We begin with a definition of Maslov's canonical operator. Let $\Lambda$ be a Lagrangian manifold in $\mathbb{R}^n$ and let $\pi: \Lambda \to X$ be the projection on the base. We denote by $C$ the set of points $x \in \Lambda$ whose rank $\operatorname{rank} d\pi_x < n$ and by $\tilde{C}$ the image of these points in $X$. $\tilde{C}$ is usually called the caustic. We will call $C$ the Maslov cycle on $\Lambda$. We will write $C = C_1 \cup C_2$ where $C_1 = \{ x \in C, \operatorname{rank} d\pi_x = n-1 \}$ and $C_2 = \{ x \in C, \operatorname{rank} d\pi_x < n-1 \}$. In a general position $C_1$ is a submanifold of codimension 1 and $C_2$ is a closed set which is a union of submanifolds of codimension $\geq 3$. $C_1$ can be oriented in a natural way so that $\Lambda$...
is well defined.

The Maslov canonical operator maps half-densities on \( \Lambda \) into half-densities on \( \mathcal{X} \). We will define it locally (on \( \Lambda \)) and patch together. The indication that what we get is not actually an operator but just a kind of approximate operator.

**Local definition of a set** \( x \in \Lambda - C \)  

Since \( (\mathcal{M})_x \) is injective \( x \) maps as rigid \( \mathcal{U} \) of \( x \), diffeomorphically onto a rigid \( \mathcal{V} \) of \( \pi(x) \). The inverse map is of the form \( x \mapsto \phi_x \), where \( \phi \) is a smooth function on \( \mathcal{V} \). Given a half-density \( \sigma \) on \( \mathcal{U} \), we may

\[
(A) \quad \sigma \mapsto \tilde{\sigma} e^{\lambda \phi}
\]
\[ \tilde{a} \text{ is the half density on } \mathcal{V} \text{ corresponding to } a \text{ on } \mathcal{U}. \] (Note: there is already an ambiguity in our definition. \( \phi \) is only determined up to an additive constant.)

Definition at a point \( x_0 \in \mathcal{C} \): Let \( \mathcal{V} \) be a neighborhood of \( x_0 \). We choose a phase function \( \psi \) on \( \mathcal{V} = \mathbb{R}^n \), \( \psi = \psi(x, \theta) \), such that \( (x, \theta) \in \mathcal{C} \) \( \rightarrow \) \( \text{grad} \, \psi \) \( \text{parametrizes a neighborhood } \mathcal{U} \) of \( x_0 \) on \( \mathbb{C} \).

The half density \( a \) on \( \mathcal{U} \) corresponds to a \( \frac{1}{2} \)-density \( \tilde{a} \) on \( \mathcal{C} \). The functions \( \frac{2\pi i}{3}, \ldots, \frac{2\pi i}{n} \) give us a canonical way of traversing the normal bundle \( N(x_0) \). Let us be the transverse measure.
on the fiber $N$. Finally let $\overline{a}$ be a half-density on $V \times \mathbb{R}^n$ which has support in a tube around $C_0$ and takes the value $\overline{a} \cdot \psi(x)$ at $x \in C_0$. On $U$ we define the Melrose operator by

$$\overline{a} \rightarrow \frac{1}{\sqrt{2\pi n}} \int \frac{\overline{a} e^{i x \cdot \theta} \psi(x, \theta)}{\sqrt{1 - \theta}} d\theta$$

This definition depends of course on the choice of $\overline{a}$. By a simple integration by parts one can show that another choice of $\overline{a}$ would change the RHS of (B) by a term of order $O(\frac{1}{n})$.

We will now describe how the expressions (A) and (B) match together. Let $\overline{w}$
be the action from $\mathbb{R}^\infty$ to $\mathbb{R}$ (in the usual $\delta x$ notation). Its restriction to $\Lambda$ is closed, and we can assume it exact on $\Omega$, i.e., $\omega|\Omega = d\phi$ for some function $\phi$ on $\Omega$. The $\phi$ then unconditionally is up to an additive constant, the same as the $\phi$ occurring in (A) at $\phi = x \in \Omega - C$.

The main result in this subject is the theorem of stationary phase:

**Theorem:** Let $\Omega_1, \ldots, \Omega_k$ be the connected components of $\Omega - C$. On $\Omega_i$, the RHS of (B) is equal to $c_i \sqrt{\pi} e^{x_i \lambda} + O(1/x)$.

Moreover, the constants $c_i$ and $c_0$ are related by $c_i = e^{x_i k_0} c_0$, where $k_0$ is the intersection no of...
We now make the following assumptions

I. The restriction of the action from $\mathcal{M}$ to $\Lambda$ is exact.

II. The "dual class" of $C$ in $\mathcal{H}^2(\Lambda)$ is zero.

Then the Maslov operator can be defined globally as follows. We choose the $\phi$ described in the paragraph above so that it is defined globally, the restriction of $\mathcal{M}$ to $\Lambda = \partial \phi$, and we choose the restriction $\pi_\Lambda$ constant in $\phi$ so that if we go around a path in $\Lambda$, we come back to where we started from. Then the formulas (A) and (B) define a map, $a \mapsto \frac{\pi a}{\Lambda}$.

(C) $\frac{1}{2}$ density on $\Lambda \rightarrow \frac{1}{2}$ density on $X$, depending on $\Lambda$, modulo $\frac{1}{2}$ densities of order $O(\frac{1}{\Lambda})$.

We can write down an explicit formula for $\frac{\pi a}{\Lambda}$. 
at $q_1 \in X-C$ as follows. Let $\omega$ be a fixed closed form in $\Lambda-C$. Suppose $x$ has $k$ special points $p_1, \ldots, p_k$ on $\Lambda$. Let $\phi: \mathbb{R}^k \to \Lambda$ be a smooth curve spanning $q_1$ to $p_1$ and intersecting $C$ transversally. Then

$$\omega|_\Lambda = \sum \bar{a}_i(x) \in \left( \int_{C} \phi'(s)^t \frac{\partial \phi}{\partial s} + \alpha + \phi(q_1) \right)$$

where $u_i$ is the intersection no. of $\Lambda$ with $y_i$ and $\bar{a}_i$ is the $i$-th density of $x$ associated with $\alpha$ and $p_1$.

**Proof.** On $\Lambda$, $\omega = d\phi$, so $\omega \left( \frac{\partial \phi}{\partial s} \right) = \frac{d\phi}{ds}(\phi(q_1))$, and the integral in the exponential $= \phi(p_1) - \phi(q_1)$.

**Remark.** By replacing $x$ by a finite no. of base points and choosing other points judiciously, (O) sometimes can be defined even when the conditions I and II don't hold.
Example: Let $Y$ be a manifold and $\pi: Y \to \mathbb{R}$ a Lagrangian manifold in the cotangent bundle of $Y$. Suppose that condition I holds for $\lambda_0 = \pi$. Let $P$ be the action from restricted to $\lambda_0 = \pi$. Let $F$ be a function on $T^*_Y$ and $\pi = \pi_P$ with corresponding Hamiltonian vector field. Let $\rho: T^*_Y \to T^*_Y$ be the flow generated by $\pi$. $\rho$ sweeps out a Lagrangian submanifold of the cotangent bundle of $\mathbb{R} \times Y$ namely the set of points $(\xi, t, \nu)$ where $(\xi, t) \in \rho(T_0)$ and $\nu = \pi_P(\xi, t)$. We'll denote this Lagrangian submanifold by $A$. It's fairly easy to see that $A$ satisfies condition I, but it needn't satisfy condition II. Nevertheless $\mathcal{X}_A$ can be defined as follows. Let $(\xi, t)$ be a point in $(\mathbb{R} \times Y) \setminus \pi$. Let $A^1$ be the set of points above it on $A$. 


Let $(0, q, \ldots, q_n)$ be the points on the backward flow at time $t = 0$. Then

$$
\dot{q}_A = -\sum_{i=1}^{n} a_i (t, q) \cdot \frac{d}{dt} \left( \phi_i (q_i) \right) + u_i + \phi_i (q_i)
$$

where $\dot{q}_A$ is the curve $s \to (s, \phi_i (q_i))$, $0 \leq s \leq t$, and $u_i$ is its intersection with $C$. 
\( \xi 2 \) In \( \nu = 0, 1, \ldots, k \) let \( \mathcal{P}(x, \theta) \) be an order PDE mapping itself densities on \( X \) into itself densities on \( X \). We want to look for asymptotic solutions as \( \nu \) gets large of the partial differential operator

\[ \mathcal{P}(x, \theta) = \sum \frac{1}{(k-x)} \mathcal{P}(x, \theta) \]

Let \( \mathcal{P}(x, \theta) \) be the function \( \sum \mathcal{P}(x, \theta) \) where \( \mathcal{P}(x, \theta) \) is the top symbol of the operator \( \mathcal{P}(x, \theta) \).

Let \( c = \sum C_i \) where \( C_i \) is the subprincipal part of the operator \( \mathcal{P}(x, \theta) \). One of Maslov's main results is the following theorem.

Thus, let \( X \) be a Lagrangian manifold on which \( \mathcal{P}(x, \theta) = 0 \). Let \( a \) be a half-density satisfying the transport equation.
\[ \Sigma_{\mathcal{E}} a + c a = 0 \]

where \( \mathcal{E} \) is the Hamiltonian vector field corresponding to \( \mathcal{E} \). Then \( \mathcal{P}(x, \mathcal{D} a) \Lambda a = O \left( \frac{1}{\hbar^2} \right) \).

Rather than trying to prove this theorem we'll derive as a corollary of it Moskow's explicit formula for the solution of the Schrödinger equation:

\[
\left( E \right) \quad \frac{\hbar}{\sqrt{-1}} \frac{\partial \psi}{\partial t} = -\hbar^2 \sum \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + V(x) \psi
\]

in \( \mathbb{R}^n \times \mathbb{R} \). The associated symbol is

\[ \mathcal{W} \mathcal{N}(\hbar^2 / 2) \]

\[ \mathcal{R} = \left( \sum \frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) \right) = \mathcal{H}(x, \hbar \frac{\partial}{\partial \hbar}) \]

and the Hamiltonian is

\[ \mathcal{H} = \frac{\partial}{\partial \hbar} - \sum \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x} \frac{\partial}{\partial V} \frac{\partial}{\partial x} \psi \]
and the action $\mathcal{A}(\Xi) = q - \sum s^2$. When $H = 0$ this is equal to $-\sum s^2 + V(x)$.

On an integral curve of $\Xi$ we have \( \dot{x} = \frac{\partial H}{\partial p} = s \)

so we can write the action integral via (1) as \[-\mathcal{A}(\Xi(t))^{\prime}\]

\[-\int_0^t \sum \left(\frac{\dot{x}(t)}{2}\right)^2 + V(x)\ dt = \int_0^t L(x, \dot{x}, \dot{t})\ dt\]

where $L$ is the classical Lagrangian.

Finally note that in the transport equation $c = 0$

since all the terms of the Schrödinger operators are self-adjoint.

Now let $q_0 \in \mathbb{C}^{\mathbb{R}^n}$ be a \( \frac{1}{2} \) density on $\mathbb{R}^n$ with $q_0$ compactly supported.
We want to find a solution \( \phi \) of the equation (E) which takes as initial data \( \psi(x,0) = \varphi_0 \subset \mathbb{R}^n \). Let \( \varphi \) be the flow associated with \( \frac{\mathbf{x}^2}{2} + V(x) \) at time 0. Consider the map \( \mathbb{R}^n \to \mathbb{R}^n \) which maps \( x \to \varphi(x) = \pi P_{\mathbb{R}}(x, (\partial \varphi)_x) \), \( \pi \) being the projection of the cotangent bundle of \( \mathbb{R}^n \) onto its base. Let \( \varphi \) be a regular value of this map and let \( \mathbf{x}_1, \ldots, \mathbf{x}_k \) be its preimages.

Thus,

\[
\psi(q,t) = \sum \left( \frac{\partial \varphi}{\partial q} \right)^{-1}(q) \varphi_0(q) \mathbf{e} \times \left( \int_0^t \mathbf{x}_i + \mathbf{e}_i \right)
\]

modulo an error term of order \( O(\frac{1}{t}) \) as

where \( t \) is small and \( \frac{1}{t} \) is over the classical path \( \varphi_\varepsilon(t) \).
2 \to \emptyset \text{ and } \emptyset_0 \text{ is the intersection of the corresponding path in } \mathbb{A} \text{ with Maslov cycle.}

Except for the expression for the amplitude above formula, the theorem is a direct corollary of the theorem above. 

To see that the expression for amplitude is right we note that it is as follows. We shift \( a_0 \) up to \( \emptyset_0 \) it by \( (d\chi)^* \) (since we want it to be the transport equation \( \frac{\partial}{\partial \varphi} a = 0 \)) and then it down. The Jacobean for this map is

and since we are mapping \( \frac{1}{2} \) dimensions

the contourment of order \( \frac{1}{2} \) \( a_0(\varphi) \) gets

\[
\left| \frac{d\varphi}{\varphi} \right|^{-\frac{1}{2}} a_0(\varphi) \]
We will use the result above to derive Maslov's asymptotic expression for the fundamental solution of the Schrödinger operator. We begin with the plane-wave expansion of the $S$-function

$$S(x-x_0) = a(x) \sum \psi_n(x-x_0) d\eta$$

where $a(x)$ is a smooth function equal to 1 at $x_0$ and having support in a small neighborhood of $x_0$. Setting $\eta = \frac{x-x_0}{2\pi}$, we get

$$(F) \quad S(x-x_0) = \frac{a(x)}{(2\pi)^n} \int e^{i\frac{(x-x_0)}{2\pi} \cdot \xi} d\xi$$

We will use this expression and the theorems above to obtain an asymptotic solution of (F) with initial data $S(x-x_0)$ at $t = 0$. Let $(y, t)$ be a point of $\mathbb{R} \times \mathbb{R}^n$. We will assume that there are only a finite no. of classical trajectories joining $y$ to $x_0$. 
and that $g$ and $x_0$ are non-conjugate along these trajectories. Clearly the same will be true for all trajectories joining $g$ to points in the segment of $a(x)$.

Consider the graph of $\lambda(x-x_0)$ in the tangent bundle of $\mathbb{R}^n$. This is just the set of $\{(x, g(x)), x \in \mathbb{R}^n\}$.

Let $\Lambda(g)$ be the set of all trajectories which hit this graph at $t = 0$. ($\Lambda(g)$ is a Lagrangian submanifold of $T^*\mathbb{R}^n$). Our assumption about $g$ guarantees that there are only a finite number of trajectories

$$\lambda_{x, g}: \mathbb{R}^n \rightarrow \Lambda(g), \quad x = 1, \ldots, N$$

whose terminal $g(x)$ lie above $(x, g)$ and whose initial $g(x)$ lie above the support of $a(x)$. Let $(\lambda_{x, g})_0$ be the initial $F_0$ of the curve $\lambda_{x, g}$. The theorem
The given is the following asymptotic formula

\[ a(x) e^{\frac{x}{R} (x-x_0)} \] at time \( t = 0 \)

\[ \sum a_x(x(x)) \left| \frac{\partial a}{\partial x} \right|^{-\frac{1}{2}} \mathcal{C} = \frac{1}{\text{ch} x} \left( \sum \frac{1}{\text{ch} x} \right) + \frac{1}{\text{ch} x} \]

where \( u_x \) is the intersection curve of \( \text{ch} x \) with the
Möbius cycle on \( \Lambda(x) \). Plugging this into (E)
we get the following expression for the fundamental
solution of (E).

\[ G(y; t, x_0) = \sum a_x(x(x)) \left| \frac{\partial a}{\partial x} \right|^{-\frac{1}{2}} \mathcal{C} \sum \frac{1}{\text{ch} x} + \ldots \]

We will try to evaluate (G) using stationary phase.
To do this we need to determine the mutual
G of the phase function.
\[ \phi(x) = \sum_{i} \left( x_i - x_i^0 \right) \cdot s_i \]

as a function of \( x \). To do so we'll need some general facts about symplectic geometry: Let \( X \) be a manifold, and \( \Lambda \) a Lagrangian submanifold of \( T^*X \). Let \( \omega \) be the action form. In each \( s \in \mathbb{R} \), let \( \gamma \) be a smooth curve in \( \Lambda \) and suppose \( \gamma \) depends smoothly on \( s \). Let \( r(s) \) and \( p(s) \) be the unitized and terminal \( \gamma \) of \( s \).

**Lemma**

\[
\frac{d}{ds} \sum_{i} \omega \left( \frac{d\gamma}{ds} \right) = \omega \left( \frac{dp}{ds} \right)
\]

**Proof**: There is a tubular neighborhood \( U \) of \( \gamma \) in \( \Lambda \) in which \( \omega \) is exact; i.e., \( \omega = df \mid U \). Thus

\[
\sum_{i} \omega = f(x(s)) - f(x(0))
\]

Differentiating with respect to \( s \) we get the assertion above. \( \therefore \) E.D.
Now let compute \( \frac{\partial}{\partial x_i} \mathcal{S}_{\xi} \). Note first that the curves \( \xi_{\xi} \) all lie in a fixed Lagrangian manifold in \( T^* \mathbb{R} \times \mathbb{R}^n \), namely the set of all trajectories that at time \( t \) lie above the \( \phi^t (a, t) \). Let \( v_{x_{\xi}} \) and \( W_{x_{\xi}} \) be the tangent vectors to the initial and terminal curves of \( \xi_{\xi} \) obtained by varying \( \xi_{\xi} \) and leaving the other coordinates of \( x_{\xi} \) fixed.

Day the lemma

\[
\frac{\partial}{\partial x_i} \mathcal{S}_{\xi} = \omega(W_{x_{\xi}}, v_{x_{\xi}}) - \omega(W_{x_{\xi}}, v_{x_{\xi}})
\]

The end point of \( \xi_{\xi} \) projects onto the fixed set \( \phi^t (a, t) \) in the base for all \( \xi \), so \( (\partial^\pi) W_{x_{\xi}} = 0 \) and hence \( \omega(W_{x_{\xi}}, v_{x_{\xi}}) = 0 \) (because of the way \( \omega \) is defined!). On the other hand...
\[ (d\gamma) V_{\gamma} = \frac{\partial x_\gamma (x)}{\partial x} \quad \Rightarrow \quad a \left( V_{\gamma_{\gamma}} \right) = \gamma \cdot \frac{\partial x_\gamma}{\partial x} \]

at \((x(x), \gamma))\), therefore we get:

\[ \frac{\partial \gamma}{\partial x} = -\gamma \cdot \frac{\partial x_\gamma}{\partial x} \]

and \( \frac{\partial \phi (x)}{\partial x} = (x(x) - x_0) \). This gives

Then, the critical \( x_0 \) of the phase function in the integral (5) are precisely those \( x \) for which \( x(x) = x_0 \), i.e., for which the integral curve \( x \) joins \( x_0 \) to \( y \).

If we apply stationary phase to (5) and use the fact that \( a(x_0) = 1 \) we get the following asymptotic formula for the RHS.
where \( q_x(t) \), \( 0 \leq t \leq t \) is a classical trajectory
going from \( x_i \) to \( x_f \), and \( x' = x_i + \int_{x_i}^{x_f} \left( \frac{\partial V}{\partial x} \right) dt \).

Maslov identifies \( x' \) with the number of conjugate \( \xi \) along the trajectory, \( \psi_x(\xi) \). I don't at the moment see why this is the relation between the number and the intersection no. of \( \xi \) with the Maslov cycle.

Maslov gives an alternative proof of the formula (4) using Symmann's integrals. He starts with Symmann's representation of the fundamental solution of the Schrödinger equation:
\( G(y, t, x_0) = \sum_{i=1}^{\infty} \int_{t}^{t^*} \frac{d}{dt} \mathbf{L}(\mathbf{r}, \dot{\mathbf{r}}, \mathbf{x}) \, dt \, \delta \mathbf{y} \)

where \( \mathbf{y}(t) \) is any path joining \( x_0 \) to \( y \)

and \( \delta \mathbf{y} \) is Feynman's measure on path space.

Let's apply stationary phase to the RHS alone (ignoring the fact that the integral is not over a finite dimensional region). The critical pts. of the phase function are just those paths \( \mathbf{y} \) for which the first variation \( \delta \mathbf{L} = 0 \), which by the principle of least action are just the classical trajectories, that is, the \( \mathbf{y}(t) \) above.

The signature of \( \delta^2 \mathbf{L} \) at each of these trajectories is, by Morse theory, equal to the no. of conjugated \( \dot{y}, \dot{x} \), along the trajectory. Therefore we obtain
asymptotic formula for the RHS of (I)

$$\sigma(x,y,t) = \sum K_\alpha \left( \frac{\delta}{2\pi} \right)^{\frac{n}{2}} L(\alpha, i\omega, s) \alpha^n + \text{other terms}$$

The $K_\alpha$ is the quotient of two infinite quantities namely $(2\pi i)^{\frac{n}{2}}$ and det $(L(\alpha, i\omega, s))$. But apparently these cancel each other out and give the finite answer computed above?
Chapter II, Singularities

§ 1

Let \( \Lambda \subset T^* X \) be a Lagrangian manifold and \( \pi: \Lambda \to X \) the projection mapping. We recall

**Def.** The Maslov cycle is the set of all \( \pi x \in \Lambda \) where \( \frac{d\pi}{dx} \) has rank \( < n \). The caustic is the image of the Maslov cycle.

Let us denote by \( S_i(\Lambda) \) the set of points \( \Lambda \) where the rank of \( \frac{d\pi}{dx} \) is \( n - i \). We will prove

**Proposition 1.1.** In general position \( S_i(\Lambda) \) is a submanifold of \( \Lambda \) of codimension \( \frac{i(i+1)}{2} \).

From this we conclude \( S_2(\Lambda) \) is of codimension 1, generally so it can occur in all dimensions, \( S_2(\Lambda) \).
is generically of codimension 3, so it occurs just in 3 dimensions (as a collection of isolated $\mathcal{F}$s). $\mathcal{S}(1)$ occurs for the first time in 6 dimensions, $\mathcal{S}_2(1)$ in 10 dimensions and so on. Note that $\overline{\mathcal{S}(1)}$ is a "pseudomanifold". Its boundary, $\overline{\mathcal{S}_2}$, has dimension equal to $\dim \mathcal{S}(1) - 2$. Therefore it supports a homology class. The dual class is the characteristic class figuring in part I.

To prove the proposition we will need:

**Lemma**: Let $S$ be the set of all symmetric, non-negative matrices, and $S_i < S$ the matrices of rank $i$. Then $S_i$ is a submanifold of $S$ of codimension $i \left( \frac{i+1}{2} \right)$

**Proof**: It is enough to prove this for the set of matrices whose first $(n-i) \times (n-i)$ minor is non-singular.
Let 

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\]

be such a matrix. Postmultiplying this by the matrix

\[
\begin{pmatrix}
I & A^{-1}B \\
0 & I
\end{pmatrix}
\]

we get

\[
\begin{pmatrix}
A & 0 \\
C & D - CA^{-1}B
\end{pmatrix}
\]

Therefore the matrix 1.1 is of rank \( n - i \) if and only if

\[
D - CA^{-1}B = 0.
\]

In the space of all \( n \times n \) matrices the equation \( D - CA^{-1}B = 0 \) consists of \( n^2 \) independent scalar equations over the matrices of corank 1 are of codimension \( \frac{n^2}{2} \). However, if (1.1) is symmetric then \( D - CA^{-1}B \) is symmetric (since \( A^{-1} \) is symmetric and \( C^* = B \)) so there are just \( \frac{n^2(n+1)}{2} \).
independent real scalar equations, and the codimension of

$$S = \frac{1}{2}$$

S. E. D.

We will now prove the theorem. We will assume for simplicity that $X$ is an open subset of $\mathbb{R}^n$ and that linear coordinates, $x_i$, are chosen on $X$ such that

$\phi \circ x^i$ are the dual coordinates, the map $(\phi, x) \to \phi$

maps $X$ diffeomorphically onto an open subset of $\mathbb{R}^n$.

Both these assumptions are valid locally (x) and the proof of the general case can be reduced to the case by a simple partition of unity argument.

Given the above assumptions there exist a function $\lambda = H(s)$ such that $A$ is the locus of

point $x^i = \frac{\partial H}{\partial s_i}$ (i.e. $A$ is the graph of $dH$)

This is some problem if $A$ intersects the zero section, $s = 0$. We will assume $A$ always lies in the complement of $s = 0$. 
The map $\pi : \mathbf{A} \rightarrow X$ has the form

$$
\pi(x) = \left( \frac{\partial f_1}{\partial x_1}, \ldots, \frac{\partial f_i}{\partial x_i} \right)
$$

Therefore, $\pi$ has corank $i$ precisely when the Hessian $\partial^2 f = \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)$ has corank $i$. By the Thom-Whitney theorem, we can perturb $f$ so that the map $x \mapsto \partial^2 f$ intersects $E$. Enlarge $f$ in this perturbed map $E_i (A)$ is a submanifold of the perturbed $A$ of codimension $\frac{r(r+1)}{2}$. Q.E.D.

The classification of the singularities of $\pi$ into the $S_i$'s can be refined further. Suppose $1$ is in general position so that $E_i (A)$ is a submanifold of $1$. For $x \in E_i (A)$ let $K_x$ be the kernel of $\pi$ in the tangent space to $1$ at $x$, and let
\( L_x \) be the tangent space to \( S_i(1) \) at \( x \).

**Definition** \( x \in S_{i,j}^{(1)} \iff \dim K_x \cap L_x = j \)

Note that \( j < i \) in the definition to make sense.

**Proposition 1.2** \( S_{i,j}^{(1)} \) is a submanifold of \( S_i(1) \) of codimension
\[
\frac{1}{6} \left( i(i+1)(i+2) - r(r+1)(r+2) \right) + r(r-1) \quad \text{where} \quad r = n-j
\]

The proof is considerably more complicated than the proof of Proposition 1.1 and we won't give it here. (See Boardman, A simple trick of Cartan, described to us by Mather reduces the Lagrangian case to the Boardman case.)
One can define inductively singular sets $S_{i}, i \geq 1$ (1) sets, for $i$ in general position.

The most important of these for us are the singularities. To describe them more efficiently we will need a more efficient way of parameterizing $X$ than the symmetric Jacobian procedure described above.

In singularity we will assume for the rest of this § that $X$ is an open subset of $\mathbb{R}^{n}$.

Lemma. Suppose $x \in S_{1}(1)$ then in a neighborhood of $x$, $X$ can be parameterized by a phase function involving just one phase variable. (i.e. a phase function, $\phi = \phi(x, \theta)$ on $X \times \mathbb{R}$.)

See for example, [2].
Let $W_k$ be the subpace of $\mathbb{R}^{n+1}$ consisting of all $n+1$-tuples $(a_1, \ldots, a_{n+1})$ with $a_1 = a_2 = \ldots = a_k = 0$. Define a function $w : X \times \mathbb{R} \to \mathbb{R}^{n+1}$ by the formula

$$w(x, \theta) = \left( \frac{\partial w}{\partial \theta_1}, \frac{\partial w}{\partial \theta_2}, \ldots, \frac{\partial w}{\partial \theta_{n+1}} \right)(x, \theta)$$

By the $\mathcal{E}$-theorem, it can be perturbed so that $w(x, \theta)$ is $\mathcal{E}$ to all the $W_k$. Let's call a phase function with this property $\mathcal{E}$-generic.

**Definition** Given a phase function $w : X \times \mathbb{R}$ let $C$ be its critical set and $A \subseteq X$ the associated Lagrangian manifold. Since the
Diagram

\[ C \xrightarrow{d\Phi} \Lambda \]

commutes, the sets \( S, \ldots, (c) \) get mapped bijectively onto the set \( S, \ldots, (\Lambda) \), so it suffices to describe \( S, \ldots, (c) \).

**Proposition 3** Let \( \mathcal{W} \) be the \( \mathcal{W} \) generic and let \( \mathcal{C} \) be its critical set. Then \( \mathcal{W}^{(\mathcal{C})} = \mathcal{W}^{\mathcal{C}} \) if and only if \( x \in S, \ldots, (c) \). Moreover, \( S, \ldots, (c) \) is of codimension \( k \) in \( \mathcal{C} \) and is parameterized by the \( k+1 \) independent equations

\[
\frac{\partial}{\partial \theta} = \cdots = \frac{\partial}{\partial \theta^k} = 0
\]

**Proof:** \( \mathcal{C} \) itself is parameterized via \( X \sim \mathbb{R} \) by the
The tangent space to \( S_1(C) \) at \((s,0)\) is the set of vectors annihilated by \( d\left(\frac{2s}{3}\right)_0\) and \( d\left(\frac{2s}{3}\right)_{x,0} \).

So the fiber of the projection \( \pi: X \times \mathbb{R} \to X \) is tangent to \( S_1(C) \) if and only if \( \frac{2s}{3} = 0 \). Extending the argument above, we see that \( S_3(C) \) is parametrized by the independent equations \( \frac{2s}{3} = \frac{2}{3s} = \frac{2}{3s^3} = 0 \).
To leave this with inductive step of the proof to the reader.

Corollary (1) in general position \( \mathcal{S}_\geq \) is a submanifold of \( \mathcal{V} \) of codimension \( k \).

When we come to looking at asymptotic properties of PDE's we will mainly be considering the shapes of PDE that occur in classical physics such as the reduced wave equation, the Klein-Gordon equation, the Schrödinger equation etc. All these examples occur in dimensions \( \leq 4 \), so it will be helpful to have a complete classification of the kinds of singularities that can occur in these low dimensions. From the results of this section we
get the following provisional classification:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>dim 1</td>
<td>$S_1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>dim 2</td>
<td>$S_2$</td>
<td>$S_{1,1}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>dim 3</td>
<td>$S_3$</td>
<td>$S_{3,1}$</td>
<td>$S_{2,1,1}$</td>
<td>$S_{2,0}$</td>
</tr>
<tr>
<td>dim 4</td>
<td>$S_4$</td>
<td>$S_{2,1}$</td>
<td>$S_{5,1,1}$</td>
<td>$S_{2,1,1,1}$</td>
</tr>
</tbody>
</table>

This classification can't be very much refined. We will show below that all the $S_i$ singularities of the same kind are isomorphic. The $S_{2,0}$ singularities are a little more complicated. In dimension 3, there are two kinds that can occur (the so-called...
all types umbilic and hyperbolic umbilic) are
dim 4. There are three kinds (the elliptic, the
generated and the hyperbolic umbilic). In particular,
there are precisely 7 distinct types of singularities
that can occur in dimension 4. There are mutal
(umbilic, the seven then) catastrophes.
Throughout this $\mathcal{E}$ we will assume $\Lambda$ is in general position so that the results of $\mathcal{E}_1$ hold. Then the Nash cycle is a pseudomanifold whose interior is the set of points $S_{\rho_0}(\Lambda)$. We will call these points fold points, notation which will be explained shortly. Our goal is to give a simple description of the pseudo-collapsing operator, $\mathcal{K}_\Lambda$, in the neighborhood of a fold point. Our starting point is the following proposition.

**Proposition 2.1.** If $x_0 \in S_{\rho_0}(\Lambda)$, we can choose a phase function $\psi = \phi(x, \theta) \in X \times \mathbb{R}$ parameterizing a neighborhood of $x_0$ in $\Lambda$ such that

\[\phi(x, \theta) = \psi_0(x) + \rho(x) \theta - \frac{\theta^3}{3} \quad \text{with} \quad \delta \rho \neq 0.\]
Note: In $\Theta$ of the form above the critical set $C$ where $\Theta = 0$ is just the set $\Theta = 0$, 
$\Theta^2 = \rho(x)$, and the caustic is the set $\rho = 0$.

If we choose coordinates $x_1, \ldots, x_n$ on $X$ such that $\rho = x_1$ and use $(\Theta, x_2, \ldots, x_n)$ as a system of coordinates on $\Lambda$, then the map $\Lambda \to X$ is given locally by

$$(\Theta, x_2, \ldots, x_n) \to (\Theta^2, x_2, \ldots, x_n)$$

In other words it is the map which folds the $\Theta < 0$ plane onto the $\Theta > 0$ plane with fold along the line $\Theta = 0$.

Assuming Proposition 2.1 for the moment we see that the Master remaining operator has the
\[
\begin{align*}
(2) \quad a(x, \theta) & \rightarrow \int a(x, \theta) e^{ik \left( x_0 + \rho \theta - \frac{\theta^2}{2} \right)} d\theta \\
\text{in the neighborhood of a fixed point. We will} & \\
\text{simplify the RHS of (2) as follows. By the} & \\
\text{Malgrange Preparation theorem we can find functions} & \\
a_0(x), \ a_1(x) \ \text{and} \ b(x, \theta) \ \text{such that} & \\
a(x, \theta) = a_0(x) + a_1(x) \theta + b(x, \theta) \left( \rho(x) - \theta^2 \right) & \\
\text{where} \ \rho(x) - \theta^2 = \frac{\partial \phi}{\partial \theta}. \ \text{Thus} \ \chi_{/\theta} \text{can be} & \\
written: & \\
a_0(x) \int e^{-ik \phi(x, \theta)} d\theta + a_1(x) \int \theta e^{-ik \phi} d\theta + \int b(x, \theta) \frac{\partial \phi}{\partial \theta} e^{-ik \phi} d\theta & \\
\text{The last term in the sum is} \int b(x, \theta) \frac{\partial \phi}{\partial \theta} e^{-ik \phi} d\theta & \\
\text{which is of order} \ \frac{1}{k}. \ \text{Integrating by part}
\end{align*}
\]
and then repeating the same argument over again we prove (with different $a_0$ and $a_1$ from the above)

**Theorem 2.2** There exist functions $a_0(x)$ and $a_1(x)$ depending on $a(x,0)$ such that

$$K_q = a_0(x) \sum e^{-kq(x,0)} \, d\theta + a_1(x) \int e^{-kq(x,0)} \, d\theta + O(K^{-n})$$

Remark: It is rather complicated actually to write down the dependence of $a_0$ and $a_1$ on $a$. This involves looking at a special case of the following problem:

Given a function $f(x,0)$ on $\mathbb{R} \times \mathbb{R}$ with $\frac{\partial^2 f}{\partial x^2}(0) = 0$ if $x < 0$ and $\frac{\partial f}{\partial x}(0) \neq 0$, then the Malgrange
Proposition asserts that for every function $a = a(x, \theta)$ there exist functions $a_0(x), \ldots, a_{n-1}(x)$ and $h(x, \theta)$ such that $a(x, \theta) = \sum a_i(x) \theta^i + h(x, \theta)$. How do the $a_i$'s and $h$ depend on $a$? If $S$ and $a$ are real analytic then $h$ and the $a_i$'s are uniquely determined (by the uniqueness part of the Weierstrass preparation theorem!) and one can show. (See for example Arnold [1] that the map

$$a \rightarrow (a_0, \ldots, a_{n-1}, h)$$

behaves in some ways like an $n$-order differential operator. If $S$ and $a$ are smooth then $h$ and the $a_i$'s may not even be uniquely determined. (See, for example, Malgrange [3].)
To simplify (2.2) further we recall the definition of the Airy function:

\[ Y(t) = \int e^{\frac{t\theta}{2}} \, d\theta \quad \text{for} \quad \theta \in \mathbb{R} \]

This is a Bessel function of type \( \frac{3}{2} \). Its properties are exhaustively discussed in the Bureau of Standard tables [1]. Among other things it can be characterized as a solution of the ordinary differential equation:

\[ Y'' + tY = 0 \]

This equation is the standard equation in one dimension which describes transitions from oscillatory behavior to exponentially damped behavior. Assuming \( t \) is approximately constant then in the region \( t > 0 \)
the solutions of \( Y'' + \lambda Y = 0 \) are approximately

\( \sin \) and \( \cos \) functions, and in the region \( \lambda > 0 \)

they are approximately exponentially increasing and

exponentially decreasing.

Differentiating (2.3) under the integral sign we get

\[
Y'(t) = \sum_{\phi} e^{\lambda t - \phi^2} \, d\phi.
\]

Therefore from (2.2) we obtain the following:

**Theorem:** if (3.5) is a fold point of \( \sigma \), then

for a half-density \( \sigma \) suggested on a sufficiently

small neighborhood of (3.5) in \( \sigma \) there exist half-density

\( \sigma_0 \) and \( \sigma_1 \) on \( \sigma \) such that
\[ X^a = e^{iK r_0} \left\{ \frac{q_0}{K^2} Y(K r_0) + \frac{a_0}{K} Y'(K r_0) \right\} + O(K^{-n}) \]

Remark: Any functions play an important role in the asymptotic theory of the Schrödinger equation (see for example Messiah, [ ])

and of the reduced wave equation (see Ludwig, [ ])

The theorem above shows that this fact has nothing to do with the special properties of these equations but only with the presence of self-singularities. In the following section we will use standard properties of any functions to describe what "illuminated regions" and "shaded regions" look like in the neighborhood of a single caustic.
Finally we will give proposition 2.1. The idea of this proof is due to Chester, Friedman, and Westfall, \[ \] (though they work with analytic rather than smooth data.)

Our starting point is the following theorem, due to Whitney.

**Lemma 1** Let \( f \) be a smooth even function on the real line. Then there exists a smooth function \( g \) on the real line such that \( f(x) = g(x^2) \). If \( f \) depends smoothly on a set of parameters, \( g \) can be chosen so that it depends smoothly on the same parameters.

See [?]

Now let \( V \) be a Lagrangian manifold.
with a fold point at \((x_0, r_0)\).

Assume for simplicity that \(X = \mathbb{R}^n\) and that \(x_0\) is the origin. Let

\[ d = d(x_0) \]

on \(X \times \mathbb{R}\) be a phase function

parametrizing \(X\) in a neighborhood of \(x_0\) and let \(C\) be its critical set. Assume that the point on \(C\) corresponding to \((x_0, r_0)\) is the origin. We will prove

\[ \text{Lemma 2} \]

There exist smooth functions \(u_0\) and \(\rho\)

on \(X\) and \(\tilde{\rho}\) on \(X \times \mathbb{R}\)

such that restricted to \(C\):

\[
\begin{aligned}
\begin{cases}
\frac{\tilde{\rho}^3}{3} - \rho \tilde{\rho} + u_0 = \rho, & \frac{\partial \tilde{\rho}}{\partial \tilde{\theta}} > 0, \\
\tilde{x}^2 - \rho = 0
\end{cases}
\end{aligned}
\]

Proof: First let us prove the assertion for the special
when the base manifold $X$ is one-dimensional.

The assumption that the origin is a fold point of $C$ means that \( \frac{\partial \Phi}{\partial \theta} = \frac{\partial^2 \Phi}{\partial \theta^2} = 0 \), and \( \frac{\partial^3 \Phi}{\partial \theta^3} \neq 0 \), at 0. (See proposition 1.3.) Since

\( \frac{\partial^3 \Phi}{\partial \theta^3} \neq 0 \), we can solve for $x$ as a function of $\theta$ on $C$ and we get $x = x(\theta)$. Since $\frac{\partial^3 \Phi}{\partial \theta^3} \neq 0$,

$x'(\theta) = 0$ and $x''(\theta) \neq 0$ so by a change of coordinates on $x$ we can assume $x = \theta^2$ on $C$.

Let $C^+$ be the part of $C$ where $\theta > 0$, and $C^-$ the part where $\theta < 0$. By the second of the two equations (2.4), we must have $s = +\sqrt{\rho}$ on $C^+$ and $s = -\sqrt{\rho}$ on $C^-$ so on $C^+$.
we have
\[ -\frac{2}{3} \rho^{\frac{3}{2}} + u_0 = \phi(\theta) \]

and on \( C^- \) we have
\[ \frac{3}{2} \rho^{\frac{3}{2}} + u_0 = \phi(-\theta) \]

Since \( \rho \) and \( u_0 \) are functions of \( x \) alone we must have

\[ u_0(x) = \frac{1}{2} (\phi(\theta) + \phi(-\theta)) \]

\[ \rho(x)^3 = \frac{1}{4} (\phi(\theta) - \phi(-\theta))^2 \]

with \( x = \theta^2 \). The expressions on the right are both even functions of \( \theta \), so \( u_0 \) and \( \rho^3 \) exist by Lemma 1. To show that the cube root of \( \rho^3 \) exists we note that since \( \phi'(\theta) = \phi''(\theta) = 0 \),
and \( \alpha''(0) \neq 0 \) the Taylor series for
\[
(f(\theta) + \theta)^2
\]
begin with a non-zero term of order 6. Thus \( \rho \) exists and is of order 2 with
respect to \( \theta \) and of order \( 2 \) with respect to \( x \).

In particular, \( s = \sqrt{\rho} \) exist on \( C \) and \( \frac{\partial s}{\partial \theta} \neq 0 \).

Now suppose \( \dim X > 1 \). Choose coordinates

\((x_1, \ldots, x_n)\) on \( X \) such that \( \frac{\partial x_i}{\partial \theta} \neq 0 \). In
\[a = (a_2, \ldots, a_n) \] let \( C_a = \) the intersection of \( C \) with
the hyperplane \( x_2 = a_2, \ldots, x_n = a_n \). Applying
the preceding argument to \( C_a \), we find function
\( \mu_0, \rho, \) and \( s \) on \( C_a \) satisfying (2.4) and
depending smoothly on \( a \). We let \( \mu_0, \rho, \) and \( s \)
be the corresponding functions on C.

Finally extend \( \tilde{z} \) from \( C \) to \( X \subseteq \mathbb{R} \) arbitrarily. This concludes the proof of lemma 2.

So far proposition 2.1 let \( \tilde{z}(x) = u_0(x) + p(x) \tilde{z}(0) - \frac{x^3}{3} \).

Then (2.4) are easily seen to be the critical set of \( \tilde{z} \) equals the critical set of \( u_0 \). Making the change of coordinates \( x \rightarrow Z \), \( \theta \rightarrow \tilde{z}(\theta, x) \) we get

a phase function of the desired form.

Remark: In \( \S 4 \) we will obtain another proof of proposition 2.1 as a special case of a much more general result. (See proposition 4.1.)
As an application of the ideas discussed above, we will examine from the Maslov point of view some results of Fokker on the reduced wave equation:

\[ \Delta u + k^2 u = 0 \] (3.1)

The pages of Fokker we will be mainly concerned with are: "Uniform asymptotic expansions at a caustic". Communications on pure and appl. math., vol. I. x 215-250 (1966).

The problem we want to look at is the following: Construct a solution of (3.1)
with prescribed boundary data on an oriented hypersurface \( S \) in \( \mathbb{R}^n \) like the kind shown in Figure 1.

![Diagram](image)

Figure 1

Suggest the family of normal lines to \( S \)

that \( C \) as an envelope (i.e., they all lie on one side of \( C \) and are tangent to \( C \)) in geometric optics give a very good approximate solution to (3.1) in the region shaded in red. (See Section 4 of these notes.) However, geometric optics...
makes some rather unfeasible assertions about what happens near $C$, e.g., the light intensity at $C$ is infinite and there is no illumination at all in the blue shaded region. On the other hand, as mentioned above, Judd and works out the predictions of physical optics concerning what happens near $C$. The picture he gets is roughly the following:

1) At points in the red shaded region whose distance from $C$ is large compared with $k^{-3}$, the approximation of geometric optics is correct to order $k^{-1}$.

2) The light intensity on the caustic itself is large but finite (of order $k^{-5}$).
I. On the dark side of a etch is an illuminated strip of width approximately $\epsilon^{3}$.

Jedrzej's results are uniform in the sense that they are valid for all $k > 0$. In other words, if we reflect a monochromatic beam of light through a lens $L$, producing an image on a screen $S$, we can predict from these results how the image changes as we change the frequency of the light.

(See Figure 2 below.)
figure 2

\[ n = 1 \]

The wave 

varies frequency of light by putting yellow, blue, red etc. filters in front of flashlight

Region of illumination on dark side of C gets thinner as one goes from infra-red to ultraviolet part of spectrum. For example, illuminated region for red filter is 1.4 times as wide as illuminated region for violet filter.
To solve equation 3.1 asymptotically with boundary data prescribed along $\partial$, we consider in the cotangent bundle of $\mathbb{R}^n$, the Lagrangian manifold consisting of all points $x + t n_x, n_x$, where $x \in \partial$ and $n_x$ is the unit normal at $x$ pointing in the direction of the orientation. The caustic $C$ in figure 4 is precisely the set of critical sets of the map $\pi: \mathcal{A} \to \mathbb{R}^n$. It is not hard to show that for $\partial$ in general position, most of these sets are fixed points; others which are not form a subset of codimension one. (See figure 5. Also compare with proof of §1)
Reflection in a parabolic mirror produces a curved caustic. All the points on E except p are gold points (we will discuss the behavior of solutions of (3.1) in the neighborhood of the ray, p, in §5.)
at a fold point the Maslov canonical operator gives a solution of \( \mathbf{3.1} \) which has the general form

\[
\begin{align*}
\sigma_k^* & \left[ \frac{\partial}{\partial \kappa} A(\kappa^3 \rho) + \frac{\partial}{\partial \kappa} A'(\kappa^3 \rho) \right] \\
& + O\left(\frac{1}{\kappa}\right)
\end{align*}
\]

Here \( \rho, \sigma, \gamma_0, \) and \( \gamma_1 \) are functions of \( k \) alone and not depending on \( \kappa \), \( A \) is the Any function and \( A' \) its derivative. (See \( \mathbf{3.2} \), page 2.) We will worry about how to determine \( \sigma, \rho, \gamma_0, \) and \( \gamma_1 \) a little later on. However we already know from \( \mathbf{3.1} \) that \( \delta \rho \neq 0 \).
\[ A(t) \approx \frac{1}{\sqrt{\pi t^4}} \cos \left( \frac{t^3}{3} \right) - \frac{t^3}{\sqrt{\pi}} \sin \left( -\frac{t^3}{3} - \frac{\pi}{4} \right) \quad \text{and} \]

\[ A'(t) \approx \frac{t}{\sqrt{\pi t^4}} \left( -\sin \left( \frac{t^3}{3} \right) - \frac{\pi}{4} \right) \]

For \( t << 0 \)

\[ A(t) \approx \frac{t^3}{2\sqrt{\pi} (-t)^{1/3}} e^{-\frac{t^3}{2\sqrt{3}}} \quad \text{and} \]

\[ A'(t) \approx \frac{1}{\sqrt{\pi t^4}} \cos \left( \frac{t^3}{3} \right) - \frac{t^3}{\sqrt{\pi}} \sin \left( -\frac{t^3}{3} - \frac{\pi}{4} \right) \quad \text{and} \]

In order to discuss the qualitative behavior of the solution (3.2) we recall some basic facts about Airy functions:
\[ (3.6) \quad A'(t) = \frac{1}{2\sqrt{\pi}} \cdot (-t)^{\frac{3}{2}} e^{-\frac{t^2}{2}} \]

\[ \text{as } t \to 0 \]

\[ (3.7) \quad A(t) \sim c_1 + c_2 t \]

where \( c_1 = .355 \) and \( c_2 = -.259 \)

\[ \text{proof:} \quad (3.3) \text{ and } (3.4) \text{ can be obtained by applying stationary phase to the integral form of the Airy function and } (3.5) \text{ and } (3.6) \text{ can be obtained from } (3.3) \text{ and } (3.4) \]

by analytic continuation. In 3.7 see the Bureau of Standards table [1].

Combining these results with (3.2) we see...
that for $k^3 \rho > 0$, the solution, $u$, of (3.1) satisfies

\[(3.8) \quad u = \frac{k^{-\frac{2}{3}} e^{ikt \rho}}{\sqrt{\pi} \left[ k^\frac{2}{3} \rho \right]^\frac{1}{4}} \left( s_0 \cos \left( \frac{k^\frac{2}{3} \rho}{3} - \frac{2\pi}{4} \right) - \frac{1}{2} \rho^\frac{1}{4} \sin \left( \frac{k^\frac{2}{3} \rho}{3} - \frac{\pi}{4} \right) \right)\]

which is identical with the solution given by geometric optics. In $k^3 \rho$ close to zero, the first term of (3.2) dominates and we get

\[(3.9) \quad u = e^{ikt \rho} \frac{s_0}{k^\frac{2}{3}} A(k^\frac{2}{3} \rho)\]

where $A$ is given by (3.1). Comparing (3.8) with (3.9), we see that the value of $u$ as $\rho \to 1$ approximately at time $t$ is value of $u$ for $\rho = 1$. 
Finally noting $k^3 \rho < 0$ we get

$$\psi \sim \frac{k^3 e^{i k \rho \cos \theta}}{\sqrt{(k^3 \rho)^2 + \rho^2 (-k^3)^2}} \left( e^{-\frac{3}{2} k^3 \rho^2} + \frac{\rho}{r} e^{-\frac{3}{2} k^3 (-\rho)^2} \right)$$

(3.10)

When $\rho$ is comparable to $k^3 \rho$ we can't

simplify the solution (3.2) very much. This is a

region of transition in which the approximation

of geometric optics breaks down and the

exact solution 3.9 is not yet valid. Notice that

there is still a significant amount of illumination

on the dark side of the caustic if $-k^3 \rho \approx 1$

The leave it as an exercise for the reader.
to read off from the expressions above the
crude qualitative behavior of \( u \) described earlier.

We will now discuss how we determine
the functions \( \sigma, \rho, \eta, \) and \( \Theta \). The
Hamburger upper bound of the equation (3.1)
is the function \( s^2 - 1 \) (see \( \xi \) of part 2)

Our general prescription for constructing asymptotic
approximations of (3.1) is to find Lagrangian
manifolds \( \mathcal{M} \) on which \( s^2 - 1 = 0 \) and on \( \mathcal{M} \) to
find other densities which are invariant under
the Hamiltonian flow associated with \( s^2 - 1 \);

\[ v \text{ only } \Rightarrow \sum \xi \frac{\partial}{\partial x^i} \]

Let's look at
what this first condition involves.

Our phase function is of the form

\[ \varphi = \psi(x, \theta) = \phi(x) + \rho(x) \theta - \frac{1}{3} \theta^3, \]

where \( \varphi \) and \( \psi \) are the same as the \( p \) and \( \sigma \) in the previous paragraph. The critical set \( C \) of \( \varphi \) is the set of all \((x, \theta)\) where \( \rho(\theta) = \theta^2 \) and the map \((x, \theta) \mapsto x, \rho(\theta)\) must map this set diffeomorphically onto our Lagrange manifold. Therefore, we must have

\[ \sum \left( \frac{\partial \psi}{\partial x_i} \right)^2 = 1 \quad \text{on} \quad C \]
\[ \frac{3}{4^2} \]

Since \( \theta^2 = \rho \) on \( \mathcal{C} \) and since \( \mathcal{L} \) and \( \theta \) are independent on \( \mathcal{C} \) with respect to functions of \( x \), the equation (3.11) breaks up with the pair of equations

\begin{align*}
(3.12) \quad & (\nabla \sigma)^2 + \rho (\nabla \rho)^2 = 1 \\
(3.13) \quad & \nabla \sigma \cdot \nabla \rho = 0
\end{align*}

These are the second equations for \( \rho \) and \( \sigma \) which Helwing derive in §1 of his paper.

In two dimensions, it is rather easy to analyze them geometrically. When \( \rho = 0 \), \( \nabla \rho \)
is perpendicular to the caustic, so $\tau_0$ is tangent to the caustic by (3.13), and by (3.12) $\sigma$ is just the arclength variable along the caustic itself. If we integrate (3.12) and (3.13) as equations of $C$, let us suppose first of all that $C$ is a circle of radius $a$ about the origin in $\mathbb{R}^2$.

Let $(\rho, \theta)$ be the polar coordinates of any point in $\mathbb{R}^2$.

Then on $C$, $\sigma = a \theta$, it is clear that for (3.12) and (3.13) to be satisfied with initial data $\rho = 0$ and $\sigma = a \theta$ on $C$, $\rho$ must be radially symmetric so we can write
\[ p = p(r) : \text{ From (3.13) we deduce that} \]
\[ \sigma \text{ is constant in the radial direction so} \]
\[ \sigma = a \Theta \text{ on a whole right of } C. \text{ Thus (3.12) reduces to the equation} \]
\[ \left( \frac{a}{r} \right)^2 + \Theta(r) \left( p'(r) \right)^2 = 1 \]
\[ \Theta(r) \left( p'(r) \right)^2 = \frac{(r-a)(r+a)}{a^2} \]

Setting \( \Theta(r) = C (r-a) + O((r-a)^2) \) in the vicinity of \( C \), we get

\[ C^3 (r-a) = \frac{(r-a)(r+a)}{a^2} \text{ at } a \]

So \[ C^3 = \frac{2}{a} \] Thus:

\[ (3.14) \]
\[ p(r) = \left( \frac{2r}{a} \right)^{\frac{1}{3}} (r-a) \]
in the vicinity of \( C \)

Now consider the case of a general caustic \( C \).

Let \( x_0 \in \mathcal{E} \) and \( r \) distance along the normal line through \( C \) at \( x_0 \). Applying the argument above to the osculating circle to \( C \) at \( x_0 \) (which has third order contact with \( C \) at \( x_0 \)) we set

\[
q \quad \text{(8.14)}
\]

holding along the normal line through \( C \) at \( x_0 \), where \( q \) is the radius of curvature at \( x_0 \).

Finally we will see what the analogue of the transport equation is in our
Our asymptotic solution of (3.1) has the form

\[ u(x) = \sum g(x, \theta) e^{i k (\sigma + \rho \theta - \frac{\theta^3}{3})} d\theta \]

where \( \sigma \) and \( \rho \) are in principle determinable by the equations (3.12) and (3.13). What about \( g \)? We will prove

**Theorem.** Let \( g \) be the phase function defined

(\( \phi(x, \theta) = \sigma + \rho \theta - \frac{\theta^3}{3} \)) in \( X \times R \)

and let

\[ q = \frac{1 - (\Theta')^2}{\Theta} \quad (\text{since } 1 - (\Theta')^2 \]

vanishes on the locus of points where \( \frac{\partial q}{\partial \theta} = 0 \)
and $\psi$ is a generic phase function, $\Theta$ is a smooth function. Therefore we have

\[ 2 (V\Theta) \cdot V\psi + \Theta (V \Theta \cdot V\psi) + \frac{2}{\Theta} (\Theta \cdot V \Theta \cdot V\psi) = 0 \mod \left( \frac{2\pi}{\Theta} \right) \]

See Ludwig's formula.

\[ (\Xi) \]

Proof: (3.15) just asserts that the half-density on $C$ associated with $\Theta$ is invariant under the Hamiltonian flow. To see this we just of all prove:

**Lemma 1.** The vector field on $C$ associated with the Hamiltonian vector field $2 \Xi \cdot \frac{2}{\Theta}$. 

\[ v = e \sum \frac{\partial \Phi}{\partial x_i} \frac{2}{3x_i} + \frac{2}{\partial \theta} \]

**Proof**

The vector field is tangent to \( C \) since it

satisfies \( \phi \) and \( \frac{\partial \Phi}{\partial \theta} \);

and at \( e \) and the Hamiltonian

vector field projects onto the same vector in the base

at each \( \theta \) namely:

\[ 2 \sum \frac{\partial \Phi}{\partial x_i} \frac{2}{x_i} \cdot \]  

Therefore, in the two vector fields agree on the set where \( \Phi \)

is objective. By continuity they agree everywhere.

Q.E.D.

Now let \( \Omega \) be an \( n \)-form on \( \mathbb{R}^{n+1} \) such

that on \( C \) \( \Omega \wedge \delta \left( \frac{\partial \Phi}{\partial \theta} \right) = dx \wedge d\theta \). Let

\( \tilde{\Omega} \) be the restriction of \( \Omega \) to \( C \). Since
\[ R \text{ is determined up to a multiple of } d \left( \frac{2\alpha}{\theta} \right) \]

\[ R_0 \text{ is continuously defined on } C \text{ and is the usual volume from there} \]

**Lemma 2**  \[ \sqrt{R_0} = \left( \Delta \varphi - \frac{2\eta}{\theta} \right) \sqrt{R_0} \]

**Proof**  All it is enough to show that

\[
(3.16) \quad \left( \sqrt{R} \right)^\alpha d \frac{2\alpha}{\theta} = 2 \left( \Delta \varphi - \frac{2\eta}{\theta} \right) d\varphi \cdot d \frac{2\eta}{\theta}
\]

Then we can get the equation above by restricting it to \( C \) and taking square roots. This gives (3.16).

We first of all note that

\[
(3.17) \quad \sqrt{\frac{2\alpha}{\theta}} = \frac{2\eta}{\theta} \frac{2\eta}{\theta}
\]
by a straightforward computation. Now

the left hand side of (3.16) can be written

\[ \mathbf{L} \times d \left( \frac{\partial \mathbf{V}}{\partial \theta} \right) - \mathbf{L} \times d \left( \mathbf{x} \cdot \frac{\partial \mathbf{V}}{\partial \theta} \right) \]

\[ = \mathbf{L} \times (dx \times d\theta) - \frac{\partial \mathbf{L}}{\partial \theta} \mathbf{L} \times d \left( \frac{\partial \mathbf{V}}{\partial \theta} \right) \]

\[ = \mathbf{L} \times (dx \times d\theta) - \frac{\partial \mathbf{L}}{\partial \theta} \mathbf{L} \times \frac{\partial \mathbf{V}}{\partial \theta} \]

by (3.17). Since \( \mathbf{L} \times (dx \times d\theta) = (\text{div} \ \mathbf{V}) \ dx \times d\theta \)

\[ (\text{div} \ \mathbf{V}) \ dx \times d\theta = (\text{div} \ \mathbf{V}) \ \mathbf{L} \times d \left( \frac{\partial \mathbf{V}}{\partial \theta} \right) \]

\[ (\mathbf{L} \times \mathbf{L}) \times d \left( \frac{\partial \mathbf{V}}{\partial \theta} \right) = (\text{div} \ \mathbf{V} - \frac{\partial \mathbf{L}}{\partial \theta}) \ \mathbf{L} \times d \left( \frac{\partial \mathbf{V}}{\partial \theta} \right) \]

which is essentially (3.16)

Q.E.D.

Combining lemmas 1 and 2, we see that for
a half density \( g \sqrt{\Omega} \) to be invariant under the transport equation \( g \) must satisfy 5.15
In order to analyze more complicated singularities, we will need results analogous to Proposition 2.1. Our goal in this section will be to derive, at least formally, such results for the iterated \( S \) singularities and for the simplest kinds of \( S \) singularities.

In the iterated \( S \) singularities we have the following result:

**Proposition 4.1** Let \( X \) be a neighborhood of the origin in \( \mathbb{R}^n \) and \( \Theta = \Theta(x, \theta) \) a phase function on \( X \times \mathbb{R} \). Suppose that at the origin \( \frac{\partial^i \Theta}{\partial \theta_h} = 0 \) for \( i = 1, \ldots, k-1 \) and \( \frac{\partial^k \Theta}{\partial \theta^k} \neq 0 \). Then there exists a function \( \Theta_x = \Theta_x(x, \theta) \) and functions of \( x \);

\[ f_0(x), \ldots, f_{k-1}(x) \]

such that \( \Theta_x(0) = f_1(0) = \ldots = f_{k-1}(0) = 0 \), \( \frac{\partial^k \Theta_x}{\partial \theta^k} \neq 0 \) at \( 0 \) and:

...
\[
\ell(x, \Theta) = f_0 + f_1 \Theta + \ldots + f_{k-2} \Theta^{k-2} + \frac{\Theta^k}{K} + \epsilon(x, \Theta)
\]

where, \( \epsilon(x, \Theta) \) vanishes to infinite order when \( x = 0 \).

Proof: Since \( \frac{d^k \ell}{d \Theta^k} = 0 \) at \( 0 \), we can write
\[
\ell(0, \Theta) = c + \frac{\Theta^k}{K} h(\Theta) \quad \text{where} \quad h(\Theta) \to 0 \quad \text{at} \quad 0
\]

Making the coordinate change \( \Theta = \Theta \sqrt{h(\Theta)} \). we get
\[
\ell(x, \Theta) = c + \frac{\Theta^k}{K} + \epsilon(x, \Theta) \quad \text{where} \quad \epsilon(x, \Theta)
\]

vanish to degree 1 in \( x \) when \( x = 0 \).

Now suppose by induction that we can write
\[
\ell(x, \Theta) = f_0(x) + f_1(x) \Theta + \ldots + f_{k-2}(x) \Theta^{k-2} + \frac{\Theta^k}{K} + \epsilon(x, \Theta)
\]

where the \( f_i \)'s are polynomials of degree \( \leq N \) in \( x \)

and \( \epsilon(x, \Theta) = O(1 \times 1^N) \). Let \( \epsilon(x, \Theta) = \)
\[ \sum_{|I|=N} \varepsilon_I(\theta) x^I + O(1|x|^{N+1}) \quad \text{We will try to} \]

Find homogeneous polynomials \( \xi_0' = \sum_{|I|=N} \xi_{I} x^I \)

and \( \Theta_1 = \Theta + \sum_{|I|=N} \eta_I(\theta) x^I \quad \text{such that} \]

\[ (4.2) \quad \sum_{m=0}^{k-2} (\xi_m + \xi_m') \Theta_1^m + \frac{\Theta_1}{k} = \lambda(x, \theta) + \varepsilon_1(x, \theta) \]

where \( \varepsilon_1(x, \theta) = O(1|x|^{N+1}) \quad \text{when} \quad x = 0 \)

Equating coefficients of \( x^I \) \( (4.2) \) reduces to the following system of equations:

\[ (4.2) \quad \xi_{I} + \xi_{I'} \Theta + \ldots + \xi_{k-I} \Theta^{k-I} + \eta_I(\theta) \Theta^{k-I} = \varepsilon_I(\theta) \]

which are easily solvable by Euler's theorem with remainder

\[ \text{Q.E.D.} \]
To study the $S_2$ singularities, we need a short discussion on the theory of cubic binary forms over the reals. Let $\phi = \phi(x, b)$ be a homogeneous polynomial of degree 3 in the variables $x$ and $b$.

We will say that $\phi$ is degenerate if the quadratic forms $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial b}$ are multiples of each other. Suppose this is the case, we suppose $\frac{\partial \phi}{\partial b} = c \frac{\partial \phi}{\partial x}$. Making the coordinate change $x_1 = x$, $b = \beta + c x$, we get

$$\frac{\partial \phi}{\partial b} = 0,$$

so $\phi = a x_1^3$. Relabeling $\alpha = \sqrt[3]{a}$, we see that either $\phi \equiv 0$ or $\phi = x_1^3$.

Now suppose $\phi$ is non-degenerate. Let $
abla$ be the one-dimensional vector space consisting of quadratic forms in $x$ and $b$ modulo linear combinations.
\[ \frac{\partial^2}{\partial x^2} \text{ and } \frac{\partial^2}{\partial y^2} \]

Let \( L \) be the two-dimensional vector space spanned by \( x \) and \( y \). Multiplication gives us a bilinear map of \( L \) with \( L \), and if we choose a basis in \( L \), this map can be viewed as a quadratic form \( \phi \) on \( L \). It is clear that this quadratic form is non-zero; otherwise, \( x^2, y^2 \) and \( xy \) would all be multiple of \( \frac{\partial^2}{\partial x^2} \) and \( \frac{\partial^2}{\partial y^2} \).

\[ \frac{\partial^2}{\partial x^2} \]

**Definition 4.2**. We will say \( \phi \) is hyperbolic if the quadratic form described above is of rank 2 and index 1, elliptic if it is of rank 2 and index 2 or 0, and parabolic if it is of rank 1.

**Proposition 4.3**. (Classification theorem for cubic binary forms)
6) \( \phi \) is hyperbolic, then by a

linear change of coordinates it can be written in the form:

\[
(4.3) \quad \phi(x, y) = \frac{x^2 - y^2}{3}.
\]

If it is parabolic it can be written in the form:

\[
(4.4) \quad \phi(x, y) = ax^2
\]

and if it is elliptic it can be written in the form:

\[
(4.5) \quad \phi(x, y) = ax^2 + by^2.
\]

Proof: We will discuss the hyperbolic case. The other two cases are handled similarly. If \( \phi(x, y) \) is
In hyperbolic we can make a linear change of coordinates such that the quadratic form just described sends the pair \((0, \varphi)\) to 0 and \((\beta, \delta)\) to \(c^2\). This means \(x^2\) and \(\delta^2\) are both linear combinations of \(\frac{\partial \varphi}{\partial x}\) and \(\frac{\partial \varphi}{\partial \delta}\).

Since \(x^2\) and \(\delta^2\) are linearly independent, \(\frac{\partial \varphi}{\partial x}\) and \(\frac{\partial \varphi}{\partial \delta}\) must also be linear combinations of \(x^2\) and \(\delta^2\), i.e.

\[
\frac{\partial \varphi}{\partial x} = a_1 x^2 + a_2 \delta^2 \quad \text{and} \quad \frac{\partial \varphi}{\partial \delta} = b_1 x^2 + b_2 \delta^2
\]

for some numbers \(a_1, a_2, b_1, b_2\). Clearly, for the first of these equations to hold, the coefficient of \(x^2\) in \(\varphi\) must be zero, and for the second to hold, the coefficient of \(\delta^2\) must be zero. This shows that

\[
\varphi = \frac{1}{3} \left(c_1 x^2 + c_2 \delta^2\right)
\]

Replacing \(x\) by \(\sqrt{c_1} x\) and \(\delta\) by \(\sqrt{c_2} \delta\), we get \(\varphi\) in the desired form.

\(\Box\)
Example. Let \( h = h(x, y) \) be a smooth function on \( \mathbb{R}^2 \). Suppose that its first and second derivatives do not vanish and the cubic term in its Taylor series is a non-degenerate cubic 2 by 2 form. If this form is hyperbolic, we can find coordinates \( x' = x'(y, \theta) \) and \( y' = y'(x, \theta) \) such that

\[
 h(x', y') = c + \alpha' x'^3 + \beta' y'^3 \quad \text{near 0}
\]  

If it is elliptic, we can find coordinates such that

\[
 h(x', y') = c + \alpha' x'^3 - \beta' y'^2 \quad \text{near 0}
\]

Clearly, we will discuss the hyperbolic case. The elliptic case is handled similarly. Clearly, we
can write

\[ h(x, \beta) = \frac{x^3 + \beta^3}{3} + \sum_{i+j=4} f_{ij}(x, \beta) x^i \beta^j. \]

Making the coordinate change \( x = x + \delta_2 \beta^2, \beta = \beta \)

we can assume that in the expression on the RHS, \( \delta_{ij} = 0 \)

This means we can write

\[ h(x, \beta) = \frac{x^3 + \beta^3}{3} + 3, x^3 + 3 \beta^3 \quad \text{where} \quad \delta(x) = \delta(0) \]

= 0. Letting \( \alpha = x \sqrt{1 + g}, \quad \text{and} \quad \beta = \beta \sqrt{1 + g} \)

we get \( h(x, \beta) = \frac{x^3 + \beta^3}{3} \) as required.

\[ \text{Q.E.D.} \]

In the parabolic singularities the situation is a little more complicated. For example, \( x^3 \) and
\[ \varphi^2 + \varphi^4 \text{ are not equivalent. We will prove} \]

**Proposition 4.4** Suppose \( h = h(\varphi, \psi) \) vanishes together with its first and second derivatives at 0. Suppose the cubic term in the Taylor series at 0 is \( \varphi^8 \).

Then if \( \frac{\partial^3 h}{\partial \psi^3} \neq 0 \) we can find a change of coordinates \( \varphi_i = \varphi_i(\varphi, \psi) \) and \( \psi_i = \psi_i(\varphi, \psi) \) at 0 such that

\[ (4.7) \quad \pm h = \varphi_i^2 \psi_i + \psi_i^4 \]

**Proof:** We can write

\[ \pm h(\varphi, \psi) = \varphi^2 \psi + \sum_{i+j=4} l_{ij} \varphi^i \psi^j \text{ with } l_{20} > 0 \]

at the origin. Setting \( \varphi_i = \varphi/\sqrt{l_{20}} \) and
\[ \alpha_i = \sqrt{\frac{1}{2} + \beta} + 8 \alpha \sqrt{\sqrt{\frac{1}{2} + \beta}} \quad \text{we get:} \]

\[ \pm h(\gamma, \delta) = \gamma^3 \beta + \beta^4 + \rho \gamma^2 \beta + \tau \gamma^4 \]

with \( \rho \) and \( \tau \) appropriate functions of \( \alpha \) and \( \beta \),

and \( \rho(0) = 0 \). Finally replacing \( \gamma \) by \( \gamma \sqrt{1 + \beta} \),

we can assume that \( \rho = \sigma \). Thus we've reduced our problem to the case when \( h(\gamma, \delta) \) has the form:

\[ (4.8) \quad \pm h(\gamma, \delta) = \gamma^2 \beta + \beta^4 + \rho \gamma^4 \]

We will now look for a coordinate change

\[ \alpha_i = \sigma_i \quad \beta_i = \beta + \omega \sigma^2 \]

and a function \( \rho \) of \( \alpha \) and \( \beta \) which vanishes at 0 such that

\[ \pm h(\gamma, \delta) = \gamma^2 \beta + \beta^4 + \rho \gamma^2 \beta \]
Substituting the expressions for \( \alpha \) and \( \beta \) in the RHS of (6.8) we get \( \pm \lambda \langle \beta \rangle \) equal to the following mess:

\[
\alpha^2 \beta + \beta^4 + (\nu + \nu \rho + 6 \nu^2 \beta^2 + 4 \nu^3 \delta^2 + 4 \nu^4 \lambda^4) \alpha^4 + \\
+ (\rho + 4 \nu \beta^2) \alpha^2 \beta
\]

If we make the substitution \( \rho = -4 \nu \beta^2 \) in the first bracketed expression, we get

\[
\nu = \nu + 2 \nu^2 \beta^2 + 4 \nu^3 \delta^2 + 4 \nu^4 \lambda^4, \quad \text{and}
\]

\[
\nu = \nu \quad \text{when} \quad \lambda = 0 \quad \text{this equation can be}
\]

solved for \( \nu \) in terms of \( \lambda, \delta \) and \( \beta \). Defining

\[
\rho \rightarrow -4 \nu \beta^2 \quad \text{we get}
\]

\[
\pm \lambda \langle \beta \rangle = \nu^2 \beta + \beta^4 + \rho \nu^2 \beta,
\]
as asserted. Finally if we replace $x_1$ by $\sqrt{1+p_1}$
we can eliminate the $p$ term.

\[ \Box \quad \Box \quad \Box \quad \Box \]

Q.E.D.

One concludes from theorem I for the $S_{3,0}$

completeness is the following:

Proposition 4.5 Get $\phi = \phi(x, y, z)$ be a phase

function on $X \times \mathbb{R}^2$. Suppose that at the origin

$\phi$ and all its first and second derivatives with

respect to $x$ and $y$ vanish. Suppose that the

quadratic term in the Taylor series expansion of $\phi(0, x, y)$
at the origin is non-degenerate. Then if it

is algebraic we can find functions $\phi = \phi(x, y, z)$,
\( \theta_i = \theta_i(x, y, z, t) \) and \( \phi_i = \phi_i(x) \), \( i = 0, 1, 2, 3 \), such that

\[
\phi_i(0) = \phi_i(0) = \dot{\phi}_i(0) = \dot{\phi}_{i2}(0) \quad \therefore \quad \frac{\partial}{\partial x} (x, y, z, t) \neq 0, \quad \text{and}
\]

\[
(4.9) \quad \phi = \phi_0 + \phi_1 \alpha + \phi_2 \beta + \phi_3 \alpha \beta + \frac{\alpha^3 + \beta^3}{3} + \epsilon(x, y, z)
\]

where \( \epsilon(x, y, z) \) vanishes to infinite order when \( x = 0 \).

If the cubic term is elliptic, we can find

\( \alpha, \beta, \phi_0, \ldots, \phi_3 \) as above such that

\[
(4.10) \quad \phi = \phi_0 + \phi_1 \alpha + \phi_2 \beta + \frac{\alpha^2 + \beta^2}{3} + \alpha^3 - \alpha^2 \beta^2 + \epsilon(x, y, z)
\]

where \( \epsilon(x, y, z) \) vanishes to infinite order when \( x = 0 \).

If the cubic term is parabolic and the hypothesis of Proposition 4.4 is satisfied by \( \epsilon(0, 0, 0) \),

we can find \( \alpha, \beta, \phi_0, \phi_1, \phi_2, \phi_3, \phi_4 \) as above such that
\[ x = \frac{f}{a} + f_1 x + f_2 \beta + f_3 \alpha^2 + f_4 \alpha \beta^2 + a^2 \beta + \beta^4 + \varepsilon(x, \alpha, \beta) \]

where \( \varepsilon(x, \alpha, \beta) \) vanishes to infinite order when \( x = 0 \).

Proof. We'll just discuss the hyperbolic case, leaving the parabolic and elliptic cases as an exercise for the reader. We will prove (4.9) by the same kind of induction argument as that in the proof of Proposition 4.1. The case \( N = 1 \) is just the corollary to Proposition 4.3, so we'll assume the case \( N-1 \) and prove the case \( N \). Our inductive assumption is that

\[ x = \frac{f}{a} + f_1 x + f_2 \beta + f_3 \alpha \beta + \frac{a^2 + \beta^3}{3} + \varepsilon(x, \alpha, \beta) \]
when the \( f_i \)'s are polynomials of degree \( < N \)

\( \text{dim } x \) , and \( e(x, \nu, \beta) = O(1|x|^N) \). Let

\[
\mathcal{E}(x, \nu, \beta) = \sum_{|I| = N} \mathcal{E}_I(\nu, \beta) x^I + O(1|x|^{N+1})
\]

We will try to find homogeneous polynomials

\[
\mathcal{F}_i = \sum_{|I| = N} \mathcal{F}_{i,I} x^I \quad \text{of degree } N \text{ in } x \text{ and }
\]

functions

\[
\begin{align*}
\alpha_i &= \alpha + \sum_{|I| = N} \mathcal{F}_{i,I}(\nu, \beta) x^I \\
\beta_i &= \beta + \sum_{|I| = N} \mathcal{W}_{i,I}(\nu, \beta) x^I
\end{align*}
\]

such that:

\[
(4.12) \quad \alpha = (\frac{\xi + \xi'}{2}) + (\frac{\xi + \xi'}{2}) \alpha_i + (\frac{\xi + \xi'}{2}) \beta_i + (\frac{\xi + \xi'}{2}) \alpha_i \beta_i + \frac{\alpha^3 + \beta^3}{3} + \mathcal{E}_3(x, \nu, \beta)
\]

\[
\text{where } \mathcal{E}_3(x, \nu, \beta) = O(1|x|^{N+1})
\]

Equating coefficients.
We get

\[ \frac{1}{4} \sum \frac{1}{2} \alpha + \frac{1}{2} \beta + \frac{1}{2} \alpha^2 + \frac{1}{4} \beta^2 = \frac{1}{4} \Delta \]

which can be easily solved for constants \( \frac{1}{2} \alpha \) etc.

and functions \( P(x, \beta) \), \( Q(x, \beta) \) using the

integral form of the Taylor series with \( \Delta \).

\[ \text{G.E.D.} \]
We will now discuss the $S_2$ singularities in a little more detail. Let $\Lambda$ be a Lagrangian subvariety in $T^*X$ and let $(k_0, z_0)$ be an $S_2$ singularity on $\Lambda$. Let $\varphi = \varphi(x, y, z)$ be a phase function on $X \times \mathbb{R}^2$ parametrizing $\Lambda$ in a neighborhood of $(k_0, z_0)$. We will assume that $(k_0, z_0)$ is the point on the critical set of $\varphi$ corresponding to $(k, z)$. Since $(k, z)$ is an $S_2$ singularity, the first and second derivatives of $\varphi(x, y, z)$ with respect to $x$ and $y$ are zero at $(k_0, z_0)$. Let $\varphi_3(x, y)$ be the cubic term in the Taylor series expansion of $\varphi(x, y, z)$ at $(0, 0)$. It is easy to see that $(k_0, z_0)$ is an $S_3$ singularity if and only if $\varphi_3$ is degenerate, and is an $S_{2,2}$ singularity if and only if $\varphi_3 \equiv 0$.
Definition 5.1: We will say that \((x, y, z)\) is a hyperbolic (elliptic, parabolic) \(S_2\) singularity if \(\Omega\) is hyperbolic (elliptic, parabolic).

We must check that this definition is independent of \(\Omega\) and is an intrinsic property of \(\Lambda\).

Let \(\mathcal{O}_x\) be the local ring of formal power series in the coordinates of \(\mathcal{C}\) at \((x_0, y_0, z_0)\) and let \(\mathcal{O}_\Lambda\) be the corresponding local ring at \((x_0, y_0, z_0)\).

Let \(\mathcal{O}_x^*\) and \(\mathcal{O}_\Lambda^*\) be the quotient rings \(\mathcal{O}_x/(x-x_0)\) and \(\mathcal{O}_\Lambda/(x-x_0)\). Since \(\Lambda\) and \(\mathcal{C}\) are diffeomorphic, \(\mathcal{O}_\Lambda\) and \(\mathcal{O}_x\) are isomorphic, and so are \(\mathcal{O}_\Lambda^*\) and \(\mathcal{O}_x^*\).

Now it is clear from Definition 5.2 that hyperbolicity etc. is an algebraic property of the
local ring of formal power series in $x$ and $\beta$ modulo the ideal generated by $\frac{d^2}{dx^2}(\alpha, \beta)$ and $\frac{d}{d\beta}(\alpha, \beta)$. However, this is just the ring $\mathbb{R}^*_\mathcal{E}$ since the defining equations of $\mathcal{E}$ are $\frac{d^2}{dx^2} = \frac{d}{d\beta} = 0$.

Therefore, hyperbolicity etc. is an algebraic property of $\mathbb{R}^*_\mathcal{E}$. Q.E.D.

We will now prove:

Proposition 5.2. In 1 in general position, $S_2(1)$ is a submanifold of codimension 3; the elliptic and hyperbolic points are open subsets of $S(1)$; the parabolic points are a codimension 2 submanifold; the $S_3^1$ points a codimension 2 submanifold; and the $S_3^2$ points a codimension 4 submanifold.
Consequently elliptic and hyperbolic points can occur for the first time in dimension 3, parabolic points for the first time in dimension 4, $S_2$ singularities for the first time in dimension 5, and $S_{2,2}$ singularities for the first time in dimension 7.

**Proof:** Let $W$ be the space of all polynomial functions on $\mathbb{R}^2$ and $\mathbb{R}$ of degree $\leq 3$. Let $W_1$ be the subspace of polynomials with zero first order term and $W_2$ the subspace of polynomials with zero first and second order terms. Let $PCW_2$ be the parabolic cubic polynomials and $DCW_2$ the degenerate cubic polynomials.

We will need
Lemma \( P \) is a codimension 1 submanifold of \( \mathbb{R} \)
and \( Q \) a codimension 2 submanifold.

Proof: Consider in \( \mathbb{R} \) the open set consisting of
generate polynomials whose 2\(^\text{nd} \) term is non-zero. Restricting such
generate polynomials to the affine subset of \( \mathbb{R} \) space defined by \( x \neq 0 \)
we get an isomorphism between the \( \mathbb{R} \) of these polynomials
and the \( \mathbb{R} \) of cubic polynomials on the real line. The
generate polynomials correspond to polynomials on \( \mathbb{R} \) with double
roots and the degenerate polynomials to polynomials with
single roots. We can view the polynomials with double roots
as a vector bundle over \( \mathbb{R} \) with fiber \( \mathbb{R}^0 \) (just assign to
each polynomial its double root) and the polynomials with
single roots as a fiber bundle over \( \mathbb{R} \) with fiber \( \mathbb{R} \). Therefore
\( P \) has dimension 3 and \( D \) has dimension 2. Q.E.D.
Now let \( \phi \) be a phase function on \( X \times \mathbb{R}^2 \).

Let \( \hat{\phi} : X \times \mathbb{R}^2 \to \mathbb{W} \) be the map which assigns to each point \( (x, y, \rho) \) the Taylor series expansion of \( \phi(x, y, \rho) \) to order 3 about the point \( (x_0, y_0) \).

We will say that \( \phi \) is \( \mathbb{W} \) generic if \( \hat{\phi} \) is to \( \mathbb{W}, \mathbb{W}_2, \mathcal{P} \) and \( \mathcal{D} \). This clear that if \( \phi \) is \( \mathbb{W} \) generic the assertions of proposition 5.2 are true for the corresponding Lagrangian manifold. However, by the Thom \( \mathcal{T} \) theorem every \( \phi \) can be partitioned into a \( \mathbb{W} \) generic \( \hat{\phi} \), so this concludes the proof of Proposition 5.2.

Q.E.D.

At each point of \( X \) we've attached the local ring \( \mathbb{K}_x \). We've already seen that from
the structure of this local ring above we can determine whether or not the singularity is elliptic, hyperbolic, or parabolic. We will now show

Proposition 5.3: If \( (x, y) \) is elliptic or hyperbolic, the dimension of \( \mathbb{R}^s \) on the real line is \( g \) and if \( (x, y) \) is parabolic, the dimension is \( \geq g \).

Proof: If \( (x, y) \) is elliptic, we can parameterize \( A \) in a neighborhood of \( (x, y) \) by a phase function of the form \( \frac{x_1^3 + x_2^3}{3} + \varepsilon(x, y, z) \) where \( \varepsilon(x, y, z) \) vanishes when \( x = x_0 \), hence \( \mathbb{R}^s \) is isomorphic to the formal power series ring in \( x \) and is divided by the ideal \( (x^2, y^2) \). As a
with $\mathfrak{g}$ over $R$ this has $1, \xi, \eta, \theta$ and $\epsilon R$ as a basis.

If $(\xi, \eta)$ is parabolic we can parameterize $\mathfrak{g}$ in a Neighborhood of $(\xi, \eta)$ by a phase function of the form $\xi \xi \eta + \epsilon(x, y, \theta)$ where $\epsilon(x, y, \theta)$ is of order $4$ in $x$ and $\theta$, whereas $\xi, \xi \xi, \theta^2$ and $\theta^3$ are all independent modulo $\mathfrak{g}_1$ and $\mathfrak{g}_4$.

G. E. D.

**Definition 5.4**: We will say a parabolic germ $(\xi, \eta) \in \mathfrak{g}(A)$ is regular if $\dim \mathfrak{g}_1^{\mathfrak{p}^0} = 5$ and exceptional if $\dim \mathfrak{g}_1^{\mathfrak{p}^0} > 5$. 

**Theorem 6.1**: The set of exceptional germs is a subset of $\mathfrak{g}(A)$.
Proposition 5.5: In a general position the set of exceptional parabolic points is a subset of codimension 1 in the set of all parabolic points. Hence, in dimension 4, all parabolic points are regular.

Proposition 5.5 is an easy consequence of the following proposition, which we leave as an exercise for the reader:

Proposition 5.6: Let \( (x, y, 0) \) be a parabolic point on \( \mathbb{A}^3 \). Let \( z = \varphi(x, y, 0) \) be a plane function parameterizing \( \mathbb{A}^3 \) in a neighborhood of \( (x, y, z) \), and having the form

\[ \varphi(x, y, z) = x^2 + \varepsilon(x, y, z) \]
when $\varepsilon(x, y, z)$ is of order 4 in $x$ and $y$.

Then $(x, y)$ is a regular parabolic point if and only if

$$\frac{\partial^2}{\partial z^2}(x, y, z) \neq 0.$$
We must still show that the canonical
from theorems we derived in \(\mathbb{E}^4\) are true \(C^\infty\)
not just formally. In the proof we will
need the Malgrange preparation theorem in its
"Grothendieck - Kashiwara" form. We recall the
statement of the theorem as it is given, for example,
in Malgrange's book (see [ ] or ).

Let \(E^n\) be the ring of germs of \(C^\infty\) functions
at the origin in Euclidean \(n\)-space. Let \(m^n\) be
its maximal ideal. Choose a mapping \(p: \mathbb{R}^n \rightarrow \mathbb{R}^p\)
onto \(\mathbb{R}^p\) mapping the origin to the origin and get
an induced map \(p^*: E^p \rightarrow E^n\). Now let
$f_1, \ldots, f_k$ be functions on $\mathbb{R}^n$ and 
$(f_1, \ldots, f_k)$ the ideal they generate in $\mathbb{E}_n$. Let $R$ be the quotient ring $\mathbb{E}_n / (f_1, \ldots, f_k)$. Because of the map $\pi^* : \mathbb{E}_p \to \mathbb{E}_n$, we can view $R$ as an $\mathbb{E}_p$ module.

Hence, $R$ is a finitely generated $\mathbb{E}_p$ module if and only if $R/m_p R$ is a finite-dimensional vector space over the real numbers.

Moreover, a collection of elements $x_1, \ldots, x_k \in R$ are a set of generators for $R$ as a module over $\mathbb{E}_p$ if and only if their images in $R/m_p R$ are a spanning set of vectors of $R/m_p R$ (as a vector space over $\mathbb{R}$).
As an application of this theorem, let us prove the Whitney theorem about even functions quoted at the end of §2. Let \( n = p = 1 \), let \( A = \mathbb{E}^2 \) and let \( \rho \) be the map \( \mathbb{R} \to \mathbb{R} \), \( x \mapsto x^2 \). Then, since \( A/(x^2)A \) is generated by 1 and \( x \), every smooth function can be written in the form \( f(x^2) + \lambda x g(x^2) \) for smooth \( f \) and \( g \). In the function to be even, \( g \) must be 0.

The usual preparation theorem for a function \( F = F(x,t) \) on \( \mathbb{R}^n \times \mathbb{R}^n \) can be obtained from the theorem above by letting \( \rho : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) be the map \( (x,t) \mapsto x \) and letting \( \mathcal{R} \) be the ring \( \mathcal{E}_{\mathbb{R}^n}/(F) \). Conversely, it is not hard to prove the theorem above, assuming the usual preparation theorem. See [1] for details.
Now let's look at the problem that came up in §4.

A phase function \( \Phi = \Phi(x, \theta) \) is given to us on \( \mathbb{R}^n \times \mathbb{R}^n \) and another phase function \( \Phi' = \Phi(x, \theta) + \xi(x, \theta) \) such that \( \xi(x, \theta) \) vanishes to infinite order when \( x = 0 \). We want to find a pair of diffeomorphisms \( \Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( \varphi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n \) such that for all \( x \in \mathbb{R}^n \) and \( \theta \in \mathbb{R}^n \), the diagram

\[
\begin{array}{ccc}
\mathbb{R}^n \times \mathbb{R}^n & \xrightarrow{\Phi} & \mathbb{R}^n \times \mathbb{R}^n \\
\downarrow \quad \Phi & & \quad \downarrow \Phi' \\
\mathbb{R}^n & \xrightarrow{\varphi} & \mathbb{R}^n 
\end{array}
\]

(6.1)

commutes and such that \( \Phi \) conjugates \( \Phi' \) with \( \varphi \).
in the following weak sense:

\[(6.2) \quad \vartheta^*(a^* + u) = \varphi \]

where \( u = u(x) \) is a function of \( x \) alone. Such a result is clearly what we need to get rid of the error terms occurring on the RHS of (6.1), (4.2), (4.1), and (4.1). Note that the presence of \( u \) causes no problems for us, because we can, in each case, absorb it in the first term on the RHS.

Yet \( \vartheta^*_e = \vartheta^*_e(x, \theta) = \varphi(x, \theta) + \epsilon \vartheta(x, \theta) \). We will try to construct \( \vartheta^*_e \) and \( u^*_e \) with the same properties as \( \vartheta, \varphi \) and \( u \) above, such that

\[(6.3) \quad \vartheta^*_e(a^*_e + u^*_e) = \varphi \quad \text{for all } x \]
$f$, $g$, and $w$, depending smoothly on $t$, and $\xi$ being the identity map when $t = 0$. Now, $t = 0$, we can write the coordinates of $f$ and $g$ in powers of $t$:

$$f(x,t) = x_i + a_i(x) t + O(t^2) \quad i = 1, \ldots, n$$

and

$$g(x,t) = x_i + b_i(x, \theta) + O(t^2) \quad i = 1, \ldots, n$$

if we plug these expressions into (6.3) and differentiate, setting $t = 0$, we get:

$$\tag{6.4} -\varepsilon(x, \theta) = a_i(x) + \sum_{i} \frac{\partial}{\partial x_i} a_i + \sum_{x} \frac{2 \partial}{\partial \theta_i} b_i$$

with $a_i = \frac{d}{dt} \left. s_i \right|_{t=0}$.
Definition 6.1 We will say that $\phi$ is unfunctionally stable if for every $\phi(x,0)$ on the LHS of (6.4) there exist $\phi_0 = \phi_0(x)$ and $\phi_1 = \phi_1(x,0)$ such that (6.4) holds in a neighborhood of the origin.

If we apply the Mather's conjecture theorem to the map: $\mathbb{R}^n \times \mathbb{R}^N \rightarrow \mathbb{R}^N; (x, \theta) \rightarrow \theta$, with $\mathbb{R} = \mathbb{E}_{n+N} / (\frac{2\pi}{\omega_1}, \ldots, \frac{2\pi}{\omega_N})$, we get the following criterion for unfunctionally stability.

Proposition 6.2 $\phi$ is unfunctionally stable if and only if for every function $\phi_0 = \phi_0(\theta)$ there exist real nos. $c_0, \ldots, c_n$ and functions $\eta_0, \ldots, \eta_n$
of $\theta$ such that

\begin{equation}
\phi(\theta) = c_0 + \sum_i c_i \frac{\partial \phi}{\partial x_i}(0, \theta) + \sum_i I_i(\theta) \frac{\partial \phi}{\partial x_i}(0, \theta)
\end{equation}

**Exercise:** Show that this criterion is satisfied for the phase function defining a simple model:

$$
\ell(x, \theta) = x \theta - \frac{x^3}{3}
$$

In the situation we are considering, our deformed phase function $\ell_t(x, \theta) = \ell(x, \theta) + t \phi(x, \theta)$ where $\phi$ vanishes to infinite order when $x = 0$; through the criterion (6.5) it is the same for $\phi_0$ and $\phi_2$. This proves
Proposition 6.3 If \( \Phi \) is unfunctionally stable then \( \Phi \) is unfunctionally stable for all \( t \).

The main goal of this section is to prove the following:

Theorem 6.4 Suppose \( \Phi = \Phi(x,0) \) is unfunctionally stable.

Let \( \Phi'(x,0) = \Phi(x,0) + \epsilon(x,0) \) where \( \epsilon \) vanishes to infinite order when \( x = 0 \). Then there exist \( \Phi_0 \) and \( \Phi_1 \) satisfying (6.1) and 6.2.

We will prove, in fact, that there exist \( \Phi_0 \) and \( \epsilon \) satisfying the condition analogous to 6.1 and satisfying (6.3) on the whole interval \( 0 \leq t \leq 1 \).
As a first step in the proof we will need:

**Lemma 6.5** If the LHS of (6.1) vanishes when \( x = 0 \), then we can choose the \( a_i \)'s and \( b_i \)'s on the RHS so that they also vanish when \( x = 0 \).

**Proof:** If \( \varepsilon(x, \theta) = 0 \) when \( x = 0 \), we can write \( \varepsilon(x, \theta) = \sum \xi_i \varepsilon_i(x, \theta) \) for smooth \( \varepsilon_i \).

In each \( i \) we can solve

\[-\varepsilon_i(x, \theta) = a_{i,1} x_1 + \frac{\partial \xi_i}{\partial x_1} a_{i,2} x_2 + \ldots + \frac{\partial \xi_i}{\partial \theta} b_{i,1}(x, \theta) + \ldots\]

Setting \( a_i = \sum a_{i,j} x_j \) and \( b_i = \sum b_{i,j} x_j \), we get a solution of (6.4) that vanishes when \( x = 0 \).

\( \text{Q.E.D.} \)
We want to determine \( f_t \), \( g_t \) and \( \phi_t \) satisfying (6.3). Differentiating (6.3) with respect to \( t \) we get

\[- \varepsilon (g_t) = u_t (g_t) + \sum_{\alpha} \frac{\partial}{\partial x_{\alpha}} (u_{\alpha} + \phi_t) (g_t) \frac{\partial}{\partial x_{\alpha}} f_t (x, t) \]

\[\quad + \sum_{\alpha} \frac{\partial}{\partial x_{\alpha}} (u_{\alpha} + \phi_t) (g_t) \phi_t (x, \theta, t) \]

Here the \( f_t \)'s and the \( g_t \)'s are the coordinates of \( f_t \) and \( g_t \) and the dots indicate differentiation with respect to \( t \). If we set

\[
\begin{cases}
  a_i (x, t) = \frac{\partial}{\partial x_i} (f_t^{-1} (x), t) & i = 1, \ldots, n \\
  b_{\alpha} (x, \theta, t) = \frac{\partial}{\partial x_{\alpha}} (g_t^{-1} (x, \theta), t) & \alpha = 1, \ldots, N \\
  a_0 (x, t) = \frac{\partial}{\partial t} + \sum \frac{\partial}{\partial x_i} a_i
\end{cases}
\]
the equation above reduces to

\[ e(x,0) = \phi_0(x,t) + \sum \frac{\partial}{\partial x_i} \phi_i + \sum \frac{\partial}{\partial \theta_i} \phi_i \]

which is identical with (6.4) except that all the terms are functions of \( t \). We will try to solve (6.7) for functions \( \phi_i \) and \( \phi_i \) which are smooth over the whole interval \( 0 \leq t \leq 1 \).

We first note that it is enough to do this in a small interval about each point in \( [0,1] \). By a partition of unity in \( t \) these solutions can be patched together to give a global solution in \( t \). By Proposition 6.3 \( \phi_i \) is unfunctionally stable so (6.7) can be solved.
If $t_0$ is fixed $t$, such that the solutions are smooth in $x$ and $\theta$. The only question is whether these solutions can be chosen to be smooth in $t$. To see this we apply the Malgrange preparation theorem to the map $\mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}$, with $\mathcal{R} = \mathcal{E}_{n+N+1} / \langle \frac{d}{d\theta_1}, \ldots, \frac{d}{d\theta_N} \rangle$.

The preparation theorem says that (6.7) can be solved with an arbitrary smooth function on the LHS if and only if (6.5) holds for $\delta_0$. We have already seen, however, (proposition 6.3) that this condition is the same for all $t$, and it holds when $t = 0$ because of the infinitesimal stability of $\delta$.

Finally, we note that if $\varepsilon(x, \theta)$ vanishes when $x = 0$
as is the case with us, we can choose the $a_i$'s and $b_i$'s to vanish when $x = 0$ (Lemma 6.5).

To conclude the proof of Theorem 6.7 we must solve the equations (6.6) with initial data

$$
\begin{cases}
\frac{\partial u}{\partial t}(x, 0) &= x, & u = 1, \ldots, n \\
\theta_k(x, \theta, 0) &= \theta_k, & \nu = 1, \ldots, N \\
\theta(x, 0) &= 0
\end{cases}
$$

(6.9)

The first pair of these equations are just ordinary differential equations in $x$ and $t$, and since the expressions on the LHS are 0 when $x = 0$, they are solvable globally in $t$ for a sufficiently small right of the origin.
in $(x, y)$ space. The rest of the equations

(6.5) can be solved by linear Hamilton-Jacobi theory

or just by integrating the vector field $(a_1(x, y), \ldots, a_n(x, y))$.

This can be done globally in $t$ for the same reasons as above.

\textbf{Q.E.D.}
We will now use the results of the preceding section to derive the canonical form theorems of §4.

We will actually prove a theorem which includes these results and is applicable to other singularities besides the elementary ones discussed in §4. This theorem is a variant of Thom's "universal unfolding" theorem.

Definition 7.1
Let \( \psi = \psi(\theta) \) be a smooth function defined on a neighborhood of the origin in \( \mathbb{R}^n \).

Let \( E_N \) be the ring of germs of smooth functions at the origin of \( \mathbb{R}^n \) and let \( (\frac{\partial \psi}{\partial \theta}) \) be the ideal in \( E_N \) generated by the first partial derivatives of \( \psi \). We will say \( \psi \) is of finite type if \( (\frac{\partial \psi}{\partial \theta}) \) is of finite codimension in \( E_N \).
Examples which we have already seen are:

\textbf{In } N = 1

1) \( \psi = \psi(0) = \frac{\Theta^k}{k+1} \)

\textbf{In } N = 2

(7.1) 2) \( \psi = \psi(\theta_1, \theta_2) = \frac{\Theta_1^3 + \Theta_2^3}{3} \)

\[ \begin{align*}
& 3) \quad \psi = \Theta_1^3 - \Theta_1 \Theta_2^2 \\
& 4) \quad \psi = \Theta_1^3 \Theta_2 + \Theta_2^4
\end{align*} \]

Suppose the codimension of \( \frac{\partial^k}{\partial \theta^k} \) in \( \mathcal{E}_N \)

is \( k+1 \). Then we can choose functions \( \psi_0, \ldots, \psi_k \in \mathcal{E}_N \) such that their images form a basis for \( \mathcal{E}_N / \frac{\partial^k}{\partial \theta^k} \). We will assume that \( \psi_0 = 1 \).

Our main result is:
Theorem 2.1

Let \( \kappa = \kappa(x, \theta) \) be a smooth function defined on a neighborhood of the origin in \( \mathbb{R}^n \times \mathbb{R}^n \) such that \( \kappa(0, \theta) = \psi(\theta) \). Then there exists smooth functions \( \xi_i^u = \xi_i^u(x) \), \( \xi_i^u = 0; \ldots, \kappa \) and \( \bar{\theta}_i = \bar{\theta}_i(x, \theta) \), \( u = 1, \ldots, N \) such that the \( \xi_i^u \)'s are all zero at the origin, \( \bar{\theta}_i(0, \theta) = \theta_i \), and

\[
\kappa(x, \theta) = f(x) + \frac{1}{2} f(x) \psi_i^1(\bar{\theta}) + \ldots + \frac{1}{2} f(x) \psi_i^k(\bar{\theta}) + \psi(\bar{\theta})
\]

If we apply this theorem to example 2) of (6.1) and let \( \frac{1}{2} \psi_i^j(\theta) = \theta_i \), \( u = 1, \ldots, k-1 \) we get the canonical form (4.1). We leave it as an exercise for
the reader to get the summed forms (4.1), (4.2) and (4.3) by applying this theorem with the examples 1), 2) and 3) of 7.1 and appropriate choices for the $\psi$'s.

Proof of theorem 7.1

We first of all prove the assertion formally. The proof is practically identical with the proofs of propositions 4. and 4'. We will argue by induction that by changing the $\theta$ coordinate we can write

$$\mathcal{A}(x, \theta) = f_0(x) + \ldots + f_k(x) \psi_k(\theta) + \psi(\theta) + \epsilon(x, \theta)$$

where the $f_i$'s are polynomials in $x$ of order $\leq r-1$. 
and $e(x, \theta) = O(1x^{1+})$ uniformly in $\theta$.

We will try to find a coordinate change

$$\bar{\theta}_i = \theta_i + \sum_{|I|=1} \xi_{iI} (\theta) x^I,$$

and polynomials

$$s_i(x) = \sum_{|I|=1} \frac{\xi_{iI}}{I} x^I$$

such that

$$\varphi(x, \theta) = \sum (s_i + \xi'_i) x^i \psi_i (\bar{\theta}) + \psi(\bar{\theta}) + \bar{e}(x, \theta)$$

where $\bar{e}(x, \theta) = O(1x^{1+})$ uniformly in $\theta$. Let

$$\varepsilon(x, \theta) = \sum_{|I|=1} \varepsilon_{iI} (\theta) x^I + O(1x^{1+})$$
Equate the coefficient of \( x^1 \) on the RHS and LHS of (7.6) we get

\[
(7.8) \quad \sum \frac{f_{n,1}}{n!} \psi_n (\theta) + \sum \frac{d^n}{d\theta^n} \tau_n (\theta) = -e_t (\theta)
\]

By hypothesis these equations are solvable and we can continue the induction.

\( \Box \) E. D.

We will now prove Theorem 7.1. Because of what we've just proved we can assume that

\[
th_t (x, \theta) = \Phi_0 (x) + \sum_{k=1}^{\infty} \Phi_k (x) \psi_k (\theta) + \psi (\theta) + e (x, \theta)
\]

where \( e (x, \theta) \) vanishes its infinite order when \( x = 0 \).
Let's also assume for the moment that $n \geq k$ and that $\psi_1, \ldots, \psi_k$ are linearly independent at the origin. Then the $\psi_i(\theta)$ are linear combinations (with constant coefficients) of the $\frac{\partial \psi_i(\theta)}{\partial \theta_j}$ and 1, so the hypotheses of proposition 6.2 are satisfied and $G(x, \theta)$ is unfunctionally stable.

Theorem 7.1, therefore, follows from theorem 6.4.

Suppose on the other hand that the $\psi_i(\theta)$ are not linearly independent. Consider on the space $\mathbb{R}^{n+k} \times \mathbb{R}^N$ the phase function

$$
\Phi(x, y, \theta) = \Phi_0(x) + \sum_{i=1}^k (\delta_i(x) + g_i) \psi_i(\theta) + \psi(\theta) + \epsilon
$$
By applying the preceding argument to this phase function we can find a change of coordinates \( \tilde{\theta} = \tilde{\theta}(x, y, \theta) \) and \( \tilde{\xi} = \tilde{\xi}(x, y) \) \( u = 0, \ldots, k \) such that

\[
\Phi(x, y, \theta) = \tilde{\Phi}(x, y) + \sum_{u=1}^{k} \tilde{\xi}(x, y) \psi_{u}(\tilde{\theta}) + \psi(\tilde{\theta})
\]

Setting \( \bar{\theta} = \tilde{\theta}(x, 0, \theta) \) we're done.
Fourier integral operators from the Radon transform point of view

D. Schaeffer

V. Guillemin
The purpose of these notes are not to discuss any new results but to describe a point of view toward some existing results; i.e. Hörmander's "Fourier integral operators" paper and a related series of papers by Donald Ludwig. The reasons for this are not entirely pedagogical. (See the concluding paragraph below.)

We recall that distributions on manifolds enjoy two types of functoriality, "push-forward" and "pull-back". "Push-forward" is simplest to describe. For a proper map \( f: X \to Y \), \( f_* \) on distributions is the dual operation to \( f^* \) on compactly supported \( \mathbb{C}^\infty \) functions. "Pull-back" is only defined for submersions, \( f: X \to Y \). Locally a submersion is just a projection map, for example the standard projection

\[
\mathbb{R}^k \times \mathbb{R}^\ell \to \mathbb{R}^k
\]
Given a \( C^0 \) density \( \rho(x, y) \, dx \, dy \) on \( \mathbb{R}^n \times \mathbb{R}^n \), we define its push-forward as

\[
(\int \rho(x, y) \, dy) \, dx
\]

The dual operation is the pull-back operation on distributional densities. The point of view toward Fourier integrals which we want to describe here is one that makes maximum use of these two functors.

We'll begin with "wave front" sets. Given a distribution \( \mu \) on a manifold \( X \), Hormander attaches to it a subset \( \text{WF}(\mu) \) of the cotangent bundle \( T^*_X \) called its wave-front set. Intuitively, \( (x_0, \xi_0) \) is not in \( \text{WF}(\mu) \) if \( \mu \) is smooth at \( x_0 \) in the direction \( \xi_0 \). Hormander's precise definition is:

There are various wave front sets. For simplicity, we'll use:

The projective wave front set:

\( (x, \xi) \in \text{WF}(\mu) \Rightarrow (\xi, \partial) \in \text{WF}(\mu) \)
\[(x_0, x_0) \notin \text{WF}(\phi) \iff \hat{\rho}(\xi)\text{ is rapidly decreasing in a conical nbhd. of } \xi_0\] where \(\hat{\cdot}\) is the Fourier transform and \(\rho\) a bump function at \(x_0\).

We'll now give an alternative definition of \(\text{WF}(\phi)\).

**Rough Definition:** \((x_0, x_0) \notin \text{WF}(\phi)\) if for every function \(\phi : X \rightarrow \mathbb{R}\) with \((d\phi)_x = 0\) the push-forward, \(\phi \ast \rho \phi\), is smooth, \(\rho\) being, as above, a bump function at \(x_0\).

The workable definition adds: if \(\phi\) depends smoothly on parameters so does \(\phi \ast \rho \phi\).

We'll show that the two definitions agree. First however, let's use the second definition.
to compute \( WF(\psi) \) for a particularly simple class of distributions.

**Definition:** A distribution \( \psi \) on \( X \) is wave-like if it is of the form \( \psi = f^* \gamma \) where \( \gamma \) is a distribution on \( \mathbb{R} \) and \( f: X \to \mathbb{R} \) a submersion.

**Lemma 1** For a wave-like distribution, \( f^* \gamma \), the wave-front set is the set of normal vectors to the surfaces \( f = c \), \( c \) being in the singular support of \( \gamma \).

**Proof** Suppose \( \gamma: X \to \mathbb{R} \) and \( (d\gamma)_x = (df)_x \).

Then we can choose coordinates \( x_1, \ldots, x_n \) centered at \( x_0 \) such that \( f = x_1 \) and \( \gamma = x_2 \). Let \( \rho \) be a bump function at \( x_0 \). Then

\[
\Theta \ast \rho f^* \gamma = \int \left( \int \gamma(x_2) \rho(x_1, \ldots, x_n) \, dx_1 \right) \, dx_3 \, dx_4 \ldots \, dx_n,
\]

which is smooth.
We'll also need the Radon transform. For simplicity we'll state this in odd dimensions.

**Lemma 2** Given we $S^{n-1}$, let $\phi_x : \mathbb{R}^n \to \mathbb{R}$ be the map $x \mapsto x \cdot x$ and let $D = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial t}$ on $\mathbb{R}$.

Then for a compactly supported distribution $\mu$, in $\mathbb{R}^n$ one has

(i) $\int_S \phi_x^* D^{n-1} (\phi_x) \ast \mu \, dw = \frac{i}{2 (2\pi)^n} \hat{\mu}$

**Proof:** See Fritz John.

Finally we'll need a trivial identity:

**Lemma 3** $(\phi_x)_* \mu (t) = \hat{\mu} (t w)$

To prove the equivalence of our two definitions we note that by lemma 3 if $\hat{\mu}$ is rapidly decreasing in the direction $w_0$, $(\phi_{w_0})_* \mu$ is smooth. Suppose now that $w_0$ is not in $\text{WF} (\mu)$.
if it exists, Hormander shows that the method of continuity works if $f$ satisfies:

(ii) $(\partial_x^m f)^\wedge \neq 0$ whenever $(\xi, \nu) \in WF(f^\wedge)$

To prove this observe first that the theorem is true

a) when $u$ is smooth. Then $f^\wedge u$ can be defined as the usual pull-back of functions.

b) when $u$ is wave-like. If $u = g^\wedge v$ for some $g : Y \to \mathbb{R}$ then the $F$ condition above just says that $g \circ f$ is a submersion, so we can define $f^\wedge u = (g \circ f)^\wedge v$.

For the general case just break $u$ into a sum of a) and b) using the Radon transform.

We'll now describe Hormander's Fourier
integrals from the "Radon" point of view. (*)

This approach requires first of all a familiarity with the classical homogeneous distributions on the real line namely:

(iii) $\delta(x), \delta'(x), \frac{1}{x^n}, x^r, (x+i0)^r$, etc.

Consider the pull-backs of these distributions with respect to submersions $f: Z \to R$. These are the classical "coaches" or "boundary layers". By a lemma their WF sets is the normal bundle to the hypersurface $f = 0$.

Finally, given a submersion $\pi: Z \to X$, consider distributions of the form

(iv) $\pi^* \rho \delta_q$

$x_i$ being a distribution on the list (iii).

(*) All of the following discussion except for the occasional use of words like "funcctor" is due to Donald Ludwig.
Proposition. The Fourier integrals of Hormander are just those distributions on $X$ which can be approximated to arbitrary order of smoothness by distributions of the type (iv).

We will now prove a theorem of Ludwig concerning the wave-front sets of distributions of the form (iv).

Theorem. Let $Z = X \times Y$ and let $\pi: Z \rightarrow X$ be the usual projection. For fixed $y \in Y$, let $S(y)$ be the hypersurface in $X$ consisting of the points $x \in X$, $f(x, y) = 0$. (Assume for simplicity that $S(y)$ has no singular points.) Let $\Lambda \subset T^*_X$ be the normal bundle of the envelope of these hypersurfaces. Then $WF(\pi) \subset \Lambda$.
Proof. We first prove

Lemma 4. Let \( \pi: \mathcal{E} \to \mathcal{X} \) be a submersion and \( \omega \) a distribution on \( \mathcal{E} \). Then \( \text{WF}(\pi_\# \omega) \) is contained in the set of \( (x,\eta) \) with the property: \( \exists \neq \zeta \in \mathcal{E}, \pi(\zeta) = x \) and \( (\pi_\# \omega)_\zeta \in \text{WF}(\eta) \).

Proof. Let \( f: \mathcal{X} \to \mathbb{R} \) with \( \partial f = \eta \) and observe that \( f_\# (\pi_\# \omega) \) smooth \( \iff \) \( f \circ \pi_\# \omega \) smooth.

To prove the theorem, we note that, as was already observed, \( \text{WF}(f^\# \eta) \) for \( f: \mathcal{X} \to \mathbb{R} \) and \( \omega \) on the list (iii) is just the normal bundle to \( f = 0 \). By lemma 4, the WF of the push forward is the set of vectors \( x, \frac{\partial f}{\partial x}(x,y) \) where \( f(x,y) = \frac{\partial f}{\partial y}(x,y) = 0 \) which, by classical surface...
theory (see Struik)

is the equation for the normal bundle to the envelop.

As an illustration of Ludwig's result let $X$ be $\mathbb{R}^n$ and let $Y$ be a hypersurface in $\mathbb{R}^n$.

Let $f : \mathbb{R}^n \times Y \to \mathbb{R}$ be the function

$$(x, y) \mapsto |x - y|^2 - c^2$$

and let $\nu$ be the delta function.

For fixed $y \in Y$, $\delta_y$ is the delta function of the sphere of radius $c$ about $y$ in $\mathbb{R}^n$. "Summing" these $\delta_y$'s, we get, by Ludwig's theorem of distributions on $\mathbb{R}^n$, whose wave front set lies on the envelop of the spheres $|x - y| = c$ as $y \in Y$, i.e. the singularities not on the envelop cancel each other out by "interference" (corroborating)
Huygen's principle! ) See figure 1

[Diagram]

Figure 1

The above results convey we hope some of the flavor of the "Radon" approach to Fourier integrals.

To conclude we note that there have been two recent developments in the theory of generalized functions having many analogies with the work of Hormander. One is the recent work of Sato on hyperfunctions and the other the work of Maslov-Leray on asymptotic functions. In both cases there are analogies of wave front sets, Fourier integrals, etc. There is some evidence that this is due to the existence of analogous push-forwards and pull-backs.

S3-DS-VG-12
Bibliography


F. John

D. Ludwig

D. Straik
Let \( p(D) \) be a constant coefficient 1st order elliptic pseudo-differential operator on \( X \) with type symbol \( p(\xi) \).

Consider \( \frac{1}{\sqrt{-1}} \frac{\partial}{\partial t} - p(D) \) on \( X \times \mathbb{R} \) with symbol \( \eta - p(\xi) \).

**Exercise A.** The bicharacteristic flow on \( T^*_X \) associated with \( p \) is given by

\[
\begin{align*}
(A) & \quad \begin{cases} 
  x \rightarrow x + t \frac{\partial p}{\partial \xi}(\xi) \\
  \xi \rightarrow \xi
\end{cases}
\end{align*}
\]

**Exercise B.** Let \( Y \) be the 2011 dimensional submanifold of \( T^*_X \times \mathbb{R} \) on which \( \eta - p(\xi) = 0 \). (This is diffeomorphic to \( T^*_X \times \mathbb{R} \) under the map \( (x, \xi, t) \rightarrow (x, \xi, t, p(\xi)) \) )

Let \( \Lambda \).
be a homogeneous Lagrangian submanifold of $\mathcal{F}$ defined by the phase function $L_x \Sigma_\xi - S(\xi)$.

Then the Lagrangian manifold $\mathcal{L}$ flowed out in $\mathcal{F}$ by the flow $(A)$ is defined by the phase function $\Sigma_\xi - S(\xi) - tP(\xi)$.

Hint: Set $x_0 = \frac{2S}{\partial S}(\xi_0)$. Then at time $t$ $(x_0, \xi_0)$ flows into $(x, \xi)$ with $x = \left( \frac{2S}{\partial S} + t \frac{\partial P}{\partial \xi} \right)(\xi_0)$.

Exercise (C) Let $w_0$ be a distribution in $\mathcal{F}$ of type $\mathcal{D}^m(\Lambda_0)$ with leading symbol of the form $a(x, \xi)$. Then there exists a solution $u_1$ of the initial value problem

$$
\left( \frac{i}{\sqrt{-1}} \frac{\partial}{\partial t} - P \right) u_1 = 0 \quad u_1(x_0) = w_0
$$

expressible as an oscillatory integral associated with $1$. 
with leading symbol of the form \( a^*(x, t, s) \)

where \( a^*(x, t, s) = a(x + t \frac{dp}{ds}, s) \)

\[ \text{Exercise (D):} \quad \text{If Exercise (B):} \quad u \]

has the form

\[ u(x, t) = \int e^{i (x \cdot s - t p(s) - s(\xi))} \chi(\xi) a^* d\xi \]

if we neglect lower order terms in the asymptotic expansion. (\( \chi \) is the usual cut-off function)

Show \( u \) can be written in the form

\[ \text{Exercise (D):} \quad \int e^{i t \left( \frac{dp}{ds} \right)^m} \left[ \int e^{i x \cdot \sigma - s(\sigma)} a^*(x, t, s) d\sigma \right] \]

\( d^s \) now being the volume form on \( \Sigma(s) = 1 \)
Exercise E In a given fixed $x$, compute the critical set of the function $x \cdot p - S(t)$, restricted to the hypersurface $p(t) = 1$, using Lagrange multipliers, i.e., solve the $n+1$ equations

\[(E) \quad x_i = \frac{\partial S}{\partial x_i} - \lambda \frac{\partial p}{\partial x_i} \quad \Rightarrow \quad p(t) = 1\]

Let $\Lambda'$ be the subset of $\Lambda_0$ on which $p = 1$.

Show that the solutions of (E) correspond 1-1 with $\gamma(t)$ such that the bicharacteristic ray $x_0 + t \frac{\partial p}{\partial s}(x_0)$ hits $x$ at $t = \gamma$.

Show that degenerate critical $\gamma(t)$ correspond to singularities of the map

\[(F) \quad \Lambda' \times \mathbb{R} \rightarrow X \]

$(\xi_0, x_0, t) \rightarrow x_0 + t \frac{\partial p}{\partial s}(x_0)$
Compare the solutions of (E) with the critical
points of the phase function in the integral (D).

Because (F) show that if a singularity of the
map (F) has "intensity" then the
governor of the second
integrand of the integral (D) is of order

\[ n^{-1+m-\frac{3}{2}+\alpha} \]

and make the relevant
conclusions about the $H^s$ space in which $u$ sits at that
point as a function of $t$. (I think we can conclude
from this that $u$ goes into a lower $H^s$ space
at the characteristic above (x,t). This is a special case of a
rather general fact.) The main insight is that since $u$ is $H^{\frac{3}{2}}$
in the $t$ direction and satisfies a hyperbolic equation, it must
be $H^s$ in the $x$ direction as well.)
On Mal{\small o}rov Part I Meromorphic operators

We begin with definition of Mal{\small o}rov's canonical operator. Let $\Lambda$ be Lagrangian manifold in $\mathbb{T}_X^*$ and let $\Pi: \Lambda \to X$ be the projection on the base. We denote by $\mathbb{C}$ the set of points $x \in \Lambda$ such that $\operatorname{rank} d_{\Pi x} < n$. Let $\overline{C}$ the image of these points in $X$. $\overline{C}$ is usually called the caustic.

We will denote $\mathbb{C}$ with Mal{\small o}rov angle on $\Lambda$. We will denote $C = C_1 \cup C_2$ where $C_1 = \left\{ x \in \mathbb{C}, \operatorname{rank} d_{\Pi x} = n-1 \right\}$ and $C_2 = \left\{ x \in \mathbb{C}, \operatorname{rank} d_{\Pi x} < n-1 \right\}$. For the general situation $C_1$ is a submanifold of codim 1 and $C_2$ is a closed exact which is a union of submanifolds of codim $\geq 3$. $C_1$ can be oriented in a natural way and $\Pi$.

\[ \overline{C} \]
is a curve in $V$ its intersection $m$ with $\gamma$

g is well defined.

The Maslov canonical operator $\gamma$ maps half-densities on $\Lambda$ into half-densities on $V$. We will define it locally (on $\Lambda$) and patch together. The

indicates that what we get is not actually an operator but just a kind of approximate operator.

### Local definition of a $\gamma$ at $x_0 \in \Lambda - C$

Since $(\pi_i)_{x_0}$ is injective, $\pi$ maps $\gamma$ into a right $\mathcal{U}$ of $x_0$

diffeomorphically onto a right $\mathcal{U}$ of $\pi(x_0)$. The inverse map is of the form $x \rightarrow d\phi$, where

$\phi$ is a smooth function on $V$ defined a half-density $a$ on $\mathcal{U}$. We may

\[
(A) \quad a \rightarrow \bar{a} e^{i\lambda \phi}
\]
where \( \sigma \) is the half density on \( V \) corresponding to \( a \) on \( U \). (Note: there is already an ambiguity in our definition. \( \sigma \) is only determined up to an additive constant.)

Definition at \( a + t \sigma \in \mathbb{C} \); let \( V \) be a neighborhood of \( \pi(\sigma) \). We choose a phase function \( \varphi \) on \( V \times \mathbb{R}^n \), \( \varphi = \psi(\xi, \theta) \), such that \( (\xi, \theta) \in \mathbb{C} \rightarrow \exp(i\varphi) \) parametrizes a neighborhood \( U \) of \( \sigma \) on \( V \).

The half density \( a \) on \( U \) corresponds to a \( \frac{1}{2} \) density \( \bar{a} \) on \( \Theta \). The functions \( \frac{\partial \varphi}{\partial \xi}, \ldots, \frac{\partial \varphi}{\partial \theta} \) give us a canonical way of trivializing the normal bundle \( N(\Theta) \) to \( U \). The Lebesgue measure
on the fiber $N_x$. Finally let $\tilde{a}$ be a half density on $X \times \mathbb{R}^n$ which has support in a tube around $e_0$ and takes the value $\tilde{a} \circ \tilde{u}(x)$ at $x \in \mathcal{C}_0$. On $U$ we define the Maslov operator by

$$a \rightarrow \frac{1}{\sqrt{2\pi \lambda^n}} \int_{\mathcal{C}_0} e^{x \cdot \nabla \tilde{u}(x,0)} \psi(x,0) \mathrm{d}x.$$ 

This definition depends of course on the choice of $\tilde{a}$.

By a simple integration by parts one can show that another choice of $\tilde{a}$ would change the RHS of (B) by a term of order $O(\frac{\lambda}{\lambda^n})$. What about different $\psi$?

We will now describe how the expressions (A) and (B) patch together. Yet...
be the action form on $T^*_X$ ($\omega = \sum x_i dx_i$ in the usual x, x notation.) Also restrict to $\Lambda$ as closed, and we can assume $\omega$ exact on $\mathcal{Y}$, i.e. $\omega|\mathcal{Y} = d\phi$ for some function $\phi$ on $\mathcal{Y}$. The $\phi$ chosen is up to an additive constant the same as the $\phi$ occurring in (4) at $\phi_k \in \mathcal{Y}$.

The main result in this subject is the theorem of stationary phase.

**Theorem:** Let $\mathcal{U}_1, \ldots, \mathcal{U}_k$ be the connected components of $\mathcal{Y} - C$. On $\mathcal{U}_k$ the RHS of (B) is equal to

$$c_k \sqrt{t} e^{\mu_k \phi} + O(t^{\frac{1}{2}}).$$

Moreover, the constants $c_k$ and $C_k$ are related by

$$c_k = e^{\mu_k t_k} c_0,$$

where $t_k$ is the intersection of $C$ with the complement of the Milnor fiber $C$ with any curve $C$ giving a $t_k$ in $\mathcal{U}_k$ at a $t_k$ in $\mathcal{U}_k$. 
We now make the following assumptions:

I. The restriction of the action form \( \omega \) to \( \Lambda \)

is exact.

II. The "dual class" of \( C \) in \( H^1(\Lambda) \) is zero.

Then the Maslov operator can be defined globally as follows. We choose the \( \phi \) described in the paragraph above so that it is defined globally as restriction of \( \omega \) to \( \Lambda \).

We choose the notation consistent with (A) so that if we go around a path in \( \Lambda \) we come back to where we started from. Then the formulas (A) and (B) define a map, \( a \rightarrow \frac{\partial}{\partial a} \).

(c) \( \frac{1}{2} \) densities on \( \Lambda \) \( \rightarrow \) \( \frac{1}{2} \) densities on \( X \), depending on \( \Lambda \), modulo \( \frac{1}{2} \) densities of order \( O(\frac{1}{a}) \).

We can write down an explicit formula for \( \frac{\partial}{\partial a} \).
at $q_0 \in X - \bar{C}$ as follows. Let $\bar{\omega}$ be a fixed base point in $\bar{C}$. Everyone $x$ has $k$ geometrically $x, \ldots, k$ on $\Lambda$. Let $\phi : [0, 1] \to \Lambda$ be a smooth curve joining $x_0$ to $p_0$ and intersecting $C$ transversely. Then

$$\omega_0(x) = \sum \bar{a}_i(x) \cdot \left( \int_0^1 \omega \left( \frac{d\phi}{dt} \right) + \psi + \phi(t) \right)$$

where $\bar{a}_i$ is the intersection no. of $\phi$ with $\Sigma$ and $\bar{a}_i$ is the $i$-density at $x$ associated with $\phi$ and $p_0$.

**Proof.** On $\Lambda$, $\omega = d\phi$, so $\omega \left( \frac{d\phi}{dt} \right) = \frac{d\phi}{dt}(\phi(t))$, and the integral in the argument $= \phi(p) - \phi(q_0)$.

**Remark.** By replacing $\phi$ by a finite no. of base points and choosing other points judiciously, (D) sometimes can be defined even when the conditions I and II don't hold.
Example: Let $Y$ be a manifold and $\Lambda_0$ a Lagrangian submanifold in the cotangent bundle of $Y$. Suppose that condition I holds for $\Lambda_0$. If the action from restricted to $\Lambda_0 = \partial H_0$. Let $\mathcal{P}$ be a function on $T^*_Y$ and $\mathcal{E} = \mathcal{E}_\mathcal{P}$ be the corresponding Hamiltonian vector field. Let $\mathcal{P}_t: T^*_Y \to T^*_Y$ be the flow generated by $\mathcal{E}$. $\mathcal{P}_t$ sweeps out a Lagrangian submanifold of the cotangent bundle of $R \times Y$, namely the set of points $(x, \xi, t, \tau)$ where $(x, \xi) \in \mathcal{P}(\Lambda_0)$ and $\tau = \mathcal{P}(x, \xi)$. We shall denote this Lagrangian submanifold by $\Lambda_t$. It's fairly easy to see that $\Lambda_t$ satisfies condition I, but it couldn't satisfy condition II. Nonetheless, $\Lambda_t$ can be defined as follows. Let $(x, \xi)$ be a point in $(R \times Y) - \mathcal{E}_\mathcal{P}$ and let $(t, p_1), \ldots, (t, p_n)$ be the set above it on $\Lambda_t$. 
(1) Let \((0, q_1), \ldots, (1, q_n)\) be the points in the  
backward flow at time \(t = 0\). Then

\[
(2) \quad \mathcal{X}_q = \sum \alpha_x (t, q) e^{i \int_0^t \gamma (s, x \cdot \delta_x (s)) ds + \mu_t + \phi (x)}
\]

where \(\delta_x\) is the curve \(s \mapsto (s, \beta (x))\), \(0 \leq s \leq t\),  
and \(\alpha_x\) is the ciphertext vector with \(C\).
2. Let $u = u_1, \ldots, u_k$ and let $\mathcal{P}(x, D)$ be
an integro-differential operator mapping self-densities on $X$
onto self-densities on $X$. We want to look for
asymptotic solutions as $\lambda$ gets large of the partial
integro-differential equation
\[ \mathcal{P}(x, D, \lambda) = \sum_{(k, l)} \mathcal{R}(k, l) \cdot \mathcal{P}(x, D) \]

Let $\mathcal{P}(x, \lambda)$ be the function $\sum \mathcal{P}(x, \lambda)$ where
$\mathcal{P}(x, \lambda)$ is the top symbol of the operator $\mathcal{P}(x, D)$

Define $c = \sum c_i$, where $c_i$ is the subprincipal
symbol of the operator $\mathcal{P}$. One of Maslov's
main results is the following theorem

Thus, let $\mathcal{P}$ be a Lagrangian manifold on
which $\mathcal{P}(x, \lambda) = 0$. Let $\nu$ be a self-dense
form satisfying the transport equation

\[ \mathcal{P}(x, D, \lambda) \]
\[ L a + c a = 0 \]

where \( L \) is the Hamiltonian vector field corresponding to \( \phi \). Then \( \mathcal{P}(x, \partial_t \phi) \wedge \phi = o(\frac{1}{\hbar^2}) \).

Rather than trying to prove this theorem we'll derive as a corollary of it Moskow's explicit formula for the solution of the Schrödinger equation.

(E) \[ \frac{i\hbar}{\sqrt{-1}} \frac{\partial \psi}{\partial t} = -\hbar^2 \sum \frac{1}{2} \frac{\partial^2 \psi}{\partial x_k^2} + \sum V(x) \psi \]

in \( \mathbb{R}^n \times \mathbb{R} \). The associated symbol is

\[ \mathcal{W} \mathcal{A}(\frac{\partial^2}{\partial x_k^2}) \quad \forall \psi - \left( \sum \frac{\hbar^2}{2} + V(x) \right) = \mathcal{H}(x, \hbar, t, \psi) \]

and the Hamiltonian is

\[ H = \sum \frac{\hbar^2}{2} + \sum \frac{\partial V}{\partial x_k} \frac{\partial}{\partial x_k} \]

\[ \mathcal{A} \frac{\partial \psi}{\partial t} - \sum \frac{\hbar^2}{2} \frac{\partial^2 \psi}{\partial x_k^2} + \sum \frac{\partial V}{\partial x_k} \frac{\partial}{\partial x_k} \psi = 0 \]
and the action \( \mathcal{A}(\bar{\Sigma}) = a' - \sum l_{\Sigma}^2 \). When \( H = 0 \) this \( \mathcal{A} \) equals to \( -\sum l_{\Sigma}^2 + V(x) \).

On an integral curve of \( \bar{\Sigma} \) we have \( \dot{x} = \frac{dx}{ds} = \xi \).

So we can write the action integral of (2) as

\[ -\mathcal{L}(x, \xi, t) \int s^t \sum (\frac{\xi(t)}{2})^2 + V(x) \, dt = \int s^t L(x, \xi, s, t) \, dt \]

where \( L \) is the classical Lagrangian.

Finally note that in the transport equation \( \xi = 0 \) since all the terms of the Schrödinger operator are self-adjoint.

Now let \( a_0 e^{-it \Delta} \phi_0 \) be a \( \frac{1}{2} \)-density on \( \mathbb{R}^n \) with \( a_0 \) compactly supported.
We want to find a solution $\mathcal{S}$ of the equation $$(E)$$

which takes on initial data $f(x,0) = \phi_0 \in C^0$ at $t=0$.

Let $\phi^t$ be the flow associated with $\frac{\partial}{2} + V(x)$ at time $t$. Consider the map $R^* \to R^n$

which maps $x \rightarrow \phi^t(x) = \phi^t \left( x, (0,0) \right), \quad \forall x \in \mathbb{R}^n$.

Using the projection of the cotangent bundle of $R^n$ onto its base, let $\theta$ be a regular value of this map and let $\omega^1, \ldots, \omega^n$ be its generator.

$$\lim_{t \to 0} \mathcal{S}(\theta, t) = \sum_{i=1}^n \left( \frac{\partial \phi}{\partial q_i} \right)^{-\frac{1}{\alpha}} q_0(q_i) \phi^t(q_i) E \left( \int_0^t \sqrt{\omega_1^2 + \omega_2^2 + \cdots + \omega_n^2} \right)$$

modulo an error term of order $O(\frac{1}{\alpha})$.

where the integral is over the classical path $\gamma(t)$.
joining 0 to 0' and 0' is the intersection

invariant of the corresponding path in S with
the Maslov angle

except for the expression of the amplitude
in the above formula the theorem is a direct corollary
of the theorem above. To see that the expression
of the amplitude is right we note that it is
derived as follows. We lift a wey to \( \Omega \)
map it by \( (d \phi) \)
(since we want it to

satisfy the transport equation, \( \frac{d}{dt} a = 0 \)) and other

project down. The Jacobean for this map is

\( \frac{1}{|Q' \Omega|} \) and since we are mapping \( \frac{1}{2} \) densities

which are \underline{contravariant} of order \( \frac{1}{2} \) \( a_0 (\Omega) \) gets

mapped into

\[ \left| \frac{d}{dt} a_0 (\Omega) \right|^{-\frac{1}{2}} a_0 (\Omega) \]
\[ s(x - x_0) = a(x) \int e^{i \frac{p}{\hbar} \cdot (x - x_0)} \, dp \]

where \( a(x) \) is a smooth function equal to 1 at \( x_0 \) and having support in a small neighborhood of \( x_0 \). Setting \( N = \frac{p}{\hbar} \), we get

\[ (E): \quad s(x - x_0) = \frac{a(x)}{i\hbar} \int e^{i \frac{p}{\hbar} \cdot (x - x_0)} \, dp. \]

We will use this expression and the theorem above to obtain an asymptotic solution of \( (E) \) with initial data \( s(x - x_0) \) at \( t = 0 \). Let \( (y, t) \) be a point of \( \mathbb{R} \times \mathbb{R}^n \). We will assume that there are only a finite no. of classical trajectories joining \( y \) to \( x_0 \).
and that \( g \) and \( x_0 \) are non-conjugate along these trajectories. Clearly the same will be true for all trajectories joining \( g \) to points in the isosceles of \( a(x) \).

Consider the graph of \( \mathcal{L}(x-x_0) \) in the cotangent bundle of \( T^n \). This in fact the set of \( \{(x, s) \in \mathbb{R}^n \} \).

Let \( \Lambda(t) \) be the set of all trajectories which hit this graph at \( t = 0 \). \( \Lambda(t) \) is a Lagrangian submanifold of \( T^n \). Our assumption about \( g \) guarantees that there are only a finite no. of trajectories

\[ \varphi_{x, s} : [0, t] \rightarrow \Lambda(t) \quad s = 1, \ldots, N \]

whose terminal \( x_s \) lie the above \( (y, t) \) and whose initial \( x_0 \) lie the above isosceles of \( a(x) \). Let \( (x_0(s), s) \) be the initial \( x_0 \) of the curve \( \varphi_{x, s} \). The theorem...
Here gives us the following asymptotic formula

\[ a(x) \cong \frac{\partial a}{\partial x} (x - x_0) \]

at time \( t = 0 \)

\[ \sum \alpha_k(x(\xi)) \left| \frac{\partial \psi}{\partial x} \right|^{-\frac{1}{2}} \leq \frac{1}{\theta h} \int_{\Omega} \left( s \right) + (k(x) - x) \frac{\partial s}{\partial x} \right) \frac{\partial x}{\partial x} \]

where \( u \) is the interaction term of \( \theta \), with the

Madelung cycle on \( \Lambda(\nu) \), Owing this into (5)

we get the following expression for the fundamental

solution of (E)

\[ G(y, t, x_0) = \sum \frac{1}{\theta h} \int_{\Omega} \alpha_k(x(\xi)) \left| \frac{\partial \psi}{\partial x} \right|^{-\frac{1}{2}} e^{ik \cdot S} d\xi \]

We will try to evaluate (6) using stationary phase.

To do this, we need to determine the mutual

value of the phase function.
\[ \phi(\xi) = \bar{S} \omega + (x(\xi) - x_0) \cdot \sigma \]

as a function of \( \xi \). To do so we'll need some general facts about symplectic geometry. Let \( X \) be a manifold, and \( L \) a Lagrangian submanifold of \( T^*_X \). Let \( \omega \) be the action form. In each \( s \in \mathbb{R} \) let \( \xi_s \) be a smooth curve in \( L \) and suppose \( \xi_s \) depends smoothly on \( s \). Let \( u(s) \) and \( v(s) \) be the initial and terminal \( \xi_s \) of \( \xi \).

Lemma \[ \frac{d}{ds} S_\omega = \omega \left( \frac{d\xi}{ds} \right) - \omega \left( \frac{d\xi}{ds} \right) \]

Proof: There is a tubular neighborhood \( U \) of \( \xi_s \) in \( T^*_X \) which \( \omega \) is exact; i.e., \( \omega = df \) in \( U \). Thus

\[ S_\omega = f(x(\xi)) - f(x(\xi)) \]

Differentiating with respect to \( s \) we get the assertion above. \( \Xi \) E.D.
Now let compute \[ \frac{\partial}{\partial \xi} \mathcal{S}_w \] Note just that the curves \( \beta \) all lie in a fixed Lagrangian manifold \( \mathcal{T}_{\mathbb{R} \times \mathbb{R}^n} \) namely the set of all trajectories that, at time 0, lie above the \( \phi^t \) (5,5). Let \( V_{3x} \) and \( W_{3x} \) be the tangent vectors to the initial and terminal curves of \( \beta_j \) obtained by varying \( \beta \) and keeping the other coordinates of \( s \) fixed. By the lemma, \[ \frac{\partial}{\partial \xi} \mathcal{S}_w = \mathcal{W}(W_{3x}) - \mathcal{W}(V_{3x}) \]
The end point of \( \beta \) projects onto the fixed \( \phi^t \) (5,5) in the base for all \( s \), so \( \partial \mathcal{W} \partial V_{3x} = 0 \) and hence \( \mathcal{W}(W_{3x}) = 0 \) (because of the way \( \mathcal{W} \) is defined!). On the other hand...
\((\text{iv})\) \( V_{ij} = \frac{\partial x_i(x)}{\partial x_j} \), so \( a(V_{ij}) = \delta_{ij} \cdot \frac{\partial x_i}{\partial x_i} \)

at \((x_0, s)\). Therefore, we get:

\[
\frac{3}{2} s \, \phi'(s) = -s \cdot \frac{\partial x_i}{\partial x_i}
\]

and \( \frac{\partial}{\partial x_j} \phi(s) = (x(s) - x_0) \). This gives

Thus, the initial \( x_0 \) of the phase function in the integral \((G)\) are precisely those \( s \) for which \( x(s) = x_0 \), i.e., for which the integral curve \( x_0 \) joins \( x_0 \) to \( y \).

If we apply stationary phase to \((G)\) and use the fact that \( a(x_0) = 1 \) we get the following asymptotic formula for the RHS.
\[ G(y, t, x_0) \sim \sum_{n} \frac{1}{(2\pi \hbar)^{\frac{d}{2}}} \left| \frac{2\hbar}{2\xi} \right|^{\frac{d}{2}} (x, \xi) e^{\frac{-i}{\hbar} \int_{0}^{t} \phi_{n}(\xi, \xi') \, dt'} \]

where \( q_{i}(r), \quad 0 \leq r \leq t \) is a classical trajectory going from \( x_{i} \) to \( y_{j} \), and \( \xi = \frac{2\hbar}{2\xi} \phi_{i}(x_{i}, \xi) \)

Madelung identifies \( \xi \) with the number of congruent \( \xi \) along the trajectory, \( \phi_{i}(r) \) at don't at the moment see why this is the relation between the number and the intersection no. of \( \xi \) with the Madelung cycle.

Madelung gives an alternative proof of the formula (H) using Lighthill's integrals: The start with Lighthill's representation of the fundamental solution of the Schrödinger operator:
\( \mathcal{G}(y, t, x_0) = \int_{x_0}^{\gamma} \mathcal{L}(y, \dot{y}, y) \, d\gamma \)

where \( \gamma(t) \) is any path joining \( x_0 \) to \( y \)

and \( \mathcal{L} \) is Lagrange's measure on path space.

Yet apply stationary phase to the RHS above

(ignoring the fact that the integral is not over a finite dimensional region.)

The critical sets of the phase function are just those paths \( \gamma \) for which the first variation \( \delta \mathcal{L} = 0 \), which, by the principle of least action are just the classical trajectories, that is, the \( \gamma(x) \) above.

The signature of 2nd \( \delta^2 \mathcal{L} \) at each of these trajectories

is, by Morse theory, equal to the no. of conjugate points \( \gamma(t) \) along the trajectory. Therefore we obtain
Asymptotic formula for the RHS of (I)

\[ \sigma(x, y, z) \approx \sum K_k \epsilon \frac{\xi^N(x^h, y^h, z^h)}{\xi^N} + \epsilon \frac{\zeta^N}{\zeta^N} \]  

where \( K_k \) is the quotient of two infinite quantities, namely \( (2\pi h)^{\frac{N}{2}} \) and \( \det (S^L) \), but apparently these cancel each other out and give the finite answer computed above.
If \( (x,0,y,1) \notin WF(K) \), then \( K \) can be extended by continuity \( C^\infty \to C^\infty \).

\[ Ku = \pi_2^* K \rho^* u, \text{ product defined by } \rho \text{ before.} \]

**Note:** \( \Lambda \subset S \times T \). If \( A \subset S \), then

\[ \Lambda(A) = \{ t \in T : \exists (s,t) \in A \text{ s.t. } s \in A \} \]

\[ \Lambda(x,y) = \text{graph of } f : S \to T. \]

**Note:** If \( \omega \subset T^* \times \mathbb{R} \), \( \omega \) define \( W' \).

**Then:** In above situation

\[ WF(Ku) \subset WF(K)' (WF(u))u A, \text{ where} \]

\[ A = \{ (x,y) : (x,\bar{x},y,0) \in WF(K)(\exists y) \}. \]

\[ WF(Ku) \subset WF(K)' (WF(u))u A, \text{ where} \]

\[ A = \{ (x,y) : (x,\bar{x},y,0) \in WF(K)(\exists y) \}. \]

**Proof:** Suppose \( K \in C^\infty(S \times T) \), \( L \in C^\infty(\mathbb{R} \times T) \), neither interacting nor zero sections. Properly supported. Then \( K \circ L \) is bracketed above case with \( K \circ L \) in \( C^\infty(S \times T) \),

\[ WF(K \circ L)' = \left( K \circ L \right)' = WF(K)' \circ WF(L)' . \]

**Note:** \( \Lambda_1 \circ \Lambda_2 = \{ (s,u) : (s,t) \in \Lambda_1, (t,u) \in \Lambda_2 \text{ } \exists t \} \)

when \( \Lambda_1 \subset S \times T, \Lambda_2 \subset T \times U. \)
Given, \( K \circ L = \bigg\{ \pi_2 * \left\{ \left\{ \pi_1 * K \right\} \left( \pi_3 * L \right) \right\} \bigg\} \), where

\[ \begin{align*}
\pi_1 &\xrightarrow{\pi_2} \pi_3 \\
X \times Y \times Z &\xrightarrow{\pi_1} X \times Y \\
X \times Y &\xrightarrow{\pi_2} X \times Z \\
Y \times Z &\xrightarrow{\pi_3} \end{align*} \]

**Compound distribution**

\( \mathbb{Z} \rightarrow \mathbb{R} \)

\( \pi \downarrow \)

\( \mathbb{X} \)

Let \( u \in \mathcal{D}(\mathbb{R}) \) be homogeneous, \( a \in C^\infty \text{ on } \mathbb{Z} \).

Consider \( \pi_* \phi^* u \).

**Example:**

1. Use Radon

\[ \mathcal{S} = \int \langle x, \omega \rangle^* u \omega \, d\omega \quad , \quad u \omega = e^{\left\{ (x+i\alpha)^{\frac{n}{2}} \pm (x+i\beta)^{\frac{n}{2}} \right\}} \]

2. Any dual of compound dist:

Chain rule: \( \frac{\partial}{\partial x_i} (\phi^* u) = \phi^* \frac{\partial}{\partial x_i} \phi u \)

3. Fund soln of elliptic eqn coeff PDE.

(Must generalize to include asymptotic series.)
Conform $\mathfrak{sp}(V)^{\circ} \cap (\mathfrak{u}_0, \mathfrak{u}_0) = \mathfrak{p}_+((u, v))$.

Lie alg: $(\mathfrak{sp}(V)^{\circ} \cap (\mathfrak{u}_0, v) + (u, \mathfrak{u}_0) = 0$.

Eigenvals of $\mathfrak{sp}(V)$ occur in pairs s.t. $\lambda_1 + \lambda_2 = 0$.

Pairing of $+\times =$ eigen space.

Claim: $\overline{X, Y} \text{ lies in } V$, then

\[ 0 \to \overline{X} \to V \xrightarrow{\mathfrak{p}} Y \to 0 \]

$\mathfrak{p} \in \text{cap}(V)$, with $\mu_{\mathfrak{p}} = 1$.

Read brackets: given $\mathfrak{p}$, etc.

Param. $\{ \lambda : \mathfrak{m} \to \overline{X} \}$ by $\mathfrak{p}$.

If $\mathfrak{p} \in \text{cap}(V)$, let $Q_\mathfrak{p}(u, v) = (\mathfrak{p}u, v) - \frac{1}{2} \mu_\mathfrak{p}(u, v)$.

Claim: $Q_\mathfrak{p}$ is symm.

Remark: $Q_\mathfrak{p}$ determines $\mathfrak{p}$.

Identify:
1. $\{ \overline{X, Y} \text{ lies in } X \}
2. \{ \mathfrak{p} \in \text{cap}(V) : \mu_\mathfrak{p} = 1, \mathfrak{p}_{\overline{X}} \to 0 \}
3. \{ \text{symm } \mathfrak{f} \text{ maps } \mathfrak{f}(\overline{X, u}) = -\frac{1}{2}(x, v) \text{ for } x \in \overline{X} \}$.

By 3, $\mathfrak{b}$ is affine space. $\approx S^2(V/\overline{X})$. 
\[ \psi^{(y)} = \frac{1}{2} k |y|^2 \]

Holographic \( 1 \text{-dim } \exp \{ -\text{NImg} \} \to \mathbb{R}^n \) \text{away.}

\[ H = \mathbb{I} - \text{II}. \text{ (Also \( \text{column} > 1 \text{, but } 2 \text{-dep. in direction.})} \]

Define index, \( z = \text{no. cut. pts} \)

Sommerfeld: \( n(\frac{\partial u}{\partial y} - i ku) \to 0, \quad u \to 0 \text{ at } \infty. \)

\[ \psi \sim \frac{1}{a} \sum_{y} \frac{e^{-\pi |y|^2/2} e^{ik u}}{\sqrt{1 + u^2 (\text{II} - \text{II} y)}} \]

(Stationary phase + Green's thm + radiation cond.)

Backward waves cancel.

Note: Many lines \( \Rightarrow \) expression indep. of small change in surface.
Z \subset \tilde{T}_X \sim \{0\} \text{ homog.}

\text{Prop. 1: } u_n \to u \text{ if } u_n \text{ is compact} \iff \exists \tilde{b} \in \tilde{C}_c \text{ s.t. } u_n \to \tilde{b} \text{ in } C_c.

\text{Prop. 2: } u_n \to u \text{ if for } \psi: \tilde{X} \times S \to \mathbb{R}, \quad \psi(x, s_0) = \varepsilon_n,

\exists \text{ bump } b \text{ m.s. } (x_0, s_0) \text{ s.t. } \psi \# b \pi^* u_n \to \psi \# b \pi^* u.

\text{Properties of } \tilde{b}:

1. \text{ Suff. to test for } \tilde{b}: X \times S^{n-1} \to \mathbb{R}, \quad \tilde{b}(x, w) = \langle x, w \rangle.

2. \quad C_c \subset \subset \subset \subset C_c

3. \quad C_c \text{ is dense (Pf: convolution.)}

\text{Lemma: Let } \psi \text{ be bump fun., } f: \tilde{X} \to \mathbb{R}. \text{ Let } \psi \# f(x_0)

\text{If supp } u \text{ compact, then } \exists \varepsilon > 0 \text{ such that } f(x, w) \to x_0

\text{Mollify } \psi \text{ to } \psi^\#
Prop: Let \( f : X \to Y \); then \( f^* : C^\infty_{\text{pt}}(Y) \to C^\infty_{\text{pt}}(X) \)

\[
f^* Z = \{ (y_0, \eta_0) : \exists x_0 \in f^{-1}(y_0) \text{ s.t. } (x_0, (df)^t_{x_0} \eta_0) \in Z \}
\]

Prop: Let \( f : X \to Y \) is subm., \( f^* : C^\infty \to C^\infty \) induces map \( C^\infty_{\text{pt}}(Y) \to C^\infty_{\text{pt}}(X) \) when

\[
f^* Z = \{ (x_0, \eta_0) : y_0 = f(x_0) \text{ s.t. } \exists \eta_0 \in Z \text{ s.t. } \xi_0 \cdot (df)^t_{x_0}(\eta_0) \}
\]

Proof.

Pf: Prob 1 (Check that not from sm. dist.)

Then for \( f : X \to Y \), \( Z \subset \text{PT}^*(Y) \). Suppose \( (y_0, \eta_0) \in Z \),

\[
+ f^*(x_0) = y_0 \Rightarrow (df)^t_{x_0} \eta_0 \neq 0. \text{ Then } f^* : C^\infty_{\text{pt}}(Y) \to C^\infty_{\text{pt}}(X)
\]

will def by continuity.

(Cond: f transversal to hypersurface to which \( \eta_0 \) is normal.)

(Think \( \eta \in C^\infty_{\text{pt}}(Y) \) as \( \eta = d\xi \); thus is F wea - like dist.

with \( \eta \) distinct surface normal to directions in \( Z \).)

Idea of Pf: decompose \( \eta \) into more like dist. Here suff if

composite \( X \xrightarrow{f} Y \xrightarrow{H} R \) is submersion.

Consider 2 classes of directions: smooth or not sm., bit the transversal.
Q: Apply to take R. form. of own general dist?

Note: Write down answer but we try to verify info.