LECTURE 10

LECTURE OUTLINE

• Min Common / Max Crossing duality theorems
• Strong duality conditions
• Existence of dual optimal solutions
• Nonlinear Farkas’ lemma

Reading: Sections 4.3, 4.4, 5.1

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DUALITY THEOREMS

• Assume that $w^* < \infty$ and that the set

$M = \{(u, w) \mid \text{there exists } w \text{ with } w \leq w \text{ and } (u, w) \in M\}$

is convex.

• Min Common/Max Crossing Theorem I: We have $q^* = w^*$ if and only if for every sequence

$\{(u_k, w_k)\} \subset M\text{ with } u_k \to 0,$

there holds

$$w^* \leq \liminf_{k \to \infty} w_k.$$  

\[ (u_k, w_k) \subset M, \; u_k \to 0, \; w^* \leq \liminf_{k \to \infty} w_k \]

\[ (u_k, w_k) \subset M, \; u_k \to 0, \; w^* > \liminf_{k \to \infty} w_k \]

• Corollary: If $M = \text{epi}(p)$ where $p$ is closed proper convex and $p(0) < \infty$, then $q^* = w^*.$)
DUALITY THEOREMS (CONTINUED)

• Min Common/Max Crossing Theorem II: Assume in addition that $-\infty < w^*$ and that

$$D = \{u \mid \text{there exists } w \in \mathbb{R} \text{ with } (u, w) \in M\}$$

contains the origin in its relative interior. Then $q^* = w^*$ and there exists $\mu$ such that $q(\mu) = q^*$.

• Furthermore, the set $\{\mu \mid q(\mu) = q^*\}$ is nonempty and compact if and only if $D$ contains the origin in its interior.

• Min Common/Max Crossing Theorem III: Involves polyhedral assumptions, and will be developed later.
PROOF OF THEOREM I

• Assume that $q^* = w^*$. Let $\{(u_k, w_k)\} \subset M$ be such that $u_k \to 0$. Then,

$$q(\mu) = \inf_{(u,w) \in M} \{w + \mu' u \} \leq w_k + \mu' u_k, \quad \forall k, \forall \mu \in \mathbb{R}^n.$$ 

Taking the limit as $k \to \infty$, we obtain $q(\mu) \leq \lim \inf_{k \to \infty} w_k$, for all $\mu \in \mathbb{R}^n$, implying that

$$w^* = q^* = \sup_{\mu \in \mathbb{R}^n} q(\mu) \leq \lim \inf_{k \to \infty} w_k.$$

Conversely, assume that for every sequence $\{(u_k, w_k)\} \subset M$ with $u_k \to 0$, there holds $w^* \leq \lim \inf_{k \to \infty} w_k$. If $w^* = -\infty$, then $q^* = -\infty$, by weak duality, so assume that $-\infty < w^*$. Steps:

• **Step 1:** $(0, w^* - \epsilon) \notin \text{cl}(M)$ for any $\epsilon > 0$. 

![Diagram showing the proof of Theorem I](image-url)
PROOF OF THEOREM I (CONTINUED)

• **Step 2:** $M$ does not contain any vertical lines. If this were not so, $(0, -1)$ would be a direction of recession of $\text{cl}(M)$. Because $(0, w^*) \in \text{cl}(M)$, the entire halfline $\{(0, w^* - \epsilon) \mid \epsilon \geq 0\}$ belongs to $\text{cl}(M)$, contradicting Step 1.

• **Step 3:** For any $\epsilon > 0$, since $(0, w^* - \epsilon) \notin \text{cl}(M)$, there exists a nonvertical hyperplane strictly separating $(0, w^* - \epsilon)$ and $M$. This hyperplane crosses the $(n + 1)$st axis at a vector $(0, \xi)$ with $w^* - \epsilon \leq \xi \leq w^*$, so $w^* - \epsilon \leq q^* \leq w^*$. Since $\epsilon$ can be arbitrarily small, it follows that $q^* = w^*$. 

![Diagram showing strictly separating hyperplane](image-url)
PROOF OF THEOREM II

• Note that \((0, w^*)\) is not a relative interior point of \(M\). Therefore, by the Proper Separation Theorem, there is a hyperplane that passes through \((0, w^*)\), contains \(M\) in one of its closed halfspaces, but does not fully contain \(M\), i.e., for some \((\mu, \beta) \neq (0, 0)\)

\[
\beta w^* \leq \mu' u + \beta w, \quad \forall (u, w) \in M,
\]

\[
\beta w^* < \sup_{(u,w) \in M} \{\mu' u + \beta w\}
\]

Will show that the hyperplane is nonvertical.

• Since for any \((u, w) \in M\), the set \(M\) contains the halfline \(\{(u, w) \mid w \leq w\}\), it follows that \(\beta \geq 0\). If \(\beta = 0\), then \(0 \leq \mu' u\) for all \(u \in D\). Since \(0 \in \text{ri}(D)\) by assumption, we must have \(\mu' u = 0\) for all \(u \in D\), a contradiction. Therefore, \(\beta > 0\), and we can assume that \(\beta = 1\). It follows that

\[
w^* \leq \inf_{(u,w) \in M} \{\mu' u + w\} = q(\mu) \leq q^*
\]

Since the inequality \(q^* \leq w^*\) holds always, we must have \(q(\mu) = q^* = w^*\).
NONLINEAR FARKAS’ LEMMA

- Let $X \subset \mathbb{R}^n$, $f : X \mapsto \mathbb{R}$, and $g_j : X \mapsto \mathbb{R}$, $j = 1, \ldots, r$, be convex. Assume that
  
  \[ f(x) \geq 0, \quad \forall \ x \in X \text{ with } g(x) \leq 0 \]

  Let

  \[ Q^* = \{ \mu \mid \mu \geq 0, \ f(x) + \mu' g(x) \geq 0, \ \forall \ x \in X \}. \]

  Then $Q^*$ is nonempty and compact if and only if there exists a vector $x \in X$ such that $g_j(x) < 0$ for all $j = 1, \ldots, r$.

- The lemma asserts the existence of a nonvertical hyperplane in $\mathbb{R}^{r+1}$, with normal $(\mu, 1)$, that passes through the origin and contains the set

  \[ \{(g(x), f(x)) \mid x \in X\} \]

  in its positive halfspace.
PROOF OF NONLINEAR FARKAS’ LEMMA

- Apply MC/MC to

\[ M = \{(u, w) \mid \text{there is } x \in X \text{ s. t. } g(x) \leq u, \ f(x) \leq w\} \]

- \(M\) is equal to \(M\) and is formed as the union of positive orthants translated to points \((g(x), f(x))\), \(x \in X\).

- The convexity of \(X\), \(f\), and \(g_j\) implies convexity of \(M\).

- MC/MC Theorem II applies: we have

\[ D = \{u \mid \text{there exists } w \in \mathbb{R} \text{ with } (u, w) \in M\} \]

and \(0 \in \text{int}(D)\), because \(((g(x), f(x)) \in M\).