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On Maslov Part I The Schrodinger operator

§ 1 We begin with a definition of Maslov's canonical operator. Let Λ be a Lagrangian manifold in T^*_X and let $\pi : \Lambda \to X$ be the projection on the base. We denote by C the set of points $x \in \Lambda$ where rank $d\pi_x < n$ and by $\bar{C}$ the image of these points in X. $\bar{C}$ is usually called the caustic. We will call C the Maslov cycle on Λ. We will write $C = C_1 \cup C_2$ where $C_1 = \{ x \in C,\ \text{rank } d\pi_x = n-1 \}$ and $C_2 = \{ x \in C,\ \text{rank } d\pi_x < n-1 \}$. For Λ in general position C_1 is a submanifold of codim 1 and C_2 is a closed set which is a union of submanifolds of codim ≥ 3. C_1 can be oriented in a natural way so that if γ

is a curve in Λ its intersection with C,

is well defined.

The Maslov canonical "operator" maps $\frac{1}{2}$ densities on Λ into half-densities on X. We will define it locally (on Λ) and patch together. The " " indicates that what we get is not actually an operator but just a kind of approximate operator.

<u>Local definition at a pt</u> $x_0 \in \Lambda - C$: Since $(d\pi)_{x_0}$ is bijective π maps a nghd $\mathcal{U}$ of x_0 diffeomorphically onto a nghd V of $\pi(x_0)$. The inverse map is of the form $x \longrightarrow d\phi_x$ where ϕ is a smooth function on V. Given a half-density a on $\mathcal{U}$ we map

(A) $$a \longrightarrow \bar{a}\, e^{i\lambda\phi}$$

(3)

where $\bar{a}$ is the half density on V corresponding to a on $\mathcal{U}$. (Note: there is already an ambiguity in our definition. ϕ is only determined up to an additive constant.)

Definition at a pt $x_0 \in C$: Let V be a nghd of $\pi(x_0)$. We choose a phase function ψ on $V \times \mathbb{R}^N$, $\psi = \psi(x, \theta)$, such that $(x, \theta) \in C_\theta \longrightarrow \text{grad}_x \psi$ parametrizes a nghd $\mathcal{U}$ of x_0 on Λ. The half density a on $\mathcal{U}$ corresponds to a $\frac{1}{2}$ density $\tilde{a}$ on C_θ. The functions $\frac{\partial \psi}{\partial \theta_1}, \dots, \frac{\partial \psi}{\partial \theta_N}$ give us a canonical way of trivializing the normal bundle $N(C_\theta)$. Let μ_x be Lebesgue measure on the fiber N_x. Finally let $\bar{a}$ be a half density on $V \times R^N$ which has support in a tube around C_ϕ and takes the value $\tilde{a} \otimes \sqrt{\mu}(x)$ at $x \in C_\phi$. On $\mathcal{U}$ we define the Maslov operator by

$$(B) \qquad a \longrightarrow \frac{1}{\sqrt{2\pi\lambda}^N} \int \bar{a}\, e^{i\lambda\psi(x,\theta)} \sqrt{d\theta}$$

This definition depends of course on the choice of $\bar{a}$. By a simple integration by parts one can show that another choice of $\bar{a}$ would change the RHS of (B) by a term of order $O(\frac{1}{\lambda})$

We will now describe how the expressions (A) and (B) patch together. Let ω

be the action form on T^*_X. ($\omega = \sum \xi_i \, dx_i$ in the usual x, ξ notation) Its restriction to Λ is closed, and we can assume its exact on $\mathcal{U}$. i.e $\omega \mid \mathcal{U} = d\phi$ for some function ϕ on $\mathcal{U}$. The ϕ here incidentally is, up to an additive constant, the same as the ϕ occuring in (4) at pts $x \in \mathcal{U} - C$.

The main result in this subject is the theorem of stationary phase.

<u>Theorem</u>: Let $\mathcal{U}_1, \ldots, \mathcal{U}_K$ be the connected components of $\mathcal{U} - C$. On $\mathcal{U}_\alpha$ the RHS of (B) is equal to $c_\alpha \, \bar{a} \, e^{i\lambda\phi} + O(\frac{1}{\lambda})$. Moreover the constants c_α and c_β are related by $c_\alpha = e^{i \, k_{\alpha\beta}} c_\beta$ where $k_{\alpha\beta}$ is the intersection no of Maslov cycle C with any curve γ joining a pt in $\mathcal{U}_\alpha$ to a pt in $\mathcal{U}_\beta$.

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We now make the following assumptions

I : The restriction of the action form ω to Λ is exact.

II The "dual class" of C in $H^1(\Lambda)$ (Z_4 coefficients) is zero.

Then the Maslov operator can be defined globally as follows. We chose the ϕ described in the paragraph above so that it defined globally i.e restriction of ω to Λ $= d\phi$. and we choose the ~~solutions~~ ~~[illegible]~~ constant in (A) so that if we go around a path in Λ we come back to where we started from. Then the formulas (A) and (B) define a map, $a \to \underset{\Lambda}{K} a$

(C) $\frac{1}{2}$ densities on Λ $\longrightarrow$ $\frac{1}{2}$ densities on X, depending on λ, modulo $\frac{1}{2}$ densities of order $O(\frac{1}{\lambda})$

We can write down an explicit formula for $\underset{\Lambda}{K} a$

at pts $x \in X - \bar{C}$ as follows. Let q_0 be a fixed base point in $\Lambda - C$. Suppose x has k preimages, $p_1, \ldots, p_k$ on Λ. Let $\gamma_i : [0,1] \to \Lambda$ be a smooth curve joining q_0 to p_i and intersecting C transversally. Then

$$(D) \qquad \mathcal{K}_\Lambda a(x) = \sum \bar{a}_i(x)\, e^{i\left(\int_0^1 \omega\left(\frac{d\gamma_i}{dt}\right) + \mu_i + \phi(q_0)\right)}$$

where μ_i is the intersection no of γ_i with $\mathcal{G}$ and $\bar{a}_i$ is the $\frac{1}{2}$ density at x associated with a and p_i.

Proof. On Λ $\omega = d\phi$, so $\omega\left(\frac{d\gamma_i}{dt}\right) = \frac{d\phi}{dt}(\gamma_i(t))$, and the integral in the exponential $= \phi(p_i) - \phi(q_0)$.

Remark. By replacing q_0 by a finite no. of base point and choosing these points judiciously, (D) sometimes can be defined even when the conditions I and II don't hold

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Example : Let Y be a manifold and Λ_0 a Lagrangian manifold in the cotangent bundle of Y. Suppose that condition I holds for Λ_0 i.e the action form restricted to $\Lambda_0 = d\phi_0$. Let p be a function on T^*_Y and $\Xi = \Xi_p$ the corresponding Hamiltonian vector field. Let $\rho_t : T^*_Y \to T^*_Y$ be the flow generated by Ξ. ρ_t sweeps out a Lagrangian submanifold of the cotangent bundle of $R \times Y$ namely the set of points (x, ξ, t, τ) where $(x,\xi) \in \rho_t(\Lambda_0)$ and $\tau = p(x,\xi)$. We'll denote this Lagrangian submanifold by Λ. It's fairly easy to see that Λ satisfies condition I, but it needn't satisfy condition II. Nevertheless $\mathcal{K}_\Lambda$ can be defined as follows. Let (t,y) be a point in $(R \times Y) - \overline{C}$, and let $(t, p_1), \ldots (t, p_N)$ be the set above it on Λ.

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Let $(0, q_1), \cdots, (0, q_N)$ be the points on the backward flow at time $t = 0$. Then

$$(D)' \quad \mathcal{X}_\Lambda a = \sum_\alpha \bar{a}_\alpha(t, y)\, e^{i\int_0^t \omega\left(\frac{d\gamma_\alpha}{dt}\right)dt + \mu_\alpha + \phi_0(q_\alpha)}$$

where γ_α is the curve $s \longrightarrow (s, \beta_s(q_\alpha))$, $0 \leq s \leq t$, and μ_α is its intersection no. with C.

§ 2 For $i = 0, 1, \ldots, k$ let $P_i(x, D)$ be an ith order P.D.E mapping half densities on X into half-densities on X. We want to look for asymptotic solutions as λ gets large of the partial differential operator
$$P(x, D, \lambda) = \sum \frac{1}{(\sqrt{-1}\,\lambda)^k} P_k(x, D)$$
Let $p(x, \xi)$ be the function $\sum p_i(x, \xi)$ where $p_i(x, \xi)$ is the top symbol of the operator $P_i(x, D)$.

Let $c = \sum c_i$ where c_i is the subprincipal part of the operator P_i. One of Maslov's main results is the following theorem

__Thm__ Let Λ be a Lagrangian manifold on which $p(x, \xi) = 0$. Let a be a half-density on Λ satisfying the transport equation

$$\mathcal{L}_{\Xi} a + c a = 0$$

where Ξ is the Hamiltonian vector field corresponding to P. Then $P(x, D, \lambda) \underset{\wedge}{\lambda a} = O\left(\frac{1}{\lambda^2}\right)$

Rather than trying to prove this theorem we'll derive as a corollary of it Maslov's explicit formula for the solution of the Shroedinger equation.

$$(E) \qquad \frac{\hbar}{\sqrt{-1}} \frac{\partial \psi}{\partial t} = -\hbar^2 \sum \frac{1}{2} \frac{\partial^2 \psi}{\partial x_i^2} + V(x)\psi$$

in $\mathbb{R}^n \times \mathbb{R}$. The associated symbol is

$$\tau - \left(\sum \frac{\xi_i^2}{2} + V(x)\right) = H(x, \xi, t, \tau)$$

and the Hamiltonian is

$$\Xi = \frac{\partial}{\partial t} - \sum \xi_i \frac{\partial}{\partial x_i} + \frac{\partial V}{\partial x_i} \frac{\partial}{\partial \xi_i} ,$$

and the action $= \omega(\Xi) = \tau - \sum \xi_i^2$. When $H = 0$ this is equal to $-\sum \xi_i^2 + V(x)$.

On an integral curve of Ξ we have $\dot{x}_i = \frac{\partial H}{\partial \xi_i} = \xi_i$ so we can write the action integral in (D) as

$\cancel{-\frac{1}{2}\int_0^t \sum (\dot{x}_i(t))^2}$

$$-\int_0^t \sum \frac{(\dot{x}_i(t))^2}{2} + V(x)\, dt = \int_0^t L(x, \dot{x}, t)\, dt.$$

where L is the classical Lagrangian.

Finally note that in the transport equation $c = 0$ since all the terms of the Shroedinger operator are self-adjoint.

Now let $a_0 e^{i\frac{1}{\hbar}\phi_0}$ be a $\frac{1}{2}$ density on $\mathbb{R}^n$ with a_0 compactly supported.

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We want to find a solution f of the equation (E) which takes on initial data $f(x,0) = a_0 e^{i \frac{1}{\hbar} \phi_0}$ at $t = 0$.

Let ρ_t be the flow associated with $\frac{\xi^2}{2} + V(x)$ at time t. Consider the map $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ which maps $x \longrightarrow Q(x) = \pi \rho_t \left(x, (d\phi_0)_x \right)$, π being the projection of the cotangent bundle of $\mathbb{R}^n$ onto its base. Let Q be a regular value of this map and let $q_1, \dots, q_K$ be its preimages.

__Thm__ $$f(Q,t) = \sum \left| \left(\frac{DQ}{\partial q} \right) \right|^{-\frac{1}{2}} (q_i)\, a_0(q_i)\, e^{i \lambda \left(\int_0^t L + \gamma_i + \phi_0 \right.}$$

modulo an error term of order $O\left(\frac{1}{\lambda}\right)$, $\lambda = \frac{1}{\hbar}$,

where the integral is over the classical path $q_i(t)$

q_1 to Q and γ_1 is the intersection of the corresponding path on Λ with Maslov cycle.

Except for the expression for the amplitude above formula the theorem is a direct corollary Theorem above. and the (D′) To see that the expression amplitude is right we note that it is as follows. We lift a_0 up to Λ_0, it by $(d\phi_{-t})^*$ (since we want it to the transport equation $\mathcal{L}_\xi a = 0$) and then down. The Jacobean for this map is and since we are mapping $\frac{1}{2}$ densities are contravariant of order $\frac{1}{2}$ $a_0(q_1)$ gets into

$$\left| \frac{DQ}{dq}(q_1) \right|^{-\frac{1}{2}} a_0(q_1)$$

§3 We will use the result above to derive Maslov's asymptotic expression for the fundamental solution of the Shroedinger operator. We begin with the plane-wave expansion of the δ-function.

$$\delta(x-x_0) = a(x) \int e^{i\eta\cdot(x-x_0)}\, d\eta$$

where $a(x)$ is a smooth function equal to 1 at x_0 and having support in a small nghd. of x_0. Setting $\eta = \frac{\xi}{\hbar}$ we get

$$(F) \qquad \delta(x-x_0) = \frac{a(x)}{(\hbar)^n} \int e^{i\frac{\xi}{\hbar}\cdot(x-x_0)}\, d\xi$$

We will use this expansion and the theorem above to obtain an asymptotic solution of (E) with initial data $\delta(x-x_0)$ at $t=0$. Let (y,t) be a point of $\mathbb{R}\times\mathbb{R}^n$. We will assume that there are only a finite no. of classical trajectories joining y to x_0

and that y and x_0 are non-conjugate along these trajectories. Clearly the same will be true for all trajectories joining y to points in the support of $a(x)$.

Consider the graph of $d \cdot (x - x_0) \cdot \xi_0$ in the cotangent bundle of $\mathbb{R}^n$. This is just the set of pts. $\{(x, \xi_0),\ x \in \mathbb{R}^n\}$

Let $\Lambda(\xi_0)$ be the set of all trajectories which hit this graph at $t = 0$. ($\Lambda(\xi_0)$ is a Lagrangian submanifold of $T^*_{\mathbb{R} \times \mathbb{R}^n}$). Our assumption about y guarantees that there are only a finite no of trajectories

$$\gamma_{\alpha, \xi_0} : [0, t] \longrightarrow \Lambda(\xi_0) \qquad \alpha = 1, \ldots, N$$

whose terminal pts lie above (y, t) and whose initial pts lie above the support of $a(x)$. Let $(x_\alpha(\xi), \xi)$ be the initial pt. of the curve $\gamma_{\alpha, \xi}$. The theorem

7) above gives us the following asymptotic formula for the solution of (E) with initial data

$a(x)\, e^{i\frac{\xi}{h}\cdot(x-x_0)}$ at time $t=0$:

$$\sum_\alpha a_\alpha(x(\xi)) \left|\frac{\partial y}{\partial x_\xi}\right|^{-\frac{1}{2}} e^{\frac{i}{h}\left(\int_{\gamma_{\alpha,\xi}} \omega + (x(\xi)-x_0)\cdot\xi\right) + \frac{\pi i}{4} m_\alpha}$$

where m_α is the intersection no of $\gamma_{\alpha,\xi}$ with the Maslov cycle on $\Lambda(\xi)$. Plugging this into (F) we get the following expression for the fundamental solution of (E).

$$(G) \qquad G(y,t,x_0) = \sum_\alpha \frac{1}{h^n} \int a_\alpha(x(\xi)) \left|\frac{\partial y}{\partial x_\xi}\right|^{-\frac{1}{2}} e^{\frac{i}{h}\int \omega + m_\alpha + \cdots}\, d\xi$$

We will try to evaluate (G) using stationary phase. To do this we need to determine the critical pts of the phase function.

$$\phi(\xi) = \int_{\gamma_{\alpha,\xi}} \omega + (x(\xi) - x_0) \cdot \xi$$

as a function of ξ. To do so we'll need some general facts about symplectic geometry: Let X be a manifold, and Λ a Lagrangian submanifold of T^*_X. Let ω be the action form. For each $s \in \mathbb{R}$ let γ_s be a smooth curve in Λ and suppose γ_s depends smoothly on s. Let $\alpha(s)$ and $\beta(s)$ be the initial and terminal pts of γ_s.

<u>Lemma</u> $$\frac{d}{ds} \int_{\gamma_s} \omega = \omega\left(\frac{d\beta}{ds}\right) - \omega\left(\frac{d\alpha}{ds}\right)$$

<u>Proof</u>: There is a tubular nghd., $\mathcal{U}$, of γ_s in Λ on which ω is exact; i.e. $\omega = df$ on $\mathcal{U}$. Thus $\int_{\gamma_s} \omega = f(\beta(s)) - f(\alpha(s))$. Differentiating with respect to s we get the assertion above. Q.E.D.

Now let compute $\frac{\partial}{\partial \xi_\alpha} \int_{\gamma_{y,\xi}} \omega$. Note first that the curves $\gamma_{y,\xi}$ all lie in a fixed Lagrangian manifold in $T^*_{R\times R^n}$ namely the set of all trajectories that at time t lie above the pt (y,t). Let $V_{y,\alpha}$ and $W_{y,\alpha}$ be the tangent vectors to the initial and terminal curves of $\gamma_{y,\xi}$ obtained by varying ξ_α and leaving the other coordinates of ξ fixed. By the lemma

$$\frac{\partial}{\partial \xi_\alpha} \int_{\gamma_{y,\xi}} \omega = \omega(W_{y,\alpha}) - \omega(V_{y,\alpha})$$

The end point of $\gamma_{y,\xi}$ projects onto the fixed pt (y,t) in the base for all ξ, so $(d\pi) W_{y,\alpha} = 0$ and hence $\omega(W_{y,\alpha}) = 0$ (because of the way ω is defined!) On the other hand

$(d\pi)\, V_{i,\alpha} = \dfrac{\partial x_\alpha(\xi)}{\partial \xi_i}$, so $\omega(V_{i,\alpha}) = \xi \cdot \dfrac{\partial x_\alpha}{\partial \xi_i}$

at $(x(\xi), \xi)$. Therefore, we get:

$$\frac{\partial}{\partial \xi_i} \int_{\gamma_{\alpha,\xi}} \omega = -\, \xi \cdot \frac{\partial x_\alpha}{\partial \xi}$$

and $\dfrac{\partial}{\partial \xi_i} \phi(\xi) = (x(\xi) - x_0)_i$. This proves

__Thm__ The critical pts. of the phase function in the integral (G) are precisely those ξ for which $x(\xi) = x_0$ i.e. for which the integral curve $\gamma_{\alpha,\xi}$ joins x_0 to y.

If we apply stationary phase to (G) and use the fact that $a(x_0) = 1$ we get the following asymptotic formula for the RHS.

$$(H)\qquad G(y,t,x_0) \sim \sum_\alpha \frac{1}{(2\pi\hbar)^{\frac{n}{2}}} \left|\frac{\partial y}{\partial \xi}\right|^{-\frac{1}{2}}(x_0,\xi_\alpha)\, e^{\frac{i}{\hbar}\int_0^t L(\dot q_\alpha, q_\alpha, t)\,d\tau + \nu_\alpha}$$

where $q_\alpha(\tau)$, $0 \le \tau \le t$ is a classical trajectory going from x_0 to y, and $\nu_\alpha = m_\alpha + \operatorname{sign}\left(\frac{\partial x_t}{\partial \xi}\right)(x_0,\xi_\alpha)$

Maslov identifies ν_α with the number of conjugate pts along the trajectory, $q_\alpha(\tau)$. I don't at the moment see why this is the relation between this number and the intersection no. of γ_α with the Maslov cycle.

Maslov gives an alternative proof of the formula (H) using Feynmann integrals: He starts with Feynmanns representation of the fundamental solution of the Shroedinger operator:

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I) $G(y, t, x_0) = \int e^{\frac{i}{\hbar}\int_0^t L(q,\dot{q},\tau)\,d\tau}\, d\mu$

where $q(\tau)$ is any path joining x_0 to y and μ is Feynmann measure on path space.

Let's apply stationary phase to the RHS above (ignoring the fact that the integral is not over a finite dimensional region.) The critical pts. of the phase function are just those paths for which the first variation $\delta \int L = 0$, which by the principal of least action are just the classical trajectories, that is, the $q_\alpha(\tau)$ above.

The signature of $\delta^2 \int L$ at each of these trajectories is, by Morse theory, equal to the no. of conjugate pt., γ_α, along the trajectory. Therefore we obtain asymptotic formula for the RHS of (I)

$$G(x,y,t) \approx \sum K_\alpha \, e^{\frac{i}{\hbar}\int_0^t L(q_\alpha, \dot{q}_\alpha, \tau)\, d\tau + i\frac{\pi}{4}\gamma_\alpha}.$$

Here K_α is the quotient of two infinite quantities, namely $(2\pi i\hbar)^{\frac{\infty}{2}}$ and $\det(\delta^2 L)$, but apparently these cancel each other out and give the finite answer computed above. ?

(1)

On Maslov, Part II, Singularities

§ 1

Let $\Lambda \subset T^*_X$ be a Lagrangian manifold, and $\pi : \Lambda \to X$ the projection mapping. We recall

Def: The Maslov cycle is the set of all pts $x \in \Lambda$ where $d\pi_x$ has rank $< n$. The caustic is the image of the Maslov cycle.

Let us denote by $S_i(\Lambda)$ the set of points on Λ where the rank of $d\pi$ is $n-i$. We will prove.

Proposition 1.1 For Λ in general position $S_i(\Lambda)$ is a submanifold of Λ of codimension $\frac{i(i+1)}{2}$.

From this we conclude: $S_1(\Lambda)$ is of codimension 1 generically so it can occur in all dimensions. $S_2(\Lambda)$

is generically of codimension 3 so it occurs first in 3 dimensions (as a collection of isolated pts.). $S_3(\Lambda)$ occurs for the first time in 6 dimensions, $S_4(\Lambda)$ in 10 dimensions and so on. Note that $\overline{S_1(\Lambda)}$ is a "~~pseudomanifold~~". Its boundary, $\overline{S}_2$, has dimension equal to $\dim S_1(\Lambda) - 2$. Therefore it support a homology class. The dual class is the characteristic class figuring in part I.

To prove the proposition we will need:

__Lemma__ Let S be the set of all symmetric $n \times n$ matrices, and $S_i \subset S$ the matrices of corank i. Then S_i is a submanifold of S of codim $\frac{i(i+1)}{2}$

__Proof__: It is enough to prove this for the set of matrices whose first $(n-i)\times(n-i)$ minor is non-singular.

Let

$$(1.1)\qquad \begin{array}{ccc} & \begin{array}{cc} n-i & i \end{array} & \\ n-i & \begin{pmatrix} A & B \\ C & D \end{pmatrix} & \\ & & i \end{array}$$

be such a matrix. Postmultiplying this by the matrix

$\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}$ we get $\begin{pmatrix} A & 0 \\ C & D-CA^{-1}B \end{pmatrix}$ Therefore

the matrix 1.1 is of rank $n-i$ if and only if

$D-CA^{-1}B=0$. In the space of all $n \times n$ matrices

the equation $D-CA^{-1}B=0$ consist of i^2

independent scalar equations so the matrices of corank i

are of codimension i^2. However, if (1.1) is symmetric

then $D-CA^{-1}B$ is symmetric (since A^{-1} is

symmetric and $C^t = B$) so there are just $i\frac{(i+1)}{2}$

independent scalar equations, and the codimension of S_i in S is $\frac{i(i+1)}{2}$. Q.E.D.

We will now prove the theorem. We will assume for simplicity that X is an open subset of $\mathbb{R}^n$ and that linear coordinates, x, are chosen on X such that if ξ are the dual coordinates the map $(x,\xi) \to \xi$ maps Λ diffeomorphically onto an open subset of ξ space. Both these assumptions are valid locally (*) and the proof of the general case can be reduced to this case by a simple partition of unity argument.

Given the above assumptions there exists a function $H = H(\xi)$ such that Λ is the locus of points: (x,ξ) $x_i = \frac{\partial H}{\partial \xi_i}$ (i.e. Λ is the graph of dH.)

(*) There is some problem if Λ intersect the zero section, $\xi = 0$. We will assume Λ always lies in the complement of $\xi = 0$.

so the map $\pi : \Lambda \to X$ has the form

$$(1.2) \qquad (\xi_1, \ldots, \xi_n) \longrightarrow \left(\frac{\partial H}{\partial \xi_1}, \ldots, \frac{\partial H}{\partial \xi_n} \right)$$

Therefore, π has corank i precisely when the Hessian $d^2 H = \left(\frac{\partial^2 H}{\partial \xi_i \partial \xi_j} \right)$ has corank i. By the Thom $\pitchfork$ theorem we can perturb H so that the map $\xi \to d^2 H$ intersect S_i $\pitchfork$ally. In this perturbed map $S_i(\Lambda)$ is a submanifold of the perturbed Λ of codimension $\frac{i(i+1)}{2}$. Q.E.D.

The classification of the singularities of π into the S_i's can be refined further. Suppose Λ is in general position so that $S_i(\Lambda)$ is a submanifold of Λ. For $x \in S_i(\Lambda)$ let K_x be the kernel of $(d\pi)_x$ in the tangent space to Λ at x and let

L_x be the tangent space to $S_i(\Lambda)$ at x.

<u>Definition</u> $\quad x \in S_{i,j}(\Lambda) \iff \dim K_x \cap L_x = j$

Note that $j \leq i$ for the definition to make sense

<u>Proposition 1.2</u> For Λ in general position $S_{i,j}(\Lambda)$ is a submanifold of $S_i(\Lambda)$ of codimension $\frac{1}{6}\left(i(i+1)(i+2) - r(r+1)(r+2)\right) - r(i-r)$ where $r = i-j$

The proof is considerably more complicated than the proof of proposition 1.1 and we won't give it here. (See Boardman, . A single trick of Porteous, described to us by Mather reduces the Lagrangian case to the Boardman case.)

One can define inductively singular sets $S_{i,j,k}(\Lambda)$, $S_{i,j,k,l}(\Lambda)$ etc. for Λ in general position. The most important of these for us are the $S_{i,i,\dots,i}^{k}$ singularities. To describe them we will need a more efficient way of parametrizing Λ than the symmetric Jacobian procedure described above. For simplicity we'll assume for the rest of this § that X is an open subset of $\mathbb{R}^n$.

<u>Lemma</u> Suppose $x_0 \in S_1(\Lambda)$. Then in a nbhd of x_0 Λ can be parametrized by a phase function involving just one phase variable. (i.e. a phase function, $\phi = \phi(x, \theta)$, on $X \times \mathbb{R}$.)

See, for example, Hormander [ ]

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Let W_K be the subspace of $\mathbb{R}^{n+1}$ consisting of all $n+1$ tuples $(a_1, \dots, a_{n+1})$ with $a_1 = a_2 = \cdots = a_K = 0$. Given a function ϕ on $X \times \mathbb{R}$ we define a map $w(\phi) : X \times \mathbb{R} \to \mathbb{R}^{n+1}$ by the formula

$$w(\phi)(x, \theta) = \left(\frac{\partial \phi}{\partial \theta}, \frac{\partial^2 \phi}{\partial \theta^2}, \dots, \frac{\partial^{n+1} \phi}{\partial \theta^{n+1}} \right)(x, \theta)$$

By the $\pitchfork$ theorem ϕ can be perturbed so that $w(\phi)$ is $\pitchfork$ to all the W_K's. Let's call a phase function with this property $\underline{W}$-generic.

~~Assumptions~~ Given a phase function ϕ on $X \times \mathbb{R}$ let C be its critical set and $\Lambda \subset T^*_X$ the associated Lagrangian manifold. Since the diagram

$$\begin{tikzcd} C \arrow[rr, "d_x\alpha"] \arrow[dr] & & \dot{\Lambda} \arrow[dl] \\ & X & \end{tikzcd}$$

commutes, the sets $S_{1,\dots,1}(C)$ get mapped bijectively onto the sets $S_{1,\dots,1}(\Lambda)$, so to describe $S_{1,\dots,1}(\Lambda)$ it suffices to describe $S_{1,\dots,1}(C)$.

<u>Proposition</u> 3 Let α be W generic and let C be its critical set. Then $\mathcal{W}(\alpha)(x) \in W^{k+1}$ if and only if $x \in S_{\overbrace{1,\dots,1}^{k}}(C)$. Moreover $S_{\overbrace{1,\dots,1}^{k}}(C)$ is of codimension k in C and is parametrized by the $k+1$ independent equations

$$\frac{\partial\alpha}{\partial\theta} = \cdots = \frac{\partial^k\alpha}{\partial\theta^k} = 0$$

<u>Proof</u>: C itself is parametrized in $X \times R$ by the

equation $\frac{\partial\phi}{\partial\theta} = 0$; so the tangent space to C at (x,θ) is the set of vectors annihilated by $d\left(\frac{\partial\phi}{\partial\theta}\right)_{(x,\theta)}$. This shows that the fiber of the projection $X \times R \to X$ is tangent to C if and only if $\frac{\partial^2\phi}{\partial\theta^2} = 0$. Thus $S_1(C)$ is parametrized in $X \times R$ by the pair of equations $\frac{\partial\phi}{\partial\theta} = \frac{\partial^2\phi}{\partial\theta^2} = 0$. $\pitchfork$ of the map $w(\phi)$ to W_2 just says that these equations are independent, so $S_1(C)$ is a submanifold of $X \times R$ of codimension 2. The tangent space to $S_1(C)$ at (x,θ) is the set of vectors annihilated by $d\left(\frac{\partial\phi}{\partial\theta}\right)_{x,\theta}$ and $d\left(\frac{\partial^2\phi}{\partial\theta^2}\right)_{x,\theta}$; so the fiber of the projection $\pi: X \times R \to X$ is tangent to $S_1(C)$ if and only if $\frac{\partial^3\phi}{\partial\theta^3} = 0$. Repeating the argument above we see that $S_{1,1}(C)$ is parametrized by the independent equations $\frac{\partial\phi}{\partial\theta} = \frac{\partial^2\phi}{\partial\theta^2} = \frac{\partial^3\phi}{\partial\theta^3} = 0$.

4

We leave the k th inductive step of the proof to the reader.

<u>Corollary</u> For Λ in general position $S_{\overbrace{1,\dots,1}^{k}}(\Lambda)$ is a submanifold of Λ of codimension k.

When we come to looking at asymptotic properties of PDE's we will mainly be considering the types of PDE that occur in classical physics such as the reduced wave equation, the Klein-Gordon equation, the Shroedinger equation etc. All these examples occur in dimensions ≤ 4, so it will be helpful to have a complete classification of the kinds of singularities that can occur in these low dimensions. From the results of this section we get the following provisional classification.

	codim of singularity: 1	2	3	4
dim 1	S_1			
dim 2	S_1	$S_{1,1}$		
dim 3	S_1	$S_{1,1}$	$S_{1,1,1}$; $S_{2,0}$	
dim 4	S_1	$S_{1,1}$	$S_{1,1,1}$; $S_{2,0}$	$S_{1,1,1,1}$

This classification can't be very much refined. We will show below that all the S_1 singularities of the same kind are isomorphic. The $S_{2,0}$ singularities are a little more complicated. In dimension 3 there are two kinds that can occur (the so-called

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elliptic umbilic and hyperbolic umbilic.) In dim 4 there are three kinds (the elliptic, the parabolic and the hyperbolic umbilic.) In particular there are precisely 7 distinct types of singularities that can occur in dimension 4. These are, mutatis mutandi, the seven Thom catastrophes.

§2

Throughout this § we will assume Λ is in general position so that the results of §1 hold. Then the Maslov cycle is a pseudomanifold whose interior is the set of points $S_{1,0}(\Lambda)$. We will call these points fold points, notation which will be explained shortly. Our goal is to give a simple description of the Maslov canonical operator, K_Λ, in the neighborhood of a fold point. Our starting point is the following proposition:

Proposition 2.1 If $x_0 \in S_{1,0}(\Lambda)$ we can choose a phase function $\varphi = \varphi(x,\theta)$ on $X \times \mathbb{R}$ parametrizing ~~[illegible]~~ a neighborhood of x_0 on Λ such that

(2.1) $$\varphi(x,\theta) = \omega_0(x) + \rho(x)\theta - \frac{\theta^3}{3} \quad \text{with } d\rho \neq 0.$$

<u>Note</u>: In ϕ of the form above the critical set C where $\frac{\partial\phi}{\partial\theta} = 0$ is just the set: (x,θ), $\theta^2 = \rho(x)$; and the caustic is the set $\rho = 0$. If we choose coordinates $x_1, \ldots, x_n$ on X such that ρ is x_1 and use $(\theta, x_2, \ldots, x_n)$ as a system of coordinates on Λ then the map $\Lambda \longrightarrow X$ is given locally by

$$(\theta, x_2, \ldots, x_n) \longrightarrow (\theta^2, x_2, \ldots, x_n)$$

In other words it is the map which folds the $\theta < 0$ plane onto the $\theta > 0$ plane with fold along the line $\theta = 0$.

Assuming proposition 2.1 for the moment we see that the Maslov canonical operator has the

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form

$$(2.2) \qquad a(x,\theta) \longrightarrow \int a(x,\theta)\, e^{iK\left(\mu_0 + \rho\theta - \frac{\theta^3}{3}\right)} d\theta$$

in the neighborhood of a fold point. We will simplify the RHS of (2.2) as follows. By the Malgrange preparation theorem we can find functions $a_0(x)$, $a_1(x)$ and $h(x,\theta)$ such that

$$a(x,\theta) = a_0(x) + a_1(x)\,\theta + h(x,\theta)\left(\rho(x) - \theta^2\right)$$

where $\rho(x) - \theta^2 = \frac{\partial\phi}{\partial\theta}$. Thus $\mathcal{X}_\Lambda a$ can be written:

$$a_0(x)\int e^{iK\phi(x,\theta)}d\theta + a_1(x)\int \theta e^{iK\phi}d\theta + \int h(x,\theta)\frac{\partial\phi}{\partial\theta}e^{iK\phi}d\theta$$

The last term in this sum is $\int h(x,\theta)\frac{1}{K\sqrt{-1}}\frac{\partial}{\partial\theta}e^{iK\phi}d\theta$ which is of order $\frac{1}{K}$. Integrating by part

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and then repeating the same argument over again we prove (with different a_0 and a_1 from the above)

<u>Proposition</u> 2.2 For every integer N there exist functions $a_0(x)$ and $a_1(x)$, depending on $a(x,\theta)$, such that

$$K_\Lambda a = a_0(x)\int e^{ik\varphi(x,\theta)}d\theta + a_1(x)\int \theta e^{ik\varphi}d\theta + O(k^{-N})$$

<u>Remark</u> It is rather complicated actually to write down the dependence of a_0 and a_1 on a. This involves looking at a special case of the following problem: Given a function $f(x,\theta)$ on $\mathbb{R}^n \times \mathbb{R}$ with $\frac{\partial^i f}{\partial\theta^i}(0) = 0$ for $i<k$ and $\frac{\partial^k f}{\partial\theta^k}(0) \neq 0$, then the Malgrange

preparation asserts that for every function $a = a(x,\theta)$ there exist functions $a_0(x), \dots, a_{n-1}(x)$ and $h(x,\theta)$ such that $a(x,\theta) = \sum a_i(x)\theta^i + hf$. How do the a_i's and h depend on a? If f and a are real analytic then h and the a_i's are uniquely determined (by the uniqueness part of the Weierstrass preparation theorem!) and one can show (See for example Arnold [ ]) that the map

$$a \longrightarrow (a_0, \dots, a_{n-1}, h)$$

behaves in some ways like an $n-1$ order differential operator. If f and a are smooth then h and the a_i's may not even be uniquely determined. (See, for example, Malgrange [ ])

To simplify (2.2) further we recall the definition of the Airy function:

$$(2.3) \qquad Y(t) = \int e^{i\left(t\theta - \frac{\theta^3}{3}\right)} d\theta \qquad t \in R,\ \theta \in R$$

This is a Bessel function of type $\frac{1}{2}$. Its properties are exhaustively discussed in the Bureau of Standard tables [ ] pp . Among other things it can be characterized as a solution of the ordinary differential equation: $Y'' + tY = 0$.

This equation is the standard equation in one dimension which describes transitions from oscillatory behavior to exponentially damped behavior: Assuming t is approximately constant then in the region $t > 0$

0).

the solutions of $Y'' + tY = 0$ are approximately sine and cosine functions, and in the region $t < 0$ they are approximately exponentially increasing and exponentially decreasing.

Differentiating (2.3) under the integral sign we get

$$Y'(t) = \int i\theta e^{i(\theta t - \frac{\theta^3}{3})} \, d\theta .$$

Therefore from (2.2) we obtain the following:

<u>Theorem</u> If (x,ξ) is a fold point of Λ, then for a half-density a supported on a ~~so~~ sufficiently small neighborhood of (x,ξ) in Λ there exist half-densities a_0 and a_1 on X such that

$$\chi_\Lambda a = e^{iK\mu_0}\left\{ \frac{a_0}{K^{\frac{1}{3}}} Y\left(K^{\frac{2}{3}}\rho\right) + \frac{a_1}{K^{\frac{2}{3}}} Y'\left(K^{\frac{2}{3}}\rho\right) \right\} + O\left(K^{-N}\right)$$

Remark: Airy functions play an important role in the asymptotic theory of the Shroedinger equation (See for example Messiah, [ ]) and of the reduced wave equation (See Ludwig, [ ]). The theorem above shows that this fact has nothing to do with the special properties of these equations but only with the presence of fold singularities. In the following section we will use standard properties of Airy functions to describe what "illuminated regions" and "shaded regions" look like in the neighborhood of a simple caustic.

Finally we will prove proposition 2.1. The idea of this proof is due to Chester, Friedman, and Ursell, [ ], (though they work with analytic rather than smooth data.)

Our starting point is the following theorem, due to Whitney.

<u>Lemma</u> 1 Let f be a smooth even function on the real line. Then there exists a smooth function g on the real line such that $f(x) = g(x^2)$.
of f depends smoothly on a set of parameters, g can be chosen so that it depends smoothly on the same parameters

See [ ]

Now let Λ be a Lagrangian manifold with a fold point at (x_0, ξ_0). Assume for simplicity that X is $\mathbb{R}^n$ and that x_0 is the origin. Let $\phi = \phi(x, \theta)$ on $X \times \mathbb{R}$ be a phase function parametrizing Λ in a neighborhood of x_0 and let C be its critical set. Assume that the point on C corresponding to (x_0, ξ_0) is the origin. We will prove

<u>Lemma 2</u> There exist smooth functions u_0 and ρ on X and ξ on $X \times \mathbb{R}$ such that restricted to C:

$$(2.4) \quad \begin{cases} \dfrac{\xi^3}{3} - \rho\xi + u_0 = \phi, & \dfrac{\partial\xi}{\partial\theta} \neq 0, \text{ and} \\ \xi^2 - \rho = 0 \end{cases}$$

<u>Proof</u>: First let's prove the assertion for the special

when the base manifold, X, is one dimensional.

The assumption that the origin is a fold point of C means that $\frac{\partial \phi}{\partial \theta} = \frac{\partial^2 \phi}{\partial \theta^2} = 0$, and $\frac{\partial \phi}{\partial \theta \partial x} \neq 0$, $\frac{\partial^3 \phi}{\partial \theta^3} \neq 0$, at 0. (See proposition 1.3.) Since $\frac{\partial \phi}{\partial \theta \partial x} \neq 0$ we can solve for x as a function of θ on C and we get $x = x(\theta)$. Since $\frac{\partial^3 \phi}{\partial \theta^3} \neq 0$ $x'(\theta) = 0$ and $x''(\theta) \neq 0$ so by a change of coordinates on X we can assume $x = \theta^2$ on C.

Let C^+ be the part of C where $\theta > 0$ and C^- the part where $\theta < 0$. By the second of the two equations (2.4), we must have $\xi = +\sqrt{\rho}$ on C^+ and $\xi = -\sqrt{\rho}$ on C^- so on C^+

we have

$$-\frac{2}{3}\rho^{\frac{3}{2}} + u_0 = \alpha(\theta)$$

and on C^- we have

$$\frac{2}{3}\rho^{\frac{3}{2}} + u_0 = \alpha(-\theta) .$$

Since ρ and u_0 are functions of x alone we must have

$$(2.5) \qquad u_0(x) = \frac{1}{2}\left(\alpha(\theta) + \alpha(-\theta)\right)$$

$$\rho(x)^3 = \frac{9}{4}\left(\alpha(\theta) - \alpha(-\theta)\right)^2$$

with $x = \theta^2$. The expressions on the right are both even functions of θ, so u_0 and ρ^3 exist by lemma 1. To show that the cube root of ρ^3 exists we note that since $\alpha'(\theta) = \alpha''(\theta) = 0$,

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and $\alpha'''(\theta) \neq 0$ the Taylor series for

$(\alpha(\theta) + \alpha(-\theta))^2$ begins with a non-zero term of order 6. Thus ρ exists and is of order 2 with respect to θ and of order 2 with respect to x. In particular, $\xi = \sqrt{\rho}$ exists on C and $\frac{\partial \xi}{\partial \theta} \neq 0$.

Now suppose $\dim X > 1$. Choose coordinates $(x_1, \dots, x_n)$ on X such that $\frac{\partial \alpha}{\partial \theta \partial x_1} \neq 0$. For $a = (a_2, \dots, a_n)$ let C_a = the intersection of C with the hyperplane $x_2 = a_2, \dots x_n = a_n$. Applying the preceding argument to C_a we find functions μ_0^a, ρ^a and ξ^a on C_a satisfying (2.4) and depending smoothly on a. We let μ_0, ρ, and ξ

be the corresponding functions on C.

Finally extend ξ from C to $X \times \mathbb{R}$ arbitrarily. This concludes the proof of lemma 2.

To prove proposition 2.1 let $\psi(x,\theta) = u_0(x) + \rho(x)\,\xi(\theta) - \frac{\xi^3(\theta)}{3}$. From (2.4) one easily sees that the critical set of ψ equals the critical set of ϕ. Making the change of coordinates $x \to x$, $\theta \to \xi(\theta, x)$ we get a phase function of the desired form.

<u>Remark</u>: In §4 we will obtain another proof of proposition 2.1 as a special case of a much more general result. (See proposition 4.1)

§3

As an application of the ideas discussed above we will examine from the Maslov point of view some results of Ludwig on the reduced wave equation:

$$(3.1) \qquad \Delta u + k^2 u = 0 \, .$$

The paper of Ludwig we will be mainly concerned with is: "Uniform asymptotic expansions at a caustic". Communications on pure and appl. math., vol XIX 215-250 (1966). The problem we want to look at is the following: Construct a solution of (3.1) with prescribed boundary data on an oriented hypersurface, S, in $\mathbb{R}^n$ like the kind shown in figure 1.

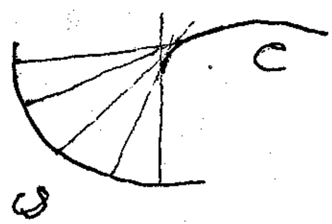


figure 1

Suppose the family of normal lines to S have C as an envelope (ie they all lie on one side of C and are tangent to C) In $k \gg 0$ geometric optics gives a very good approximate solution to (3.1) in the region shaded in red. (See part 1 of these notes.) However geometric optics makes some rather implausible assertions about what happens near C, e.g. the light intensity at C is infinite and there is no illumination at all in the blue shaded region. In the paper we mentioned above Ludwig works out the predictions of physical optics concerning what happens near C. The picture he gets is roughly the following

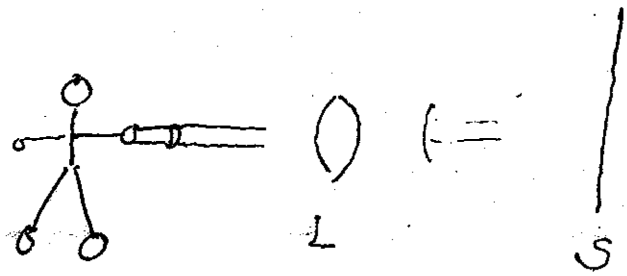
a) At points in the red shaded region whose distance from C is large compared with $k^{-\frac{2}{3}}$ the approximation of geometric optics is correct to order $\frac{1}{K}$.

b) The light intensity on the caustic itself is large but finite (of order $k^{\frac{1}{6}}$).

c) On the dark side of C there is an illuminated strip of width approximately $k^{-\frac{2}{3}}$

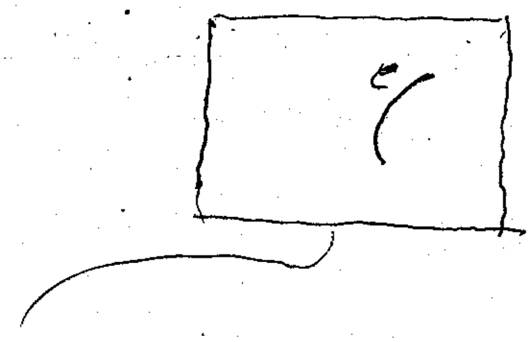
Ludwig's results are uniform in the sense that they are valid for all $k >> 0$. In other words if we refract a monochromatic beam of light through a lens L producing an image on a screen S we can predict from these results how the image changes as we change the frequency of the light (See figure 2 below.)

figure 2



?

man varies frequency of light by putting yellow, blue, red etc. filters in front of flashlight



Region of illumination on dark side of C gets thinner as one goes from infra red to ultraviolet part of spectrum. For example illuminated region for red filter ~ 1.4 time as wide as illuminated region for violet filter.

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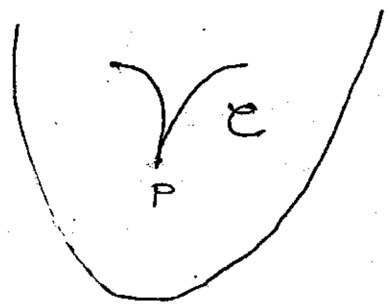
To solve equation 3.1 asymptotically with boundary data prescribed along S we consider in the cotangent bundle of $\mathbb{R}^n$ the Lagrangian manifold consisting of all points $(x + t n_x, n_x)$ where $x \in S$ and n_x is the unit normal at x pointing in the direction of the orientation. The caustic C in figure 2 is precisely the set of critical pts of the map $\pi : \Lambda \longrightarrow \mathbb{R}^n$. It is not hard to show that for S in general position most of these pt are fold points; those which are not form of subset of codimension one. (See figure 3. Also compare with prop of § 1) figure 3



Reflection in a parabolic mirror produces a cusped caustic. All the point on C except P are fold points (We will discuss the behavior of solutions of (3.1) in the nghd. of the cusp, P, in §5.)

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at a fold point the Maslov canonical operator gives a solution of 3.1 which has the general form

$$(3.2) \qquad e^{i\kappa\sigma}\left\{\frac{g_0}{\kappa^{\frac{1}{3}}} A\left(\kappa^{\frac{2}{3}}\rho\right) + \frac{g_1}{i\kappa^{\frac{2}{3}}} A'\left(\kappa^{\frac{2}{3}}\rho\right)\right\} + O\left(\frac{1}{\kappa}\right)$$

Here ρ, σ, g_0 and g_1 are function of x alone not depending on κ, A is the Airy function and A' its derivative. (See §2, prop 2.) We will worry about how to determine σ, ρ, g_0 and g_1 a little later on. However we already know from §1 that $d\rho \neq 0$

near C and $\rho = 0$ on C, so ρ can be regarded as a normal coordinate for C.

In order to discuss the qualitative behavior of the solution (3.2) we recall some basic facts about Airy functions:

<u>Theorem</u> For $t \gg 0$

(3.3) $$A(t) \sim \frac{1}{\sqrt{\pi}\, t^{\frac{1}{4}}} \cos\left(\frac{2}{3} t^{\frac{3}{2}} - \frac{\pi}{4}\right)$$ and

(3.4) $$A'(t) \sim \frac{t^{\frac{1}{4}}}{\sqrt{\pi}} \left(-\sin\left(\frac{2}{3} t^{\frac{3}{2}} - \frac{\pi}{4}\right)\right)$$

For $t \ll 0$

(3.5) $$A(t) \sim \frac{1}{2\sqrt{\pi}\,(-t)^{\frac{1}{4}}}\, e^{-\frac{2}{3}(-t)^{\frac{3}{2}}}$$ and

(3.6) $$A'(t) \sim \frac{1}{2\sqrt{\pi}} (-t)^{\frac{1}{4}} e^{-\frac{2}{3}(-t)^{\frac{3}{2}}}$$

For t close to 0

(3.7) $$A(t) \approx c_1 + c_2 t$$

where $c_1 \doteq .355$ and $c_2 \doteq -.259$

Proof (3.3) and (3.4) can be obtained by applying stationary phase to the integral form of the Airy function and (3.5) and (3.6) can be obtained from (3.3) and (3.4) by analytic continuation. For 3.7 see the Bureau of Standards tables [ ]

Combining these result with (3.2) we see that for $k^{\frac{2}{3}}\rho >> 0$, the solution, u, of (3.1) satisfies

$$(3.8) \quad u \sim \frac{k^{-\frac{1}{3}} e^{ik\sigma}}{\sqrt{\pi}\,[k^{\frac{2}{3}}\rho]^{\frac{1}{4}}} \left(g_0 \cos\left(\frac{2k}{3}\rho^{\frac{3}{2}} - \frac{\pi}{4}\right) - g_1 \rho^{\frac{1}{2}} \sin\left(\frac{2k}{3}\rho^{\frac{3}{2}} - \frac{\pi}{4}\right)\right)$$

which is identical with the solution given by geometric optics. For $K^{\frac{2}{3}}\rho$ close to zero the first term of (3.2) dominates and we get

$$(3.9) \quad u \sim e^{ik\sigma} \frac{g_0}{K^{\frac{1}{3}}} A\left(K^{\frac{2}{3}}\rho\right)$$

where A is given by 3.7. Comparing (3.8) with (3.9) we see that the value of u on the caustic is approximately $k^{\frac{1}{6}}$ times its value for $\rho \sim 1$.

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Finally when $k^{\frac{2}{3}}\rho << 0$ we get

$$u \sim \frac{k^{-\frac{1}{3}} e^{iK\sigma}}{\sqrt{\pi}\left[k^{\frac{2}{3}}(-\rho)\right]^{\frac{1}{4}}} \left(g_0 e^{-\frac{2}{3}k(-\rho)^{\frac{3}{2}}} + g_1 (-\rho)^{\frac{1}{2}} e^{-\frac{2}{3}k(-\rho)^{\frac{3}{2}}} \right) \tag{3.10}$$

When ρ is comparable to $k^{\frac{2}{3}}$ we can't simplify the solution (3.2) very much. This is a region of transition in which the approximation of geometric optics breaks down and the extremely simple solution 3.9 is not yet valid. Notice that there is still a significant amount of illumination on the dark side of the caustic if $-k^{\frac{2}{3}}\rho \simeq 1$.

We leave it as an exercise for the reader to read off from the expressions above the rough qualitative behavior of u described earlier.

We will now discuss how one determines the functions σ, ρ, g_0 and g_1. The ~~Hamiltonian~~ symbol of the equation 3.1 is the function $\xi^2 - 1$ (See § of part I) Our general prescription for constructing asymptotic solutions of (3.1) is to find Lagrangian manifolds, Λ, on which $\xi^2 - 1 = 0$ and on Λ to find half densities which are invariant under the Hamiltonian flow associated with $\xi^2 - 1$; namely $2 \sum \xi_i \frac{\partial}{\partial x_i}$. Lets look at

(1)

what this first condition involves.

Our phase function is of the form

$$\varphi = \varphi(x, \theta) = \sigma(x) + \rho(x)\,\theta - \frac{1}{3}\theta^3,$$

where ρ and σ are the same as the ρ and σ in the previous paragraph. The critical set, C, of φ is the set of all (x, θ) where $\rho(x) = \theta^2$ and the map $(x, \theta) \to x, (d\varphi)_x$ must map this set diffeomorphically onto our Lagrangian manifold. Therefore, we must have

(3.11) $$\sum \left(\frac{\partial \varphi}{\partial x_i}\right)^2 = 1 \quad \text{on} \quad C$$

43 (should be 42)

Since $\theta^2 = \rho$ on C and since 1 and θ are independent on C with respect to functions of x, the equation (3.11) breaks up into the pair of equations

$$(\nabla\sigma)^2 + \rho(\nabla\rho)^2 = 1 \tag{3.12}$$

and

$$\nabla\sigma \cdot \nabla\rho = 0 \tag{3.13}$$

These are the eikonal equations for ρ and σ which Ludwig derive in § 1 of his paper. In two dimensions it rather easy to analyze them geometrically. When $\rho = 0$, $\nabla\rho$

is perpendicular to the caustic, so $\nabla\sigma$ is tangent to the caustic by (3.13), and by (3.12) σ is just the arclength variable along the caustic itself. To interpret (3.12) and (3.13) as equations for ρ, let's suppose first of all that C is a circle of radius a about the origin in $\mathbb{R}^2$.

Let (r, θ) be the polar coordinates of any point in $\mathbb{R}^2$. Then on C, $\sigma = a\theta$. It is clear that for (3.12) and (3.13) to be satisfied with initial data $\rho = 0$ and $\sigma = a\theta$ on C, ρ must be radially symmetric so we can write

$\rho = \rho(r)$: From (3.13) we deduce that σ is constant in the radial direction so $\sigma = a\theta$ in a whole nghd of C. Thus (3.12) reduces to the equation

$$\left(\frac{a}{r}\right)^2 + \rho(r)\left(\rho'(r)\right)^2 = 1 \qquad \text{or}$$

$$\rho(r)\left(\rho'(r)\right)^2 = \frac{(r-a)(r+a)}{r^2}$$

Setting $\rho(r) = c(r-a) + O((r-a)^2)$ in the vicinity of C we get

$$c^3(r-a) = \frac{(r-a)(r+a)}{r^2} \quad \text{at } a$$

or $c^3 = \frac{2}{a}$ Thus:

$$\rho(r) \doteq \left(\frac{2}{a}\right)^{\frac{1}{3}}(r-a) \tag{3.14}$$

in the vicinity of C

Now consider the case of a general caustic, C. Let $x_0 \in C$ and r distance along the normal line through C at x_0. Applying the argument above to the oscillating circle to C at x_0 (which has third order contact with C at x_0) we get (3.14) holding along the normal line ~~???~~ through C at x_0, where a is the radius of curvature at x_0.

Finally we will see what the analogue of the transport equation is in our set-up. Our asymptotic solution of (3.1) has the form

$$u(x) = \int g(x,\theta)\, e^{ik\left(\sigma + \rho\theta - \frac{\theta^3}{3}\right)}\, d\theta$$

where σ and ρ are in principle determinable by the equations (3.12) and (3.13). What about g? We will prove

Proposition. Let φ be the phase function above (i.e $\varphi(x,\theta) = \sigma + \rho\theta - \frac{\theta^3}{3}$) on $X \times \mathbb{R}$ and let $\tau = \dfrac{1-(\nabla\varphi)^2}{\frac{\partial\varphi}{\partial\theta}}$ (Since $1-(\nabla\varphi)^2$ vanishes on the locus of points where $\frac{\partial\varphi}{\partial\theta} = 0$

and φ is a generic phase function, q is a smooth function.). Then we have

$$(3.15)\quad 2\,(\nabla\varphi)\cdot\nabla g + (\Delta\varphi)\,g + \frac{\partial}{\partial\theta}(q\,g) = 0 \bmod \left(\frac{\partial\varphi}{\partial\theta}\right)$$

See Ludwig, pg , formula.

Proof: (3.15) just asserts that the half-density on C associated with g is invariant under the Hamiltonian flow. To see this we first of all prove:

Lemma 1 The vector field on C associated with the Hamiltonian vector field $2\sum \xi_i \frac{\partial}{\partial x_i}$

on Λ is the vector field

$$V = 2 \sum \frac{\partial \varphi}{\partial x_i} \frac{\partial}{\partial x_i} + \varphi \frac{\partial}{\partial \theta}$$

Proof This vector field is tangent to C since it kills φ and $\frac{\partial \varphi}{\partial \theta}$; and it and the Hamiltonian vector field project onto the same vector on the base at each pt; namely: $2 \sum \frac{\partial \varphi}{\partial x_i} \frac{\partial}{\partial x_i}$. Therefore, the two vector field agree on the set where $d\pi$ is bijective. By continuity they agree everywhere.

Q.E.D.

Now let Ω be an n form on $\mathbb{R}^{n+1}$ such that on C, $\Omega \wedge d\left(\frac{\partial \varphi}{\partial \theta}\right) = dx \wedge d\theta$. Let Ω_0 be the restriction of Ω to C. Since

Ω is determined up to a multiple of $d\left(\frac{\partial \phi}{\partial \theta}\right)$

Ω_0 is intrinsically defined on C and is the usual volume form there.

<u>Lemma 2</u> $\mathcal{L}_V \sqrt{\Omega_0} = \left(\Delta\phi - \frac{\partial \rho}{\partial \theta}\right)\sqrt{\Omega_0}$

<u>Proof</u> It is enough to show that

(3.16) $$\left(\mathcal{L}_V \Omega\right) \wedge d\frac{\partial \phi}{\partial \theta} = 2\left(\Delta\phi - \frac{\partial \rho}{\partial \theta}\right) d\Omega \wedge d\frac{\partial \phi}{\partial \theta},$$

for then we can get the equation above by restricting to C and taking square roots. To prove (3.16) we first of all note that

(3.17) $$\mathcal{L}_V \frac{\partial \phi}{\partial \theta} = \frac{\partial \rho}{\partial \theta}\frac{\partial \phi}{\partial \theta}$$

by a straightforward computation. Now the left hand side of (3.16) can be written

$$\mathcal{L}_V\left(\Omega \wedge d\left(\frac{\partial f}{\partial \theta}\right) - \Omega \wedge d\left(\mathcal{L}_V \frac{\partial f}{\partial \theta}\right)\right)$$

$$= \mathcal{L}_V\left(dx \wedge d\theta\right) - \frac{\partial g}{\partial \theta}\ \Omega \wedge \frac{d\partial f}{\partial \theta} \quad \text{on } C$$

by (3.17). Since $\mathcal{L}_V\left(dx \wedge d\theta\right) =$

$$(\operatorname{div} V)\, dx \wedge d\theta = (\operatorname{div} V)\, \Omega \wedge d\frac{\partial f}{\partial \theta}$$

$$(\mathcal{L}_V \Omega) \wedge d\left(\frac{\partial f}{\partial \theta}\right) = \left(\operatorname{div} V - \frac{\partial g}{\partial \theta}\right) \Omega \wedge d\left(\frac{\partial f}{\partial \theta}\right)$$

which is essentially 3.16

Q.E.D.

Combining lemmas 1 and 2 we see that for a half density $g\sqrt{\Omega_0}$ to be invariant under the transport equation g must satisfy 3.15

In order to analyze more complicated singularities we will need results analogous to proposition 2.1. Our goal in this section will be to derive, at least formally, such results for the iterated S_1 singularities and for the simplest kinds of S_2 singularities:

In the iterated S_1 singularities we have the following result.

<u>Proposition 4.1</u> Let X be a neighborhood of the origin in $\mathbb{R}^n$ and $\phi = \phi(x,\theta)$ a phase function on $X \times \mathbb{R}$. Suppose that at the origin $\frac{\partial^i \phi}{\partial \theta^i} = 0$ for $i = 1, \dots, k-1$ and $\frac{\partial^k \phi}{\partial \theta^k} \neq 0$. Then there exists a function $\theta_1 = \theta_1(x,\theta)$ and functions of x; $f_0(x), \dots, f_{k-2}(x)$ such that $\theta_1(0) = f_1(0) = \dots = f_{k-2}(0) = 0$, $\frac{\partial \theta_1}{\partial \theta} \neq 0$ at 0 and:

$$(4.1) \qquad d(x,\theta) = f_0 + f_1\,\theta_1 + \cdots + f_{K-2}\,\theta_1^{K-2} + \frac{\theta_1^K}{K} + \varepsilon(x,\theta)$$

where, $\varepsilon(x,\theta)$ vanishing to infinite order when $x = 0$

Proof: Since $\dfrac{\partial^K d}{\partial\theta^K} \neq 0$ at 0, we can write

$d(0,\theta) = c + \dfrac{\theta^K}{K} h(\theta)$ where $h(\theta) \neq 0$ at 0

Making the coordinate change $\theta_1 = \theta \sqrt[K]{h(\theta)}$. we

get $d(x,\theta) = c + \dfrac{\theta_1^K}{K} + \varepsilon(x,\theta)$ where $\varepsilon(x,\theta)$

vanishes to degree 1 in x when $x = 0$.

Now suppose by induction that we can write:

$$d(x,\theta) = f_0(x) + f_1(x)\,\theta + \cdots + f_{K-2}(x)\,\theta^{K-2} + \frac{\theta^K}{K} + \varepsilon(x,\theta)$$

where the f_i's are polynomials of degree $< N$ in x

and $\varepsilon(x,\theta) = O(|x|^N)$ Let $\varepsilon(x,\theta) =$

$\sum_{|I|=N} \varepsilon_I(\theta)\, x^I + O\left(|x|^{N+1}\right)$ .. We will try to find homogeneous polynomials $f_u' = \sum_{|I|=N} f_{u,I}'\, x^I$ and $\theta_1 = \theta + \sum_{|I|=N} \nu_I(\theta)\, x^I$ such that

$$(4.2) \qquad \sum_{u=0}^{K-2} \left(f_u + f_u'\right)\theta_1^u + \frac{\theta_1^K}{K} = d(x,\theta) + \varepsilon_1(x,\theta)$$

where $\varepsilon_1(x,\theta)$ is $O\left(|x|^{N+1}\right)$ when $x=0$.

Equating coefficients of x^I (4.2) reduces to the following system of equations:

$$(4.2)_I \qquad f_{0,I} + f_{1,I}\,\theta + \dots + f_{k-2,I}\,\theta^{k-2} + \nu_I(\theta)\,\theta^{k-1} = \varepsilon_I(\theta)$$

which are easily solvable by Taylor's theorem with remainder.

Q.E.D.

To study the S_2 singularities, we need a short digression on the theory of cubic binary forms over the reals. Let $\phi = \phi(\alpha, \beta)$ be a homogeneous polynomial of degree 3 in the variables α and β.

We will say that ϕ is <u>degenerate</u> if the quadratic forms $\frac{\partial \phi}{\partial \alpha}$ and $\frac{\partial \phi}{\partial \beta}$ are multiples of each other. Suppose this is the case, i.e. suppose $\frac{\partial \phi}{\partial \beta} = c \frac{\partial \phi}{\partial \alpha}$. Making the coordinate change $\alpha_1 = \alpha$, $\beta_1 = \beta + c\alpha$, we get $\frac{\partial \phi}{\partial \beta_1} = 0$, so $\phi = a\alpha_1^3$. Relabelling $\alpha = \sqrt[3]{a}\,\alpha_1$ we see that either $\phi \equiv 0$ or $\phi = \alpha^3$.

Now suppose ϕ is non-degenerate. Let L_ϕ be the one dimensional vector space consisting of quadratic forms in α and β modulo linear combinations of $\frac{\partial\phi}{\partial\alpha}$ and $\frac{\partial\phi}{\partial\beta}$. Let K be the two dimensional vector space spanned by α and β. Multiplication gives us a bilinear map of K into L_ϕ; and if we choose a basis in L_ϕ this map can be viewed as a quadratic form ~~=~~ on K. It is clear that this quadratic form is non-zero; otherwise α^2, β^2 and $\alpha\beta$ would all be multiples of $\frac{\partial\phi}{\partial\alpha}$ and $\frac{\partial\phi}{\partial\beta}$

Definition 4.2 We will say ϕ is hyperbolic if the quadratic form described above is of rank 2 and index 1, elliptic if it is of rank 2 and index 2 or 0, and parabolic if it is of rank 1

Proposition 4.3 (Classification theorem for cubic binary forms

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over R.) If ϕ is hyperbolic, then by a linear change of coordinates it can be written in the form:

(4.3) $$\phi(\alpha,\beta) = \frac{\alpha^3+\beta^3}{3}$$

If it is parabolic it can be written in the form:

(4.4) $$\phi(\alpha,\beta) = \alpha^2\beta$$

And if it is elliptic it can be written in the form:

(4.5) $$\phi(\alpha,\beta) = \alpha^3-\alpha\beta^2$$

Proof: We will discuss the hyperbolic case. The other two cases are handled similarly. If $\phi(\alpha,\beta)$ is hyperbolic we can make a linear change of coordinates such that the quadratic form just described sends the pair (α,α) to 0 and (β,β) to 0. This means α^2 and β^2 are both linear combinations of $\frac{\partial\phi}{\partial\alpha}$ and $\frac{\partial\phi}{\partial\beta}$. Since α^2 and β^2 are linearly independent, $\frac{\partial\phi}{\partial\alpha}$ and $\frac{\partial\phi}{\partial\beta}$ must also be linear combinations of α^2 and β^2; i.e

$\frac{\partial\phi}{\partial\alpha} = a_1\alpha^2 + a_2\beta^2$ and $\frac{\partial\phi}{\partial\beta} = b_1\alpha^2 + b_2\beta^2$ for real numbers a_1, a_2 and b_1, b_2. Clearly, for the first of these equations to hold, the coefficient of $\alpha^2\beta$ in ϕ must be zero, and for the second to hold the coefficient of $\alpha\beta^2$ must be zero. This shows that

$\phi = \frac{1}{3}\left(c_1\alpha^3 + c_2\beta^3\right)$. Replacing α by $\sqrt[3]{c_1}\,\alpha$ and β by $\sqrt[3]{c_2}\,\beta$ we get ϕ in the desired form.

Q.E.D.

<u>Corollary</u> Let $h = h(\alpha, \beta)$ be a smooth function on $\mathbb{R}^2$. Suppose that its first and second derivatives at 0 vanish and the cubic term in its Taylor series is a non-degenerate cubic binary form. If this form is hyperbolic we can find coordinates $\alpha_1 = \alpha_1(\alpha, \beta)$ and $\beta_1 = \beta_1(\alpha, \beta)$ such that

$$h(\alpha,\beta) = c + \frac{\alpha_1^3 + \beta_1^3}{3} \quad \text{near } 0 \tag{4.6}$$

If it is elliptic we can find coordinates such that

$$h(\alpha,\beta) = c + \alpha_1^3 - \alpha_1\beta_1^2 \quad \text{near } 0 \tag{4.7}$$

<u>Proof</u>: We will discuss the hyperbolic case. The elliptic case is handled similarly. Clearly we

(9).

can write

$$h(\alpha,\beta) = \frac{\alpha^3+\beta^3}{3} + \sum_{i+j=4} f_{i,j}(\alpha,\beta)\,\alpha^i\beta^j .$$

Making the coordinate change $\alpha_1 = \alpha + f_{2,2}\,\beta^2$, $\beta_1 = \beta$

we can assume that in the expression on the RHS, $f_{2,2} = 0$

This means we can write

$$h(\alpha,\beta) = \frac{\alpha^3+\beta^3}{3} + g_1\alpha^3 + g_2\beta^3 \quad \text{where } g_1(0) = g_2(0)$$

$= 0$. Setting $\alpha_1 = \alpha\sqrt[3]{1+g_1}$ and $\beta_1 = \beta\sqrt[3]{1+g_2}$,

we get $h(\alpha,\beta) = \dfrac{\alpha_1^3+\beta_1^3}{3}$ as required

Q.E.D.

In the parabolic singularities the situation is a little more complicated. In example $\alpha^2\beta$ and

$\alpha^2\beta + \beta^4$ are not equivalent. We will prove

<u>Proposition</u> 4.4 Suppose $h = h(\alpha,\beta)$ vanishes together with its first and second derivatives at 0. Suppose the cubic term in its Taylor series at 0 is $\alpha^2\beta$. Then if $\frac{\partial^4 h}{\partial \beta^4} \neq 0$ we can find a change of coordinates $\alpha_1 = \alpha_1(\alpha,\beta)$ and $\beta_1 = \beta_1(\alpha,\beta)$ at 0 such that

$$(4.7) \qquad \pm h = \alpha_1^2\beta_1 + \beta_1^4$$

<u>Proof</u>: We can write

$$\pm h(\alpha,\beta) = \alpha^2\beta + \sum_{i+j=4} \delta_{i,j}\,\alpha^i\beta^j \quad \text{with} \quad \delta_{4,0} > 0$$

at the origin. Letting $\alpha_1 = \alpha \big/ \sqrt[8]{\delta_{0,4}}$ and

$\beta_1 = \sqrt[4]{f_{0,4}}\,\beta + \delta_{1,3}\,\alpha \Big/ 4\sqrt[4]{f_{0,4}}$ we get:

$$\pm h(\alpha,\beta) = \alpha_1^2\beta_1 + \beta_1^4 + \rho\,\alpha_1^2\beta_1 + r\,\alpha_1^4$$

with ρ and r appropriate functions of α and β, and $\rho(0) = 0$. Finally replacing α_1 by $\alpha_1\sqrt{1+\rho}$, we can assume that $\rho = 0$. Thus we've reduced our problem to the case when $h(\alpha,\beta)$ has the form:

$$\pm h(\alpha,\beta) = \alpha^2\beta + \beta^4 + r\alpha^4 \tag{4.8}$$

We will now look for a coordinate change $\alpha_1 = \alpha$, $\beta_1 = \beta + \mu\,\alpha^2$, and a function ρ of α and β which vanishes at 0 such that

$$\pm h(\alpha,\beta) = \alpha_1^2\beta_1 + \beta_1^4 + \rho\,\alpha_1^2\beta_1$$

Substituting the expressions for α_1 and β_1 in the RHS of (4.8) we get $\pm h(\alpha,\beta)$ equal to the following mess:

$$\alpha^2\beta + \beta^4 + \left(\mu + \mu\rho + 6\mu^2\beta^2 + 4\mu^3\alpha^2\beta + \mu^4\alpha^4\right)\alpha^4 + \cdots$$
$$+ \left(\rho + 4\mu\beta^2\right)\alpha^2\beta$$

If we make the substitution $\rho = -4\mu\beta^2$ in the first bracketed expression, we get

$\nu = \mu + 2\mu^2\beta^2 + 4\mu^3\alpha^2 + \mu^4\alpha^4$, Since $\nu = \mu$ when α and $\beta = 0$ this equation can be solved for μ in terms of ν, α and β. Defining ρ to be $-4\mu\beta^2$ we get

$$\pm h(\alpha,\beta) = \alpha_1^2\beta_1 + \beta_1^4 + \rho\,\alpha_1^2\beta_1$$

as asserted. Finally if we replace α_1 by $\sqrt{1+\rho}\,\alpha_1$ we can eliminate the ρ term.

Q.E.D.

Our canonical form theorem for the $S_{2,0}$ coingularities is the following:

<u>Proposition 4.5</u> Let $\phi = \phi(x,\alpha,\beta)$ be a phase function on $X \times \mathbb{R}^2$. Suppose that at the origin ϕ and all its first and second derivatives with respect to α and β vanish. Suppose that the cubic term in the Taylor series expansion of $\phi(0,\alpha,\beta)$ at the origin is non-degenerate. Then if it is hyperbolic we can find functions $\alpha_1 = \alpha_1(x,\alpha,\beta)$,

$\beta_1 = \beta_1(\alpha,\beta,x)$ and $f_i = f_i(x)$, $i = 0, 1, 2, 3$, such that

$\alpha_1(0) = \beta_1(0) = f_0(0) = \cdots = f_3(0)$, $\dfrac{\partial(\alpha_1,\beta_1)}{\partial(\alpha,\beta)} \neq 0$, and

$$(4.9) \qquad \phi = f_0 + f_1\alpha_1 + f_2\beta_1 + f_3\alpha_1\beta_1 + \frac{\alpha_1^3+\beta_1^3}{3} + \varepsilon(x,\alpha,\beta)$$

where $\varepsilon(x,\alpha,\beta)$ vanishes to infinite order when $x = 0$

If the cubic term is elliptic we can find α_1, β_1, $f_0, \ldots, f_3$ as above such that

$$(4.10) \qquad \phi = f_0 + f_1\alpha_1 + f_2\beta_1 + f_3(\alpha_1^2+\beta_1^2) + \alpha_1^3 - \alpha_1\beta_1^2 + \varepsilon(x,\alpha,\beta)$$

where $\varepsilon(x,\alpha,\beta)$ vanishes to infinite order when $x = 0$.

If the cubic term is parabolic and the hypothesis of proposition 4.4 is satisfied by $\phi(0,\alpha,\beta)$ we can find $\alpha_1, \beta_1, f_0, f_1, f_2, f_3, f_4$ as above such that

$$(4.11) \quad \pm d = f_0 + f_1\alpha_1 + f_2\beta_1 + f_3\alpha_1^2 + f_4\beta_1^2 + \alpha_1^2\beta_1 + \beta_1^4 + \varepsilon(x,\alpha,\beta)$$

where $\varepsilon(x,\alpha,\beta)$ vanishes to infinite order when $x = 0$.

<u>Proof</u> We'll just discuss the hyperbolic case, leaving the parabolic and elliptic cases as an exercise for the reader. We will prove (4.9) by the same kind of induction argument as that in the proof of proposition 4.1. The case $N = 1$ is just the corollary to proposition 4.3, so we'll assume the case $N-1$ and prove the case N. Our inductive assumption is that

$$d = f_0 + f_1\alpha + f_2\beta + f_3\alpha\beta + \frac{\alpha^3 + \beta^3}{3} + \varepsilon(x,\alpha,\beta)$$

where the f_u's are polynomials of degree $< N$ in x, and $e(x, \alpha, \beta) = O(|x|^N)$. Let

$$\varepsilon(x, \alpha, \beta) = \sum_{|I| = N} \varepsilon_I(\alpha, \beta)\, x^I + O\left(|x|^{N+1}\right).$$

We will try to find homogeneous polynomials

$$f_u' = \sum_{|I| = N} f_{u,I}'\, x^I$$

of degree N in x and functions

$$\begin{cases} \alpha_1 = \alpha + \sum_{|I| = N} \nu_I(\alpha, \beta)\, x^I \\ \beta_1 = \beta + \sum_{|I| = N} \mu_I(\alpha, \beta)\, x^I \end{cases}$$

such that:

$$(4.12) \quad \alpha = (f_0 + f_0') + (f_1 + f_1')\alpha_1 + (f_2 + f_2')\beta_1 + (f_3 + f_3')\alpha_1\beta_1 + \frac{\alpha_1^3 + \beta_1^3}{3} + \varepsilon_1(x, \alpha, \beta)$$

where $\varepsilon_1(x, \alpha, \beta) = O\left(|x|^{N+1}\right)$. Equating coefficients of x^I we get

$$(4{:}12)_I \qquad \delta'_{0,I} + \delta'_{1,I}\,\alpha + \delta'_{2,I}\,\beta + \delta'_{3,I}\,\alpha\beta + \nu_I\,\alpha^2 + \mu_I\,\beta^2 = \varepsilon_I(\alpha,\beta)$$

which can be easily solved for constants $\delta'_{0,I}$ etc. and functions $\nu_I(\alpha,\beta)$, $\mu_I(\alpha,\beta)$ ~~[illegible]~~ using the integral form of the Taylor series with Remainder.

Q.E.D.

§5 We will now discuss the S_2 singularities in a little more detail. Let Λ be a Lagrangian manifold in T^*_X and let (x_0, ξ_0) be an S_2 singularity on Λ. Let $\varphi = \varphi(x, \alpha, \beta)$ be a phase function on $X \times \mathbb{R}^2$ parametrizing Λ in a neighborhood of (x_0, ξ_0). We will assume that $(x_0, 0, 0)$ is the point on the critical set of φ corresponding to (x_0, ξ_0). Since (x_0, ξ_0) is an S_2 singularity the first and second derivatives of $\varphi(x, \alpha, \beta)$ with respect to α and β are zero at $(x_0, 0, 0)$. Let $\varphi_3(\alpha, \beta)$ be the cubic term in the Taylor series expansion of $\varphi(x_0, \alpha, \beta)$ at $(0,0)$. It is easy to see that (x_0, ξ_0) is an $S_{2,1}$ singularity if and only if φ_3 is degenerate, and is an $S_{2,2}$ singularity if and only if $\varphi_3 \equiv 0$.

<u>Definition</u> 5.1 We will say that (x_0, ξ_0) is a hyperbolic (elliptic, parabolic) S_2 singularity if d_3 is hyperbolic (elliptic, parabolic).

We must check that this definition is independent of q and is an intrinsic property of Λ:

Let $\mathcal{R}_C$ be the local ring of formal power series in the coordinates of C at $(x_0, 0, 0)$ and let $\mathcal{R}_\Lambda$ be the corresponding local ring at (x_0, ξ_0). Let $\mathcal{R}_C^{\#}$ and $\mathcal{R}_C^{\#}$ be the quotient rings $\mathcal{R}_C/(x-x_0)$ and $\mathcal{R}_\Lambda/(x-x_0)$. Since Λ and C are diffeomorphic $\mathcal{R}_\Lambda$ and $\mathcal{R}_C$ are isomorphic, and so are $\mathcal{R}_\Lambda^{\#}$ and $\mathcal{R}_C^{\#}$.

Now it is clear from definition 4.2 that hyperbolicity etc. is an algebraic property of the local ring of formal power series in α and β, modulo the ideal generated by $\frac{\partial\phi}{\partial\alpha}(x_0,\alpha,\beta)$ and $\frac{\partial\phi}{\partial\beta}(x_0,\alpha,\beta)$. However, this is just the ring $R^{\#}_{C}$ since the defining equations of C are $\frac{\partial\phi}{\partial\alpha}=\frac{\partial\phi}{\partial\beta}=0$. Therefore, hyperbolicity etc. is an algebraic property of $R^{\#}_{\Lambda}$. Q.E.D.

We will now prove:

Proposition 5.2 For Λ in general position, $S_2(\Lambda)$ is a submanifold of codimension 3; the elliptic and hyperbolic points are open subset of $S_2(\Lambda)$, the parabolic points are a codimension 1 submanifold, the $S_{2,1}$ points a codimension 2 submanifold, ~~the~~ and the $S_{2,2}$ points a codimension 4 submanifold.

Consequently elliptic and hyperbolic points can occur for the first time in dimension 3, parabolic points for the first time in dimension 4, $S_{2,1}$ singularities for the first time in dimension 5, and $S_{2,2}$ singularities for the first time in dimension 7.

Proof: Let W be the space of all polynomial functions in α and β of degree ≤ 3. Let W_1 be the subspace of polynomials with zero first order terms and W_2 the subspace of polynomials with zero first and second order terms. Let $P \subset W_2$ be the parabolic cubic polynomials and $D \subset W_2$ the degenerate cubic polynomials.

We will need

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Lemma P is a codimension 1 submanifold of W_2 and Q a codimension 2 submanifold.

Proof: Consider in W_2 the open set consisting of polynomial whose x^3 term is non-zero. Restricting such a polynomial to the affine subset of projective space defined by $x \neq 0$ we get an isomorphism between the set of these polynomials and the set of cubic polynomials on the real line. The parabolic polynomials correspond to polynomials on $\mathbb{R}$ with double roots and the degenerate polynomials to polynomials with triple roots. We can view the polynomials with double roots as a vector bundle over $\mathbb{R}$ with fiber $\mathbb{R}^2$ (just assign to each polynomial its double root) and the polynomials with triple roots as a fiber bundle over $\mathbb{R}$ with fiber $\mathbb{R}$. Therefore P has dimension 3 and D has dimension 2. Q.E.D.

Now let ϕ be a phase function on $X \times R^2$.

Let $\tilde{\phi} : X \times R^2 \longrightarrow W$ be the map which assigns to each point (x_0, α_0, β_0) the Taylor series expansion of $\phi(x_0, \alpha, \beta)$ to order 3 about the point (α_0, β_0).

We will say that ϕ is W generic if it is π to W_1, W_2, P and D. It is clear that if ϕ is W generic the assertions of proposition 5.2 are true for the corresponding Lagrangian manifold. However, by the Thom π theorem every ϕ can be perturbed to a W generic ϕ; so this concludes the proof of Proposition 5.2 Q.E.D.

To each point of Λ we've attached the local ring R^{**}_{Λ}. We've already seen that from the structure of this local ring alone we can determine whether an $S_{2,0}$ singularity is elliptic hyperbolic or parabolic. We will now show

<u>Proposition</u> 5.3 If (x_0, ξ_0) is elliptic or hyperbolic the dimension of $\mathcal{R}^{\#}_{\Lambda}$ over the reals is 4 and if (x_0, ξ_0) is parabolic the dimension is ≥ 5.

<u>Proof</u>: If (x_0, ξ_0) is elliptic we can parametrize Λ in a neighborhood of (x_0, ξ_0) by a phase function of the form $\frac{\alpha^3 + \beta^3}{3} + \varepsilon(x, \alpha, \beta)$ where $\varepsilon(x, \alpha, \beta)$ vanishes when $x = x_0$. Therefore $\mathcal{R}^{\#}_{\Lambda}$ is isomorphic to the formal power series ring in α and β divided by the ideal (α^2, β^2). As a

vector space over $\mathbb{R}$ this has $1, \alpha, \beta$ and $\alpha\beta$ as a basis.

If (x_0, ξ_0) is parabolic we can parametrize Λ in a nghd of (x_0, ξ_0) by a phase function of the form $\alpha^2\beta + \varepsilon(x, \alpha, \beta)$ where $\varepsilon(x_0, \alpha, \beta)$ is of order 4 in α and β. Therefore $1, \alpha, \beta, \beta^2$ and β^3 are all independent modulo $\frac{\partial \phi}{\partial \alpha}$ and $\frac{\partial \phi}{\partial \beta}$.

Q.E.D.

<u>Definition</u> 5.4 We will say a parabolic point $(x_0, \xi_0) \in S_2(\Lambda)$ is regular if $\dim \mathcal{R}^{\#}_{\Lambda} = 5$ and exceptional if $\dim \mathcal{R}^{\#}_{\Lambda} > 5$.

~~Proposition The set of exceptional points is a subset of~~

Proposition 5.5. In Λ in general position the set of exceptional parabolic points is a subset of codimension 1 in the set of all parabolic points.

Therefore in dimension 4 all parabolic points are regular.

Proposition 5.5 is an easy consequence of the following proposition which we leave as an exercise for the reader:

Proposition 5.6. Let (x_0,ξ_0) be a parabolic point on Λ. Let $\varphi = \varphi(x,\alpha,\beta)$ be a phase function parametrizing Λ in a nghd of (x_0,ξ_0) and having the form

$$\varphi(x,\alpha,\beta) = \alpha^2\beta + \varepsilon(x,\alpha,\beta)$$

where $\varepsilon(x_0, \alpha, \beta)$ is of order 4 in α and β.

Then (x_0, ξ_0) is a regular parabolic point if and only if

$$\frac{\partial^4 d}{\partial \beta^4}(x_0, 0, 0) \neq 0 .$$

§6

We must still show that the canonical form theorems we derived in §4 are true C^∞, not just formally. In the proof we will need the Malgrange preparation theorem in its "Grothendieck - Houzel" form. We recall the statement of this theorem as its given, for example, in Malgrange's book (See [ ] 00). Let $\mathcal{E}_n$ be the ring of germs of C^∞ functions at the origin in Euclidean n-space. Let m_n be its maximal ideal. Given a mapping ρ of R^n into R^p mapping the origin to the origin we get an induced map $\rho^*: \mathcal{E}_p \longrightarrow \mathcal{E}_n$. Now let

②

$f_1, \ldots, f_k$ be functions on $\mathbb{R}^n$ and $(f_1, \ldots, f_k)$ the ideal they generate in $\mathcal{E}_n$. Let $\mathcal{R}$ be the quotient ring $\mathcal{E}_n / (f_1, \ldots, f_k)$. Because of the map $\rho^* : \mathcal{E}_p \to \mathcal{E}_n$ we can view $\mathcal{R}$ as an $\mathcal{E}_p$ module

__Theorem__ $\mathcal{R}$ is a finitely generated $\mathcal{E}_p$ module if and only if $\mathcal{R}/m_p\mathcal{R}$ is a finite dimensional vector space over the real no. Moreover, a collection of elements $\alpha_1, \ldots, \alpha_k \in \mathcal{R}$ are a set of generators for $\mathcal{R}$ as a module over $\mathcal{E}_p$ if and only if their images in $\mathcal{R}/m_p\mathcal{R}$ are a spanning set of vectors for $\mathcal{R}/m_p\mathcal{R}$ (as a vector space over $\mathbb{R}$.)

As an application of this theorem, let's prove the Whitney theorem about even functions quoted at the end of §2. Let $n = p = 1$, let $\mathcal{R} = \mathcal{E}_1$ and let ρ be the map : $\mathbb{R} \to \mathbb{R}$, $x \to x^2$. Then, since $\mathcal{R}/(x^2)\mathcal{R}$ is generated by 1 and x, every smooth function can be written in the form $f(x^2) + x\,g(x^2)$ for smooth f and g. For the function to be even, g must be 0.

The usual preparation theorem for a function $F = F(x,t)$ on $\mathbb{R}^n \times \mathbb{R}$ can be obtained from the theorem above by letting $\rho : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be the map $(x,t) \to x$ and letting $\mathcal{R}$ be the ring $\mathcal{E}_{n+1}/(F)$. Conversely it is not hard to prove the theorem above, assuming the usual preparation theorem. See [ ] for details

Now lets look at the problem that came up in § 4.

A phase function $\varphi = \varphi(x, \theta)$ is given to us on $\mathbb{R}^n \times \mathbb{R}^N$ and also a perturbed phase function $\varphi' = \varphi(x,\theta) + \varepsilon(x,\theta)$ such that $\varepsilon(x,\theta)$ vanishes to infinite order when $x = 0$. We want to find a pair of diffeomorphisms $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^n \times \mathbb{R}^N$ each defined on a nghd of the origin and having the origin as a fixed point such that f and g make the diagram

$$(6.1) \qquad \begin{array}{ccc} \mathbb{R}^n \times \mathbb{R}^N & \xrightarrow{g} & \mathbb{R}^n \times \mathbb{R}^N \\ \downarrow & & \downarrow \\ \mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^n \end{array}$$

commute and such that g conjugates φ' into φ

in the following weak sense:

(6.2) $\quad g^*(\alpha' + u) = \varphi$

where $u = u(x)$ is a function of x alone. Such a result is clearly what we need to get rid of the error terms occurring on the RHS of (4.), (4.), (4.) and (4.). Note that the presence of u causes no problems for us, because we can, in each case, absorb it in the first term on the RHS.

Let $\alpha_t = \alpha_t(x,\theta) = \alpha(x,\theta) + t\,\varepsilon(x,\theta)$. We will try to construct f_t, g_t and u_t with the same properties as f, g and u above such that

(6.3) $\quad g_t^*(\alpha_t + u_t) = \alpha \qquad$ for all t

(6)

f_t, g_t and u_t depending smoothly on t and g_t being the identity map when $t=0$. Near $t=0$ we can write the coordinates of f_t and g_t in powers of t:

$$f_i(x,t) = x_i + a_i(x)\,t + O(t^2) \qquad i = 1, \dots, n$$

and

$$g_\alpha(x,\theta,t) = \theta_\alpha + b_\alpha(x,\theta) + O(t^2) \qquad \alpha = 1, \dots, N,$$

If we plug these expressions into (6.3) and differentiate, setting $t=0$, we get:

$$(6.4) \qquad -\varepsilon(x,\theta) = a_0(x) + \sum_i \frac{\partial d}{\partial x^i} a_i + \sum_\alpha \frac{\partial d}{\partial \theta_\alpha} b_\alpha$$

with $a_0 = -\left.\dfrac{du_t}{dt}\right|_{t=0}$.

7)

Definition 6.1 We will say that α is *infinitesimally stable* if for every $\varepsilon(x, \theta)$ on the LHS of (6.4) there exist $a_1 = a_1(x)$ and $b_1 = b_1(x, \theta)$ such that (6.4) holds in a nghd of the origin.

If we apply the Malgrange preparation theorem to the map : $\mathbb{R}^n \times \mathbb{R}^N \longrightarrow \mathbb{R}^n$; $(x, \theta) \to x$, with $\mathcal{R} = \mathcal{E}_{n+N} / \left(\frac{\partial \alpha}{\partial \theta_1}, \ldots, \frac{\partial \alpha}{\partial \theta_N} \right)$, we get the following criterion for infinitesimal stability

Proposition 6.2 α is infinitesimally stable if and only if for every function $\lambda = \lambda(\theta)$ there exist real nos. $c_0, \ldots, c_n$ and functions $\tau_1, \ldots, \tau_N$

of θ such that

$$(6.5) \qquad \lambda(\theta) = c_0 + \sum_i c_i \frac{\partial \phi}{\partial x_i}(0,\theta) + \sum_\alpha T_\alpha(\theta) \frac{\partial \phi}{\partial \theta_\alpha}(0,\theta)$$

Exercise : Show that this criterion is satisfied for the phase function defining a simple caustic:

$$\phi(x,\theta) = x\theta - \frac{\theta^3}{3}$$

In the situation we are considering, our deformed phase function $\phi_t(x,\theta)$ is of the form $\phi(x,\theta) + t\,\varepsilon(x,\theta)$ where ε vanishes to infinite order when $x = 0$; therefore the criterion (6.5) is the same for ϕ and for ϕ_t. This proves

Proposition 6.3 If $\mathcal{L}$ is infinitesimally stable then $\mathcal{L}_t$ is infinitesimally stable for all t.

The main goal of this section is to prove the following:

Theorem 6.4 Suppose $\mathcal{L} = \mathcal{L}(x,\theta)$ is inf. stable. Let $\mathcal{L}'(x,\theta) = \mathcal{L}(x,\theta) + \varepsilon(x,\theta)$ where ε vanishes to infinite order when $x = 0$. Then there exist f, g and u satisfying (6.1) and 6.2

We will prove, in fact, that there exist f_t, g_t and u_t satisfying the conditions analogous to 6.1 and satisfying (6.3) on the whole interval $0 \leq t \leq 1$.

As a first step in the proof we will need:

<u>Lemma</u> 6.5 If the LHS of (6.4) vanishes when $x = 0$ then one can choose the a's and b's on the RHS so that they also vanish when $x=0$

<u>Proof</u>: If $\varepsilon(x,\theta) = 0$ when $x=0$ we can write $\varepsilon(x,\theta) = \sum x_i \varepsilon_i(x,\theta)$ for smooth ε_i.

In each i we can solve

$$-\varepsilon_i(x,\theta) = a_{0,i} + \frac{\partial d}{\partial x_1} a_{1,i} + \dots + \frac{\partial d}{\partial \theta_1} b_{1,i}(x,\theta) + \dots$$

Setting $a_j = \sum a_{j,i} x_i$ and $b_j = \sum b_{j,i} x_i$ we get a solution of 6.4 that vanishes when $x=0$.

Q.E.D

(11)

We want to determine f_t, g_t and u_t satisfying (6.3)

Differentiating (6.3) with respect to t we get

$$- \varepsilon(g_t) = \dot{u}_t(g_t) + \sum_i \frac{\partial}{\partial x_i}(u_t + d_t)(g_t)\, \dot{f}_i(x,t)$$

$$+ \sum_\alpha \frac{\partial}{\partial \theta_\alpha}(u_t + d_t)(g_t)\, \dot{g}_\alpha(x,\theta,t).$$

Here the f_i's and the g_α's are the coordinates of f_t and g_t, and the dots indicate differentiation with respect to t. If we set

$$(6.6)\quad \begin{cases} a_i(x,t) = \dot{f}_i(f_t^{-1}(x), t) & i = 1,\dots,n \\ b_\alpha(x,\theta,t) = \dot{g}_\alpha(g_t^{-1}(x,\theta), t) & \alpha = 1,\dots,N \\ a_0(x,t) = \dfrac{\partial u}{\partial t} + \sum \dfrac{\partial u}{\partial x_i}\, a_i & \end{cases}$$

the equation above reduces to

$$(6.7) \quad -\varepsilon(x,\theta) = a_0(x,t) + \sum_i \frac{\partial d_t}{\partial x_i} a_i + \sum_\alpha \frac{\partial d_t}{\partial \theta_\alpha} b_\alpha$$

which is identical with (6.4) except that all the terms are functions of t. We will try to solve (6.7) for functions a_i and b_α which are smooth over the whole interval $0 \le t \le 1$. We first note that it is enough to do this in a small interval about each point on $[0,1]$; for, by a partition of unity in t these solutions can be patched together to give a global solution in t. By proposition 6.3 d_t is infinitesimally stable so 6.7) can be solved for fixed t such that the solutions are smooth in x and θ. The only question is whether these solutions can be chosen to be smooth in t. To see this we apply the Malgrange preparation theorem to the map: $R^n \times R^N \times R \to R^n \times R$;

$$(x, \theta, t) \to (x, t), \quad \text{with} \quad \mathcal{R} = \mathcal{E}_{n+N+1} \Big/ \left(\frac{\partial \alpha_t}{\partial \theta_1}, \ldots, \frac{\partial \alpha_t}{\partial \theta_N}\right).$$

The preparation theorem says that (6.7) can be solved with an arbitrary smooth function on the LHS if and only if (6.5) holds for α_t. We have already seen, however, (proposition 6.3) that this condition is the same for all t, and it hold when $t = 0$ because of the infinitesmal stability of α.

Finally we note that if $\varepsilon(x, \theta)$ vanishe when $x = 0$

as is the case with us, we can choose the a_i's and b_α's to vanish when $x = 0$ (Lemma 6.5).

To conclude the proof of theorem 6.4 we must solve the equations (6.6) with initial data

$$(6.9) \quad \begin{cases} f_i(x,0) = x_i & i = 1, \dots, n \\ g_\alpha(x,\theta,0) = \theta_\alpha & \alpha = 1, \dots N \\ \mu(x,0) = 0 \end{cases}$$

The first pair of these equations are just ordinary differential equations in f and g, and since the expressions on the LHS are 0 when $x = 0$ they are solvable globally in t for a sufficiently small nghd of the origin

(15)

in (x,θ) space. The last of the equations

(6.6) can be solved by linear Hamilton - Jacobi theory

i.e just by integrating the vector field $(a_1(x,t), \ldots, a_n(x,t))$.

This can be done globally in t for the same reasons as above.

Q.E.D

§7 We will now use the results of the preceding section to derive the canonical form theorems of §4. We will actually prove a theorem which includes those results and is applicable to other singularities besides the elementary ones discussed in §4. The theorem is a variant of Thom's "universal unfolding" theorem.

<u>Definition</u> 7.1 Let $\psi = \psi(\theta)$ be a smooth function defined on a nghd. of the origin in $\mathbb{R}^N$. Let $\mathcal{E}_N$ be the ring of germs of smooth functions at the origin of $\mathbb{R}^N$ and let $\left(\frac{\partial \psi}{\partial \theta}\right)$ be the ideal in $\mathcal{E}_N$ generated by the first partial derivatives of ψ. We will say ψ is of <u>finite type</u> if $\left(\frac{\partial \psi}{\partial \theta}\right)$ is of finite codimension in $\mathcal{E}_N$.

Examples which we have already seen are :

In $N = 1$

a) $\psi = \psi(\theta) = \dfrac{\theta^{K+1}}{K+1}$

In $N = 2$

(7.1) b) $\psi = \psi(\theta_1, \theta_2) = \dfrac{\theta_1^3 + \theta_2^3}{3}$

c) $\psi = \theta_1^3 - \theta_1 \theta_2^2$

d) $\psi = \theta_1^3 \theta_2 + \theta_2^4$

Suppose the codimension of $\left(\frac{\partial \psi}{\partial \theta}\right)$ in $\mathcal{E}_N$ is $K+1$. Then we can choose functions $\psi_0, \ldots, \psi_K$ $\in \mathcal{E}_N$ such that their images form a basis for $\mathcal{E}_N / \left(\frac{\partial \psi}{\partial \theta}\right)$. We will assume that $\psi_0 = 1$.

Our main result is :

Theorem 7.1 Let $\phi = \phi(x, \theta)$ be a smooth function defined on a nghd of the origin in $\mathbb{R}^n \times \mathbb{R}^N$ such that $\phi(0, \theta) = \psi(\theta)$. Then there exists smooth functions $f_i = f_i(x)$, $i = 0, \dots, k$ and $\bar{\theta}_i = \bar{\theta}_i(x, \theta)$ $i = 1, \cdots, N$; such that the f's are all zero at the origin, $\bar{\theta}_i(0, \theta) = \theta_i$, and

$$(7.2) \quad \phi(x, \theta) = f_0(x) + f_1(x)\,\psi_1(\bar{\theta}) + \dots + f_k(x)\,\psi_k(\bar{\theta}) + \psi(\bar{\theta})$$

If we apply this theorem to example a) of (7.1) and let $\psi_i(\theta) = \theta^i$ $i = 1, \dots, k-1$ we get the canonical form (4.). We leave it as an exercise for the reader to get the canonical forms (4.), (4.) and (4.) by applying this theorem with the examples b) c) and d) of 7.1 and appropriate choices for the ψ_i's.

<u>Proof</u> of <u>theorem</u> 7.1 We first of all prove the assertion formally. The proof is practically identical with the proofs of propositions 4. and 4. . We will argue by induction that by changing the θ coordinate we can write

$$(7.3) \quad \alpha(x, \theta) = f_0(x) + \dots + f_k(x)\,\psi_k(\theta) + \psi(\theta) + e(x, \theta)$$

where the f_i's are polynomials in x of order $\leq r-1$

and $\varepsilon(x,\theta)$ is $O(|x|^r)$ uniformly in θ.

We will try to find a coordinate change

$$(7.4)\quad \bar{\theta}_i = \theta_i + \sum_{|I|=r} \gamma_{i,I}(\theta)\, x^I \quad , \quad i = 1, \dots, N$$

and polynomials

$$(7.5)\quad f_i'(x) = \sum_{|I|=r} f_{i,I}\, x^I$$

such that

$$(7.6)\quad \alpha(x,\theta) = \sum (f_i + f_i')(x)\, \psi_i(\bar{\theta}) + \psi(\bar{\theta}) + \bar{\varepsilon}(x,\theta)$$

where $\bar{\varepsilon}(x,\theta)$ is $O(|x|^{r+1})$ uniformly in θ. Let

$$(7.7)\quad \varepsilon(x,\theta) = \sum_{|I|=r} \varepsilon_I(\theta)\, x^I + O(|x|^{r+1})$$

(21)

Equating the coefficient of x^I in the RHS and LHS of (7.6) we get.

$$(7.8) \qquad \sum f_{i,I}\, \psi_i(\theta) + \sum \frac{\partial \psi}{\partial \theta_i}\, v_{i,I}(\theta) = -\varepsilon_I(\theta)$$

By hypothesis these equations are solvable and we can continue the induction.

Q.E.D.

We will now prove theorem 7.1. Because of what we've just proved we can assume that

$$\ell(x,\theta) = f_0(x) + \sum_{i=1}^{K} f_i(x)\, \psi_i(\theta) + \psi(\theta) + \varepsilon(x,\theta)$$

where $\varepsilon(x,\theta)$ vanishes to infinite order when $x = 0$.

Let's also assume for the moment that $n \geq k$ and that $df_1, \ldots, df_k$ are linearly independent at the origin. Then the $\psi_i(\theta)$ are linear combinations (with constant coefficients) of the $\frac{\partial \phi}{\partial \theta_i}(0, \theta)$ and 1, so the hypotheses of proposition 6.2 are satisfied and $\phi(x, \theta)$ is infinitesimally stable. Theorem 7.1, therefore, follows from theorem 6.4.

Suppose on the other hand that the df_i's are not linearly independent. Consider on the space $\mathbb{R}^{n+k} \times \mathbb{R}^N$ the phase function

$$\Phi(x, y, \theta) = f_0(x) + \sum_{i=1}^{k} \left(f_i(x) + y_i \right) \psi_i(\theta) + \psi(\theta) + \varepsilon$$

(23)

By applying the preceding argument to this phase function we can find a change of coordinates $\tilde{\theta} = \tilde{\theta}(x, y, \theta)$ and $\tilde{f}_i = \tilde{f}_i(x, y)$ $i = 0, \ldots, k$, such that

$$\Phi(x, y, \theta) = \tilde{f}_0(x, y) + \sum_{i=1}^{k} \tilde{f}_i(x, y)\, \psi_i(\tilde{\theta}) + \psi(\tilde{\theta})$$

Setting $\bar{\theta} = \tilde{\theta}(x, 0, \theta)$ we're done.

Fourier integral operators from the Radon transform point of view

D. Schaeffer

V. Guillemin

The purpose of these notes are not to discuss any new results but to describe a point of view toward some existing results: i.e. Hormander's "Fourier integral operators" paper and a related series of papers by Donald Ludwig. The reasons for this are not entirely pedagological. (See the concluding paragraph below.)

We recall that distributions on manifolds enjoy two types of functoriality, "push-forward" and "pull-back". "Push-forward" is simplest to describe. For a proper map $f : X \longrightarrow Y$, f_* on distributions is the dual operation to f^* on compactly supported C^∞ functions. "Pull-back" is only defined for submersions, $f : X \to Y$. Locally a submersion is just a projection map, for example the standard projection

$$\mathbb{R}^k \times \mathbb{R}^\ell \longrightarrow \mathbb{R}^k$$

Given a C_0^∞ density $\rho(x,y)\,dx\,dy$ on $\mathbb{R}^k \times \mathbb{R}^\ell$ we define its push-forward as

$$\left(\int \rho(x,y)\,dy \right) dx$$

The dual operation is the pull-back operation on distributional densities. The point of view toward Fourier integrals which we want to describe here is one that makes maximum use of these two functors.

We'll begin with "wave front" sets. Given a distribution u on a manifold X Hormander attaches to it a subset $WF(u)$ of the cotangent bundle T^*_X called its wave-front set (*) Intuitively (x_0, ξ_0) is not in $WF(u)$ if u is smooth at x_0 in the direction ξ_0. Hormander's precise definition is.

(*) There are various wave front sets. For simplicity we'll use the projective wave front set: $(x,\xi) \in WF(u) \Rightarrow (x, \lambda\xi) \in WF(u)$ for $\lambda \neq 0$.

$(x_0, \xi_0) \notin WF(u) \iff \widehat{\rho u}(\xi)$ is rapidly decreasing in a ~~[illegible]~~ conical nghd. of ξ_0, where $\wedge$ is the Fourier transform and ρ a bump function at x_0.

We'll now give an alternative definition of $WF(u)$.

<u>Rough</u> <u>Definition</u>: $(x_0, \xi_0) \notin WF(u)$ if for every function $f : X \longrightarrow \mathbb{R}$ with $(df)_{x_0} = \xi_0$ the push-forward, $f_* \rho u$, is smooth, ρ being, as above, a bump function at x_0.

The workable definition adds: if f depends smoothly on parameters so does $f_* \rho u$

We'll ~~[illegible]~~ show that the two definitions agree. First however lets use the second definition to compute $WF(u)$ for a particularly simple class of distributions.

<u>Definition</u>: A distribution, u, on X is <u>wave-like</u> if it is of the form $u = f^*\nu$ where ν is a distribution on $\mathbb{R}$ and $f: X \longrightarrow \mathbb{R}$ a submersion.

<u>Lemma 1</u> For a wave-like distribution, $f^*\nu$, the wave-front set is the set of normal vectors to the surfaces $f = c$, c being in the singular support of ν.

<u>Proof</u> Suppose $g: X \longrightarrow \mathbb{R}$ and $(dg)_{x_0} \neq (df)_{x_0}$. Then we can choose coordinates $x_1, \ldots, x_n$ centered at x_0 such that $f = x_1$ and $g = x_2$. Let ρ be a bump function at x_0. Then

$$g_* \rho f^* \nu = \int \left(\int \nu(x_1) \rho(x_1, \cdots x_n)\, dx_1 \right) dx_3\, dx_4 \cdots dx_n$$

which is smooth,

We'll also need the Radon ~~transform~~ inversion formula. For simplicity we'll state this in odd dimensions.

<u>Lemma 2</u> Given $\omega \in S^{n-1}$, let $\phi_\omega : \mathbb{R}^n \to \mathbb{R}$ be the map $x \to x \cdot \omega$ and let $D = \frac{1}{\sqrt{-1}} \frac{d}{dt}$ on $\mathbb{R}$. Then for a compactly supported distribution, μ, on $\mathbb{R}^n$ one has

$$(i) \qquad \int \phi_\omega^* \, D^{n-1} (\phi_\omega)_* \mu \, d\omega = \frac{1}{2(2\pi)^n} \mu$$

Proof: See Fritz John,

Finally we'll need a trivial identity:

<u>Lemma 3</u> $\widehat{(\phi_\omega)_* \mu}(t) = \hat{\mu}(t\omega)$

To prove the equivalence of our two definitions we note that by lemma 3 if $\hat{\mu}$ is rapidly decreasing in the direction ω_0, $(\phi_{\omega_0})_* \mu$ is smooth. Suppose now that ω_0 is not in $WF(\mu)$

if it exists. Hormander shows that the method of continuity works if f satisfies:

(ii) $\quad (df)_x^t \xi \neq 0 \quad$ whenever $(f(x), \xi) \in WF(u)$.

To prove this observe first that the theorem is true

a) when u is smooth. Then $f^* u$ can be defined as the usual pull-back of functions.

b) when u is wave-like. If $u = g^* \nu$ for some $g : Y \to \mathbb{R}$ then the π condition above just says that $g \circ f$ is a submersion, so we can define $f^* u = (g \circ f)^* \nu$.

For the general case just break u into a sum of a) and b) using the Radon transform.

We'll now describe Hormander's Fourier

integrals from the "Radon" point of view. (*)

This approach requires first of all a familiarity with the classical homogeneous distributions on the real line namely:

(iii) $\delta(x)$, $\delta^{(j)}(x)$, $\frac{1}{x^n}$, x_+^λ, $(x+i0)^\lambda$, etc.

Consider the pull-backs of these distributions with respect to submersions $f: Z \longrightarrow R$. These are the classical "couches" or "boundary layers". By lemma 1 their WF set is the normal bundle to the hypersurface $f = 0$

Finally given a submersion $\pi: Z \longrightarrow X$, consider distributions of the form

(iv) $\pi_* \rho f^* u$

u being a distribution on the list (iii)

(*) All of the following discussion except for the occasional use of words like "functor" is due to Donald Ludwig.

Proposition The Fourier integrals of Hormander are just those distributions on X which can be approximated to arbitrary order of smoothness by distributions of the type (iv).

We will now prove a theorem of Ludwig concerning the wave-front sets of distributions of the form (iv).

Theorem Let $Z = X \times Y$ and let $\pi : Z \to X$ be the usual projection. For fixed $y \in Y$ let $S(y)$ be the hypersurface in X consisting of the points : $x \in X$, $f(x,y) = 0$. (Assume for simplicity that $S(y)$ has no singular points) Let $\Lambda \subset T^*_X$ be the normal bundle of the envelop of these hypersurfaces. Then $WF(u) \subset \Lambda$

Proof We first prove.

Lemma 4 Let $\pi : Z \to X$ be a submersion and μ a distribution on Z. Then $WF(\pi_* \mu)$ is contained in the set of (x, η) with the property: $\exists\, z \in Z$, $\pi(z) = x$ and $(d\pi)_z^t \eta \in WF(\mu)$

Proof: Let $f : X \to \mathbb{R}$ with $(df)_x = \eta$ and observe that $f_* (\pi_* \mu)$ smooth $\Leftrightarrow$ $(f \circ \pi)_* \mu$ smooth.

To prove the theorem we note that, as was already observed, $WF(f^* \nu)$ for $f : X \to \mathbb{R}$ and ν on the list (iii) is just the normal bundle to $f = 0$. By lemma 4 the WF of the push forward is the set of vectors $x, \frac{\partial f}{\partial x}(x,y)$ where $f(x,y) = \frac{\partial f}{\partial y}(x,y) = 0$ which, by classical surface theory (see Struik,)

is the equation for the normal bundle to the envelop.

As an illustration of Ludwig's result let X be $\mathbb{R}^n$ and let Y be a hypersurface in $\mathbb{R}^n$. Let $f: \mathbb{R}^n \times Y \longrightarrow \mathbb{R}$ be the function $(x,y) \longrightarrow |x-y|^2 - c^2$ and let ν be the delta function. For fixed $y \in Y$ $f_y^* \nu$ is the delta function of the sphere of radius c about y in $\mathbb{R}^n$. "Summing" these $f_y^* \nu$'s, we get, by Ludwig, a distribution on $\mathbb{R}^n$ whose wave front set lies on the envelop of the spheres $|x-y| = c$ $y \in Y$. i.e the singularities not on the envelop cancel each other out by "interference" (corroborating

Huygen's principle!) See figure 1

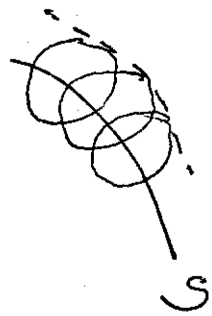

figure 1

The above results convey we hope some of the flavor of the "Radon" approach to Fourier integrals

To conclude we note that there have been two recent developments in the theory of generalized functions having many analogies with the work of Hormander. One is the recent work of Sato on hyperfunctions and the other the work of Maslov-Leray on "asymptotic functions" In both cases there are analogues of wave front sets, Fourier integrals, etc. There is some evidence that this is due to the existence of analogous push-forwards and pull-backs.

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On singularities and H^s spaces Part I

Let $P(D)$ be a const. coefficient 1st order elliptic ψdo differential operator on X with top symbol $p(\xi)$.

Consider $\frac{1}{\sqrt{-1}} \frac{\partial}{\partial t} - P(D)$ on $X \times \mathbb{R}$ with symbol $\tau - p(\xi)$.

Exercise A. The bicharacteristic flow on T^*_X associated with p is given by

$$(A) \quad \begin{cases} x \longrightarrow x + t \frac{\partial p}{\partial \xi}(\xi) \\ \xi \longrightarrow \xi \end{cases}$$

Exercise B Let Y be the $2n+1$ dimensional submanifold of $T^*_{X \times \mathbb{R}}$ on which $\tau - p(\xi) = 0$ (This is diffeomorphic to $T^*_X \times \mathbb{R}$ under the map $(x, \xi, t) \longrightarrow (x, \xi, t, p(\xi))$) Let Λ_0

be a homogeneous Lagrangian submanifold of T^*_X defined by the phase function $\sum x_i \xi_i - S(\xi)$. Then the Lagrangian manifold Λ flowed out in Y by the flow (A) is defined by the phase function: $\sum x_i \xi_i - S(\xi) - t\,p(\xi)$.

Hint: Set $x_0 = \frac{\partial S}{\partial \xi}(\xi_0)$. Then at time t (x_0, ξ_0) flows into (x, ξ_0) with $x = \left(\frac{\partial S}{\partial \xi} + t\frac{\partial p}{\partial \xi}\right)(\xi_0)$

Exercise (C) Let u_0 be a distribution on X of type $\mathcal{D}^m(\Lambda_0)$ with leading symbol of the form $a(x,\xi)$. Then there exists a solution, u, of the initial value problem

$$\left(\frac{1}{\sqrt{-1}}\frac{\partial}{\partial t} - P\right)u = 0 \qquad u(x,0) = u_0$$

expressible as an oscillatory integral associated with Λ

with leading symbol of the form $a^*(x,t,\xi)$

where $a^*(x,t,\xi) = a(x + t\frac{\partial p}{\partial \xi}, \xi)$

Exercise (D) By exercise (B) u

has the form

(D) $$u(x,t) = \int e^{i(x\cdot\xi - tp(\xi) - S(\xi))}\chi(|\xi|)\, a^*\, d\xi$$

if we neglect lower order terms in the asymptotic expansion. (χ is the usual cut-off function)

Show u can be written in the form

(D_1) $$\int e^{-it\lambda} \underbrace{\lambda^{(n-1)+m}}_{\chi(\lambda)} d\lambda \left[\int_{p(\xi)=1} e^{i\lambda(x\cdot\xi - S(\xi))} a^*(x,t,\xi)\, d\xi \right]$$

$d\xi$ now being the volume form on $p(\xi) = 1$

Exercise E For a given fixed x compute the critical pts of the function $x \cdot \xi - S(\xi)$, restricted to the hypersurface $p(\xi) = 1$, using Lagrange multipliers, i.e. solve the $n+1$ equations

$$(E) \qquad x_i = \frac{\partial S}{\partial \xi_i} - \lambda \frac{\partial p}{\partial \xi_i}, \quad p(\xi) = 1$$

Let Λ_0' be the subset of Λ_0 on which $p = 1$. Show that the solutions of (E) correspond 1-1 with pts $(x_0, \xi_0) \in \Lambda_0'$ such that the bicharacteristic ray $x_0 + t \frac{\partial p}{\partial \xi}(\xi_0)$ hits x at $t = \lambda$.

Show that degenerate critical pts correspond to singularities of the map

$$(F) \qquad \Lambda_0' \times \mathbb{R} \longrightarrow X$$

$$(x_0, \xi_0), t \longrightarrow x_0 + t \frac{\partial p}{\partial \xi}(\xi_0)$$

Compare the solutions of (E) with the critical points of the phase function in the integral (D)

Exercise (F) Show that if a singularity of the ray (F) has "intensity" α then the power of ~~[illegible]~~ η in the second integrand of the integral (D_1) is of order

$\eta^{n-1+m-\frac{n}{2}+\alpha}$ and make the relevant conclusions about the H^s space in which u sits at that point, as a function of t. (I think one can conclude from this that u jumps into a lower H^s space at the characteristic above (x,t). This is a special case of a rather general fact. The main idea is that since u is $H^{s-\alpha}$ in the t direction and satisfies a Hyperbolic equation for which the t direction is non characteristic its got to be $H^{s-\alpha}$ in the x direction as well)

Guillemin
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(1)

On Maslov Part I ~~Schrodinger operator~~

§ 1 We begin with definition of Maslov's canonical operator. Let Λ Lagrangian manifold in T^*_X and let $\pi : \Lambda \to X$ be the projection on the base. We denote by C the set of points $x \in \Lambda$ where rank $d\pi_x < n$, by $\bar{C}$ the image of these points in X. $\bar{C}$ is usually called the *caustic*. We will call C the *Maslov cycle* on Λ. We will write $C = C_1 \cup C_2$ where $C_1 = \{ x \in C$, rank $d\pi_x = n-1$ and $C_2 = \{ x \in C$, rank $d\pi_x < n-1 \}$. For in general position C_1 is a submanifold of codim 1 and C_2 is a closed set which is a union of submanifolds of codim ≥ 3 -

C_1 can be oriented in a natural way that if γ

is a curve in Λ its intersection with C_1 is well defined.

The Maslov canonical "operator" maps $\frac{1}{2}$ densities on Λ into half-densities on X. We will define it locally (on Λ) and patch together. The " " indicates that what we get is not actually an operator but just a kind of approximate operator.

<u>Local definition at a pt</u> $x_0 \in \Lambda - C$: Since $(d\pi)_{x_0}$ is bijective π maps a nghd. $\mathcal{U}$ of x_0 diffeomorphically onto a nghd V of $\pi(x_0)$. The inverse map is of the form $x \longrightarrow d\phi_x$ where ϕ is a smooth function on V. Given a half-density a on $\mathcal{U}$ we map

(A) $$a \longrightarrow \bar{a}\, e^{i\lambda\phi}$$

③

where $\bar{a}$ is the half density on V corresponding to a on $\mathcal{U}$. (Note: there is already an ambiguity in our definition. ϕ is only determined up to an additive constant.)

<u>Definition at a pt</u> $x_0 \in C$: Let V be a nghd of $\pi(x_0)$. We choose a phase function ψ on $V \times \mathbb{R}^N$, $\psi = \psi(x, \theta)$; such that $(x, \theta) \in C_\theta \longrightarrow \mathrm{grad}_x \psi$ parametrizes a nghd $\mathcal{U}$ of x_0 on Λ. The half density a on $\mathcal{U}$ corresponds to a $\frac{1}{2}$ density $\tilde{a}$ on C_θ. The functions $\frac{\partial \psi}{\partial \theta_1}, \ldots, \frac{\partial \psi}{\partial \theta_N}$ give us a canonical way of trivializing the normal bundle $N(C_\theta)$. Let μ_x be Lebesgue measure

(4).

on the fiber N_x. Finally let $\bar{a}$ be a half density on $V \times R^N$ which has support in a tube around C_θ and takes the value $\tilde{a} \otimes \sqrt{u}(x)$ at $x \in C_\theta$. On $\mathcal{U}$ we define the Maslov operator by

$$(B) \qquad a \longrightarrow \frac{1}{\sqrt{2\pi\lambda}^N} \int \bar{a}\, e^{i\lambda\psi(x,\theta)} \sqrt{d\theta}$$

This definition depends of course on the choice of $\bar{a}$. By a simple integration by parts one can show that another choice of $\bar{a}$ would change the RHS of (B) by a term of order $O(\frac{1}{\lambda})$. What about different ψ?

We will now describe how the expressions (A) and (B) patch together. Let ω

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be the action form on T^*_X. ($\omega = \sum \xi_i \, dx_i$ in the usual x, ξ notation) Its restriction to Λ is closed, and we can assume it exact on $\mathcal{U}$. i.e $\omega \mid \mathcal{U} = d\phi$ for some function ϕ on $\mathcal{U}$. The ϕ here incidentally is, up to an additive constant, the same as the ϕ occuring in (A) at pts $x \in \mathcal{U} - C$.

The main result in this subject is the theorem of stationary phase:

<u>Theorem</u>: Let $\mathcal{U}_1, \ldots, \mathcal{U}_K$ be the connected components of $\mathcal{U} - C$. On $\mathcal{U}_\alpha$ the RHS of (B) is equal to $c_\alpha \, \bar{q} \, e^{i\lambda\phi} + O(\frac{1}{\lambda})$. Moreover the constants c_α and c_β are related by $c_\alpha = e^{i k_{\alpha\beta}} c_\beta$ where $k_{\alpha\beta}$ is the intersection no of the Maslov cycle C with any curve γ joining a pt in $\mathcal{U}_\alpha$ to a pt in $\mathcal{U}_\beta$.

We now make the following assumptions

I : The restriction of the action form ω to Λ is exact.

II The "dual class" of C in $H^1(\Lambda)$ (Z_4 coefficients) is zero.

Then the Maslov operator can be defined globally as follows. We chose the ϕ described in the paragraph above so that it defined globally i.e restriction of ω to Λ $= d\phi$. and we choose the ~~[illegible]~~ ~~[illegible]~~ constant in (A) so that if we go around a path in Λ we come back to where we started from. Then the formulas (A) and (B) define a map, $a \to \underset{\Lambda}{K} a$

(C) $\frac{1}{2}$ densities on Λ $\longrightarrow$ $\frac{1}{2}$ densities on X, depending on λ, modulo $\frac{1}{2}$ densities of order $O\left(\frac{1}{\lambda}\right)$

We can write down an explicit formula for $\underset{\Lambda}{K} a$

(7.) at pts $x \in X - \bar{C}$ as follows. Let q_0 be a fixed base point in $\Lambda - C$. Suppose x has k preimages, $p_1, \dots, p_k$ on Λ. Let $\gamma_i : [0,1] \to \Lambda$ be a smooth curve joining q_0 to p_i and intersecting C transversally. Then

$$(D) \qquad \mathcal{K}_\Lambda a(x) = \sum \bar{a}_i(x)\, e^{i\left(\int_0^1 \omega\left(\frac{d\gamma_i}{dt}\right) + \mu_i + \phi(q_0)\right)}$$

where μ_i is the intersection no of γ_i with ζ and $\bar{a}_i$ is the $\frac{1}{2}$ density at x associated with a and p_i.

<u>Proof</u> On Λ $\omega = d\phi$, so $\omega\left(\frac{d\gamma}{dt}\right) = \frac{d\phi}{dt}(\gamma_i(t))$, and the integral in the exponential $= \phi(p_i) - \phi(q_0)$.

<u>Remark</u> By replacing q_0 by a finite no. of base point and choosing these points judiciously, (D) sometimes can be defined even when the conditions I and II don't hold

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Example: Let Y be a manifold and Λ_0 a Lagrangian manifold in the cotangent bundle of Y. Suppose that condition I holds for Λ_0 i.e. the action form restricted to $\Lambda_0 = d\phi_0$. Let p be a function on T^*_Y and $\Xi = \Xi_p$ the corresponding Hamiltonian vector field. Let $\rho_t : T^*_Y \to T^*_Y$ be the flow generated by Ξ. ρ_t sweeps out a Lagrangian submanifold of the cotangent bundle of $R \times Y$ namely the set of points (x, ξ, t, τ) where $(x,\xi) \in \rho_t(\Lambda_0)$ and $\tau = p(x,\xi)$. We'll denote the Lagrangian submanifold by Λ. It's fairly easy to see that Λ satisfies condition I, but it needn't satisfy condition II. Nevertheless $\mathcal{K}_\Lambda$ can be defined as follows. Let (t,y) be a point in $(R \times Y) - \bar{C}$, and let $(t, p_1), \dots (t, p_N)$ be the pt above it on Λ.

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Let $(0, q_1), \cdots, (0, q_N)$ be the points on the backward flow at time $t = 0$. Then

$$(D)' \quad \mathcal{X}_1 q = \sum_\alpha \bar{a}_\alpha(t, y)\, e^{i\int_0^t \omega\left(\frac{d\gamma_\alpha}{dt}\right)dt + m_\alpha + \phi_0(q_\alpha)}$$

where γ_α is the curve $s \longrightarrow (s, \rho_s(q_\alpha))$, $0 \le s \le t$, and m_α is its intersection no. with C.

§2 For $i = 0, 1, \dots, k$ let $P_i(x, D)$ be an ith order P.D.E mapping half densities on X into half-densities on X. We want to look for asymptotic solutions as λ gets large of the partial differential operator
$$P(x, D, \lambda) = \sum \frac{1}{(\sqrt{-1}\,\lambda)^k} P_k(x, D)$$
Let $p(x,\xi)$ be the function $\sum p_i(x,\xi)$ where $p_i(x,\xi)$ is the top symbol of the operator $P_i(x, D)$. Let $c = \sum c_i$ where c_i is the subprincipal part of the operator P_i. One of Maslov's main results is the following theorem

<u>Thm</u> Let Λ be a Lagrangian manifold on which $p(x,\xi) = 0$. Let a be a half-density on Λ satisfying the transport equation

$$\mathcal{L}_{\Xi} a + c a = 0$$

where Ξ is the Hamiltonian vector field corresponding to P. Then $P(x, D, \lambda)\, \mathcal{K}_{\Lambda} a = O\left(\frac{1}{\lambda^2}\right)$

Rather than trying to prove this theorem we'll derive as a corollary of it Maslov's explicit formula for the solution of the Shroedinger equation

$$\text{(E)} \qquad \frac{h}{\sqrt{-1}} \frac{\partial \psi}{\partial t} = - h^2 \sum \frac{1}{2} \frac{\partial^2 \psi}{\partial x_i^2} + V(x)\psi$$

in $\mathbb{R}^n \times \mathbb{R}$. The associated symbol is

$$\tau - \left(\sum \frac{\xi_i^2}{2} + V(x) \right) = H(x, \xi, t, \tau)$$

and the Hamiltonian is

$$\Xi = \frac{\partial}{\partial t} - \sum \xi_i \frac{\partial}{\partial x_i} + \frac{\partial V}{\partial x_i} \frac{\partial}{\partial \xi_i} ,$$

and the action $= \omega(\Xi) = \tau - \sum \xi_i^2$. When $H = 0$ this is equal to $-\sum \xi_i^2 + V(x)$.

On an integral curve of Ξ we have $\dot{x}_i = \frac{\partial H}{\partial \xi_i} = \xi_i$ so we can write the action integral in (D) as

$$-\int_0^t \sum \frac{(\dot{x}_i(t))^2}{2} + V(x)\,dt = \int_0^t L(x,\dot{x},t)\,dt.$$

where L is the classical Lagrangian.

Finally note that in the transport equation $c = 0$ since all the terms of the Shroedinger operator are self-adjoint.

Now let $a_0 e^{i\frac{1}{\hbar}\phi_0}$ be a $\frac{1}{2}$ density on $\mathbb{R}^n$ with a_0 compactly supported.

We want to find a solution $\mathcal{S}$ of the equation (E) which takes on initial data $\mathcal{S}(x,0) = a_0 e^{i \frac{1}{\hbar} \phi_0}$ at $t=0$

Let ρ_t be the flow associated with $\frac{\xi^2}{2} + V(x)$ at time t. Consider the map $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ which maps $x \longrightarrow Q(x) = \pi \rho_t \left(x, (d\phi_0)_x \right)$, π being the projection of the cotangent bundle of $\mathbb{R}^n$ onto its base. Let Q be a regular value of this map and let $q_1, \dots, q_K$ be its preimage pts.

<u>Thm</u> $$\mathcal{S}(q,t) = \sum \left| \left(\frac{DQ}{\partial q} \right) \right|^{-\frac{1}{2}} (q_i) \, a_0(q_i) \, e^{i\lambda \left(\int_0^t L + \gamma_i + \phi_0 \right.}$$

modulo an error term of order $O\left(\frac{1}{\lambda}\right)$ $\quad \lambda = \frac{1}{\hbar}$.

where the integral is over the classical path $q_i(t)$

(14) joining q_i to Q and σ_i is the intersection

number of the corresponding path on Λ with

the Maslov cycle.

Except for the expression for the amplitude

in the above formula the theorem is a direct corollary

of the Theorem above. and the (D') To see that the expression

for the amplitude is right we note that it is

derived as follows. We lift a_0 up to Λ_0,

map it by $(d\Phi_{-t})^*$ (since we want it to

satisfy the transport equation $\mathcal{L}_\xi a = 0$) and then

project down. The Jacobean for this map is

$\left|\frac{DQ}{dq}\right|$ and since we are mapping $\frac{1}{2}$ densities

which are <u>contravariant</u> of order $\frac{1}{2}$ $a_0(q_i)$ gets

mapped into $\left|\frac{DQ}{dq}(q_i)\right|^{-\frac{1}{2}} a_0(q_i)$

§3 We will use the result above to derive Maslov's asymptotic expression for the fundamental solution of the Shroedinger operator. We begin with the plane-wave expansion of the δ-function.

$$\delta(x-x_0) = a(x) \int e^{i\eta\cdot(x-x_0)}\, d\eta$$

where $a(x)$ is a smooth function equal to 1 at x_0 and having support in a small nghd. of x_0. Setting $\eta = \frac{\xi}{\hbar}$ we get

$$(F) \qquad \delta(x-x_0) = \frac{a(x)}{(\hbar)^n} \int e^{\frac{i\xi}{\hbar}\cdot(x-x_0)}\, d\xi$$

We will use this expansion and the theorem above to obtain an asymptotic solution of (E) with initial data $\delta(x-x_0)$ at $t=0$. Let (y,t) be a point of $\mathbb{R}\times\mathbb{R}^n$. We will assume that there are only a finite no. of classical trajectories joining y to x_0

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and that y and x_0 are non-conjugate along these trajectories. Clearly the same will be true for all trajectories joining y to points in the support of $a(x)$.

Consider the graph of $d(x-x_0)\cdot\xi_0$ in the cotangent bundle of $\mathbb{R}^n$. This is just the set of pts. $\{(x,\xi_0), x\in\mathbb{R}^n\}$

Let $\Lambda(\xi_0)$ be the set of all trajectories which hit this graph at $t=0$. ($\Lambda(\xi_0)$ is a Lagrangian submanifold of $T^*_{\mathbb{R}\times\mathbb{R}^n}$). Our assumption about y guarantees that there are only a finite no of trajectories

$$\gamma_{\alpha,\xi_0} : [0,t] \longrightarrow \Lambda(\xi_0) \qquad \alpha = 1,\dots,N$$

whose terminal pts lie above (y,t) and whose initial pts lie above the support of $a(x)$. Let $(x_\alpha(\xi),\xi)$ be the initial pt. of the curve $\gamma_{\alpha,\xi}$. The theorem above gives us the following asymptotic formula for the solution of (E) with initial data $a(x)e^{i\frac{\xi}{h}\cdot(x-x_0)}$ at time $t=0$:

$$\sum_\alpha a_\alpha(x(\xi)) \left|\frac{\partial y}{\partial x_\xi}\right|^{-\frac{1}{2}} e^{\frac{i}{h}\left(\int_{\gamma_{\alpha,\xi}} \omega + (x(\xi)-x_0)\cdot\xi\right)+\frac{\pi i}{4}m_\alpha}$$

where m_α is the intersection no of $\gamma_{\alpha,\xi}$ with the Maslov cycle on $\Lambda(\xi)$. Plugging this into (F) we get the following expression for the fundamental solution of (E).

$$(G) \qquad G(y,t,x_0) = \sum_\alpha \frac{1}{ih^n} \int a_\alpha(x(\xi)) \left|\frac{\partial y}{\partial x_\xi}\right|^{-\frac{1}{2}} e^{\frac{i}{h}\int\omega + m_\alpha + \cdots} d\xi$$

We will try to evaluate (G) using stationary phase. To do this we need to determine the critical pts of the phase function:

$$\phi(\xi) = \int_{\gamma_{x,\xi}} \omega + (x(\xi) - x_0)\cdot\xi$$

as a function of ξ. To do so we'll need some general facts about symplectic geometry: Let X be a manifold, and Λ a Lagrangian submanifold of T^*_X. Let ω be the action form. For each $s \in \mathbb{R}$ let γ_s be a smooth curve on Λ and suppose γ_s depends smoothly on s. Let $\alpha(s)$ and $\beta(s)$ be the initial and terminal pts of γ_s.

<u>Lemma</u> $$\frac{d}{ds}\int_{\gamma_s}\omega = \omega\left(\frac{d\beta}{ds}\right) - \omega\left(\frac{d\alpha}{ds}\right)$$

<u>Proof</u>: There is a tubular nghd, $\mathcal{U}$, of γ_s in Λ on which ω is exact; i.e. $\omega = df$ on $\mathcal{U}$. Thus $\int_{\gamma_s}\omega = f(\beta(s)) - f(\alpha(s))$. Differentiating with respect to s we get the assertion above. Q.E.D.

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Now let compute $\frac{\partial}{\partial \xi_1} \int_{\gamma_{\alpha,\xi}} \omega$. Note first that the curves $\gamma_{\alpha,\xi}$ all lie in a fixed Lagrangian manifold in $T^*_{\mathbb{R}\times\mathbb{R}^n}$ namely the set of all trajectories that at time t lie above the pt (y,t). Let $V_{1,\alpha}$ and $W_{1,\alpha}$ be the tangent vectors to the initial and terminal curves of $\gamma_{\alpha,\xi}$ obtained by varying ξ_1 and leaving the other coordinates of ξ fixed. By the lemma

$$\frac{\partial}{\partial \xi_1} \int_{\gamma_{\alpha,\xi}} \omega = \omega(W_{1,\alpha}) - \omega(V_{1,\alpha})$$

The end point of $\gamma_{\alpha,\xi}$ projects onto the fixed pt (y,t) in the base for all ξ, so $(d\pi)W_{1,\alpha} = 0$ and hence $\omega(W_{1,\alpha}) = 0$ (because of the way ω is defined!) On the other hand

$(d\pi)\, V_{i,\alpha} = \frac{\partial x_\alpha(\xi)}{\partial \xi_i}$, so $\omega(V_{i,\alpha}) = \xi \cdot \frac{\partial x_\alpha}{\partial \xi_i}$

at $(x(\xi), \xi)$. Therefore, we get:

$$\frac{\partial}{\partial \xi_i} \int_{\gamma_{\alpha,\xi}} \omega = - \xi \cdot \frac{\partial x_\alpha}{\partial \xi}$$

and $\frac{\partial}{\partial \xi_i} \phi(\xi) = (x(\xi) - x_0)_i$: This proves

Thm The critical pts. of the phase function in the integral (G) are precisely those ξ for which $x(\xi) = x_0$ i.e. for which the integral curve $\gamma_{\alpha,\xi}$ joins x_0 to y.

If we apply stationary phase to (G) and use the fact that $a(x_0) = 1$ we get the following asymptotic formula for the RHS.

$$(H) \qquad G(y,t,x_0) \sim \sum_\alpha \frac{1}{(2\pi i\hbar)^{\frac{n}{2}}} \left|\frac{\partial y}{\partial \xi}\right|^{-\frac{1}{2}}(x_0,\xi_\alpha)\, e^{\frac{i}{\hbar}\int_0^t L(\dot q_\alpha, q_\alpha, t)\,d\tau + \nu_\alpha}$$

where $q_\alpha(\tau)$, $0 \le \tau \le t$ is a classical trajectory going from x_0 to y, and $\nu_\alpha = \mu_\alpha + \operatorname{sign}\left(\frac{\partial x_\alpha}{\partial \xi}\right)(x_0,\xi_\alpha)$

Maslov identifies ν_α with the number of conjugate pts along the trajectory, $q_\alpha(\tau)$. I don't at the moment see why this is the relation between this number and the intersection no. of γ_α with the Maslov cycle.

Maslov gives an alternative proof of the formula (H) using Feynmann integrals: He starts with Feynmanns representation of the fundamental solution of the Shroedinger operator:

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$$G(y,t,x_0) = \int e^{\frac{i}{\hbar}\int_0^t L(q,\dot q,\tau)\,d\tau}\,d\mu$$

where $q(\tau)$ is any path joining x_0 to y and μ is Feynmann measure on path space.

Let's apply stationary phase to the RHS above (ignoring the fact that the integral is not over a finite dimensional region.) The critical pts. of the phase function are just those paths for which the first variation $\delta \int L = 0$, which by the principal of least action are just the classical trajectories, that is, the $q_\alpha(\tau)$ above.

The signature of $\delta^2 \int L$ at each of these trajectories is, by Morse theory, equal to the no. of conjugate pt, ν_α, along the trajectory. Therefore we obtain asymptotic formula for the RHS of (I)

$$G(x,y,t) \approx \sum K_\alpha \, e^{\frac{i}{\hbar}\int_0^t L(q_\alpha, \dot{q}_\alpha, \tau)\, d\tau + i\frac{\pi}{4}\nu_\alpha}.$$

Here K_α is the quotient of two infinite quantities, namely $(2\pi i \hbar)^{\frac{\infty}{2}}$ and $\det(\delta^2 L)$, but apparently these cancel each other out and give the finite answer computed above. ?

Vasy 6 March

Thm. If $(x,0,y,\eta) \notin WF(K)$, then $\mathcal{K}$ can be extended by continuity $C_c^{-\infty} \longrightarrow C^{-\infty}$.

Pf: $\mathcal{K}u = \pi_* K \rho^* u$, product defined by before.

Not^n: $\Lambda \subset S \times T$. If $A \subset S$, then

$\Lambda(A) = \{ t \in T : \exists (s,t) \in \Lambda \text{ s.t. } s \in \Lambda \}$.

Ex: $\Lambda =$ graph of $f: S \longrightarrow T$.

Not^n: If $W \subset T^*_{X \times Y}$, define W'.

Thm: In above situation

$$WF(\mathcal{K}u) \subset WF(K)'(WF(u)) \cup A, \text{ where}$$

$$A = \{(x,\xi) : (x,\xi,y,0) \in WF(K)\ (\exists y)\}.$$

Prop: Suppose $K \in C^{-\infty}(X \times Y)$, $L \in C^{-\infty}(Y \times Z)$, neither intersecting nl zero sections. Properly supported. Then $\mathcal{K} \circ \mathcal{L}$ is kernel op assoc with ker in $C^{-\infty}(X \times Z)$, +

$$WF(K \circ L)' = WF'(K) \circ WF'(L).$$

Not^n: $\Lambda_1 \circ \Lambda_2 = \{ (s,u) : (s,t) \in \Lambda_1, (t,u) \in \Lambda_2 \ (\exists t) \}$ when $\Lambda_1 \subset S \times T$, $\Lambda_2 \subset T \times U$.

Dfn: $K \circ L = \cancel{\pi_2^*} \; \pi_{2*}\left\{ (\pi_1^* K)(\pi_3^* L) \right\}$, where

$$
\begin{tikzcd}
 & X \times Y \times Z \arrow[dl, "\pi_1"'] \arrow[d, "\pi_2"] \arrow[dr, "\pi_3"] & \\
X \times Y & X \times Z & Y \times Z
\end{tikzcd}
$$

~~Dfn:~~
Compound distributions

$$
\begin{tikzcd}
Z \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \mathbb{R} \\
X &
\end{tikzcd}
$$

φ : 0 not a crit value
π : a fibration.

Let $u \in \mathcal{D}'(\mathbb{R})$ be homog, a a C^∞ density on Z.

Consider $\pi_* \, a \, \varphi^* u$,

Ex: ① δ-fcn (Use Radon)

$$\delta = \int_{S^{n-1}} \langle x, \omega \rangle^* u_\delta \, d\omega \quad , \quad u_\delta = c\left\{ (x+i0)^{-n} \pm (x-i0)^{-n} \right\}$$

② Any der. of compound dist:
Chain rule: $\dfrac{\partial}{\partial x_i}(\varphi^* u) = \dfrac{\partial \varphi}{\partial x_i} \varphi^* u'$

③ Fund soln of elliptic const coeff PDE.
(Must generalize to include asymptotic series.)

Shlomo

Conforma $\subset Sp(V)$: $(Au, Av) = \mu_A(u, v)$.

Lie alg: $(sp(V)$: $(Au, v) + (u, Av) = 0$.

Eigenvalues of $sp(V)$ occur in pairs s.t. $\lambda_1 + \lambda_2 = 0$.

Pairing of + + − Eigenspaces.

Claim: X, Y Lag. in V, trans.

$$0 \longrightarrow X \longrightarrow V \xrightarrow{P} Y \longrightarrow 0$$

$\underline{P} \in csp(V)$, with $\mu_P = 1$.

Read backwards: given P, etc.

Param. $\{\Lambda : \text{tr to } X\}$ by P.

If $P \in csp(V)$, let $Q_P(u, v) = (Pu, v) - \frac{1}{2}\mu_P(u, v)$.

Claim: Q_P is symm.

Remark: Q_P determines P.

Identify: (1) $\{Y : Y \text{ Lag}, \text{tr to } X\}$

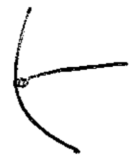
(2) $\{P \in csp(V) \quad \mu_P = 1, P_{|X} = 0$

(3) $\{$ symm of forms, $Q(x, v) = -\frac{1}{2}(x, v)$ for $x \in X\}$.

By 3, which is affine space. $\cong S^2(V/X)$.

Lin alg: Cong. doesn't change sign eigenvalues. Pivot test for pos. def.

$\boxed{\text{Solitons}}$ $\int z\, e^{-\lambda z^2}\, dz = \int$

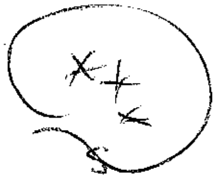
 $\psi^{x}(y) = \frac{1}{2}|x-y|^2$

Hessian non-deg iff exp $N(M) \to \mathbb{R}^n$ non-deg.

$H = \mathrm{I} - s\,\mathrm{II}$. (Also codim ≥ 1, but 2 dep. on direction.)

Define index. i = no. crit. pts.

Sommerfeld: $r\left(\frac{\partial u}{\partial r} - iku\right) \to 0$, $u \to 0$ at ∞.



j

$$u(p) \sim \frac{1}{2} \sum \frac{e^{-\# \pi i/2}\, e^{ikr}\, u(y)}{\sqrt{|\det(\mathrm{I}_y - r\,\mathrm{II}_y)|}}$$

(1)

(Stationary phase + Green's thm + radiation cond.)

Backward waves cancel.

Note: Morse index thm $\Rightarrow$ expression indep of small change in surface.

Victor 28 Feb

~~Suppose~~

?

A. Kiva Gabor Trans AMS 1971

$Z \subset T^*_X \sim \{0\}$ homog.

$C^{-\infty}_Z(X)$

Ex: $Z = \pi^{-1}(F)$, $F \subset X$ closed.

Dfn: $u_n \to u$ if cnvgs as dist $+$ if $x \notin F$ $\exists$ bump b s.t.

$bu_n \to bu$ in C^∞.

Genlz^n: $u_n \to u$ if for $\varphi : X \times S \to \mathbb{R}$, $d_x\varphi(x_0, \xi_0) = \xi_0$,

$\exists$ bump b near (x_0, s_0) s.t. $\varphi^\#_* b\pi^* u_n \to \varphi^\#_* b\pi^* u$.

Properties of top:

(1) Suff to test for $\varphi : X \times S^{n-1} \to \mathbb{R}$, $\varphi(x, \omega) = \langle x, \omega \rangle$.

(2) $C^{-\infty}_Z$ ~~closed~~ completely in $C^{-\infty}$

(3) C^∞ is dense (Pf: convolution.)

Lemma: Let φ be bump fn, $f : X \xrightarrow{(x_1, \ldots x_n) \to x_1} \mathbb{R}$. Let $\varphi^\# = f_*(\varphi\, dy)$

If supp u compact, then $f_* \underset{\uparrow}{u_\varepsilon} = \underbrace{(f_* u)_\varepsilon}_{\varphi^\#}$

Mollifier φ

Prop ~~Remarks~~: If $f : X \to Y$, then $f_* : C^{-\infty}_{\cancel{Z}} \to C^{-\infty}_{f_* Z}$

$$f_* Z = \{ (y_0, \eta_0) : \exists x_0 \in f^{-1}(y_0) + (x_0, (df)^t_{x_0} \eta_0 \in Z \}.$$

Prop: If $f : X \to Y$ is submersion $f^* : C^{-\infty} \to C^{-\infty}$ induces map $C^{-\infty}_Z(Y) \to C^{-\infty}_{f^* Z}(X)$ where

$$f^* Z = \{ (x_0, \xi_0) : y_0 = f(x_0) + \exists \eta_0 \in Z \text{ s.t. } \xi_0 = (df)^t_{x_0}(\eta_0) \}$$

~~Proto-Prop~~ ?.

Pf. Prob (Check that cont for sm. dist.)

~~Th: For any map $f : X \to Y$, $f^* : C^{-\infty}_Z(Z) \to C^{-\infty}_{f^* Z}($~~

Thm: Let $f : X \to Y$, $Z \subset PT^*(Y)$. Suppose $(y_0, \eta_0) \in Z$,

+ $f(x_0) = y_0 \Rightarrow (df)^t_{x_0} \eta_0 \neq 0$. Then $f^* : C^{-\infty}_Z(Y) \to C^{-\infty}_{f^* Z}(X)$

well def. by continuity.

(Cond: f transversal to hypersurface to which η_0 is normal.)

(Think of $u \in C^{\infty}_Z$ as ~~superposition~~ synthesis of wave-like dist with ~~normal~~ sing. surfaces. ~~given by~~ normal to directions in Z.)

Idea of pf: decompose u into wave like dist. Here suff if composite $X \xrightarrow{f} Y \xrightarrow{\varphi} \mathbb{R}$ is submersion.

Consider 2 classes of directions: smooth or not sm, but then transversal.

Q: Apply to take R. transf. of more general dist?

Note: Write down answer but use top to verify indep. of coord