LECTURE OUTLINE

• Min-Max Duality
• Existence of Saddle Points

Given $\phi : X \times Z \mapsto \mathbb{R}$, where $X \subset \mathbb{R}^n$, $Z \subset \mathbb{R}^m$
consider

$$\begin{align*}
\text{minimize} \quad & \sup_{z \in Z} \phi(x, z) \\
\text{subject to} \quad & x \in X
\end{align*}$$

and

$$\begin{align*}
\text{maximize} \quad & \inf_{x \in X} \phi(x, z) \\
\text{subject to} \quad & z \in Z.
\end{align*}$$
REVIEW

• **Minimax inequality** (holds always)

\[
\sup_{z \in Z} \inf_{x \in X} \phi(x, z) \leq \inf_{x \in X} \sup_{z \in Z} \phi(x, z)
\]

Important issue is whether minimax equality holds.

• **Definition:** \((x^*, z^*)\) is called a *saddle point* of \(\phi\) if

\[
\phi(x^*, z) \leq \phi(x^*, z^*) \leq \phi(x, z^*), \quad \forall x \in X, \forall z \in Z
\]

• **Proposition:** \((x^*, z^*)\) is a saddle point if and only if the minimax equality holds and

\[
x^* \in \arg \min_{x \in X} \sup_{z \in Z} \phi(x, z), \quad z^* \in \arg \max_{z \in Z} \inf_{x \in X} \phi(x, z)
\]

• **Connection w/ constrained optimization:**
  - Strong duality is equivalent to

\[
\inf_{x \in X} \sup_{\mu \geq 0} L(x, \mu) = \sup_{\mu \geq 0} \inf_{x \in X} L(x, \mu)
\]

where \(L\) is the Lagrangian function.

  - Optimal primal-dual solution pairs \((x^*, \mu^*)\) are the saddle points of \(L\).
MC/MC FRAMEWORK FOR MINIMAX

- Use MC/MC with $M = \text{epi}(p)$ where $p : \mathbb{R}^m \mapsto [-\infty, \infty]$ is the perturbation function

$$p(u) = \inf_{x \in X} \sup_{z \in Z} \{ \phi(x, z) - u'z \}, \quad u \in \mathbb{R}^m$$

- Important fact: $p$ is obtained by partial min.
- Note that $w^* = p(0) = \inf \sup \phi$ and $\phi(\cdot, z)$: convex for all $z$ implies that $M$ is convex.
- If $-\phi(x, \cdot)$ is closed and convex, the dual function in MC/MC is

$$q(z) = \inf_{x \in X} \phi(x, z), \quad q^* = \sup \inf \phi$$
MINIMAX THEOREM I

Assume that:

(1) $X$ and $Z$ are convex.

(2) $p(0) = \inf_{x \in X} \sup_{z \in Z} \phi(x, z) < \infty$.

(3) For each $z \in Z$, the function $\phi(\cdot, z)$ is convex.

(4) For each $x \in X$, the function $-\phi(x, \cdot) : Z \mapsto \mathbb{R}$ is closed and convex.

Then, the minimax equality holds if and only if the function $p$ is lower semicontinuous at $u = 0$.

Proof: The convexity/concavity assumptions guarantee that the minimax equality is equivalent to $q^* = w^*$ in the min common/max crossing framework. Furthermore, $w^* < \infty$ by assumption, and the set $M$ [equal to $M$ and $\text{epi}(p)$] is convex.

By the 1st Min Common/Max Crossing Theorem, we have $w^* = q^*$ iff for every sequence $\{(u_k, w_k)\} \subset M$ with $u_k \to 0$, there holds $w^* \leq \liminf_{k \to \infty} w_k$. This is equivalent to the lower semicontinuity assumption on $p$:

$$p(0) \leq \liminf_{k \to \infty} p(u_k), \quad \text{for all } \{u_k\} \text{ with } u_k \to 0$$
MINIMAX THEOREM II

Assume that:

(1) $X$ and $Z$ are convex.

(2) $p(0) = \inf_{x \in X} \sup_{z \in Z} \phi(x, z) > -\infty$.

(3) For each $z \in Z$, the function $\phi(\cdot, z)$ is convex.

(4) For each $x \in X$, the function $-\phi(x, \cdot) : Z \mapsto \mathbb{R}$ is closed and convex.

(5) 0 lies in the relative interior of $\text{dom}(p)$.

Then, the minimax equality holds and the supremum in $\sup_{z \in Z} \inf_{x \in X} \phi(x, z)$ is attained by some $z \in Z$. [Also the set of $z$ where the sup is attained is compact if 0 is in the interior of $\text{dom}(p)$.]

**Proof:** Apply the 2nd Min Common/Max Crossing Theorem.

- Counterexamples of strong duality and existence of solutions/saddle points can be constructed from corresponding constrained min examples.
EXAMPLE I

• Let \( X = \{(x_1, x_2) \mid x \geq 0\} \) and \( Z = \{z \in \mathbb{R} \mid z \geq 0\} \), and let

\[
\phi(x, z) = e^{-\sqrt{x_1 x_2}} + z x_1,
\]

which satisfy the convexity and closedness assumptions. For all \( z \geq 0 \),

\[
\inf_{x \geq 0} \left\{ e^{-\sqrt{x_1 x_2}} + z x_1 \right\} = 0,
\]

so \( \sup_{z \geq 0} \inf_{x \geq 0} \phi(x, z) = 0 \). Also, for all \( x \geq 0 \),

\[
\sup_{z \geq 0} \left\{ e^{-\sqrt{x_1 x_2}} + z x_1 \right\} = \begin{cases} 1 & \text{if } x_1 = 0, \\ \infty & \text{if } x_1 > 0, \end{cases}
\]

so \( \inf_{x \geq 0} \sup_{z \geq 0} \phi(x, z) = 1 \).

• Here

\[
p(u) = \inf_{x \geq 0} \sup_{z \geq 0} \left\{ e^{-\sqrt{x_1 x_2}} + z(x_1 - u) \right\}
\]
EXAMPLE II

• Let $X = \mathbb{R}$, $Z = \{z \in \mathbb{R} \mid z \geq 0\}$, and let
  \[ \phi(x, z) = x + zx^2, \]
  which satisfy the convexity and closedness assumptions. For all $z \geq 0$,

  \[ \inf_{x \in \mathbb{R}} \{x + zx^2\} = \begin{cases} 
  -1/(4z) & \text{if } z > 0, \\
  -\infty & \text{if } z = 0,
  \end{cases} \]

  so $\sup_{z \geq 0} \inf_{x \in \mathbb{R}} \phi(x, z) = 0$. Also, for all $x \in \mathbb{R}$,

  \[ \sup_{z \geq 0} \{x + zx^2\} = \begin{cases} 
  0 & \text{if } x = 0, \\
  \infty & \text{otherwise},
  \end{cases} \]

  so $\inf_{x \in \mathbb{R}} \sup_{z \geq 0} \phi(x, z) = 0$. However, the sup is not attained, i.e., there is no saddle point.

• Here

  \[ p(u) = \inf_{x \in \mathbb{R}} \sup_{z \geq 0} \{x + zx^2 - uz\} \]

  \[ = \begin{cases} 
  -\sqrt{u} & \text{if } u \geq 0, \\
  \infty & \text{if } u < 0.
  \end{cases} \]
SADDLE POINT ANALYSIS

• The preceding analysis indicates the importance of the perturbation function

\[ p(u) = \inf_{x \in \mathbb{R}^n} F(x, u), \]

where

\[ F(x, u) = \begin{cases} 
\sup_{z \in Z} \{ \phi(x, z) - u'z \} & \text{if } x \in X, \\
\infty & \text{if } x \notin X.
\end{cases} \]

It suggests a two-step process to establish the minimax equality and the existence of a saddle point:

1. **Show that** \( p \) **is closed and convex**, thereby showing that the minimax equality holds by using the first minimax theorem.

2. **Verify that the inf of** \( \sup_{z \in Z} \phi(x, z) \) **over** \( x \in X \), **and the sup of** \( \inf_{x \in X} \phi(x, z) \) **over** \( z \in Z \) **are attained**, thereby showing that the set of saddle points is nonempty.
SADDLE POINT ANALYSIS (CONTINUED)

• Step (1) requires two types of assumptions:
  
  (a) Convexity/concavity/semicontinuity conditions of Minimax Theorem I (so the MC/MC framework applies).
  
  (b) Conditions for preservation of closedness by the partial minimization in

\[ p(u) = \inf_{x \in \mathbb{R}^n} F(x, u) \]

  e.g., for some \( u \), the nonempty level sets

\[ \{ x \mid F(x, u) \leq \gamma \} \]

  are compact.

• Step (2) requires that either Weierstrass’ Theorem can be applied, or else one of the conditions for existence of optimal solutions developed so far is satisfied.
CLASSICAL SADDLE POINT THEOREM

- Assume convexity/concavity/semicontinuity of $\phi$ and that $X$ and $Z$ are compact. Then the set of saddle points is nonempty and compact.

- **Proof:** $F$ is convex and closed by the convexity/concavity/semicontinuity of $\phi$, so $p$ is also convex. Using the compactness of $Z$, $F$ is real-valued over $X \times \mathbb{R}^m$, and from the compactness of $X$, it follows that $p$ is also real-valued and therefore continuous. Hence, the minimax equality holds by the first minimax theorem.

  The function $\sup_{z \in Z} \phi(x, z)$ is equal to $F(x, 0)$, so it is closed, and the set of its minima over $x \in X$ is nonempty and compact by Weierstrass’ Theorem. Similarly the set of maxima of the function $\inf_{x \in X} \phi(x, z)$ over $z \in Z$ is nonempty and compact. Hence the set of saddle points is nonempty and compact. **Q.E.D.**
ANOTHER THEOREM

- Use the theory of preservation of closedness under partial minimization.

- Assume convexity/concavity/semicontinuity of \( \phi \). Consider the functions

\[
t(x) = F(x, 0) = \begin{cases} 
\sup_{z \in Z} \phi(x, z) & \text{if } x \in X, \\
\infty & \text{if } x \notin X,
\end{cases}
\]

and

\[
r(z) = \begin{cases} 
-\inf_{x \in X} \phi(x, z) & \text{if } z \in Z, \\
\infty & \text{if } z \notin Z.
\end{cases}
\]

- If the level sets of \( t \) are compact, the minimax equality holds, and the min over \( x \) of

\[
\sup_{z \in Z} \phi(x, z)
\]

[which is \( t(x) \)] is attained. (Take \( u = 0 \) in the partial min theorem to show that \( p \) is closed.)

- If the level sets of \( t \) and \( r \) are compact, the set of saddle points is nonempty and compact.

- Various extensions: Use conditions for preservation of closedness under partial minimization.
ASSUME the convexity/concavity/semicontinuity conditions, and that any one of the following holds:

(1) $X$ and $Z$ are compact.

(2) $Z$ is compact and there exists a vector $z \in Z$ and a scalar $\gamma$ such that the level set \( \{ x \in X \mid \phi(x, z) \leq \gamma \} \) is nonempty and compact.

(3) $X$ is compact and there exists a vector $x \in X$ and a scalar $\gamma$ such that the level set \( \{ z \in Z \mid \phi(x, z) \geq \gamma \} \) is nonempty and compact.

(4) There exist vectors $x \in X$ and $z \in Z$, and a scalar $\gamma$ such that the level sets

\[
\{ x \in X \mid \phi(x, z) \leq \gamma \}, \quad \{ z \in Z \mid \phi(x, z) \geq \gamma \},
\]

are nonempty and compact.

Then, the minimax equality holds, and the set of saddle points of $\phi$ is nonempty and compact.