nonvertical hyperplane that contains the epigraph of a function in its “upper” halfspace, provides lower bounds to the function values.
- The epigraph of a proper convex function does not contain a vertical line, so it appears plausible that it is contained in the “upper” halfspace of some nonvertical hyperplane.
NONVERTICAL HYPERPLANE THEOREM

- Let $C$ be a nonempty convex subset of $\mathbb{R}^{n+1}$ that contains no vertical lines. Then:

  (a) $C$ is contained in a closed halfspace of a nonvertical hyperplane, i.e., there exist $\mu \in \mathbb{R}^n$, $\beta \in \mathbb{R}$ with $\beta \neq 0$, and $\gamma \in \mathbb{R}$ such that $\mu' u + \beta w \geq \gamma$ for all $(u, w) \in C$.

  (b) If $(u, w) \notin \text{cl}(C)$, there exists a nonvertical hyperplane strictly separating $(u, w)$ and $C$.

**Proof:** Note that $\text{cl}(C)$ contains no vert. line [since $C$ contains no vert. line, $\text{ri}(C)$ contains no vert. line, and $\text{ri}(C')$ and $\text{cl}(C)$ have the same recession cone]. So we just consider the case: $C$ closed.

  (a) $C$ is the intersection of the closed halfspaces containing $C$. If all these corresponded to vertical hyperplanes, $C$ would contain a vertical line.

  (b) There is a hyperplane strictly separating $(u, w)$ and $C$. If it is nonvertical, we are done, so assume it is vertical. “Add” to this vertical hyperplane a small $\epsilon$-multiple of a nonvertical hyperplane containing $C$ in one of its halfspaces as per (a).