Optimization of Sub-10 Femtosecond Titanium Sapphire Lasers

by

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Abstract

Kerr-lens mode locked Ti:sapphire lasers are the backbone of most ultrafast and attosecond pulse generation systems. As such when building such systems the experimentalist desires an easy to use and robust system to produce femtosecond pulses for use in experiments. Towards this end I will discuss the issues involved in perfecting Kerr-lens mode locked Ti:sapphire laser technology and present experimental results on a new type of output coupler designed using chirped mirror technology. This new type of output coupler promises improved stability and improved spatial and spectral pulse qualities. In addition I will also discuss the phenomenon of Ince-Gaussian modes, which constitute a more general solution to the paraxial wave equation than the regular Hermite and Laguerre Gaussian solutions. By studying these modes, we hope to discover a relationship between the observed Ince-Gaussian mode patterns in the continuous wave beam of the laser and use this to determine the correct operating alignment for the laser cavity.

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Chapter 1

Introduction

Since their introduction in 1986 [26], Titanium Sapphire lasers have become indispensable to the field of ultrafast optics. With their introduction, they quickly replaced dye lasers as the laser of choice for ultrashort pulse generation due to their compact size and relative ease of operation. Their superior frequency stability makes them an ideal choice in frequency metrology. One of the most exciting applications has been their use as a frequency “ruler” to measure the Doppler shift of stellar spectra in the search for extra-solar planets [24]. In addition, Ti:Sapph lasers are an essential component in the latest experiments in the generation of attosecond EUV pulses [16] which opens up an entirely new realm of physics to explore.

In this thesis I attempt to evaluate the design of a new type of output coupler which we have termed the inverse-gain output coupler [InvGOC], for Ti:Sapphire lasers which promises to improve their performance and beam qualities. With these new output couplers, Ti:Sapphire lasers should become more robust, have better spectral qualities, and be easier to mode lock.

This thesis is organized as follows: Chapter 2 begins with a review of the phenomenon of mode locking which is a mode of laser operation characterized by the relative phase locking of longitudinal modes of a laser cavity. The chapter then continues with a short introduction to the Master Equation of Mode Locking and concludes with a short discussion on dispersion managed mode-locking. Chapter 3 then begins where Chapter 2 leaves off and describes Kerr-lens mode locking, the
mechanism which allows Ti:Sapph lasers to be mode locked and the special design considerations that need be taken into account when designing or operating a Kerr-lens mode locked laser. The chapter then concludes with a description of the physical laser setup we used in this work. Chapter 4 describes the double chirped mirror [DCM] mirror technology which provides precise dispersion compensation in the laser cavity thus allowing for the generation of sub 10-fs pulses as well as introduces the design of the inverse gain output coupler. Chapter 5 discusses the experiments we performed to compare the inverse gain output couplers with the traditional flat output coupler design by studying characteristics such as beam quality, spectral quality (its smoothness as well as width), ease of mode locking, output power, and the dependence of output power and spectrum on pump power. Chapter 6 then takes a slightly different route and explores the phenomenon of Ince-Gaussian laser cavity modes in an attempt at better understanding the relationship between cavity stability and the observed modes emitted from a laser cavity. We conclude the thesis in Chapter 6 with a few closing comments.
Chapter 2

Fundamentals of Mode Locking

Before discussing Ti:Sapphire lasers, it is helpful to answer the question: Exactly what is a mode-locked laser? I attempt to answer this question by showing how a mode-locked laser can be thought of as a superposition of laser cavity modes whose relative phases are controlled precisely and "locked" with respect to each other. From there we will examine the famous Master Equation of Mode Locking introduced by Herman Haus in 1975 to analytically describe the behavior of mode-locked pulses [15].

2.1 Superposition of Cavity Modes

It is generally known that any laser cavity can support many different longitudinal and transverse modes. As for the transverse modes, the most well-known of these modes is is the TEM\textsubscript{00} transverse mode which is nothing more than the fundamental Gaussian mode of the laser cavity. Since the losses associated with this transverse mode is almost always the lowest of the transverse modes then it is a perfectly reasonable to consider only a laser which oscillates in the TEM\textsubscript{00} transverse mode. However, despite only oscillating in one transverse mode, every laser cavity can support a very large number of longitudinal modes whose frequencies satisfy the condition

\[ \nu_m = \frac{mc}{2 \sum_{i} n_i(\nu_m) L_i} \equiv \frac{mc}{2 n(\nu_m) L} \]  

(2.1)
where \( m \) is a positive integer and \( n_i(\nu_m)L_i \) is the optical pathlength at the frequency \( \nu_m \) of the cavity element \( i \) of length \( L_i \). We can define an effective index of refraction \( n(\nu_m) \) such that \( n(\nu_m)L \) is the optical pathlength of the cavity where \( L \) is the geometrical pathlength of the cavity. If we make the unrealistic assumption that the cavity is dispersionless then the mode separation \( \Delta = \nu_{m+1} - \nu_m = c/(2nL) \) is a constant. Despite making such a blatantly incorrect assumption we will see later that the result still holds even for the real case of a cavity with non-zero dispersion.

Using these assumptions we then consider a laser oscillating in one transverse mode and \( N \) longitudinal modes whose frequencies are given by \( \omega_m = \omega + 2\pi \Delta m \) for integer \( m \). For such a laser the complex electric field (where the real field is obtained by taking the real part of the result) can be written as

\[
E^+(t) = \frac{1}{2} E e^{i\omega t} + \frac{1}{2} E_0 e^{i\omega t} \sum_{m=(1-N)/2}^{(N-1)/2} e^{2\pi i mt + \phi_m}
\]  

(2.2)

In this equation \( \phi_m \) is the phase of mode \( m \) which is random for a free-running laser. Furthermore, the E-field is periodic with a period of \( 1/\Delta \) which we refer to as the round-trip time \( \tau_{RT} = 1/\Delta \).

Normally the phases \( \phi_m \) are random and hence the average laser intensity is just \( N \) times the average intensity of a single mode, i.e. \( \langle I \rangle = N E_0^2/(2\sqrt{\mu_0/\epsilon}) \). However if we somehow manage to “lock” the phases at 0 (or any other constant value so long as the phases are all the same) then the summation can be calculated analytically yielding the result

\[
E^+(t) = \frac{1}{2} E e^{i\omega t} + \frac{1}{2} E_0 e^{i\omega t} \frac{\sin(N\pi t)}{\sin(\pi t)}
\]  

(2.3)

which is illustrated graphically in Fig. (2-1). We see that for \( t \to 0 \) the modes coherently add to yield a peak value of \( NE_0/2 \). Thus for large \( N \) the sum of cavity modes becomes a train of pulses spaced by \( \tau_{RT} = 1/\Delta \) of duration \( \tau_p \) where

\[
\tau_p \approx \frac{1}{N\Delta}
\]  

(2.4)
Figure 2-1: Illustration of the concept of mode-locking as the coherent sum of cavity modes of equal “locked” relative phase. Here we add 10 sine waves whose frequencies all differ by an integer multiple of a fixed frequency \( \Delta \) all with the same relative phase to produce the pulse train above. At the location of the pulse peak (green arrow) the sine waves add coherently, whereas in between pulse peaks (red arrow) the sine waves add up destructively. Specifically at the peaks, the individual modes sum to a peak amplitude of \( N \mathcal{E}_0 \) where \( \mathcal{E}_0 \) is the amplitude of a single mode.
This establishes the inverse relationship between the spectral width roughly equivalent to \( N \Delta \) and the pulsewidth \( \tau_p \). Therefore assuming this model we would need over \( 10^6 \) longitudinal modes to generate the 5 fs pulses out of an 85 MHz mode-locked laser. Furthermore, since the peak value is \( N \) times the amplitude of the single mode, we see that the peak intensity is \( N \) times the average intensity of a free-running laser, hence in the case of 5 fs pulses we have a peak intensity of over one million times greater than the free running laser.

Unfortunately, this simple frequency domain picture of constant evenly spaced longitudinal modes with constant phase does not explain the whole picture. The most important flaw in this picture is that it does not account for dispersion. If we include dispersion then the effective index of refraction of the laser cavity will vary with frequency and hence by Eq. (2.1) the mode spacing will no longer be a constant. Without a constant mode spacing, the whole mode superposition breaks down and mode locked pulses are no longer possible by this explanation. However it turns out that for real lasers with dispersion the modes are equally spaced throughout the pulse bandwidth to 3.0 parts in \( 10^{17} \) [33]. This apparent contradiction can be solved if we consider the fact that real mode-locked lasers have nonlinear pulse shaping processes such as self-phase modulation (SPM) which balance out the effects of dispersion to produce equally spaced mode locked pulses with a constant repetition frequency.

In 1975, H. Haus derived the Master Equation of Mode Locking to describe the dynamics of pulse formation in mode locked lasers which is a time domain picture which includes nonlinear effects such at SPM to explain mode locking [15]. Though the model is very simple and leaves out many important details (such as diffraction effects) it allows us to gain important insight into the operation of mode-locked lasers and is a more accurate picture than the dispersionless frequency domain picture discussed earlier. I will present an overview of Haus' equation and highlight some of the key features which help in determining the correct design considerations for mode locked lasers.
2.2 Master Equation of Mode Locking

The key in the derivation of Haus' master equation for mode locking is in the assumption that the change in the pulse profile per round trip will be small. This allows us to define two separate time scales in which to view the pulse dynamics, one on the scale of the round-trip time, $T_R$, and the other on the timescale of the pulsewidth, $\tau$. We can parameterize these two timescales with the variables $T$ and $t$ respectively and treat the two variables as independent. Using this parameterization the change in pulse envelope in one round trip can be then be approximated by the expression

$$\Delta A(T, t) = A(T, t) - A(T - T_R, t) \approx T_R \frac{\partial A(T, t)}{\partial T}$$ (2.5)

where $A(T, t)$ is the pulse envelope which is normalized so that $|A(t)|^2$ equals instantaneous power, $P(t)$. Since $T$ and $t$ are treated as independent variables, we can equivalently consider the pulse as a pulse spectrum that evolves on the time scale, $T$, hence we can take the Fourier-transform of the pulse via the definitions

$$A(T, t) = \int e^{-i\omega t} A(\omega, T) d\omega$$ (2.6)

$$\tilde{A}(T, t) = \frac{1}{2\pi} \int e^{i\omega t} A(t, T) dt$$ (2.7)

To account for the gain we assume that the gain is not fast enough to respond to the instantaneous profile of the pulse and assume a gain medium with a Lorentzian gain profile where $g(T)$ represents the saturated gain averaged over a round trip.

$$g(\omega, T) = \frac{g(T)}{1 + \left(\frac{\omega}{\Omega_g}\right)^2}$$ (2.8)

Therefore after one round trip, the pulse spectrum becomes

$$\Delta \tilde{A}(\omega, T) = e^{g(\omega, T)} \tilde{A}(\omega, T) \approx g(T) \left[1 - \left(\frac{\omega}{\Omega_g}\right)^2\right] \tilde{A}(\omega, T)$$ (2.9)

where in making the approximation, we have assumed that the pulse spectrum is
narrow compared to the gain bandwidth $\Omega_g$. We can then move into the time domain by making the replacement $i\omega \to \partial / \partial t$.

\[
\Delta A(t, T)_{gain} \approx g(T) \left[ 1 + \frac{1}{\Omega_g^2} \partial^2 \partial t^2 \right] A(t, T) \tag{2.10}
\]

where $g(T)$ is the gain averaged over a round trip.

To add the effect of linear (frequency-independent) loss, we just write

\[
\Delta A(t, T)_{loss} = -l_0 A(t, T) \tag{2.11}
\]

where $l_0$ is the loss per round trip.

### 2.2.1 Saturable Absorber

In order for the system to favor pulse operation, it must be energetically favorable to operate in pulse operation over continuous mode operation. To do this, many lasers, including the Ti:Sapphire, rely on a fast saturable absorber whose losses are inversely proportional to the peak intensity of the pulse. In Ti:Sapphire lasers this absorber is provided by the Kerr-lens effect which we will discuss in the next chapter. To model the fast saturable absorber (i.e. the soft aperture in the case of Ti:Sapphire lasers) we assume that the loss saturates instantaneously with the power

\[
q(t) = q_0 + \frac{1}{1 + |A(t, T)|^2} \tag{2.12}
\]

which in the limit of small saturation, can be expanded to lowest order in $|A(t, T)|^2$ as

\[
\Delta A(t, T)_{sat.loss} = (q_0 - \gamma |A(t, T)|^2) A(t, T) \tag{2.13}
\]

where $\gamma$ is the self amplitude modulation (SAM) coefficient. This saturable absorber promotes mode locking by making the total losses per round trip exceed the gain for the leading and trailing edges of the pulse while yielding a positive net gain for the peak of the pulse (refer to Fig.(2-2)).
Adding all of the effects discussed so far together gives us the simple form of the master equation

$$ T_R \frac{\partial A(t, T)}{\partial T} = (g - l)A(t, T) + \frac{g}{\Omega_g^2} \frac{\partial^2}{\partial t^2} A(t, T) - \gamma |A(t, T)|^2 A(t, T) \quad (2.14) $$

where $g \equiv g(T)$ and $l = l_0 + q_0$. To get the steady state solution, we just set the left hand side to zero and solve for $A(t, T)$. Surprisingly, in this simple form the equation can be solved exactly with the solution

$$ A(t) = A_0 \text{sech}(t/\tau) \quad (2.15) $$

where $1/\tau^2 = \gamma A_0^2 \Omega_g^2 / 2g$ and $l - g = g/\Omega_g^2 \tau^2$.

Looking at the solution for this simple case we note that we should expect shorter pulses for situations with a larger self-amplitude modulation (represented by the coefficient $\gamma$) and gain bandwidth. As for the analytic shape of the pulses it turns out that for real mode-locked lasers, the pulses tend to be very close to that of a
2.2.2 Self-Phase Modulation and Dispersion

Though equation (2.14) can tell us a lot about the shape of the laser pulses and their dependence on the gain and linear loss, it is still not the complete picture. Due to the high peak intensities present in these lasers, the nonlinear self-phase modulation effect (SPM) becomes non-negligible. This becomes more apparent with the revelation that the widest bandwidths produced by Kerr-lens locked lasers is wider than the gain medium itself and hence the pulse shortening cannot be due to the gain bandwidth alone.

To derive the effects of dispersion and SPM, we first decompose the electric field into it’s Fourier components.

\[ E(z, t) = \mathfrak{R} \left[ \frac{1}{2\pi} \int_0^\infty \tilde{E}(\Omega) e^{i(\Omega t - K(\Omega)z)} d\Omega \right] \quad (2.16) \]

In a dispersive medium, \( K(\Omega) \) depends non-trivially on \( \Omega \) with the relation \( K(\Omega) = n(\Omega)\Omega/c_0 \) where \( c_0 \) is the speed of light in vacuum. We then assume that the spectrum is centered about some carrier frequency \( \omega_0 \) and we make the linear transformation \( \omega = \Omega - \omega_0, k(\omega) = K(\Omega) - K(\omega_0) \) and write the electric field as

\[ E(z, t) = \mathfrak{R} \left[ A(z, t)e^{i(\omega_0 t - K(\omega_0)z)} \right] \quad (2.17) \]

where

\[ A(z, t) = \frac{1}{2\pi} \int_0^\infty \tilde{E}(\Omega)e^{i(\omega t - (K(\Omega) - K(\omega_0))z)} d\Omega \equiv \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} \tilde{A}(\omega)e^{i(\omega t - k(\omega)z)} d\omega \quad (2.18) \]

Taking the derivate of this equation with respect to \( z \) then yields

\[ \frac{\partial A(z, t)}{\partial z} = \frac{1}{2\pi} \int_0^\infty -i [K(\Omega) - K(\omega_0)] \tilde{E}(\Omega)e^{i(\omega t - (K(\Omega) - K(\omega_0))z)} d\Omega \quad (2.19) \]

\[ = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} -ik(\omega)\tilde{A}(\omega)e^{i(\omega t - k(\omega)z)} d\omega \quad (2.20) \]
To take into account SPM due to the optical Kerr effect in the laser crystal, we assume that the index of refraction is intensity dependent and can be written in the form

\[ n(\Omega, I) = n(\Omega) + n_2 I \]  

(2.21)

where \( I(z, t) = |A(z, t)|^2 / A_{\text{eff}} \), \( A_{\text{eff}} \) being the laser mode cross sectional area. The expression for the wave vector then becomes

\[ K(\Omega) = n(\Omega) \frac{\Omega}{c_0} + n_2 I \frac{\Omega}{c_0} \]

(2.22)

\[ = n_2 I \frac{\omega}{c_0} + n_2 I \frac{\omega_0}{c_0} + \frac{\partial k(\omega)}{\partial \omega} \left| \omega + \frac{1}{2} \frac{\partial^2 k(\omega)}{\partial \omega^2} \right|_0 \omega^2 + \frac{1}{6} \frac{\partial^3 k(\omega)}{\partial \omega^3} \right|_0 \omega^3 + O(\omega^4) \]

where we define

\[ \frac{1}{v_g} = \frac{\partial k(\omega)}{\partial \omega} \bigg|_0 \equiv \text{Group Velocity}, \]  

(2.23)

\[ D_2 = \frac{1}{2} \frac{\partial^2 k(\omega)}{\partial \omega^2} \bigg|_0 \equiv 2^{\text{nd}} \text{ Order Dispersion}, \]  

(2.24)

\[ D_n = \frac{1}{n!} \frac{\partial^n k(\omega)}{\partial \omega^n} \bigg|_0 \equiv n^{\text{th}} \text{ Order Dispersion} \]  

(2.25)

If we assume a narrow spectrum such that \( \omega << \omega_0 \) then we can ignore the term proportional to \( I \omega \). Fourier transforming back to the time domain using \( \omega^n \leftrightarrow (-i \frac{\partial}{\partial t})^n \)

we obtain

\[ \frac{\partial A(z, t)}{\partial z} = \left[ i \sum_{n=1} D_n \left( -i \frac{\partial}{\partial t} \right)^n + i \frac{\omega_0}{c} n_2 \frac{|A(z, t)|^2}{A_{\text{eff}}} \right] A(z, t) \]  

(2.26)

where \( D_n = \frac{1}{n!} \frac{\partial^n k(\omega)}{\partial \omega^n} \bigg|_{\omega_0} \) is the \( n^{\text{th}} \) order dispersion coefficient. Since the first order term is just the reciprocal of the group velocity, \( v_g = 1 / D_1 \), we can transform to a reference frame traveling at velocity \( v_g \) such that the pulse is stationary using the transform \( t' = t - z / v_g, z = z' \). If we only consider the term in \( D_2 \) we get the famous
nonlinear Schrodinger equation (NLSE)

\[
\frac{\partial A(z,t)}{\partial z} = \left[ -iD_2 \frac{\partial^2}{\partial t^2} + i \frac{\omega_0}{c} n_2 \frac{|A(z,t)|^2}{A_{\text{eff}}} \right] A(z,t)
\]  

(2.27)

The solution of this equation describes the propagation of a soliton, which is a pulse which propagates in a nonlinear dispersive medium without changing its shape. This stability is due to a balance in the SPM and the dispersion where the SPM acts to shorten the pulse by spectrally broadening the spectrum, while the dispersion acts to broaden the pulse by adding a linear chirp to the pulse. So long as neither effect is drastically larger than the other the two competing phenomena balance.

Starting with Eq. 2.26, we can include the effects of dispersion and SPM into the master equation by transforming the longitudinal coordinate \(z\) in equation (2.26) using the transformation \(T = z/v_g \rightarrow \partial/\partial z = TR/L \partial/\partial T\). Therefore, the master equation, including the effects of gain, loss, saturable loss, dispersion, and SPM becomes

\[
TR^2 \frac{\partial A}{\partial T} = (g - l)A + (D_f - iD_2) \frac{\partial^2}{\partial T^2} A + (\gamma + i\delta)|A|^2 A
\]  

(2.28)

where \(D_f = g/\Omega_g^2\) is the effective gain curvature. The Kerr coefficient is \(\delta = (2\pi/\lambda_C)n_2L/A_{\text{eff}}\) where \(\lambda_C\) is the carrier wavelength, \(n_2\) is the nonlinear index in \(\text{cm/W}\), and \(A_{\text{eff}}\) is the effective mode cross-section area in \(\text{cm}\).

Assuming that the recovery time of the gain is slow, we can solve this equation to yield a steady state solution [25].

\[
A(t,T) = A_0 \text{sech}^{(1-i\beta)} \left( \frac{t}{\tau} \right) e^{-i\theta}
\]  

(2.29)

The overall shape of the pulse is the same as in the case where we neglected SPM and dispersion, however by including these two effects, the pulse gains an additional chirp, \(\beta\). If we want transform limited pulses we must then use an external pulse compressor to remove this chirp. In addition to the chirp, we also note that the pulse gains a phase slip per round trip \(\theta\). This phase slip, \(\theta\), leads to a change in relative phase of the carrier wavelength with respect to the phase of the pulse envelope. As a
result of this phase slip, the frequency of the longitudinal modes becomes

\[ \nu_m = \nu_0 + m\Delta \]  

(2.30)

for integer \( m \) where \( \nu_0 < \Delta \). This differs from \( \nu_m = m\Delta \) as suggested by Eq. (2.1) which assumes no dispersion. For pulsewidths much longer than a carrier cycle, this phase slip is not important. However when the pulsewidth approaches the single-cycle limit, a relative phase slip can have large effect on the peak intensity of the pulse. For 5fs pulses, which is common for Kerr-lens mode locked lasers, this phase slip is indeed non-negligible and is usually stabilized using the 1f-2f method [19].

2.3 Dispersion Managed Mode-Locking

Equation (2.26) offers us a lot of insight into the inner workings of a mode-locked laser, however an underlying assumption in its derivation is that the dispersive and nonlinear elements are distributed evenly throughout the length of the laser. Unfortunately, for a real solid state laser such as the Ti:Sapphire, these elements are placed discretely and thus the ordering of the elements of the resonator greatly affects the resulting pulse width and behavior. However, it has been shown that for a resonator with close to overall net zero dispersion, soliton-like mode locking can still be achieved [11]. To take into account this “lumpiness” of the dispersive and nonlinear elements, we must modify the master equation by making the coefficients functions of the longitudinal variable \( z \). The coefficients \( D_2, \delta, g, l, \) and \( q \) are now defined per unit length. The modified equation for mode-locking then reads as

\[
\frac{\partial A(z, t)}{\partial z} = \left[ -iD_2(z) \frac{\partial^2}{\partial t^2} + i\delta(z)|A(z, t)|^2 + g(z) \left( 1 + \frac{1}{\Omega_2^2} \frac{\partial^2}{\partial t^2} \right) - l(z) - q(z) \right] A(z, t)
\]  

(2.31)

In order to have steady-state pulse formation, we first realize that in order to have sufficient peak intensity the pulse must be the shortest, hence chirp-free, inside of the crystal. To do this, the cavity dispersion elements must be positioned on either side
of the laser crystal so that the net dispersion from a round trip in one side of the laser cavity leads to net zero dispersion. So long as the point of net zero trip dispersion occurs inside of the laser crystal, the pulse can have a stable mode lock.

In Fig. (2-3) we illustrate the idea of dispersion managed mode-locking. The behavior of the pulse inside the laser can be understood from simulations by considering that a chirp free pulse starting out inside the crystal is spectrally broadened due to SPM and disperses in time due to the GDD, imparting a linear chirp to the pulse. When the pulse leaves the crystal and travels through sections of negative GDD, the pulse gains a chirp of the opposite sign, canceling out the positive chirp and recompressing the pulse. The output coupler of the laser can then be positioned so that a nearly transform limited pulse is output from the laser. The pulse then propagates back towards the crystal and arrives back at the crystal with zero net chirp and the process then repeats. This temporal and spectral stretching and compressing leads to a steady state solution where the pulsewidth in some parts of the laser can be much larger than the chirp free pulsewidth, as evidenced in Fig. (2-4).

Since the gain medium contributes positive group-delay dispersion (GDD), we must insert passive elements with a net negative dispersion, so that the we have a point of zero net dispersion inside of the crystal. One can achieve negative dispersion by using either gratings, prism pairs, or chirped mirror pairs. Broadband grating losses tend to be too high to make grating pairs practical, while prism pairs are inherently alignment sensitive. On the other hand, high reflectivity chirped mirror pairs, which we will cover shortly, can be engineered for a custom dispersion profile and compared to prism pairs, are relatively insensitive to alignment.

In the next chapter I will introduce the physical setup of the Ti:Sapphire laser as used in the experiments as well as introduce some phenomenon which apply in particular to solid state lasers such as the soft aperture effect and astigmatism compensation.
Figure 2-3: Simplified diagram of a laser resonator with dispersive and nonlinear (SPM) elements. Typically the cavity is balanced so that the overall net chirp added to a pulse via dispersion (GDD) in one trip going from the center to the crystal, reflecting off a mirror and then returning to the center of the crystal is zero.

Figure 2-4: Numerical simulation of master equation of a dispersion managed solid state laser. As evidenced by the plot, the width of the pulse changes periodically as it passes through the laser, leading to a characteristic "breathing" behavior. Particularly, the pulse is most narrow at the middle of the crystal (the positive dispersion region) and at the end mirrors of the laser. Figure taken from [20].
Chapter 3

Ti:Sapphire Laser

In the previous chapter we introduced the phenomenon of mode locking and introduced the Master Equation of mode locking. The crucial component required of mode locked solid state lasers is the addition of a saturable absorber. This can either be a real saturable absorber such as semiconductors or dyes [21] [32] [18], or it can be artificial. In the case of Ti:Sapphire lasers we use Kerr-lens mode locking which relies on the self lensing effect of the optical Kerr effect to generate an artificial absorber. To create such an artificial absorber great care is required in designing and setting up the laser for mode lock operation. We will discuss some of these design and operation issues as well as describe our physical laser design.

3.1 Kerr Lens Mode Locking

To see how Kerr lens mode locking works we first consider that a typical pulse in a solid state laser has a Gaussian profile

\[ I(r) = \frac{2P}{\pi w^2} e^{-2r^2/w^2} \]  

(3.1)
In a Kerr medium of sufficiently high pulse intensities the index of refraction becomes dependent upon the pulse intensity via

$$n = n_0 + n_2 I$$  \hspace{1cm} (3.2)

Therefore due to the Gaussian transverse dependence of the pulse intensity the index of refraction in the Kerr medium will have a transverse spatial dependence. Within the paraxial approximation the index of refraction in the Kerr medium can then be approximated by a parabola

$$n(r) = n'_0 \left(1 - \frac{1}{2} \gamma^2 r^2\right)$$  \hspace{1cm} (3.3)

where

$$n'_0 = n_0 + n_2 \frac{2P}{\pi w^2}, \gamma = \frac{1}{w^2} \sqrt{\frac{8n_2 P}{n'_0 \pi}}$$  \hspace{1cm} (3.4)

If the Kerr medium has a thickness $t$, then the ABCD matrix describing paraxial propagation at normal incidence through the Kerr medium is given by

$$M_k = \begin{pmatrix} \cos \gamma t & \frac{1}{n'_0 \gamma} \sin \gamma t \\ -n'_0 \gamma \sin \gamma t & \cos \gamma t \end{pmatrix}$$  \hspace{1cm} (3.5)

Thus for $t \to 0$ we can approximate the Kerr medium as a thin lens of focal length $f = 1/n'_0 \gamma^2 t$.

Since the optical Kerr effect is very fast with typical response times being less than 5 fs, it produces a near-instantaneous absorber which follows the pulse, and with very high peak intensities we can obtain significant self-lensing effects. Since mode lock pulses have peak intensities over a million times greater than free running intensities, we in effect have a cavity element which is only present while the laser is mode locked.

To take advantage of the intensity dependence we design the laser cavity such that the addition of the Kerr lens induces a change in the size of the beam waist in the laser crystal. Specifically, we design it so that the addition of a Kerr lens reduces
Figure 3-1: Artificial saturable absorber using Kerr lensing. The laser cavity is designed so that the addition of a Kerr lens reduces the size of the beam waist in the Kerr medium. The mode locked beam then has a greater overlap with the pump beam and experiences a higher gain.

the beam waist size of the TEM$_{00}$ in the laser crystal. Then if we focus the pump laser such that its waist in the crystal matches the TEM$_{00}$ for the mode locked laser then the pump laser will have a better overlap with the mode locked TEM$_{00}$ mode than the free running TEM$_{00}$ mode and thus have a higher gain. We illustrate this graphically in Fig. (3-1).

3.1.1 Cavity Stability

In order to take advantage of the artificial saturable absorber action we need to maximize the contrast between the mode locked and free running beam waists in the laser crystal. To do this we use an asymmetric linear cavity design as pictured in Fig. (3-3). Since the Kerr effect is strongly dependent on the peak intensity and hence the beam size in the crystal, it is necessary to study the stability of the laser cavity using ray transfer matrix methods which we have performed for the laser in our experiment in Fig. (3-2). If we refer to the figure, we see that due to the asymmetry of the linear cavity, the cavity possesses two stability regions. Since the beam size goes to zero at the boundaries of the stability region, we get the strongest nonlinearity and hence largest contrast in beam size for the continuous wave versus mode locked beam.
Figure 3-2: Stability curves for laser used in this work. Due to astigmatism, the beam waists of the tangential (blue) and sagittal planes (green) are different. The laser is characterized by two stability regions. In order to maximize the Kerr effect, one typically operates the laser in one of the two regions denoted by the red rectangles. In this work, we exclusively operated the laser in the left region.

size. Therefore we typically mode lock the laser at the edges of the stability regions denoted by the red rectangles in Fig. (3-2). For our specific experiments we operated the laser in the region of the left red rectangle for sake of consistency.

3.1.2 Astigmatism Compensation

Due the presence of Brewster plates and curved mirrors at non-normal incidence in the laser cavity, there is significant astigmatism present in the beam. Therefore if we perform an ABCD stability calculation of the laser cavity, it is necessary to do so for both the tangential and sagittal planes of the laser cavity. In general, the stability regions for the two orthogonal planes do no coincide at all points along the stability
Fortunately, it is possible to compensate for the astigmatism by adjusting the angle of incidence of the beam on the curved mirrors [23]. To find this angle, we consider that the focal length of the curved mirrors at an angle of incidence \( \theta \) is given by

\[
f_s = f / \cos \theta \tag{3.6}
\]
\[
f_t = f \cdot \cos \theta \tag{3.7}
\]

On entering the Brewster plate, the beam waists in either plane become

\[
w_s = w_s
\]
\[
w_t = w_t \cos \theta_r \cos \theta_B = w_t \sin \theta_B = n w_t \tag{3.9}
\]

where \( \theta_B = \arctan n \) is the Brewster angle and \( \theta_r = 90^\circ - \theta_B \) is the angle of refraction. Then for a plate of thickness \( t \) the geometrical pathlength in the plate is

\[
\chi = \frac{t}{\cos \theta_r} = \frac{t \sqrt{1 + n^2}}{n} \tag{3.10}
\]

Using the propagation equations for Gaussian beams we compute the beam size in the two orthogonal planes after propagating a distance of \( \chi \)

\[
w_s = w_0 \sqrt{1 + \left( \frac{x}{n \pi w_0^2} \right)^2} = w_0 \sqrt{1 + \left( \frac{\lambda \chi}{n \pi w_0^2} \right)^2} \tag{3.11}
\]
\[
w_t = n w_0 \sqrt{1 + \left( \frac{x}{n^3 \pi w_0^2} \right)^2} = n w_0 \sqrt{1 + \left( \frac{\lambda \chi}{n \pi w_0^2} \right)^2} \tag{3.12}
\]

Thus the Brewster plate is equivalent to propagation in free space of distances equal to

\[
d_s = d \frac{\sqrt{1 + n^2}}{n^2} \tag{3.13}
\]
\[
d_t = d \frac{\sqrt{1 + n^2}}{n^4} \tag{3.14}
\]
The beam after passing through the Brewster plate will then reflect again off of a mirror of radius $R$ at angle $\theta$ and will be collimated without astigmatism if the difference between the two distances $d_s$ and $d_t$ compensate for the difference in focal distances $f_s$ and $f_t$ which gives the condition

$$d_s - 2f_s = d_t - 2f_t$$  \hspace{1cm} (3.15)

Substituting $f = R/2$ we obtain

$$R\sin \theta \tan \theta = Nt, \text{ where } N = \sqrt{n^2 + \frac{n^2 - 1}{n^4}}$$  \hspace{1cm} (3.16)

This yields the quadratic equation

$$\cos^2 \theta + \frac{Nt}{R} \cos \theta - 1 = 0$$  \hspace{1cm} (3.17)

Solving for $\cos \theta$ and throwing away the negative root we obtain the solution

$$\theta = \arccos \left[ \sqrt{1 + \left( \frac{Nt}{2R} \right)^2} - \frac{Nt}{2R} \right]$$  \hspace{1cm} (3.18)

Therefore if we set the curved mirrors at this angle, we can completely compensate for the dispersion in the cavity.

### 3.2 Experimental Laser

The basic design of the laser we used which satisfies these requirements for Kerr lens mode locking and which we used in our experiments consists of the cavity in Fig. (3-3). The gain medium is a 2 mm long Brewster-cut Titanium-sapphire crystal which lies between two curved mirrors with a radius of curvature of 7.5 cm placed at the astigmatism compensating angle. We pump the crystal using a commercial 532 nm 6 Watt single-mode frequency doubled Nd:YAG laser and focus the incoming pump light using a lens so that the pump beam is roughly 10% smaller than the waist of
Figure 3-3: Laser used in InvOC studies. Numbered mirrors correspond to DCM mirrors where even numbered mirrors are of one type of the DCM pair while the odd numbered mirrors are of the complementary type hence 1 and 2 constitute a DCM pair as well the pair of 3 and 4 and the pair of 5 and 6. For the DCM11 tests, we used a custom made chirped mirror produced in our group. The dispersion of this mirror is engineered to correspond to a positive amount of fused silica. In order to get zero net cavity dispersion, we insert small fused silica wedges (labeled "compensating wedges" in the figure) to adjust the net cavity dispersion to zero.

the TEM\textsubscript{00} mode of the cavity to provide the needed soft aperture. All of the mirrors inside the cavity except for the end mirrors are type I (blue) or type II (green) double-chirped mirrors [DCM] which provide the needed dispersion compensation. For the non-output end mirror we used either a broadband silver mirror for the broadband type DCM (which we refer to as DCM7) laser or a specially designed highly reflective positive dispersion chirped mirror for the narrowband DCM (which we refer to as DCM11) laser. To tune the net dispersion of the cavity to zero, we used pieces of Barium Fluoride or fused silica glass inserted either into the long or short arm of the cavity. The optomechanics and other mechanical components of the laser are from a commercial 85 MHz ultra-broadband Ti:sapphire laser (IdestaQE Octavius-85M). Even though the mechanics are commercial, all the optics used in the experiment were designed in our group.

In the next chapter I will introduce the chirped mirror designs which we use to control the dispersion in dispersion-managed mode-locked Ti:sapphire lasers. Using
this technology as a starting point I will then introduce our novel output coupler design which takes advantage of double-chirped mirror technology to create output couplers which reduce the gain filtering term in Eq.2.31.
Chapter 4

Mirrors and Output Couplers

As we just saw in our derivation of the mode-locking equation, adequate dispersion management inside the laser is key to maintaining a steady state pulsed operation. Therefore, since the gain crystal adds positive dispersion to the total round trip dispersion, we must add cavity elements which possess negative dispersion to cancel out the positive dispersion of the gain crystal to give us an overall net zero dispersion. Grating pairs and prism pairs both allow negative dispersion but grating pairs suffer from high losses which makes them unsuitable for use inside of a laser cavity and prism pairs are too sensitive to alignment. In addition both grating and prism pairs suffer from higher order dispersion making higher order dispersion compensation difficult. If the desired reflectivity bandwidth of the mirror is smaller than the typical reflectivity bandwidth of a normal Bragg-mirror ($\Delta f = 0.23f_c$ for a quarter wave stack of Silicon Dioxide and Titanium Dioxide with $n_{SiO_2} = 1.48$ and $n_{TiO_2} = 2.4$) then a multi-cavity filter design consisting of Bragg stacks with an additional dispersion-control layer can be used to approximate the desired phase and amplitude properties [14]. However for fractional bandwidths over $\Delta f/f_c = 0.4$ such a simple modified Bragg stack will not suffice. To get around the problems posed by prisms and gratings Robert Szipoecs and Ferenc Krausz [31] proposed a new type of element known as chirped mirrors.
4.1 Chirped Mirrors

The basic design of a chirped mirror consists of Bragg stack of an alternating high and low index quarter wavelength thick layers which result in strong Bragg-reflection. The Bragg wavelength is chirped so that different wavelengths penetrate at different depths into the layer stack before reflecting back out of the stack. This variable distance results in a wavelength dependent group delay. Thus, with the proper choice of layers, one could conceivably engineer a chirped mirror to give an arbitrary dispersion profile.

Unfortunately when trying to compensate for dispersion for large fractional bandwidths up to $\Delta f/f_0 = 0.4$ the mirror group delay exhibits large fluctuations which prove troublesome for mode-locking. The origin of these fluctuations comes from the impedance mismatch between reflections at the air-mirror surface and reflections from inside the layer stack as well as impedance mismatch between different reflections from different depths inside the stack. These fluctuations can be reduced by gradually “turning on” the chirped layer by gradually increasing the thickness of the alternating layers of the layered structure resulting in a “double-chirped” mirror (chirped once for varying the Bragg wavelength and then again for impedance matching). However, in order to eliminate the reflection at the air-mirror boundary, a high quality AR-coating must be applied with an amplitude reflection of $r < 0.01$. Unfortunately, such broadband AR-coatings do not exist.

To overcome the unrealistic requirements imposed on the AR-coating and achieve adequate dispersion control and wide-reflectivity bandwidths, Kättner et. al proposed the use of double-chirped mirror pairs. To understand how double-chirped mirror pairs work we note that the AR coating plus Bragg stack is similar to a Gires-Tournois interferometer in that the reflections add up coherently when multiple reflections occur inside the laser over one round trip. After repeated reflections off of the chirped mirrors pre- and post pulses form in the cavity if the mode locking is not strong enough to suppress them. To minimize this, it is sufficient to add a $\pi$ phase shift between the pre- and post pulses by using a pair of a DCMs in which an additional phase shift of $\pi$ between the AR coating and the back mirror is added to one of the
mirrors. This additional phase shift then acts to cancel the phase shift between the pre- and post pulses leading to a single post pulse with reduced GDD fluctuations. By taking all the parameters into account: Bragg chirping, impedance matching, AR-coating, and phase shift layer on one of the mirrors in the pair, the parameters can be re-optimized for the mirror pair resulting in greatly reduced group delay fluctuations for the pair leading to a cancellation in the size of the oscillations up to $r^2$ in the reflection amplitude of the AR-coating.

### 4.2 DCM Mirror Designs

The mirrors that we used for our experiments were developed using an algorithm developed in our group by Jonathan Birge [7] which uses a transfer-matrix formalism to do the necessary layer optimization for the chirped mirror stack. When choosing the dispersion and reflective bandwidth of such mirrors, it’s important not only to choose as wide a bandwidth as possible, but it is also important to choose a design such that the ratio between the 3rd order and 2nd order dispersion reasonably matches that of Titanium-Sapphire plus any glass we add to the cavity. In addition, we typically choose the mirror dispersion such that the cavity will have an overall net negative dispersion. Then we can fine tune the dispersion of the cavity to zero by inserting wedged glass into the laser beam. With these thoughts in mind we used two different mirror designs for testing the design of the InvGOC output couplers. One type, which we will refer to as DCM7’s, is a relatively broadband mirror with a reflectivity bandwidth from 600nm to well above 1100nm. The second design, which we will refer to as the DCM11 design, is a relatively narrow bandwidth mirror with a reflectivity bandwidth of 600nm to 1025nm. However the group delay of the DCM11’s is only smooth in a bandwidth of 600nm to 950nm whereas the DCM7 group delay is smooth from 600-1100nm. Though the DCM11 design has a shorter usable bandwidth, the group delay has smaller ripples, as evident from Fig. (4-1) and the dispersion is designed to compensate that of fused silica, a relatively cheap and inexpensive material. The DCM7 design on the other hand has larger ripples and has a lower
Figure 4-1: Reflection coefficient (black - left axis) and group delay (red, green, and blue - right axis) for DCM11 (top) and DCM7 (bottom) type double-chirped mirrors. The blue and green curves correspond to each mirror in the DCM pair. When the two mirrors are used together as a pair, the total group delay is given by the sum of the group delays of the two different mirrors. The red curve shows the average between these two group delays. From the group delay curves the reduction in group delay ripples is clearly evident. Also note how the DCM7 mirror features a much wider bandwidth, extending from 600 all the way to 1100 nm. The DCM11 type mirror sacrifices bandwidth in order to have reduced ripples.
Figure 4-2: Group delay dispersion of DCM11 type (top) and DCM7 type (bottom) mirrors. Also plotted are the negative of the GDD of 1.84 mm of fused silica (top) and 1.62 mm of Barium Fluoride (bottom). Since the DCM7 mirror has predominately second order dispersion, Barium Fluoride is more suitable for pairing with DCM7s due to Barium Fluoride’s low third order to second order dispersion ratio. DCM11 mirrors on the other hand were designed to be used with fused silica due to fused silica’s lower cost and being easier to polish than Barium Fluoride. Notice the particularly good overlap in the GDD in DCM11 in the 700-950 nm range.
amount of third order dispersion and hence these mirrors must be compensated with a material such as Barium Fluoride, which has a lower third order to second order dispersion ratio, but which is a more expensive and harder to polish material than fused silica (see Fig. (4-2)). In addition to the DCM7 and DCM11 designs, we will also use another design known as DCM8, which is similar to the design of the DCM7's, but with a higher third order dispersion.

4.3 Inverse-Gain Output Couplers

From Eq. (2.15) we saw that the pulsewidth, ignoring the effects of SPM and GDD goes as

\[ \tau \propto \frac{1}{\Omega_g} \] (4.1)

and since \(1/\Omega_g\) is proportional to the strength of the gain-filtering this suggests that if we somehow reduce the magnitude \(1/\Omega_g\) and hence, reduce the effect of the gain filtering effect, we can expect a decrease in pulsewidth. To better understand where this gain-filtering term comes from, we first consider the gain profile in the frequency domain as shown in Fig. (4-3) (a), which depicts the case of a Gaussian gain profile depicted with the linear loss of the cavity, depicted as a horizontal red line. Only frequency components such that the net gain is positive will experience gain. Therefore for a Gaussian shaped pulse, the wings of the pulse spectrum which lie in the negative gain region will be suppressed. Typically to overcome this gain filtering, one operates the laser with high pump power and low output coupling, which has the effect of lowering the horizontal line and pushing the gain curve upwards giving us a broader range of positive net gain. However, not only does this demand high pump power and sacrifice output power, but the self-amplitude modulation [SAM] must be maximized as well. Since SAM is a maximum in regions where the the cavity is near the edge of its stability region, this makes cavity alignment critical to achieving the widest spectrum. In addition, traditional octave spanning lasers suffer from a high mode-locking threshold, low laser efficiency, less robust operation, and greatly reduced beam quality.
Positive net gain

Gain

Negative net gain

Loss

Figure 4-3: Cartoon depicting the general premise behind the inverse-gain output coupler [InvGOC]. In (a) we see the case of a normal flat output coupler. Since the loss is frequency independent, spectral components far from the center of the gain curve see a negative net gain and are thus suppressed during mode-locking. However in (b) we see that if we engineer the loss of the output coupler to follow that of the gain, then the net gain will be approximately frequency independent and have slightly positive net gain over the entire gain curve.

To overcome this filtering problem, Chen et al [10] proposed using a type of output coupler where the output transmission curve is of the same shape as that of the gain profile. For such a system, the loss profile will have the same shape as the gain, and therefore the net gain will be positive over the entire bandwidth of the gain medium. For such a system, one would expect that it would be easier to obtain octave spanning spectra, due to the absence of the gain filtering effect in the wings of the spectrum. In particular, it should be possible to operate the laser with a higher output coupling and with the laser operating more inside the CW mode stability region, which should improve laser stability as well as beam quality.

Armed with the double-chirped mirror technology, our group designed such an output coupler by first measuring the fluorescence spectrum of the Ti:sapphire crystal and using a rescaled version as the design goal in the mirror coating optimization algorithm as developed in [8]. As an additional input to the design program, we chose the group delay design goal to match that of either fused silica or a negative amount of Barium Fluoride so that we can balance the cavity dispersion by either removing or adding glass of the appropriate type. In Fig. (4-4) we show one particular design which has a peak output coupling of 8%. In total we manufactured 4 different
Figure 4-4: Reflectivity of output coupler (black) and GDD (solid blue) of an 8% inverse-gain output coupler. The shape of the reflection curve is shaped so that the transmission curve matches the shape of gain profile of Titanium Sapphire. In addition to optimizing for a specific gain profile the design also attempts to give a dispersion profile which matches that of fused silica so that the dispersion added by the output coupler can be easily compensated by removing or adding an equivalent amount of fused silica to the laser cavity. In this particular case for the 8% InvGOC, the dispersion profile follows the shape of the GDD of 1 mm of fused silica (blue dotted line).

types of these “inverse-gain output couplers” [InvGOC] with transmittance values of 1, 4, 8, and 10%. A major part of this work will be devoted to testing the performance of this new type of output coupler and compare the results with the traditional flat output coupler design.
Chapter 5

Experimental Evaluation of Inverse Gain Output Couplers

In the previous chapter we introduced the design of the inverse gain output couplers [InvGOC]. Due to their unique shape we expect gain filtering to be reduced significantly leading to improved laser characteristics such as ease of mode-locking, beam quality, and laser efficiency. To test this hypothesis, we measured various properties of the laser such as spectral width, pulse energy, and the geometric beam profile of various inverse-gain output couplers. I compare the results of the various output couplers with each other as well as compare them with traditional flat output couplers to determine if the InvGOC’s do indeed improve laser performance.

5.1 Inverse Gain versus Flat Output Coupler

As an initial test of the InvGOC’s, I first compared the performance of one of the 10% InvOCs to that of a commercially available 8% flat output coupler from LayerTech GmbH. Since the 10% InvGOC has the dispersion equivalent of roughly 1.4mm of fused silica I chose to compare them using DCM11’s as cavity mirrors due to the DCM11’s having a dispersion profile equivalent to a negative amount of fused silica within the reflectivity bandwidth of the mirrors. Since the flat output coupler has roughly zero dispersion over the entire bandwidth then to switch between the InvGOC
to the flat output coupler, we need to add roughly 1.4mm more fused silica to the cavity to balance the dispersion as well readjust the alignment to account for the displacement caused by placing a Brewster plate in the path of the beam.

Because of the reduced gain filtering, we should expect that a smaller non-linearity and hence weaker Kerr-lensing is required to generate a broadband spectrum. As a result we should expect improved spatial beam quality for the InvGOC. Therefore as a starting test we used a CCD camera along with optical bandpass filters to take intensity profile images of the beam of the mode locked laser and compare the output beams for the flat OC versus InvGOC cases (see Fig. (5-2)). Right away we notice that the InvGOC has a nice Gaussian center spot for all four of the measured wavelengths, whereas the flat output coupler shows irregular structure in the 700 and 900 nm photos with a central spot deviating from that of a Gaussian. This supports the notion that a reduced nonlinearity leads to an improvement in beam quality of the laser output.

In addition to improved beam quality, we suspect that the InvGOC will improve
the power efficiency of the laser which we test by measuring the laser characteristics as we vary the 532nm pump power. To measure this relationship, we adjust the curved mirror separation, crystal position, and alignment and try to optimize for spectral width and output power. Then starting at the lowest pump power at which we could maintain a stable mode lock we measured the spectrum using an Ando Electronics broadband spectrum analyzer and recorded the output mode-locked power. We then repeat the measurements at different pump powers, increasing the pump power by 0.1W each time until continuous-wave (CW) breakthrough occurs, which is characterized by a narrow frequency spike centered at the carrier frequency. Since SPM is power dependent its effect, and hence the SPM phase, increases with increasing cavity power. In order to cancel out this increasing phase and maintain short pulses, we must gradually increase the amount of positive dispersion in the cavity. We accomplish this by pushing a wedged piece of fused silica glass further into the beam, increasing the optical pathlength in the glass until the spectrum is as broad as possible. We then used measured spectral information and output power as well as measured transmission curves for the InvGOC’s to derive the intracavity energy and

Figure 5-2: Beam Profiles of a DCM11 laser with a flat output coupler (flat 10%) (top), and one with a 10% InvGOC (bottom) at various wavelengths.
Fourier-limited output pulse width.

We then repeat the same measurements for the flat 8% but keep the curved mirror separation the same as the InvGOC so as to leave the beam size in the crystal the same and hence the SPM roughly the same. However we slightly adjust the crystal position due to the change in relative position of the temporal focus with respect to the spatial focus caused by changing the group delay associated with changing the output coupler. Lastly, in order to calculate the intracavity energy for the flat 8% output coupler we just assumed a spectrally flat constant transmission of 8% which is a reasonable approximation within the reflectivity bandwidth of the output coupler.

In Fig. (5-3) we compare the results of the calculated intracavity energy, Fourier-limited output pulse width, and output power vs pump power for the two output couplers. In particular we notice that the output power and pulse energy is much larger than that of the flat output coupler. However the pulsewidth is almost 2 fs shorter for the flat output coupler at all pump powers. The shorter pulsewidth but lower power suggests that we are more efficiently generating a broadband spectrum by widening the spectrum to the edges of the gain bandwidth. Since the average output coupling (averaged over the spectrum) averages around 7.5% for the InvGOC 10% output coupler, then we know that the increased power is not simply due to a higher output coupling percentage. Furthermore, if we examine the curves in Fig. (5-3) we note that the pulsewidth roughly goes as $1/W$ where $W$ is the pulse energy, which is characteristic of soliton-like mode locking. In particular, the curve for the InvGOC is shifted much farther from the origin than the flat curve. For soliton mode locking with strictly 2nd order dispersion, the pulsewidth, energy, and cavity dispersion are related as $\tau W \propto D_2$ where $D_2$ is the second order dispersion. Since a dispersion-managed mode-locked laser operates in a soliton-like region [11], it is reasonable to suppose that the energy-pulsewidth curve is shifted farther out for the InvGOC due to insufficient dispersion compensation in 2nd and/or 3rd order.

If we compare the spectra of the two output couplers side-by-side, we indeed notice a wider logscale spectrum for the flat 8% with light at 600nm only 20 dB below the peak at 850nm whereas for the InvGOC, light below 625nm is already more than 50
Figure 5-3: (a) Intracavity energy vs pump power, (b) output pulsewidth vs pump power, (c) output power vs pump power, and (d) output pulsewidth vs intracavity energy for InvGOC 10% (red crosses) and flat 8% (blue boxes).
dB down. However, if we examine Fig. (5-1) we see that below about 650 nm, the transmission of the flat output coupler is greater than 15%, while the transmission of the InvGOC is less than 5%. Therefore the increased spectrum is most likely due to the greater transmission of the flat output coupler in that spectral range. In the IR we see a similar situation, with light at 1100 nm only 30 dB down from the light at 850 nm for the flat versus 50 dB for the InvGOC output coupler, which is again most likely due to the fact that the transmission of the InvGOC between 900 - 1100 nm is less than 5%, while that of the flat is 8%. Overall with the exception of the large peak at 650 nm which is likely due to it lying just outside the bandwidth of the flat output coupler, the spectrum for the flat 8% is more desirable due to it’s larger bandwidth and smoother ripples in the region between 700-900 nm.

Finally to test the dependence of the laser operation on output coupling percentage, we measured the pump dependence of the 4, 8, and 10% InvGOC’s. Again, to
Figure 5-5: (a) Intracavity energy vs pump power, (b) output pulsewidth vs pump power, (c) output power vs pump power, and (d) output pulsewidth vs intracavity energy for InvGOC for 4% (red), 8% (blue), and 10% (green) output coupling.

get a fair comparision, we keep the curved mirror separation the same for each output coupler and only tweak the crystal position so as to account for the change in relative position of the temporal and spatial focus caused by the different group delays for each output coupler. Comparing results in Fig. (5-5) we see that the output power scales linearly with output coupling while the output pulsewidth still stays well below 10fs, and in Fig. (5-6) we note little perceptable change of the laser spectrum for increasing output coupling percentages. This suggests that it may be possible to go to much higher output coupling and obtain close to 500mW of mode-locked sub-10fs pulses.
Figure 5-6: Output spectrum of InvGOC for 4% (red), 8% (blue), and 10% (green) output coupling.
5.2 Dual Output Coupler Configuration

One of the goals of the inverse-gain output couplers is to obtain high output power sub-10 femtosecond pulses without significantly sacrificing spectral bandwidth. Since the output power of the 4, 8, and 10% InvGOC's scaled linearly with increasing transmittance it should be possible to increase the transmittance further to obtain greater output power to fulfill this goal. We then simulate higher output coupling percentages by putting two output couplers at either end of the laser with the assumption that the behavior of the laser will be very close to that of a laser with a single output coupler with a transmittance equal to the sum of the transmittance of the two individual output couplers.

5.2.1 DCM11 Laser

We first installed 4% InvGOC's on either end of a DCM11 laser and measured the spectrum output from either end which we plot in Fig. (5-7). From the figure we see that the two spectra are nearly identical with 95mW mode-locked power coming out of either end with 3W pump power giving us a total of 190mW mode-locked power. We then measured the pump power dependence of the two 4% InvGOC output laser and compared the results with the 8% InvGOC using the same methods as stated previously. From Fig. (5-8) we see that the total output power out of both ends of the dual 4% InvGOC laser is roughly the same of that of the 8% output coupler (about 200mW) and the pulsewidths are very similar. The only noticable discrepancies we find are in the internal energy (Fig.(5-8 (a)) and the pulsewidth vs intra-cavity power curve (Fig. (5-8) (b)). However we must note that the group delay dispersion (GDD) of the 4% InvGOC is close to that of -1.1mm of BaF₂ while the GDD of the 8% InvGOC is close to that of 1mm of fused silica. Therefore since the ratio of third order to second order dispersion is higher for fused silica, the third order dispersion compensation for the dual 4% laser will differ from that of the single 8%. Specifically since the intra-cavity energy versus pulsewidth curve is shifted closer to the origin for the dual 4%, this suggests that the overall dispersion compensation (including second
Figure 5-7: Spectrum out of either arm of a DCM11 laser with a two InvGOC 4% output couplers.
Figure 5-8: Operating parameters for DCM11 based laser with 10% and 8% (red pluses), 10% and 4% (blue squares), 4% and 4% (green triangles), and 8% (black diamonds) InvGOC's. The output power is the sum of the power out of the two outputs.

We then measure the pump dependence for two more configurations: 4 and 10% and a 8 and 10% which we show in Fig. (5-8). We see that for all the configurations, the output pulsewidth remains below 9fs. In particular we note that the 8%, two 10 and 4%, and 10 and 8% pulsewidth vs energy curve lie very close to each other, suggesting that the overall dispersion compensation is roughly the same for the three cases. Encouragingly, we also obtain 450mW of sub-10fs pulses for the 10 and 8% configuration.
Figure 5-9: Operating parameters for DCM7 based laser with a single DCM8 pair for dual 8% InvGOC (red pluses), dual 10% InvGOC (blue squares), 8% flat OC (green triangles), 10% flat OC (black diamonds), and 8 and 10% flat OC’s (pink circles). The output power is the sum of the power out of the two outputs.

### 5.2.2 DCM7 Laser

Since the DCM11 bandwidth is much narrower than then DCM7 bandwidth we repeated the dual output coupler measurement in a laser using DCM7 mirrors. However as we mentioned previously, the 8 and 10% InvGOC possess greater third order dispersion. Therefore to try and counteract this, we substituted a pair of DCM7’s in the long arm of the cavity with a pair of DCM8’s. The reflection bandwidth of the DCM8’s is roughly the same as the DCM7’s with the only major difference being a slightly greater negative third order dispersion which should help cancel out the third order dispersion for the InvOCs. We then measure the pump dependence for the configurations: two 8% InvGOC’s, two 10% InvGOC’s, a 8% flat OC, a 10% flat OC,
Figure 5-10: Spectra for DCM7 lasers with the following configurations: one 8% InvGOC at either end of the linear laser cavity (red), a single 8% flat at one end of the cavity and a silver mirror at the other end (blue), and an 8% flat at one end of the laser cavity with a 10% flat at the other (green). The spectra are measured only from one output of the laser. For the configuration with a 8% flat at one end and a 10% flat at the other the spectrum is measured out of the end with the 8% OC.

and a 10 and 8% flat OC which we show in Fig. (5-9). For all of the configurations we note that we successfully generate sub-8 fs pulses. We also note a very high output coupling of about 600mW for the dual 10%. However, both the dual 10% and the dual 8% have the greatest pulsewidths of all the configurations. In addition the output pulsewidth versus intra-cavity energy curves are further shifted out from the origin for the InvGOC’s suggesting inadequate dispersion compensation for the dual InvGOC’s. Since the flat OC’s possess negligible GDD this suggests that the dispersion of the InvGOC’s is not being adequately compensated by the mirrors.

Fig. (5-10) compares the spectra of the dual 8%, 8% flat, and 8 and 10% flat configurations. We notice that even though we measured the spectrum out of the 8%
for both the single 8% and 10 and 8% configuration, the two configurations have very
different spectra with the 10 and 8% having significant spectra out to 1150 at just
25dB below the main peak (ignoring the spike near 700nm). Since the 10 and 8% dual
output laser is obtained from the single flat 8% laser by replacing a silver end mirror
with a 10% flat output coupler, the overall dispersion in the two lasers are roughly
the same since the 10 and 8% flat output couplers have negligible dispersion. Thus
we can conclude that dispersion cannot be the only contributor to the differences
between the two laser configurations.

5.2.3 DCM7 plus DCM11

Since our previous results suggest a significant dependence on dispersion compensa-
tion, as a final test we decided to replace the DCM8 pair in the long arm of the laser
cavity with a pair of DCM11's and measure the spectrum with a single flat 8% OC.
Since Ti:sapphire has a higher third order dispersion than Barium Flouride, then the
DCM11 pair might be able to better compensate for the third order dispersion. In
Fig. (5-11) we see that for this configuration we get a fairly flat spectrum from 650-
900nm with an absence of a sharp peak at 650nm and a gentle roll-off past 900nm.
Thus we conclude that with careful dispersion compensation very flat, broad spectra
can be possible.

5.3 Higher Transmission Output Couplers

The results in the previous section suggest that by manufacturing output couplers
with a higher transmittance, it should be possible to obtain sub-10 femtosecond pulses
with up to 0.5 W of output power. Therefore the next logical step is to manufacture
output couplers with a greater transmittance. Specifically we designed inverse-gain
output couplers with 15, 20, 25, and 30% output transmission.

In addition as we noted in our previous results comparing output couplers that
direct comparison of output couplers is impeded by the difference in dispersion for
each output coupler. Specifically the 8 and 10% inverse-gain output couplers possess
A pair of DCM11 mirrors is used in the long arm to get better dispersion compensation. For this configuration we were able to get 210 mW of 4.84 fs (Fourier-limited) pulses with a pump power of 3.5 W.

considerable third order dispersion due to the design choice of making their dispersion match that of roughly 1.0 and 1.4 mm respectively of fused silica. To obtain better results with the DCM7 laser, which has relatively low third order dispersion, it is then desirable to have output couplers with low third order dispersion to match that of the DCM7 design. For this purpose, we chose to design our new output couplers so that their dispersion closely matches that of 1.1 mm BaF2.

Furthermore in our previous results in which we compared commercially available flat output couplers with the InvGOC’s, we noted a significant difference in the reflection window of the commercial flat OC’s versus the InvGOC’s. For a fair comparison between flat and inverse-gain output couplers, we then designed flat output couplers with a near identical reflection window and dispersion curve to that of the inverse-gain output couplers (Fig. (5-12)) with transmissions of 10, 15, and 20%.

Since the dispersion of the output couplers are nearly the same with a dispersion equivalent to roughly 1.1 mm of BaF2, we tested the new output couplers in a DCM7
Table 5.1: Parameters corresponding to spectra in Table (5.1) for the high output inverse gain output couplers.

<table>
<thead>
<tr>
<th>Type</th>
<th>15%</th>
<th>20%</th>
<th>25%</th>
<th>30%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pump Power (W)</td>
<td>3.5</td>
<td>4.7</td>
<td>4.6</td>
<td>5.7</td>
</tr>
<tr>
<td>Spectrum FWHM (nm)</td>
<td>106.1</td>
<td>120.4</td>
<td>113.5</td>
<td>146.7</td>
</tr>
<tr>
<td>Output Power (mW)</td>
<td>231</td>
<td>447</td>
<td>481</td>
<td>344</td>
</tr>
<tr>
<td>Intra-cavity Pulse Energy (nJ)</td>
<td>24.00</td>
<td>36.26</td>
<td>29.77</td>
<td>21.04</td>
</tr>
<tr>
<td>Effective Output Coupling</td>
<td>11.32%</td>
<td>14.50%</td>
<td>19.01%</td>
<td>19.24%</td>
</tr>
<tr>
<td>Output Pulsewidth (fs)</td>
<td>6.21</td>
<td>5.88</td>
<td>6.73</td>
<td>4.46</td>
</tr>
</tbody>
</table>

laser with a silver mirror in the long arm and the output coupler placed in the short arm. To change output couplers all that is required is a realignment of the short arm and a slight increase or decrease of the glass wedge dispersion in the long arm to account for the minor dispersion differences between output couplers. By choosing to match the dispersion and reflection window as closely as possible, it should be possible to operate the laser in roughly the same operating point by keeping the crystal and curved mirror separation the same for each output coupler.

5.3.1 Initial Tests

As an initial test, we attempted to obtain the broadest spectrum as possible out of each output coupler while taking care to maximize the output power. In order to do this we adjusted the crystal position, curved mirror separation, pump power, and alignment so as to maximize the spectral width and output power. We increased the pump power as much as possible so as to maximize the power and spectral width without obtaining continuous-wave breakthrough. The best spectra are shown in Fig. (5-13) with corresponding operating parameters in Table (5.1) for the InvGOC's and Table (5.2) for the flat OC's.

From Fig. (5-13) we see that the 30% InvGOC has an extremely broadband spectrum with spectrum at 1150 nm only about 10 dB down from the maximum. Unfortunately for the 30% it is extremely difficult to mode lock the laser and the mode locking range (in terms of regions in which the laser will mode lock for a given
Table 5.2: Parameters corresponding to spectra in Table (5.2) for the high output flat output couplers.

<table>
<thead>
<tr>
<th>Type</th>
<th>10%</th>
<th>15%</th>
<th>20%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pump Power (W)</td>
<td>4.0</td>
<td>4.6</td>
<td>4.5</td>
</tr>
<tr>
<td>Spectrum FWHM (nm)</td>
<td>151.7</td>
<td>196.5</td>
<td>123.8</td>
</tr>
<tr>
<td>Output Power (mW)</td>
<td>325</td>
<td>330</td>
<td>465</td>
</tr>
<tr>
<td>Intra-cavity Pulse Energy (nJ)</td>
<td>38.00</td>
<td>25.54</td>
<td>27.71</td>
</tr>
<tr>
<td>Effective Output Coupling</td>
<td>10.06%</td>
<td>15.20%</td>
<td>19.74%</td>
</tr>
<tr>
<td>Output Pulsewidth (fs)</td>
<td>5.37</td>
<td>4.35</td>
<td>6.35</td>
</tr>
</tbody>
</table>

curved mirror separation and crystal position) is extremely small. Increasing the pump power has no affect. In addition, we note that the output power of the 30% InvGOC is about 100 mW less than the 20 and 25% InvGOC and the intra-cavity energy is the lowest among the InvGOC's. All of this leads us to conclude that the transmission of the 30% InvGOC is too high.

In terms of spectral width we note that the flat 15% and InvGOC 30% have the widest spectra and hence shortest Fourier-limited pulses with the spectral component at 1150 nm only 20 dB down from the peak for the flat 15% and 15 dB down from the peak for the InvGOC 30% while the Fourier-limited output pulse widths are both less than 5 fs. It is curious that the spectrum is the widest for the 15% flat, the flat OC with the second smallest output coupling, while the widest spectrum of the InvGOC's is the OC with the highest output coupling. It should be noted that for both of these output couplers, the output beam was very irregular and distorted, indicating a very high KLM action producing a strong nonlinearity. For all the other OC's, the spatial qualities of the beam are much better, hinting at a smaller nonlinearity.

In terms of maximizing power while maintaining short pulses, the InvGOC 25% and the flat 20% produce the best results with output power greater than 450 mW of sub 7 fs Fourier-limited pulses.

To test the beam quality of the inverse-gain output couplers, we then took CCD images of the output beams for the InvGOC 20% and flat 20% (Fig. (5-14)) at various wavelengths using bandpass filters with differing centering wavelengths. We
operated the lasers at the same operating point using the same crystal position and
curved mirror separation for both output couplers so as to provide a fair comparison
between output couplers.

Since the reflection curve is virtually the same for the two output couplers with
the exception of the inverse-gain dip (Fig. 5-12) the beam profiles are much more
alike than in Fig. 5-2 and the difference between the inverse-gain and flat output
coupler beam profile is less pronounced. Despite this, we still note a cleaner and less
distorted beam profile for the inverse-gain beam profile.

5.3.2 Pump Dependence

To test the pump dependence of these new output couplers we picked an operating
point (curved mirror separation, crystal position, and alignment) so as to obtain
good, stable operation with a wide spectrum, reasonably high output power, and
good beam quality. Using the same method as in previous chapters, we measured the
laser spectra and power for the 15, 20, and 25% InvGOC and 10, 15, 20, and 20%
flat output coupler as a function of pump power increasing the pump power until
continuous-wave breakthrough occurs.

We see in Fig. (5-15) that the spectra have very similar features on a logscale as
a result of the output couplers all having the same reflectivity window and dispersion.
Interestingly though if we examine the pulse characteristics in Fig. (5-16) we see that
the power remains well below 500mW for all of the output couplers, which conflicts
with the results of the dual output coupler study (Fig. (5-9)) which suggested that we
should be able to get well over 600 mW for the 20% InvGOC. Since the discrepancy
is well over 100 mW this forces us to conclude that our assumption that the output
power of two output couplers just simply “add” is incorrect. However, despite this,
the pulses still have Fourier-limited pulsewidths well below 8 fs and deliver a power of
up to 450mW at 85MHz. Due to the difficulty of mode locking of the 30% InvGOC,
we can conclude that this represents the limit of the amount of power we can obtain
from this laser using the InvGOC’s. We did not have a flat output coupler with a
transmission higher than 20%, but since the pulse characteristics were very similar for
the inverse gain and flat output couplers we suspect that a flat output coupler with a transmission greater than 20% would also experience difficulties in mode locking.

Due to the discrepancy we decided to test a dual output coupler setup using two of the 15% InvGOC's. We noticed right away that it was much easier to mode lock the double 15% InvGOC setup than the single 30% setup. Furthermore from Table (5.1) we found that for the 30% InvGOC pumped at 5.7 W, the intracavity energy is only about 21nJ. However in Fig. (5-17) we see that the intracavity energy is higher than 21nJ for all pump powers above 3.5W. Lastly we note that the intracavity energy is higher for the other output couplers (Fig. (5-16)). The relatively low intracavity energy for such a high pump power could indicate that the intracavity pulse energy is too low to allow for easy mode locking.
Figure 5-12: (a) Design group delay and (b) design transmission of high transmission output couplers which we designed to directly compare flat and inverse-gain output couplers as well as determine the maximum output power of the laser. The design was chosen carefully such that the group delay reflection and bandwidth were roughly the same with the only variation being the transmission window between 600 and 1000nm. An intental dip in the reflectivity curve is placed at 1150nm to enhance the 1150nm component of the output beam for 1f-2f self-stabilization.
Figure 5-13: Optimum spectra for DCM7 laser with high transmission (a) flat and (b) inverse-gain output couplers obtained by varying the crystal position, curved mirror separation, pump power, and cavity alignment.
Figure 5-14: Beam profiles for (a) InvGOC 20% and (b) flat OC 20% taken with different center wavelength bandpass filters and a CCD camera.
Figure 5-15: Output spectrum of DCM7 laser with high transmission output couplers.
Figure 5-16: Operating parameters for DCM7 based laser with high transmission output couplers.
Figure 5-17: Operating parameters for DCM7 based laser with double 15% inverse gain output couplers. Plot (c) is the total output power (sum of power out of both output couplers) vs. pump power.
Chapter 6

Ince Gaussians

In chapter 3 we saw the importance in determining the stability region of the laser cavity for maximizing self-amplitude modulation and the Kerr effect. Furthermore, in order to ensure sufficient astigmatism compensation, we must have a way of precisely determining the astigmatism compensation angle as discussed in section 3.1.2. We thus desire an experimental method of determining the cavity alignment as well as an indication of where in the region of cavity stability the laser is currently operating.

To aid in this task, it is useful to study the mode structure of the continuous wave beam output from the laser. While in continuous mode operation, the Kerr-effect from the crystal is effectively zero and we can model the laser in a straightforward fashion. By studying the laser modes numerically as a function of alignment and curved mirror separation, we can predict theoretically the type of mode structure we should see out of the laser output for a given alignment and curved mirror separation and in turn use this information to better align our Ti:Saph lasers.

In this chapter we investigate the continuous wave laser cavity modes of the Ti:Sapphire laser which we experimentally show to strongly resemble a family of elliptical solutions to the paraxial wave equation known as Ince-Gaussian modes (IG) [4]. I will then outline the mathematical properties of these modes showing them to be a more general solution to the paraxial wave equation than the traditional Hermite and Laguerre Gaussian mode solutions. Then I will present an overview of the results I obtained from numerical simulations of the laser cavity in attempt at recreating the
IG modes seen in the experimental laser cavity.

6.1 Experimental Observations

In order to study the CW modes of a Ti:Saph laser, I first studied the output beam of a 10GHz ring cavity laser as diagramed in Fig. 6-1. The Ti:Saph crystal is Brewster cut with a geometrical pathlength of 2 mm and with curved mirrors with radius of curvature of 25 mm. The beam is output of one of the flat mirrors which is partially reflecting and serves as the output coupler.

To study the mode structure, I took CCD images of the output beam as I varied the cavity alignment. We note that the modes we obtained in Fig. 6-2 are neither Hermite Gaussians with rectangular symmetry nor Laguerre Gaussians with circular symmetry, but possess elliptical symmetry. If we decrease the curved mirror spacing, we obtain higher order mode structures with more elliptic and hyperbolic node lines, and if we decrease the angle of incidence of the beam on one of the curved mirrors, we obtain a beam with a much higher degree of ellipticity such as in Fig. 6-2 (f).

“Good” mode locking (hallmarked by a wide spectrum and high output power) of the laser occurs when the laser is roughly aligned such that the CW mode is the most circular and lowest order as in 6-2 (a).

Despite the peculiarity of these modes they have been studied extensively by Bandres et al. [5], [4], [3], and [6] as well as been observed experimentally in circularly symmetric resonators [29]. They represent a third family of solutions to the paraxial wave equation for elliptic symmetry. In the next section we will discuss some of the
Figure 6-2: CW modes observed from a 10GHz ring cavity laser. (a) is the mode profile for a typically aligned “well behaved” laser with good mode locking qualities. By slightly varying the mirror alignment we can get higher order modes (b) and (c). (d) and (e) are obtained by reducing the curved mirror separation, while (f) is obtained by reducing the angle of incidence on one of the curved mirrors.
properties of these modes much of which is taken from [4]. I will save the mathematical
details behind the analytic computation of the Ince Gaussian modes for the appendix.

6.2 Theory of Ince-Gaussian Modes

To derive the Ince-Gaussian modes we first assume a paraxial light field which can be
written as \( U(x, y, z) = \Psi(x, y, z)e^{ikz} \) where \((x, y)\) are the transverse coordinates and
\(|(\partial/\partial z)\Psi| \ll k\Psi\) such that it satisfies the paraxial wave equation (PWE)

\[
\left( \nabla^2_t + 2ik \frac{\partial}{\partial z} \right) \Psi(r) = 0 \tag{6.1}
\]

where \( \nabla^2_t \) is the transverse Laplacian for \( r \). The well-known lowest order solution of
this equation is the fundamental Gaussian beam (GB) which we write as

\[
\Psi_G(r) = \frac{w_0}{w(z)} \exp \left[ -\frac{r^2}{w^2(z)} + \frac{kr^2}{2R(z)} - i\psi_{GS}(z) \right] \tag{6.2}
\]

where \( r \) is the radius, \( w^2(z) = w_0^2(1 + z^2/z_R^2) \) is the beam radius, \( R(z) = z + z_R^2/z \)
the radius of curvature of the phase front, \( \psi_{GS}(z) = \arctan(z/z_R) \) is the Guoy phase
shift, \( z_R = kw_0^2/2 \) is the Rayleigh range, and \( w_0 \) is the minimum beam waist.

One could then proceed from here and derive the well known [30] higher order
solutions in rectangular coordinates (Hermite Gaussians (HG)) or in cylindrical coor-
dinates (Laguerre Gaussians (LG)). However a third family of solutions in elliptical
coordinates exists. To derive these solutions, we assume a PWE wave whose complex
envelope is a modulated GB

\[
IG(r) = E(\xi)M(\eta)e^{iZ(z)}\Psi_G(r) \tag{6.3}
\]

where \( E, N, \) and \( Z \) are real functions.

We define the elliptic coordinates as \( x = f(z) \cosh(\xi) \cos(\eta), \ y = f(z) \sinh(\xi) \sin(\eta), \)
and \( z \), where \( \xi \in [0, \infty) \) and \( \eta \in [0, 2\pi) \) are the radial and angular elliptic variables
respectively. Curves of constant \( \xi \) are confocal ellipses, and curves of constant \( \eta \) are
confocal hyperbolas with semifocal separation $f$ given by $f(z) = f_0 w(z)/w_0$, where $f_0$ is the semifocal separation at the waist plane $z = 0$. Thus we see that the semifocal separation of the ellipses and hyperbolas diverge in the same way as the width of the GB.

To look for elliptic solutions, we substitute our ansatz for the IG beam solution into the PWE. We obtain three ordinary differential equations which must be satisfied

$$\frac{d^2 E}{d\xi^2} - \epsilon \sinh 2\xi \frac{dE}{d\xi} - (a - p\epsilon \cosh 2\xi)E = 0 \quad (6.4)$$

$$\frac{d^2 N}{d\eta^2} + \epsilon \sin 2\eta \frac{dN}{d\eta} + (a - p\epsilon \cos 2\eta)N = 0 \quad (6.5)$$

$$- \left( \frac{z^2 + z_R^2}{z_R} \right) \frac{dZ}{dz} = p \quad (6.6)$$

where $p$ and $a$ are separation constants, and $\epsilon = 2f_0^2/w_0^2$ is the ellipicity parameter. We can readily solve the last equation to yield the solution for the excess phase $Z(z) = -p \arctan(z/z_R)$.

At first glance, equations 6.4 and 6.5 appear formidable; however, these equations are in fact known under the name Ince equations after E. G. Ince who first studied the equations in 1923 [17]. The Ince equations are a special case of the more general Hill equation and have been studied extensively by F. M. Arscott [1] [2]. The notation I use for the Ince-Gaussian modes is taken from Arscott.

To solve Eq. 6.4 and Eq. 6.5, we first realize that Eq. 6.5 may be obtained from Eq. 6.4 by making the argument imaginary and writing $i\xi$ in place of $\eta$. Then by treating $\epsilon$ as fundamental and $a$ and $p$ as disposable parameters we can compute the solutions as a power series in $\sin \eta$ and $\cos \eta$ and choose $a$ and $p$ such that the power series converge which is outlined in Appendix A. These solutions are known as the even and odd Ince polynomials of order $p$ and degree $m$ and usually denoted as $C^m_p(\eta, \epsilon)$ and $S^m_p(\eta, \epsilon)$ respectively. $m$ and $p$ follow the constraints $0 \leq m \leq p$ for even functions and $1 \leq m \leq p$ for odd functions and the indices $(p, m)$ for a given mode always have the same parity, i.e. $(-1)^{p-m} = 1$ and $\epsilon$ is the ellipicity parameter defined earlier. Once we have solutions for Eq. 6.4 and Eq. 6.5 we find expressions
for Eq 6.3 by looking for products of $E$ and $N$ which satisfy the continuity in the whole of space. This restriction requires that $E$ and $N$ have like parity in $\xi$ and $\eta$ which leads to two sets of even and odd solutions which can be written as

$$\text{IG}^e_{p,m}(r,\epsilon) = \frac{C w_0}{w(z)} C^m_p(i\xi, \epsilon) C^m_p(\eta, \epsilon) \exp \left[ \frac{-r^2}{w^2(z)} \right] \times \exp \left[ k z + \frac{kr^2}{2R(z)} - (p + 1) \Psi_{text GS}(z) \right]$$

(6.7)

$$\text{IG}^o_{p,m}(r,\epsilon) = \frac{S w_0}{w(z)} S^m_p(i\xi, \epsilon) S^m_p(\eta, \epsilon) \exp \left[ \frac{-r^2}{w^2(z)} \right] \times \exp \left[ k z + \frac{kr^2}{2R(z)} - (p + 1) \Psi_{text GS}(z) \right]$$

(6.8)

where $C$ and $S$ are normalization constants and the superindices $e$ and $o$ refer to even and odd modes respectively. Note that if we refer to Fig. 6-3, $m$ corresponds to the number of hyperbolic node lines, whereas $(p - m)/2$ corresponds to the number of elliptic node lines without counting the node line at $\xi = 0$ for the odd modes. Since $C^0(\eta, \epsilon) = 1$, then mode $(0,0)$ just corresponds to the fundamental Gaussian beam.

### 6.2.1 Properties of Ince Gaussian Modes

IG modes have many of the same physical properties as HG and LG modes. First of all, since the IG waves and $\Psi_G$ have paraboloidal wave fronts with the same radius of curvature $R(z)$, Ince Gaussians are focused in the same way by lenses and mirrors and thus we can use the ABCD matrix formalism as we do for HG modes and make use of the $q(z)$ parameter. Thus, for an input IG beam of parameter $q_{in}$, the output $q(z)$ parameter is given by the well-known equation

$$q_{out} = \frac{Aq_{in} + B}{Cq_{in} + D}$$

(6.9)

Furthermore the width of the IG beam is proportional to $w(z)$ so that the intensity pattern goes as $w_0/w(z)$ with an otherwise invariant shape. Since the elliptic
Figure 6-3: Transverse electric field distribution of several IG modes for (a) even and (b) odd parity with \( \epsilon = 2 \).

coordinate system evolves with \( z \) according to \( f(z) = f_0 w(z)/w_0 \) as the beam propagates, then the eccentricity of the elliptic and hyperbolic nodal lines is invariant under propagation and the beam maintains its shape as a function of \( z \). As a result, IG beams focus and propagate in the same manner as fundamental Gaussian beams and propagate without changing shape.
IG Modes as a Complete Solution Set

Just like the HG and LG modes, the IG modes also constitute a complete set of solutions which are orthogonal with respect to \( p, m, \) and the parity

\[
\int \int_{-\infty}^{\infty} \overline{IG_{p,m}^\sigma} IG_{p',m'}^{\sigma'} dS = \delta_{\sigma \sigma'} \delta_{p p'} \delta_{m m'}
\]  

(6.10)

where the overbar denotes complex conduction, \( \delta \) is the Kronecker delta function, \( dS \) is the differential surface element, and \( \sigma = e, o \) is the parity. Since the IG solutions constitute a full orthonormal solution set to the PWE, then it is possible to express the HG and LG solutions as an expansion in IG modes such as

\[
LG_{n,l}^\sigma (r, \phi) = \sum_m D_m IG_{p=2n+1,m}^{\sigma} (\xi, \eta, \epsilon)
\]  

(6.11)

where

\[
D_m = \int \int_{-\infty}^{\infty} LG_{n,l}^{\sigma} \overline{IG_{2n+l,m}^{\sigma}} dS
\]  

(6.12)

and likewise for the HG modes.

Since we can express LG and HG modes in terms of IG modes and vice versa we can consider IG modes to be another valid independent set of solutions to the PWE. However we can in fact consider them to be a more general set of solutions to the PWE with HG and LG modes being limiting cases. To see this, we must examine the limiting values of the parameter \( \epsilon \). When the ellipticity of an IG mode approaches zero, i.e. when \( f_0 \rightarrow 0 \) the elliptic coordinate system becomes a circularly symmetric coordinate system and hence the IG mode must become a circularly symmetric LG mode. Since the Guoy phase shift must remain constant as a function of \( \epsilon \) and the Guoy phase shift of an LG mode is given by \( \Psi_{LG} = -(n_x + l + 1)\Psi_{GB} \) then this requires that the mode numbers of the IG and corresponding LG mode in the limit of \( \epsilon \rightarrow 0 \) be \( m = l \) and \( p = 2n + l \). Likewise Hermite Gaussian modes (HG) are the limit of Ince Gaussian modes in the limit of \( \epsilon \rightarrow \infty \). To find the mode number of the corresponding HG mode from a given IG mode of mode number \( p, m \) we note that the Guoy phase of a given HG mode is given by \( \Psi_{HG} = -(n_x + n_y)\Psi_{GB} \) and
Figure 6-4: IG modes for $p = 6$ and $\varepsilon = 2$ (middle row) with the limiting HG modes (top row) and LG modes (bottom row) for the limit for $\varepsilon = \infty$ and $\varepsilon = 0$ respectively.

hence the mode numbers of the HG and corresponding IG mode must be related by $p = n_x + n_y$. Next by equating the parity across the $x$ axis of the two modes this requires that for even modes $n_x = m$ and $n_y = p - m$ while for odd IG modes $n_x = m - 1$ and $n_y = p - m + 1$. Hence, from this we see that IG modes can be thought of as a more general PWE solution for finite, non-zero values of $\varepsilon$ in which we can obtain the well-known HG and LG solutions just by setting $\varepsilon$ to $\infty$ and 0 respectively as we show in Fig. 6-4 for the even modes of $p = 6$.

6.3 Numerical Simulation

With the modes of the laser cavity identified, we then proceeded to study the laser numerically. To do this I implemented a simulation in MATLAB based off of Endo’s simulation [13] which simulates a monochromatic single or multi-mode oscillation in a laser cavity. The technique is very similar to the Fox-Li numerical calculation of the lowest order mode of unstable resonators, but it can also be applied to stable resonators of high Fresnel number as well.

The goal of the numerical simulations was to determine if IG modes were indeed eigenmodes of a typical Ti:Saph laser cavity, and if so, try and excite these modes in
our model using a selective gain technique which we discuss in Sec. 6.3.3. After we outline our numerical model, I will present the results I obtained for both a circularly symmetric case in which we leave out astigmatism as well as the full astigmatic case.

6.3.1 Theory

In order to successfully model the laser cavity, we must account for the effects of the focusing due to the curved mirror (which we can model as a thin lens), linear loss, gain, and free space propagation.

To model the free space propagation we first consider that the complex electric field in a laser cavity at a plane \( P_1 \) can be written as \( u(x, y) \) normalized such that \(|u(x, y)|^2 \) is the intensity. In the Fresel region, the complex electric field at a plane \( P_2 \) a distance \( l \) away can be found by computing the Fresnel-Kirchhoff integral

\[
  u(x, y) = \frac{j}{\lambda l} \int_{P_1} \int_{P_1} u(x_0, y_0) e^{-jk[(x-x_0)^2+(y-y_0)^2]^{1/2}} dx_0 dy_0
\]

which in the space-frequency domain becomes

\[
  U(k_x, k_y, l) = (2\pi)^2 H(k_x, k_y, l) U_0(k_x, k_y)
\]

where

\[
  H(k_x, k_y, l) = \frac{1}{(2\pi)^2} e^{j(k_x^2 + k_y^2)/2k} l
\]

and \( U(x, y) \) is just the Fourier Transform of \( u \) defined by

\[
  U(k_x, k_y) = \mathcal{F}[u(x, y)] = \left( \frac{1}{2\pi} \right)^2 \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy u(x, y) e^{j(k_x x + k_y y)}
\]

Therefore we can model the free space propagation of the electric field in a laser cavity by scalar multiplication in the spatial frequency domain using Eq. 6.15. Similarly, if we wish to model the propagation of the electric field through a thin lens, we note that a lens imparts a parabolic phase onto the scalar electric field. Hence we
can model the propagation of the electric field through a thin lens by the equation

\[ u'(x, y) = u(x, y) \exp \left[ \frac{k}{2j} (x^2 + y^2) \right] \]  

(6.17)

where \( f \) is the focal length of the thin lens and \( u(x, y) \) and \( u'(x, y) \) are the transverse scalar electric fields immediately before and after the thin lens. Likewise, we can model loss as a scalar attenuation at a specified plane

\[ u'(x, y) = u(x, y)e^{-i} \]  

(6.18)

To simulate the gain medium we divide the gain region into several infinitesimally thin sheets spaced equally apart by \( dz \) which amplify the field according to the equation

\[ u'(x, y) = u(x, y)e^{g_i dz} \]  

(6.19)

where \( g_i \) is the saturated gain at sheet \( i \) which we approximate by the equation

\[ g_i(x, y) = \frac{g_{i0}(x, y)}{1 + \frac{\tilde{I}_i^+(x, y) + \tilde{I}_i^-(x, y)}{I_s(x, y)}} \]  

(6.20)

where \( g_{i0} \) is the small signal gain, \( I_s \) the saturation intensity, and \( \tilde{I}_i^+ \) and \( \tilde{I}_i^- \) are the average right-going and left-going optical intensities defined as

\[ \tilde{I}_i^+ = (1 - \alpha) \sum_{i=0}^{q} \alpha^i I_i^+(q - i) \]  

(6.21)

\[ \tilde{I}_i^- = (1 - \alpha) \sum_{i=0}^{q} \alpha^i I_i^-(q - i) \]  

(6.22)

\( I_i^+(q) \) and \( I_i^-(q) \) are the intensities of the \( q \)th iteration step and \( \alpha \) is a summation over a period of the cavity’s photon decay time. For a two mirror resonator with mirrors of reflectivity \( R_1 \) and \( R_2 \) then \( \alpha = R_1 R_2 \). To simulate the cross-sectional spatial dependence of the gain region caused by a tightly focused pump beam we approximate the active gain region in the crystal to be an elliptically shaped cylinder.
Figure 6-5: Equivalent system used to simulate continuous laser operation. The small red region of length $2z_R$ is the active gain region which we subdivide into 10 equally spaced gain sheets. The effects of refraction are lumped into the virtual propagation, lens, propagation, lens, propagation system outside the crystal.

of length $2z_R$ centered at the beam waist of the pump beam with its axis parallel to the optical axis. Furthermore, we assume that the major and minor axes of the elliptical cross section to be aligned with the sagittal and tangential axes of the laser cavity. We assume an elliptical shape in order to make use of the active gain technique of exciting IG order modes, which we will introduce in the next subsection.

With the exception of the gain and losses, the rest of the laser cavity can be converted to an equivalent periodic lens array due to the fact that any arbitrary $ABCD$ ray transfer matrix can be converted into an equivalent system of free space propagation, lens-transfer, and free space propagation by the equation

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
1 & L \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
-\frac{1}{L} & 1
\end{pmatrix} \begin{pmatrix}
1 & L \\
0 & 1
\end{pmatrix}
$$

Equating sides gives us

$$
L = \frac{A-1}{C} \quad (6.24)
$$

$$
f = -\frac{1}{C} \quad (6.25)
$$

Therefore, using the above equations, we can lump the effects of refraction on the surfaces of the Brewster cut gain crystal, propagation, and reflection off a curved surface into an equivalent propagation, lens, propagation, lens, and propagation system.
Using these relations we can compute the propagation of an electric field distribution in a laser cavity by scalar multiplication in the space-frequency domain using Eq. 6.14 and transforming back into the real domain and performing the scalar multiplication upon the electric field to account for propagation through a thin lens, gain, or loss. After enough round trips the field at any given point along the optical axis will converge

\[ u^{n+1}(x, y) \approx u^n(x, y) \] (6.26)

### 6.3.2 Partially Coherent Input Field

For unstable resonators, computing the steady state field distribution will always yield the lowest resonator mode. However, for the case of a stable resonator higher order modes can survive many round trips with negligible attenuation. For such a situation, the periodic lens focusing system produces perfect coherence of the input electric field value and is preserved after each round trip due to the repeated application of linear transformations represented by the lens and free space propagation. Real experiments differ in that radiation eventually fills a closed cavity after enough round trips. Therefore we desire an input field which is dispersive enough to fill the cavity.

To accomplish this we use a partially coherent random input field given by

\[ u_{in}(x, y) = E_0 \exp\{j2\pi[\text{floor}(x, y) - 1/2]\} \] (6.27)

where \(0 < \text{floor}(x, y) < 1\) is a computer-generated random number. Thus we see that each grid point has a random phase from 0 to \(2\pi\). By using such a partially-coherent input field, we avoid possible dependence between any initial field selection and the final field distribution in the cavity.

### 6.3.3 Exciting IG Modes

In order to excite desired IG modes, we must create the condition such that the losses of the desired IG mode \(IG_{p,m}^r\) is greater than any other \(IG_{p',m'}^r\). To do this I adopted the approach taken by [12] which relies on using a small elliptic gain region as pictured...
in Fig. 6-6 which we shape to only overlap with the most intense spot of the pattern on the desired IG mode. For IG modes with \( p = m \), the most intense spots of the pattern are the two outermost spots of the pattern; while for modes with \( p > m \) the most intense spots are in the two outermost spots of the innermost elliptical nodal line. Since IGMs of odd parity are rarely observed in real experiments [27], [29], [28] I opted to try and excite only IG modes of even parity.

6.3.4 Results Without Astigmatism

Since the real ring cavity laser which we wish to model in Fig. 6-1 is inherently astigmatic, our virtual lens relay consists of two independent relays for the sagittal and tangential planes. However, as a first test I assume circular symmetry and use the same virtual cavity setup for both planes, effectively leaving out astigmatism.

To test IG mode excitation I used the gain size parameters from Table 6.1 to excite three of the even modes belonging to \( p = 6 \). For sufficiently high gain, the distributions converge after a few hundred round trips as in Fig. 6-7. For each of the chosen gain patches, the initial partially incoherent field distributions converge to the expected IG modes as shown in Fig. 6-8 which proves the validity of this method of IG mode excitation.
Figure 6-7: Evolution of input field through multiple round trips in the laser starting from a partially coherent random field. After about 250 round trips, the distribution has fully converged.

Table 6.1: Selective gain parameters used to generate the IG modes shown in Fig. 6-8.

<table>
<thead>
<tr>
<th></th>
<th>IG₃₂</th>
<th>IG₆₄</th>
<th>IG₆₆</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0.19</td>
<td>0.15</td>
<td>0.45</td>
</tr>
<tr>
<td>b</td>
<td>0.22</td>
<td>0.35</td>
<td>0.4</td>
</tr>
<tr>
<td>dₓ</td>
<td>0.68</td>
<td>1.3</td>
<td>2.2</td>
</tr>
</tbody>
</table>

6.3.5 Results With Astigmatism

When we go to the real case of a laser with astigmatism, several of our assumptions that we used in the derivation of the IG modes break down. In particular in the real laser cavity, the fundamental mode is not a circularly symmetric Gaussian, but an astigmatic Gaussian with sagittal and tangential beam waists of different sizes hence our ansatz solution in the derivation of IG modes in Eq. 6.3 is immediately called into question.

A more important consideration which we note when going to the general case is that any IG mode can be written in terms of a linear combination of HG modes with the same parity about the x axis and with the same Guoy phase (see appendix). Particularly the Guoy phase of the Hermite Gaussians can be written as $\Phi_{nₓ,nᵧ} = -(nₓ + nᵧ) \arctan(z/z_R)$ where $z_R$ is the Raleigh range of the fundamental Gaussian. Therefore the component HG modes of the IG mode in question must satisfy the equation $p = nₓ + nᵧ$, and hence more than one HG mode can satisfy this condition for $p$ due to the degeneracy. If we go to the astigmatic case, then the resonance
Figure 6-8: (top row) final steady state transverse electric field distributions obtained using gain parameters from Table 6.1 with their corresponding theoretical IG modes (bottom row) for several IG modes with $p = 6$. (a) and (d) correspond to $IG_{6,2}$, (b) and (e) to $IG_{6,4}$, and (c) and (f) to $IG_{6,6}$ with $\epsilon$ values of roughly 1, 3, and 10 respectively.

frequencies of the HG modes are no longer degenerate and hence the Guoy phase of a given HG mode becomes $\Phi_{nx,ny} = -n_x \arctan(z/z_{Rz}) - n_y \arctan(z/z_{Ry})$. Since $z_{Rz} \neq z_{Ry}$ in general for the astigmatic case, then it is not possible to write an IG mode as a linear combination of more than one HG mode. This reasoning suggests that for the astigmatic laser, such as a Ti:Saph laser, it should not be possible to see IG modes. However experimentally as Fig. 6-2 shows, it is clearly possible for an astigmatic laser to produce IG modes.

Despite the above reasoning, I tried to excite IG modes in the astigmatic case in the same way that I did for the circularly symmetric case. Unfortunately we could find no analytical formulas of IG eigenmodes for the case of astigmatic cavities in the literature, and hence we have no way of knowing what IG mode a given gain patch size and position will excite. I chose the same gain patches as the case of the non-astigmatic case in which we choose the gain patch major and minor axes to be $a_z w_{0z}$ and $b_y w_{0x}$ respectively where $w_{0z}$ is the beam waist of the fundamental Gaussian in
the tangential plane and with a displacement along the tangential plane of $d_x w_{0x}$. The results I obtained for these gain patches did not produce any IG modes (see Fig. 6-9). The only clear mode I obtained was an HG mode in Fig. 6-9 (c). Other gain patches just result in other unrecognizable multi-mode patterns or HG modes as expected.

Figure 6-9: Final steady state transverse electric field distributions obtained using gain parameters from Table 6.1 with (a) using the parameters for $IG_{6,2}^c$, (b) using $IG_{6,4}^c$, and (c) using $IG_{6,6}^c$.

In summary, I was not able to generate the IG modes as seen in experiments in my computational model. The major obstacle that remains to be solved is how to derive IG modes from the PWE whose fundamental mode is an astigmatic Gaussian rather than a pure circularly symmetric Gaussian. With this knowledge, it may then be possible to directly generate the IG modes in the model as seen in experiments and use this information to better determine the region of cavity alignment in which we should operate the laser.
Chapter 7

Conclusion

7.1 Summary

To conclude I have outlined the various physical phenomena at work in Kerr-lens mode locked Ti:Sapphire lasers. Specifically, the heart of Kerr-lens mode locking is the optical Kerr effect. By using this effect we are able to create an artificial saturable absorber which is necessary for mode locking to occur. We require a large contrast between the mode locked and continuous wave beam diameters in the crystal in order to maximize the saturable absorber effect as well as tight focusing in the crystal to generate the Kerr nonlinearities in the crystal of sufficient magnitude. These requirements necessitate the need for the laser to be operated at the edge of the stability region and as a result make the laser alignment-sensitive. All of these considerations must be taken into account when seeking to maximize the performance of Kerr-lens mode locked Ti:Sapph lasers.

To address these challenges we proposed the novel design of a new type of output coupler which we referred to as the inverse gain output coupler [InvGOC] which attempts to reduce the gain filtration caused by the Lorentzian shape of the crystal gain. By reducing this filtration, the spectral width and output power should increase as well as improve spatial beam quality due to fact that one no longer needs to pump the laser as hard to achieve the same spectral bandwidth. With this design we hoped to be able to reduce the amount of nonlinearity needed for mode locking and as a
result be able to operate the laser further inside the stability region of the cavity. This would reduce the alignment sensitivity of the laser as well as improve the geometric qualities of the output beam.

We then outlined the experiments I performed which evaluate the design of the InvGOC output couplers. As a ruler for comparison we used spectral width, spectral smoothness, output power, spatial mode quality, and ease of mode locking as merits for comparison between various output coupler designs. We then compared InvGOC's with traditional flat output couplers and demonstrated their improved spatial beam qualities but were unable to definitely see which design performed better for the other merits of consideration. In an additional experiment we varied the transmittance of the output couplers and showed that the data suggested that we could go to higher output coupling percentages to gain higher output coupling. Motivated by these results, we designed new output couplers which possess higher output coupling percentages as well as possessed the same dispersion and transmittance windows. In addition we designed flat output couplers with the dispersion and transmittance windows as the new, higher output coupling InvGOC's in order to eliminate bandwidth and dispersion compensation discrepancies when comparing flat output couplers to inverse gain output couplers.

Finally we looked at the phenomenon of Ince Gaussian modes which we experimentally demonstrated occur in real Ti:Sapph lasers. By studying the Ince Gaussian modes numerically in our Ti:Sapph laser systems we hoped to gain a method of determining the beam alignment within the crystal (and hence the size of the saturable absorber effect) just by observing the output of the continuous wave beam. Though we were able to numerically simulate Ince-Gaussian modes in cylindrical symmetrical laser cavities, we were unable to excite Ince-Gaussian modes for the real case of an astigmatic laser cavity.
7.2 Future Work

In our experiments with the inverse-gain output couplers, we tried to remove the effects of dispersion and differing transmission windows of OC's by designing the high transmission InvGOC's to have identical dispersion profiles and transmission windows. To vary the total output coupling percentage of the InvGOC's we varied the peak of the transmission peak of the output coupler's transmittance curve as in Fig. 5-12. However, the transmission values at 600 and 1000 nm stayed roughly the same. To attempt to obtain better reduction in the gain filtration of the laser, we should also vary the transmission at 600 and 1000 nm as well as the peak as 800 nm. In this way we effectively vary the offset of the gain-shaped loss curve as well as vary the difference between the peak and valley of the transmittance.

As for the Ince-Gaussian simulations, we saw in Chapter 6 that since the mathematical formalism for the derivation of Ince-Gaussian modes in a laser cavity rests on the assumption of a cylindrically symmetric beam, we were unable to directly apply the analytic expressions for Ince-Gaussian beams to the astigmatic laser cavity. Because of this, we are unable to effectively use the gain selection method of section 6.3.3 to excite single Ince-Gaussian modes in the laser simulation. Therefore in order to avoid a blind search of the parameter space, it is necessary to develop an analytic or numerical way of computing the Ince-Gaussian eigenmodes of an asyrtigmatic laser cavity. With knowledge of the exact Ince-Gaussian eigenmodes of the cavity, we can determine precisely what alignment conditions are necessary to generate a given eigenmode.
Appendix A

Mathematical Properties of Ince Gaussians

Due to the esoterity of Ince Polynomials and the lack of mathematical libraries which compute their numerical values, I have dedicated this appendix to outlining their derivation and computation, the details of such can be obtained from [2] [1]. Much of what follows is taken from [4] which provides a nice summary of the mathematical properties of Ince Gaussians.

A.1 Ince Polynomials

The Ince Equation which is written as

$$\frac{d^2 y}{dx^2} + \epsilon \sin 2x \frac{dy}{dx} + (a - p \epsilon \cos 2x)y = 0$$

is a periodic linear second-order differential equation that has two families of independent solutions which are either even or odd in the variable $x$. We write these two sets of solutions as $C_p^m(x, \epsilon)$ and $S_p^m(x, \epsilon)$ respectively and are known as the Ince polynomials of order $p$ and degree $m$. In order for the solutions to satisfy the requirement of being periodic with period $2\pi$ this requires that $a$ take on very specific discrete values. These values of $a$ become the eigenvalues of the equation which are denoted $a_p^m(\epsilon)$ and...
$b^m_p(\epsilon)$ for $C^m_p$ and $S^m_p$ respectively. For a given $p$ and $\epsilon > 0$ the eigenvalues form a finite set of real values that have the additional property $a^0_p < b^1_p < a^1_p < b^2_p < \ldots < a^p_p < b^p_p$.

Since $C^m_p$ and $S^m_p$ are periodic with period $2\pi$, we can expand them into a Fourier series and divide the solutions into four classes according to their symmetry or antisymmetry about $x = 0$ and $x = \pi/2$ respectively.

\[
\begin{align*}
C_{2n}^{2k}(x, \epsilon) &= \sum_{r=0}^{n} A_r \cos 2rx, \quad k = 0, \ldots, n, \\
C_{2n+1}^{2k+1}(x, \epsilon) &= \sum_{r=0}^{n} A_r \cos(2r+1)x, \quad k = 0, \ldots, n \\
S_{2n}^{2k}(x, \epsilon) &= \sum_{r=1}^{n} B_r \sin 2rx, \quad k = 1, \ldots, n, \\
S_{2n+1}^{2k+1}(x, \epsilon) &= \sum_{r=0}^{n} B_r \sin(2r+1)x, \quad k = 0, \ldots, n
\end{align*}
\]  

(A.2)

where $A_r$ and $B_r$ are the Fourier coefficients of $C^m_p$ and $S^m_p$ respectively.

To solve for the Fourier coefficients, we substitute the Fourier expansions into Eq. A.1 to obtain three term recurrence relations between the coefficients which we write below.

For $C_{2n}^{2k}(x, \epsilon)$, $p = 2n$:

\[
\begin{align*}
(p/2 + 1)\epsilon A_1 &= aA_0 \\
(p/2 + 2)\epsilon A_2 &= -p\epsilon A_0 - (4 - a)A_1 \\
(p/2 + 2r + 2)\epsilon A_{r+2} &= [a - 4(r + 1)^2]A_{r+1} + \left(\frac{r - p}{2}\right)\epsilon A_r
\end{align*}
\]  

(A.3) (A.4) (A.5)

For $S_{2n}^{2k}(x, \epsilon)$, $p = 2n$:

\[
\begin{align*}
(p/2 + 2)\epsilon B_2 &= (a - 4)B_1 \\
(p/2 + 2r + 2)\epsilon B_{r+2} &= [a - 4(r + 1)^2]B_{r+1} + \left(\frac{r - p}{2}\right)\epsilon B_r
\end{align*}
\]  

(A.6) (A.7)

For $C_{2n+1}^{2k+1}(x, \epsilon)$, $p = 2n + 1$:

\[
\begin{align*}
\frac{\epsilon}{2}(p + 3)A_1 &= \left[a - \frac{\epsilon}{2}(p + 1) - 1\right]A_0 \\
\frac{\epsilon}{2}(p + 2r + 3)A_{r+1} &= [a - (2r + 1)^2]A_r + \frac{\epsilon}{2}(2r - p - 1)A_{r-1}
\end{align*}
\]  

(A.8) (A.9)
For $S_{2n+1}^2(x, \epsilon)$, $p = 2n + 1$:

$$\frac{\epsilon}{2}(p + 3)B_1 = [a + \frac{\epsilon}{2}(p + 1) - 1]B_0 \quad (A.10)$$

$$\frac{\epsilon}{2}(p + 2r + 3)B_{r+1} = [a - (2r + 1)^2]B_r + \frac{\epsilon}{2}(2r - p - 1)B_{r-1} \quad (A.11)$$

where $r = 1, 2, \ldots, n$ for all four cases.

To solve these equations we construct tridiagonal matrices from the recurrence relationship and compute the eigenvalues and eigenvectors of the resulting matrix. The eigenvalues then correspond to the eigenvalues $a_p, b_p$ of the Ince equation and the eigenvectors correspond to the coefficients $A_r, B_r$ of the Fourier series.

### A.2 Relations of Ince Gaussians with Hermite and Laguerre Gaussian Modes

It is well-known that the paraxial wave equation has two sets of solutions. The first set of solutions, known as the Hermite Gaussians, is the solution set obtained from solving the paraxial wave equation in rectangular coordinates and can be written in the form

$$HG_{n_x,n_y}(x, y, z) = \left(\frac{1}{2^{n_x+n_y-1\pi n_x!n_y!}}\right)^{1/2} \frac{1}{w(z)}$$

$$\times H_{n_x} \left(\frac{\sqrt{2x}}{w(z)}\right) H_{n_y} \left(\frac{\sqrt{2y}}{w(z)}\right) \exp \left[-\frac{-r^2}{w^2(z)}\right] \times \expi \left[kz + \frac{k^2r^2}{2R(z)} - (n_x + n_y + 1)\Psi_GS(z)\right] \quad (A.12)$$

where $H_n(\ . \ )$ are the nth order Hermite polynomials.

The second set of solutions, known as the Leguerre Gaussians, is obtained when
solving the paraxial wave equation in cylindrical coordinates and can be written as

\[
\text{LG}_{n,l}^{\sigma}(r, \phi, z) = \left[ \frac{4n!}{(1 + \delta_{0,l})\pi(n + l)!} \right]^{1/2} \frac{1}{w(z)} \left( \frac{\cos l\phi}{\sin l\phi} \right) \\
\times \left[ \frac{\sqrt{2r}}{w(z)} \right]^l L_n^l \left( \frac{2r^2}{w(z)^2} \right) \exp \left[ \frac{-r^2}{w^2(z)} \right] \\
\times \exp \left[ kz + \frac{k^2r^2}{2R(z)} - (2n + l + 1)\Psi_{GS}(z) \right]
\] \hspace{1cm} (A.13)

where \( L_n^l(\cdot) \) are the generalized Laguerre polynomials.

As we noted in Chapter 6, since the Laguerre, Hermite, and Ince mode Gaussians each constitute a complete solution set for the paraxial wave equation, we can write any mode from one family of solutions in terms of the solutions of another family. For example, we can write an IG mode in the form

\[
\text{IG}_{p,m}^{\sigma}(\xi, \eta, \epsilon) = \sum_{l,n} D_{l,n} \text{LG}_{n,l}^{\sigma}(r, \phi)
\] \hspace{1cm} (A.14)

where \( \sigma = \{e, o\} \) is parity and the summation is performed over all LG modes of parity \( \sigma \) such that \( p = 2n + l \). Likewise for Hermite Gaussian modes

\[
\text{IG}_{p,m}^{\sigma}(\xi, \eta, \epsilon) = \sum_{n_x,n_y} C_{n_x,n_y} \text{HG}_{n_x,n_y}^{\sigma}(x, y)
\] \hspace{1cm} (A.15)

where again the summation is performed over all HG modes such that parity of \( n_y \) matches that of \( \sigma \) and \( p = n_x + n_y \). We can compute the coefficients \( D \) and \( C \) by using the overlap integral between an IGM and LGM mode or IGM and HGM mode respectively. To compute the overlap integral between the IGM and LGM modes we can apply group theory techniques \([9]\) to yield

\[
\int \int_{-\infty}^{\infty} \text{LG}_{n,l}^{\sigma} \overline{\text{IG}_{p,m}^{\sigma'}} dS = \delta_{\sigma',\sigma} \delta_{p,2n+l}(-1)^{n+l+(p+m)/2} \\
\times \sqrt{(1 + \delta_{0,l})\Gamma(n + l + 1)n!A_{l+\delta_{x,\epsilon}}^{\sigma}/2(a_p^m)}
\] \hspace{1cm} (A.16)
where $A_{(l+\delta, o, p)}/2(a_p^m)$ is the $(l + \delta, o, p)/2$th Fourier coefficient of the $C_p^m$ or $S_p^m$ Ince polynomial.

As a consequence of the requirement that modes in the expansion of a given IG mode must all possess the same parity and Gouy phase, the summations of all the expansions among the three families of modes must involve a finite number of modes whose indices $(n, l)$ and $(n_x, n_y)$ satisfy the constraints $p = 2n + l = n_x + n_y$ for a given $p$. Each of these groups of modes then naturally split into subsets of degenerate modes of equal Gouy phase shifts and the same parity about the positive $x$. We denote these degenerate subgroups as $L_g^p$, $I_g^p$, and $H_g^p$ whose Gouy phase shifts all equal $\Psi_p(z) = (p + 1)\Psi_{GS}(z)$. Counting the number of modes in each subset we find that the number of modes in each degenerate subset can be written as

$$N_p = \begin{cases} 
\frac{(p + 2\delta, o, p)}{2}, & \text{if } p \text{ is even,} \\
\frac{(p + 1)}{2}, & \text{if } p \text{ is odd,}
\end{cases} \quad (A.17)$$

Each group of degenerate modes thus forms a complete subbasis of orthonormal modes under which any Gaussian field with Gouy shift $\Psi_p(z)$ can be expanded, and there exist linear transformation matrices for transforming from one subgroup to another which we write formally as

$$L_g^p = [L_T^p]L_g^p \quad (A.18)$$

$$H_g^p = [H_T^p]H_g^p \quad (A.19)$$

$$L_g^p = [H_T^p]H_g^p \quad (A.20)$$

where the $N_p \times N_p$ transformation matrices $[A_B T_p^o]$ are real unitary matrices that satisfy $[A_B T_p^o]^{-1} = [A_B T_p^o]^T = [B_A T_p^o]$ and whose columns (and rows) form a basis of $N_p$ orthonormal vectors of the $N_p$th-dimensional vector space. In the above light we see that we can think of the IG, LG, and HG modes as nothing more than different orthonormal basis sets which span the same set of all possible Gaussian fields.

If we wanted to compute the IG modes in terms of HG modes (for example, if we
wished to compute a given IG mode on a rectangular grid) then we could compute the overlap integral directly using the Hermite Gaussian version of Eq. A.16. However since \[ [H_I T_p^o] = [L_I T_p^o][H_L T_p^o] \] then only two of the matrices are independent. As a result, instead of computing the overlap integral directly to compute the matrix \[ [H_I T_p^o] \] we can instead compute \[ [L_I T_p^o] \] using Eq. A.16 and use the known equations [22] for the overlap integral between the HG and LG modes to compute \[ [H_G T_p^o] \] and compute \[ [H_I T_p^o] \] by computing the product of the two resulting matrices.
Bibliography


