LECTURE 11

LECTURE OUTLINE

- Min Common/Max Crossing Th. III
- Nonlinear Farkas Lemma/Linear Constraints
- Linear Programming Duality
- Convex Programming Duality

Reading: Sections 4.5, 5.1-5.3

Recall the MC/MC Theorem II: If $-\infty < w^*$ and

$0 \in D = \{ u \mid \text{there exists } w \in \mathcal{R} \text{ with } (u, w) \in M \}$

then $q^* = w^*$ and there exists $\mu$ such that $q(\mu) = q^*$.
Consider the MC/MC problems, and assume that $-\infty < w^*$ and:

1. $M$ is a “horizontal translation” of $\tilde{M}$ by $-P$,
   
   $$M = \tilde{M} - \{(u,0) \mid u \in P\},$$

   where $P$: polyhedral and $\tilde{M}$: convex.

2. We have $\text{ri}(\tilde{D}) \cap P \neq \emptyset$, where
   
   $$\tilde{D} = \{u \mid \text{there exists } w \in \mathbb{R} \text{ with } (u,w) \in \tilde{M}\}$$

Then $q^* = w^*$, there is a max crossing solution, and all max crossing solutions $\mu$ satisfy $\mu'd \leq 0$ for all $d \in R_P$.

- **Comparison with Th. II:** Since $D = \tilde{D} - P$, the condition $0 \in \text{ri}(D)$ of Theorem II is

   $$\text{ri}(\tilde{D}) \cap \text{ri}(P) \neq \emptyset$$
Proof of MC/MC Th. III

- Consider the *disjoint* convex sets $C_1 = \{(u, v) \mid v > w \text{ for some } (u, w) \in \tilde{M} \}$ and $C_2 = \{(u, w^*) \mid u \in P \}$ [if $u \in P$ and $(u, w) \in \tilde{M}$ with $w^* > w$ contradicts the definition of $w^*$]

- Since $C_2$ is polyhedral, there exists a separating hyperplane not containing $C_1$, i.e., a $(\mu, \beta) \neq (0, 0)$ such that

$$\beta w^* + \mu' z \leq \beta v + \mu' x, \quad \forall (x, v) \in C_1, \forall z \in P$$

$$\inf_{(x,v) \in C_1} \{ \beta v + \mu' x \} < \sup_{(x,v) \in C_1} \{ \beta v + \mu' x \}$$

Since $(0, 1)$ is a direction of recession of $C_1$, we see that $\beta \geq 0$. Because of the relative interior point assumption, $\beta \neq 0$, so we may assume that $\beta = 1$. 
• Hence,

\[ w^* + \mu'z \leq \inf_{(u,v) \in C_1} \{v + \mu'u\}, \quad \forall z \in P, \]

so that

\[ w^* \leq \inf_{(u,v) \in C_1, z \in P} \{v + \mu'(u - z)\} \]
\[ = \inf_{(u,v) \in \tilde{M} - P} \{v + \mu'u\} \]
\[ = \inf_{(u,v) \in M} \{v + \mu'u\} \]
\[ = q(\mu) \]

Using \( q^* \leq w^* \) (weak duality), we have \( q(\mu) = q^* = w^* \).

Proof that all max crossing solutions \( \mu \) satisfy \( \mu'd \leq 0 \) for all \( d \in \mathbb{R}_P \): follows from

\[ q(\mu) = \inf_{(u,v) \in C_1, z \in P} \{v + \mu'(u - z)\} \]

so that \( q(\mu) = -\infty \) if \( \mu'd > 0 \). \textbf{Q.E.D.}

• Geometrical intuition: every \((0, -d)\) with \( d \in \mathbb{R}_P \), is direction of recession of \( M \).
MC/MC TH. III - A SPECIAL CASE

- Consider the MC/MC framework, and assume:

  (1) For a convex function \( f : \mathbb{R}^m \mapsto (-\infty, \infty] \), an \( r \times m \) matrix \( A \), and a vector \( b \in \mathbb{R}^r \):

  \[
  M = \{(u, w) \mid \text{for some } (x, w) \in \text{epi}(f), \ Ax - b \leq u \}
  \]

  so \( M = \tilde{M} + \text{Positive Orthant} \), where

  \[
  \tilde{M} = \{(Ax - b, w) \mid (x, w) \in \text{epi}(f)\}
  \]

  (2) There is an \( x \in \text{ri}(\text{dom}(f)) \) s. t. \( Ax - b \leq 0 \). Then \( q^* = w^* \) and there is a \( \mu \geq 0 \) with \( q(\mu) = q^* \).

- Also \( M = M \approx \text{epi}(p) \), where \( p(u) = \inf_{Ax-b \leq u} f(x) \).
- We have \( w^* = p(0) = \inf_{Ax-b \leq 0} f(x) \).
Let $X \subset \mathbb{R}^n$ be convex, and $f : X \mapsto \mathbb{R}$ and $g_j : \mathbb{R}^n \mapsto \mathbb{R}$, $j = 1, \ldots, r$, be linear so $g(x) = Ax - b$ for some $A$ and $b$. Assume that

$$f(x) \geq 0, \quad \forall \ x \in X \text{ with } Ax - b \leq 0$$

Let

$$Q^* = \{ \mu \mid \mu \geq 0, \ f(x) + \mu'(Ax - b) \geq 0, \ \forall \ x \in X \}.$$ 

Assume that there exists a vector $\overline{x} \in \text{ri}(X)$ such that $A\overline{x} - b \leq 0$. Then $Q^*$ is nonempty.

**Proof:** As before, apply special case of MC/MC Th. III of preceding slide, using the fact $w^* \geq 0$, implied by the assumption.
(LINEAR) FARKAS’ LEMMA

• Let $A$ be an $m \times n$ matrix and $c \in \mathbb{R}^m$. The system $Ay = c$, $y \geq 0$ has a solution if and only if

$$A'x \leq 0 \quad \Rightarrow \quad c'x \leq 0. \quad (*)$$

• Alternative/Equivalent Statement: If $P = \text{cone}\{a_1, \ldots, a_n\}$, where $a_1, \ldots, a_n$ are the columns of $A$, then $P = (P^*)^*$ (Polar Cone Theorem).

**Proof:** If $y \in \mathbb{R}^n$ is such that $Ay = c$, $y \geq 0$, then $y'A'x = c'x$ for all $x \in \mathbb{R}^m$, which implies Eq. $(*)$.

Conversely, apply the Nonlinear Farkas’ Lemma with $f(x) = -c'x$, $g(x) = A'x$, and $X = \mathbb{R}^m$. Condition $(*)$ implies the existence of $\mu \geq 0$ such that

$$-c'x + \mu'A'x \geq 0, \quad \forall \ x \in \mathbb{R}^m,$$

or equivalently

$$(A\mu - c)'x \geq 0, \quad \forall \ x \in \mathbb{R}^m,$$

or $A\mu = c$. 
LINEAR PROGRAMMING DUALITY

• Consider the linear program

\[
\begin{align*}
\text{minimize} & \quad c'x \\
\text{subject to} & \quad a'_j x \geq b_j, \quad j = 1, \ldots, r,
\end{align*}
\]

where \( c \in \mathbb{R}^n \), \( a_j \in \mathbb{R}^n \), and \( b_j \in \mathbb{R}, j = 1, \ldots, r \).

• The dual problem is

\[
\begin{align*}
\text{maximize} & \quad b'\mu \\
\text{subject to} & \quad \sum_{j=1}^r a_j \mu_j = c, \quad \mu \geq 0.
\end{align*}
\]

• Linear Programming Duality Theorem:

(a) If either \( f^* \) or \( q^* \) is finite, then \( f^* = q^* \) and both the primal and the dual problem have optimal solutions.

(b) If \( f^* = -\infty \), then \( q^* = -\infty \).

(c) If \( q^* = \infty \), then \( f^* = \infty \).

Proof: (b) and (c) follow from weak duality. For part (a): If \( f^* \) is finite, there is a primal optimal solution \( x^* \), by existence of solutions of quadratic programs. Use Farkas’ Lemma to construct a dual feasible \( \mu^* \) such that \( c'x^* = b'\mu^* \) (next slide).
• Let $x^*$ be a primal optimal solution, and let $J = \{j \mid a'_j x^* = b_j\}$. Then, $c' y \geq 0$ for all $y$ in the cone of “feasible directions”

$$D = \{y \mid a'_j y \geq 0, \forall j \in J\}$$

By Farkas’ Lemma, for some scalars $\mu^*_j \geq 0$, $c$ can be expressed as

$$c = \sum_{j=1}^{r} \mu^*_j a_j, \quad \mu^*_j \geq 0, \forall j \in J, \quad \mu^*_j = 0, \forall j \notin J.$$ 

Taking inner product with $x^*$, we obtain $c' x^* = b' \mu^*$, which in view of $q^* \leq f^*$, shows that $q^* = f^*$ and that $\mu^*$ is optimal.
LINEAR PROGRAMMING OPT. CONDITIONS

A pair of vectors \((x^*, \mu^*)\) form a primal and dual optimal solution pair if and only if \(x^*\) is primal-feasible, \(\mu^*\) is dual-feasible, and

\[
\mu_j^* (b_j - a_j^* x^*) = 0, \quad \forall \ j = 1, \ldots, r. \quad (\ast)
\]

**Proof:** If \(x^*\) is primal-feasible and \(\mu^*\) is dual-feasible, then

\[
b'\mu^* = \sum_{j=1}^{r} b_j \mu_j^* + \left( c - \sum_{j=1}^{r} a_j \mu_j^* \right) x^* \\
= c' x^* + \sum_{j=1}^{r} \mu_j^* (b_j - a_j^* x^*) \quad (\ast\ast)
\]

So if Eq. \((\ast)\) holds, we have \(b'\mu^* = c' x^*\), and weak duality implies that \(x^*\) is primal optimal and \(\mu^*\) is dual optimal.

Conversely, if \((x^*, \mu^*)\) form a primal and dual optimal solution pair, then \(x^*\) is primal-feasible, \(\mu^*\) is dual-feasible, and by the duality theorem, we have \(b'\mu^* = c' x^*\). From Eq. \((\ast\ast)\), we obtain Eq. \((\ast)\).
CONVEX PROGRAMMING

Consider the problem

\[
\begin{align*}
\text{minimize} \quad & f(x) \\
\text{subject to} \quad & x \in X, \ g_j(x) \leq 0, \ j = 1, \ldots, r,
\end{align*}
\]

where \( X \subset \mathbb{R}^n \) is convex, and \( f : X \mapsto \mathbb{R} \) and \( g_j : X \mapsto \mathbb{R} \) are convex. Assume \( f^* : \text{finite} \).

- Consider the Lagrangian function

\[
L(x, \mu) = f(x) + \mu'g(x),
\]

the dual function

\[
q(\mu) = \begin{cases} 
\inf_{x \in X} L(x, \mu) & \text{if } \mu \geq 0, \\
-\infty & \text{otherwise}
\end{cases}
\]

and the dual problem of maximizing \( \inf_{x \in X} L(x, \mu) \) over \( \mu \geq 0 \).

- Recall this is the max crossing problem in the MC/MC framework where \( M = \text{epi}(p) \) with

\[
p(u) = \inf_{x \in X, \ g(x) \leq u} f(x)
\]
STRONG DUALITY THEOREM

• Assume that $f^*$ is finite, and that one of the following two conditions holds:

  1. There exists $x \in X$ such that $g(x) < 0$.

  2. The functions $g_j$, $j = 1, \ldots, r$, are affine, and there exists $x \in \text{ri}(X)$ such that $g(x) \leq 0$.

Then $q^* = f^*$ and the set of optimal solutions of the dual problem is nonempty. Under condition (1) this set is also compact.

• Replace $f(x)$ by $f(x) - f^*$ so that $f(x) - f^* \geq 0$ for all $x \in X$ w/ $g(x) \leq 0$. Apply Nonlinear Farkas’ Lemma. Then, there exist $\mu_j^* \geq 0$, s.t.

$$f^* \leq f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x), \quad \forall \ x \in X$$

• It follows that

$$f^* \leq \inf_{x \in X} \left\{ f(x) + \mu^* g(x) \right\} \leq \inf_{x \in X, g(x) \leq 0} f(x) = f^*.$$  

Thus equality holds throughout, and we have

$$f^* = \inf_{x \in X} \left\{ f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x) \right\} = q(\mu^*)$$
QUADRATIC PROGRAMMING DUALITY

• Consider the quadratic program

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} x' Q x + c' x \\
\text{subject to} & \quad Ax \leq b,
\end{align*}
\]

where \( Q \) is positive definite.

• If \( f^* \) is finite, then \( f^* = q^* \) and there exist both primal and dual optimal solutions, since the constraints are linear.

• Calculation of dual function:

\[
q(\mu) = \inf_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} x' Q x + c' x + \mu' (A x - b) \right\}
\]

The infimum is attained for \( x = -Q^{-1}(c + A' \mu) \), and, after substitution and calculation,

\[
q(\mu) = -\frac{1}{2} \mu' A Q^{-1} A' \mu - \mu' (b + A Q^{-1} c) - \frac{1}{2} c' Q^{-1} c
\]

• The dual problem, after a sign change, is

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} \mu' P \mu + t' \mu \\
\text{subject to} & \quad \mu \geq 0,
\end{align*}
\]

where \( P = A Q^{-1} A' \) and \( t = b + A Q^{-1} c \).