Vignettes in String Theory: Using Geometry to Probe the Worldsheet and Strongly Coupled Physics

by

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Abstract

In this thesis, we construct hybrid linear models in which the chiral anomaly of a gauged linear sigma model is canceled by the classical anomaly of a gauged WZW model. Semi-classically, this corresponds to fibering the WZW model over the naïve target space of the sigma model. When the gauge group is abelian, we recover known non-Kähler compactifications; non-abelian models describe novel quasi-geometric flux vacua of the heterotic string.

Second, we also investigate sigma models that break worldvolume Lorentz invariance. Specifically, we calculate the one loop beta function for a target space metric whose worldvolume scales space and time differently, with dynamical exponent $z = 2$. We find, as in the isotropic case, the beta function is proportional to the Ricci curvature so that conformal invariance demands Ricci-flatness. We extend this analysis to the case where space and time derivatives come with different target space metrics. We also speculate about coupling the theory to gravity.

Finally, we continue the investigation of the recently discovered holographic correspondence between Reissner-Nordstrom black holes in AdS_4 and fermion correlation functions describing Non Fermi Liquids. We numerically study the effects of adding magnetic and electric dipole couplings for the fermions in the bulk. In general, the low energy physics is controlled by an emergent AdS_2 conformal dimension. We find that adding the dipole couplings changes the attainable dimensions. We also find that these couplings can drastically change the locations of fermi surfaces in momentum space.

Thesis Supervisor: John McGreevy
Title: Assistant Professor

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Chapter 1

Introduction: A Brief Tour of String Theory From Worldsheet to Target Space To Applications

It is worth asking: "What is the use of string theory?" It has long been known that the theory provides a consistent way to quantize gravity without the usual field theory divergences. It was once thought and hoped that the unique vacuum of the theory would be found, that this vacuum would contain the Standard Model and we would have a unified theory of gravity and the other forces. Since, it has been realized that there are, unfortunately, a very large number of vacua, making it very difficult to locate the "one" that is our universe. Some of the sections of this thesis will involve investigating some of these possible vacua.

Many have given up hope of locating the one, true, vacuum in this extensive landscape. All is not lost, however; string theory teaches us rather general things about the quantization of gravity and the nature of spacetime. However, it might be worth asking whether or not string theory has any other uses.

It is a beautiful fact that, indeed, the answer seems to be "yes." As we will review, through gauge gravity duality, string theory can provide theoretical tools for strongly coupled quantum systems without gravity, like Yang-Mills theory. In addition, there is recent evidence that such dualities can help us understand strongly coupled condensed matter phenomena that have resisted theoretical attack. In this way, string theory should be viewed as a tool that not only might give us information about the one "true" vacuum, but also as a calculational tool for strongly coupled systems. Only time will tell what other uses this mysterious theory will acquire in the future.

In this chapter we give a brief, whirlwind tour of string theory and gauge gravity duality. This tour is by no means intended to be complete or historical, but rather, to adequately motivate the projects discussed in subsequent chapters. The exposition here will follow parts of [1, 2] [3], [4] and [5] .

1.1 The Bosonic String

1.1.1 The Polyakov Action

Bosonic string theory is defined by a two dimensional quantum field theory of scalars, $X^\mu(\sigma_1, \sigma_2)$, $\mu = 1, \dots D$. Here σ_1 and σ_2 are the coordinates on the two dimensional spacetime (σ_1 is Euclidean time). We regard this spacetime as the worldsheet Σ that a string sweeps out as it evolves in time. The X^μ can be regarded as coordinates on a different spacetime, called the target space, X. In this way the fields are an embedding of the worldsheet into the target space

$$X^\mu : \Sigma \to X \tag{1.1}$$

For now, we do not know much about the target space X. Consistency of the 2d quantum field theory will eventually fix its dimension. The (Polyakov) action of the bosonic string is just these scalars coupled to 2d gravity

$$\begin{aligned} S &= S_p + S_\chi \\ &= \frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{g}\,g^{ab}\partial_a X^\mu \partial_b X_\mu + \frac{\lambda}{4\pi}\int d^2\sigma\,\sqrt{g}\,R \end{aligned} \tag{1.2}$$

for some coupling λ. $a = \{1,2\}$ runs over the worldsheet coordinates. α' is a dimensionful quantity which sets the string tension/scale of the interactions. 2-d gravity is "trivial" in the sense that the variation of S_χ is zero–it is a purely topological term. In fact, the term that multiplies λ is just the Euler number, χ, of the worldsheet (for simplicity, we work with closed strings, ignoring possible boundary terms). This action has a large set of symmetries. By construction, it has two dimensional diffeomorphism invariance under which we can change coordinates to $\sigma'_a(\sigma)$ with fields transforming as

$$\begin{aligned} X'^\mu(\sigma'_1, \sigma'_2) &= X^\mu(\sigma_1, \sigma_2) \\ \frac{\partial\sigma^{c'}}{\partial\sigma^a}\frac{\partial\sigma^{d'}}{\partial\sigma^b} g'_{cd}(\sigma'_1, \sigma'_2) &= g_{ab}(\sigma_1, \sigma_2) \end{aligned} \tag{1.3}$$

In addition this theory, in two dimensions only, possesses a local Weyl symmetry under which the coordinates do not change, but by which we can "blow up" the metric by a local factor

$$\begin{aligned} X'^\mu(\sigma_1, \sigma_2) &= X^\mu(\sigma_1, \sigma_2) \\ g'_{ab}(\sigma_1, \sigma_2) &= e^{2\omega(\sigma_1,\sigma_2)}\, g_{ab}(\sigma_1, \sigma_2) \end{aligned} \tag{1.4}$$

for any function ω.

1.1.2 Interactions as Sums Over Worldsheets

So far, as a quantum field theory, the Polyakov action (1.2) looks completely free–where are the string interactions? Actually, they are already contained in (1.2). To define the quantum theory, we must look at the path integral

$$\begin{aligned} Z &\equiv \int [dX\,dg]\; e^{-S} \\ &= \int e^{-\lambda\chi}\,[dX\,dg]\; e^{-S_p} \end{aligned} \quad (1.5)$$

where we sum over all scalar and metric configurations. We should also include a sum over distinct worldsheet topologies. This sum over topologies *is* the string interactions. For example, if we are interested in the string propagator, the first topology we must include is the free closed string–the cylinder worldsheet. We know that this propagator must get corrected quantum mechanically by interactions. These corrections correspond to worldsheets that are asymptotically cylindrical, but whose topology is distinct from the cylinder. Said differently, the corrections are just cylinders with handles added. The first correction, a worldsheet with one handle, correspond to a string that splits and then rejoins. The addition of each handle adds a genus to the worldsheet and decreases the Euler number by 2. Therefore the path integral is weighted by an extra factor of $e^{2\lambda}$. Our experience with Feynman diagrams tells us to associate this diagram with two factors of a closed string coupling g_c. We have learned that

$$g_c \sim e^{\lambda} \quad (1.6)$$

Of course, it should be mentioned (though it will not be elaborated upon), that the reason string theory is a promising grand unified theory is that this model of interactions yields all amplitudes finite in perturbation theory and cures the usual UV divergences associated with quantizing gravity as a quantum field theory.

Therefore, in some sense, the problem of string theory has been reduced to doing a path integral in a two dimensional quantum field theory (1.5).

1.1.3 Gauge Fixing and $2d$ Conformal Field Theory

Before evaluating, (1.5), we must gauge fix the Diff $\times$ Weyl symmetry; these local symmetries introduce a large number of redundancies in the path integral. To do so, one must follow the Fadeev-Popov procedure and add ghosts. Heuristically, a two dimensional metric g_{ab} has three local degrees of freedom, while the two dimensional diffeomorphism group has two degrees of freedom and the Weyl group one. Therefore, Diff $\times$ Weyl is just enough local symmetry to gauge gravity away and set $g_{ab} = \delta_{ab}$. Using this freedom, the gauge fixed action in "conformal coordinates," $z \equiv \sigma^1 + i\sigma^2$, is (ignoring ghosts and the topological contribution)

$$S_p = \frac{1}{2\pi\alpha'} \int d^2z \, \partial X^\mu \bar{\partial} X_\mu \tag{1.7}$$

with $\partial \equiv \partial/\partial z$ and $\bar{\partial} \equiv \partial/\partial \bar{z}$. There is still actually an infinity of unfixed Diff $\times$ Weyl under which the metric remains unchanged. We can make holomorphic coordinate changes

$$\begin{aligned} z' &= f(z) \\ \bar{z}' &= [f(z)]^* \end{aligned} \tag{1.8}$$

as long as we also make a compensating Weyl transformation $e^{2\omega} = |\partial f|^2$. In this gauge fixed form, (1.7) is the Lagrangian of the two dimensional free boson conformal field theory, while (1.8) are the associated infinite dimensional local conformal transformations. Therefore the study of the Polyakov action after gauge fixing has led us to the study of a two dimensional conformal field theory.

1.1.4 The Weyl Anomaly and the Critical Dimension

Of course, in order for (1.5) to be well defined, the gauge symmetry, Diff $\times$ Weyl, cannot be quantum mechanically anomalous. After gauge fixing, this means that there cannot be an anomaly in the conformal symmetry. Typically, however, there is such an anomaly, called the Weyl anomaly, which is proportional to the central charge of the CFT,

$$\langle T^a_a \rangle = -\frac{c}{12} R \tag{1.9}$$

where R is the scalar curvature of the worldsheet. Thus, in order for gauge group of string theory to be anomaly free, we must have $c = 0$. Actually, the full CFT of string theory is a sum of the ghost CFT used to gauge fix (which we have ignored until now) and the CFT of the D free bosons X^μ. Given a careful treatment, one finds that $c_G = -26$ and $c_X = D$. We thus must have that $D = 26$, i.e. in order for bosonic string theory to be consistent, it must be formulated in a target space with 26 dimensions. This is sometimes called the critical dimension of bosonic string theory.

1.1.5 General Backgrounds and Non Linear Sigma Models

When one properly quantizes (1.2), one finds that the lowest energy states are a scalar with negative mass (the tachyon), a massless symmetric tensor (the graviton), a massless antisymmetric tensor (the B field), and a massless scalar mode (the dilaton). The tachyon signals an instability; this will be ignored in this discussion because we're really just warming up for superstring theory, which is tachyon free.

The action we've been considering (1.2) is an action formulated in *flat* target space, meaning that there is no target space diffeomorphism invariance $X^\mu \to X^{\mu'}(X)$, only a global Lorentz invariance. Said differently, the X's are contracted with $\eta_{\mu\nu}$, not some general curved metric $G_{\mu\nu}(X)$. Why not reconsider an action with some general curved $G_{\mu\nu}(X)$? This corresponds to considering strings in some coherent state vacuum of gravitons. While we're at it, we might as well consider strings in a coherent state background of the other massless fields

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \left[\left(g^{ab} G_{\mu\nu}(X) + i\epsilon^{ab} B_{\mu\nu}(X) \right) \partial_a X^\mu \partial_b X^\nu + \alpha' R \Phi(X) \right] \tag{1.10}$$

In addition to the worldsheet gauge symmetries (Diff $\times$ Weyl), this action has new global symmetries. One of these is target space diffeomorphisms, under which G and B transform as two tensors, while Φ transforms as a scalar. There is also a new gauge symmetry, $B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \xi_\nu(X) - \partial_\nu \xi_\mu(X)$, under which S changes by a total derivative.

Actions on curved target spaces such as (1.10) are called *non linear sigma models* (NLSM's) and will play an important role in the rest of this thesis.

We again must require that the Weyl anomaly vanish. In this curved background, careful calculation reveals the Weyl anomaly to be

$$\langle T^a_{\ a} \rangle = -\frac{1}{2\alpha'} \beta^G_{\mu\nu} g^{ab} \partial_a X^\mu \partial_b X^\nu - \frac{i}{2\alpha'} \beta^B_{\mu\nu} \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu - \frac{1}{2} \beta^\Phi R \tag{1.11}$$

Where the β's are the β functions for the couplings in the two dimensional NLSM (1.10)

$$\begin{aligned}
\beta^G_{\mu\nu} &= \alpha' R_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\lambda\omega} H_\nu^{\ \lambda\omega} + O(\alpha'^2) \\
\beta^B_{\mu\nu} &= -\frac{\alpha'}{2} \nabla^\omega H_{\omega\mu\nu} + \alpha' \nabla^\omega \Phi H_{\omega\mu\nu} + O(\alpha'^2) \\
\beta^\Phi &= \frac{D-26}{6} - \frac{\alpha'}{2} \nabla^2 \Phi + \alpha' \nabla_\omega \Phi \nabla^\omega \Phi - \frac{\alpha'}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + O(\alpha'^2)
\end{aligned} \tag{1.12}$$

where $H_{\omega\mu\nu} \equiv \partial_\omega B_{\mu\nu} + \partial_\mu B_{\nu\omega} + \partial_\nu B_{\omega\mu}$ is the gauge invariant field strength. Note that such a perturbative expansion only makes sense if $\alpha'^{1/2} R_C^{-1} \ll 1$ where R_C^{-1} is the radius of curvature of the target space. That is, the target space is weakly curved compared to the string scale. Thus, for conformal invariance to hold, we must have, order by order in worldsheet perturbation theory $\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\Phi = 0$. At lowest order in the dilaton β function, this again requires $D = 26$. Remarkably, in the absence of matter, $\Phi = H = 0$, the first equation of (1.12) gives us Einstein's equations on the target space, $R_{\mu\nu} = 0$. We will see in the next section that in the presence of matter, the vanishing of the β functions can also be understood as being derived from some covariant gravity + matter action in 26 dimensions. We will reproduce this classic computation that $\beta^G_{\mu\nu} \sim R_{\mu\nu}$ (for $\Phi = H = 0$) in Chapter 3.

1.1.6 The Low Energy Effective Action

The conformal invariance equations to lowest order, (1.12) can be derived as the classical equations of motion of an action

$$S = \frac{1}{2\kappa^2} \int d^{26}x \, (-G)^{1/2} e^{-2\Phi} \left[R - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + 4 \partial_\mu \Phi \partial^\mu \Phi \right] \tag{1.13}$$

known as the *low energy effective action.* A classical Einstein-Hilbert action is regained by making a field redefinition on G. This action, as a quantum theory, also correctly reproduces string scattering amplitudes to lowest order in α'. Comparing the amplitudes to (1.13) can be used to derive the Newton's constant κ in terms of g_s and α'.

1.1.7 Anisotropy on the Worldvolume?

The emergence of Einstein's equations is a triumph of string theory. It is worth asking how much we can tweak the worldvolume theory without doing violence to this beautiful result. To this end, in Chapter 3 we consider NLSM's that explicitly break Lorentz invariance on the worldvolume. While seemingly crazy, there has been great recent success in importing anisotropic ideas from condensed matter theory into particle physics. For example, Horava has recently written down ([6], [7]) a naively renormalizable theory of gravity in four dimensions that treats space and time anisotropically.

The NLSM's that we consider are of the form

$$S = \int dt \, d^2\sigma \, \left[G_{\mu\nu}(X) \, \partial_t X^\mu \partial_t X^\nu - \alpha G_{\mu\nu}(X) (\triangle X^\mu + \Gamma^\mu_{\rho\sigma} \partial_a X^\rho \partial^a X^\sigma)(\triangle X^\nu + \Gamma^\nu_{\alpha\beta} \partial_b X^\alpha \partial^b X^\beta) \right] \tag{1.14}$$

with flat worldvolume metric and $\triangle = \partial_a \partial^a$. These models retain diffeomorphism symmetry in the target space and also have an anisotropic conformal symmetry on the worldsheet under which $t \longrightarrow \lambda^2 t$ and $x \longrightarrow \lambda x$ (this is called the $z = 2$ anisotropic conformal symmetry with $t \longrightarrow \lambda^z t$ being the more general case). Note that this symmetry holds only in three dimensions, which is the reason for considering a worldvolume of membranes instead of strings. We examine this anisotropic conformal symmetry at the one loop quantum level by computing the β function for $G_{\mu\nu}$ We find, quite remarkably, that at one loop

$$\beta^G_{\mu\nu} \sim R_{\mu\nu} \tag{1.15}$$

once again. So while isotropy is broken on the worldvolume, target space diffeomorphism invariance still holds. Of course, this is not yet a fully dynamical theory of membranes (notice that we did not include worldvolume gravity). In Chapter 3 we also write down a classical theory of this NLSM anisotropically coupled to Horava gravity. Unfortunately, we are unable to say much about the membrane theory at the quantum level because unlike in the worldsheet case, we can not fully gauge gravity away.

1.1.8 Compactification

Returning to Lorentz invariant string theories, we observe a universe that is locally 4d Minkowski space, M_4, and presumably non-compact. This means that the extra dimensions required by string theory must be "curled up" into a compact manifold represented by some curved, Ricci flat, non linear sigma model. M_4 is represented by the free CFT of four bosons with $c = 4$. Therefore, the extra dimensions must be some NLSM that flows to a $c = 22$ CFT. In fact, we can be much more general and replace the "extra dimensions" by *any* $c = 22$ 2d CFT, even if it has no "geometric" interpretation as an NLSM. $c = 22$ is necessary so that we can non-anomalously couple things to worldsheet gravity. The requirement that this more general CFT be "compact" just means that the CFT should have a discrete spectrum. This attitude frees us from the shackles of thinking of the "extra dimensions" as geometric objects and will be an important motivation for the material in Chapter 2.

1.2 The Superstring

While the bosonic string is an interesting system, it suffers from the tachyon instability. Even if we could somehow ignore this, it has only bosonic excitations–a phenomenological problem; the universe has fermions. It turns out that we will get spacetime fermions by adding 2d fermions to the worldsheet CFT. In the following sections we show how to do this, whilst maintaining supersymmetry between the bosonic and fermionic operators. This development of the superstring proceeds in parallel with the previous section.

1.2.1 Type II Strings and $(1,1)$ Superconformal Invariance

The starting point for the bosonic string was scalars coupled to two dimensional gravity. The superstring is just the bosonic string plus supersymmetry, and so its starting point is scalars and fermions coupled to 2d supergravity (SUGRA) (the relevant amount of supersymmetry here will be non-chiral $(1,1)$, we consider the chiral $(0,1)$ in a subsequent section). We do not write out the full SUGRA action here, though once again there is a Diff $\times$ Weyl symmetry that we can use to gauge fix the graviton and the gravitino. One is again left with a CFT–this time of free bosons *and* fermions. In fact, one is left with a superconformal field theory; a CFT that satisfies the enlarged $(1,1)$ superconformal algebra (a joining of the conformal symmetry with SUSY). The gauge fixed action is

$$S = \frac{1}{4\pi}\int d^2z \left(\frac{2}{\alpha'}\partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial}\psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu\right) \tag{2.16}$$

with $\mu = 1 \ldots D$. The equations of motion imply that ψ is a function of z only, i.e. it is holomorphic while $\tilde{\psi}$ is antiholomorphic–these are left and right moving Majorana-Weyl spinors. Each spinor contributes $1/2$ to the right/left CFT central charge. Therefore the total central charge of this SCFT is $c_X = D + D/2$. However, we must also carefully analyze the ghosts needed to gauge fix SUGRA away. As it turns out, these ghosts are the superconformal completion of the bosonic ghost CFT with central charge is $c_G = 15$. Therefore, the vanishing of the Weyl anomaly implies that $D = 10$, the critical dimension of superstring theory.

1.2.2 Excitations of the Type II String

For closed strings, we have a choice of boundary conditions for $\psi^\mu, \tilde{\psi}^\mu$ as we go 2π around the string. If the fermion comes back to itself, these are Ramond (R) boundary conditions; if they return with a minus sign, Neveau-Schwartz (NS). Having set the boundary conditions, we can use the mode expansions of $\psi^\mu, \tilde{\psi}^\mu$ and X^μ to construct the Hilbert spaces of both the left handed and right handed CFTs. Let's for a moment just focus on the left handed CFT. When one quantizes the NS Hilbert space, one discovers that the lowest lying mode is a tachyon with negative mass (have no fear, this tachyon will soon be projected out of the spectrum). The next excited state is obtained by applying the lowest mode operator of ψ^μ to this vacuum. It is massless and a spacetime vector (it transforms in the $\mathbf{8}_V$ of the spacetime little group, $SO(8)$). All other excited modes are massive. Note that since the vacuum is a spacetime (target space) boson, all excitations are also spacetime bosons (there is ample opportunity for confusion between worldsheet/target space). Now, the mode expansion for the Ramond sector ψ^μ has a zero mode which does not annihilate the vacuum. Instead, these zero modes form a representation of the spacetime Clifford algebra on the vacuum. Indeed, the Ramond vacuum state is a 10d Dirac spinor. This representation is reducible to two Weyl spinors, the $\mathbf{8}$ and $\mathbf{8}'$ of $SO(8)$. These vacua are massless while all higher excitations in the R sector are massive.

One now can define a worldsheet fermion operator, F, eigenvalues ± 1, which grades whether or not states are worldsheet fermions or bosons. For consistency with the ghost CFT, the NS vacuum needs to have $F = 1$, while its first excitation has $F =$

-1. In the R sector, the $\mathbf{8}$ has $F = 1$ while the $\mathbf{8}'$ has $F = -1$. It is entirely consistent to consider $F = \pm 1$ separately as conformal field theories. Thus, we seemingly have a large number of choices as to which states to include in our string theory. Not only do we have the choice of NS or R on both sides, but also whether we include $F = \pm 1$ on the left and $\tilde{F} = \pm 1$ on the right. As it turns out, there are only two consistent choices for closed strings, type IIB and IIA.

Type IIB

The IIB theory is defined by keeping the R and NS sectors on both sides, but keeping only $F = \tilde{F} = 1$. Thus, we end up with $SO(8)$ representations

$$(\mathbf{8}_v + \mathbf{8}) \times (\mathbf{8}_v + \mathbf{8}) = \underbrace{[0] + [2] + (2)}_{NS-NS} + \underbrace{[0] + [2] + [4]_+}_{R-R} + \underbrace{(\mathbf{8}')^2 + (\mathbf{56})^2}_{NS-R/R-NS} \qquad (2.17)$$

The brackets represent antisymmetric representations of $SO(8)$ while the (2) is the traceless, symmetric representation. The NS-NS sector has bosonic excitations that are very much like the bosonic string–a dilaton, a B field and a metric. The RR sector has new bosonic excitations (fermion $\times$ fermion=boson), differential forms C_0, C_2, and C_4 corresponding to gauge invariant field strengths F_1, F_3, F_5, where F_5 is self dual. The NS-R and NS-R sector have fermions $\mathbf{8}'$'s and $\mathbf{56}$'s (fermion $\times$ boson=fermion), known as the dilatinos and gravitinos. This resulting theory actually has spacetime supersymmetry with 32 supercharges, two spinors of the same chirality. The dilatinos and gravitions are the supersymmetric partners of the NS-NS and R-R bosons. Notice that this theory is spacetime chiral–it does not contain a right handed spinor for every left handed spinor. Actually, the SUSY is enough to fix the low energy effective field theory uniquely; it is type IIB supergravity in ten dimensions with (bosonic) action

$$\begin{aligned} S_{IIB} &= S_{NS} + S_R + S_{CS} \\ S_{NS} &= \frac{1}{2\kappa^2} \int d^{10}x \, (-G)^{1/2} e^{-2\Phi} \left(R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}|H_3|^2 \right) \\ S_R &= -\frac{1}{4\kappa^2} \int d^{10}x \, (-G)^{1/2} \left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2 \right) \\ S_{CS} &= -\frac{1}{4\kappa^2} \int C_4 \wedge H_3 \wedge F_3 \end{aligned} \tag{2.18}$$

with $H_3 = dB_2$ as usual and $\tilde{F}_3 \equiv F_3 - C_0 \wedge H_3$ and $\tilde{F}_5 \equiv F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3$.

Type IIA

The IIA theory also keeps both R and NS sectors on both sides, but keeps $F = 1$ on the left hand side and $\tilde{F} = 1$ for the NS sector and $\tilde{F} = -1$ for the R sector on the right. We get

$$(\mathbf{8}_v + \mathbf{8}) \times (\mathbf{8}_v + \mathbf{8}') = \underbrace{[0] + [2] + (2)}_{NS-NS} + \underbrace{[1] + [3]}_{R-R} + \underbrace{\mathbf{8} + \mathbf{8}' + \mathbf{56} + \mathbf{56}'}_{NS-R/R-NS} \tag{2.19}$$

Again the NS-NS sector has the same excitations, while the R-R sector now has forms C_1 and C_3 corresponding to even rank field strengths F_2, F_4. This theory, too, is spacetime supersymmetric with 32 supercharges, but here there are two spinors of opposite chiralities and the theory is nonchiral. The NS-R/R-NS sector are again dilatinos and gravitinos and are the supersymmetric completion of the other sectors. Again, the SUSY fixes the low energy effective action to be that of type IIA supergravity, with bosonic action

$$\begin{aligned} S_{IIA} &= S_{NS} + S_R + S_{CS} \\ S_{NS} &= \frac{1}{2\kappa^2}\int d^{10}x\,(-G)^{1/2}e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H_3|^2\right) \\ S_R &= -\frac{1}{4\kappa^2}\int d^{10}x\,(-G)^{1/2}\left(|F_2|^2 + |\tilde{F}_4|^2\right) \\ S_{CS} &= -\frac{1}{4\kappa^2}\int B_2\wedge F_4\wedge F_4 \end{aligned} \tag{2.20}$$

with $\tilde{F}_4 \equiv dC_3 - C_1 \wedge F_3$.

1.2.3 The Heterotic String

The heterotic string, is, in some sense, a joining (or *heterosis*) of the bosonic and type II strings. In everything up until now, we have put the same CFT or SCFT on the left and on the right–the theories are non-chiral on the worldsheet. The heterotic string changes this and takes the ghosts/constrains on the left side from the bosonic string and the ghosts/constrains on the right side from the type II string. Therefore, for vanishing of the Weyl anomaly, we need a $c = 26$ matter CFT on the left and a $\tilde{c} = 15$ CFT on the right. The heterotic string keeps the ten bosons, $X^\mu(z, \bar{z})$ (which we still interpret as target space coordinates) and ten right moving fermions $\tilde{\psi}^\mu(\bar{z})$. This choice preserves the right handed portion of the SUSY/SCFT algebra and so the heterotic string will typically have only $(0, 1)$ superconformal symmetry. This sector is a $(c, \tilde{c}) = (10, 15)$ theory and so we must supplement it with a $c = 16$ CFT. The easiest way is to add 32 free left moving fermions $\lambda^A, A = 1 \ldots 32$. This sector has a global $SO(32)$ symmetry and so the naive global symmetry of this model is $SO(8)_{spin} \times SO(32)$. We still interpret the $SO(8)$ as the spacetime Lorentz symmetry, while the $SO(32)$ will turn out to be a spacetime gauge symmetry. On both left moving and right moving fermions, we must investigate the R and NS vacua and decide which F and $\tilde{F}$ to project.

1.2.4 Excitations of the Heterotic String

We will keep both NS and R states on both sides of the heterotic string and keep only states with $F = \tilde{F} = 1$. The analysis of the right moving R and NS sectors proceeds as in the previous section. Because of the different gauge fixing/constraints on the left, however, the analysis of the NS and R vacua are different. The R vacuum turns out to be massive, and so we will forget about it. The NS ground state is a tachyon and has $F = 1$, but since closed string states must have left and right states with the same mass, it gets projected out. The first excited state gets projected out because $F = -1$. There are two states at the next level which are massless. One is obtained by applying a raising operator from X^μ and is an $SO(8)$ vector. The other is obtained by applying two raising operators from λ^A and is an antisymmetric tensor of $SO(32)$, the adjoint representation. All other states are massive. Labeling both $SO(8)$ and $SO(32)$ quantum numbers, the vector is an $(\mathbf{8}_v, \mathbf{1})$ while the antisymmetric tensor is a $(\mathbf{1}, \mathbf{496})$. Tensoring this with the right handed CFT, we get

$$\begin{aligned}
[(\mathbf{8}_v, \mathbf{1}) + (\mathbf{1}, \mathbf{496})] \times (\mathbf{8}_v + \mathbf{8}) &= \underbrace{(\mathbf{1}, \mathbf{1}) + (\mathbf{28}, \mathbf{1}) + (\mathbf{35}, \mathbf{1}) + (\mathbf{56}, \mathbf{1}) + (\mathbf{8}', \mathbf{1})}_{\text{Type I SUGRA multiplet}} \\
&+ \underbrace{(\mathbf{8}_v, \mathbf{496}) + (\mathbf{8}, \mathbf{496})}_{\text{Type I gauge multiplet}}
\end{aligned} \tag{2.21}$$

As indicated, this theory is supersymmetric and has the massless content of type I SUGRA (a theory with one chiral supercharge) coupled to an $SO(32)$ gauge multiplet. A different choice of projection on F actually gives type I SUGRA coupled to the larger gauge symmetry, $E_8 \times E_8$. The low energy effective action is thus fixed to be that of type I supergravity, (bosonic action shown only, as usual)

$$S_{het} = \frac{1}{2\kappa^2} \int d^{10}x \, (-G)^{1/2} e^{-2\Phi} \left(R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}|\tilde{H}_3|^2 - \frac{\kappa^2}{g^2} Tr\left(|F_2|^2\right) \right) \tag{2.22}$$

where F_2 is the gauge field strength and $\tilde{H}_3 \equiv dB_2 - \frac{\kappa^2}{g^2}\omega_3$ and ω_3 is the Chern-Simons form of the gauge field.

1.2.5 Heterotic Calabi-Yau Compactifications

We will briefly discuss the standard heterotic Calabi-Yau compactification. We assume spacetime to be of the form $M_4 \times M$ where now M is some six dimensional, compact manifold and search for solutions to the low energy effective theory, Type I SUGRA. Compactification will, in general, break the supersymmetries, but we wish to preserve $N = 1$ SUSY in four dimensions. The reason for this is two fold–a little evidence and a lot of wishful thinking has led us to believe that there might be TeV scale supersymmetry. Also, the SUGRA equations of motion will automatically be satisfied if we can solve the simpler first order BPS equations. We should also impose the Bianchi identity

$$d\tilde{H}_3 = \frac{\alpha'}{4}\left[tr(R_2 \wedge R_2) - Tr(F_2 \wedge F_2)\right] \qquad (2.23)$$

Without details, for SUSY to hold we must have $\delta(\text{fermions}) = 0$ where δ indicates a SUSY transformation on the gravitino, the dilatino and the gaugino. It is still terribly difficult to make progress with these equations and so we focus on a particular simple class of backgrounds with $\tilde{H} = 0$ (sometimes called "torsion free") and $\Phi = 0$. Doing a thorough analysis of the SUSY equations, one finds that M is required to be Ricci flat, and also that there must exist a covariantly constant spinor on M (which in turn, means that M is a manifold of $SU(3)$ holonomy). With this covariantly constant spinor in hand, one can then construct an integrable complex structure, a Kahler form and a nowhere vanishing holomorphic top form. This implies that M should be a complex, Kahler manifold with vanishing first Chern class, also known as a Calabi-Yau 3-fold (3 is the number of complex dimensions). Yau's theorem tells us that such manifolds admit Ricci flat metrics (for each Kahler class), and so there exist solutions to the SUGRA equations. Unfortunately, there are no known 3d CY metrics in closed form. Nevertheless, the fact that we know that M is topologically CY gives valuable information about the low energy theory. It should be mentioned that (2.23) is a very hard equation to solve, even for $\tilde{H} = 0$. One usually takes a "trivial" solution, where the spin connection is embedded in the gauge connection. For solutions with nonzero $d\tilde{H}$ this equation has resisted a solution in SUGRA until very recently. We will have more to say about this in Chapter 2.

1.2.6 CY's From the Worldsheet Perspective

The previous section discussed compactification from a supergravity/target space perspective, but a proper discussion of compactification should take a worldsheet view. From this perspective the "compact dimensions" are interpreted as some $\tilde{c} = 9$ CFT, for example a $(0,2)$ NLSM on a target space M. This NLSM is a theory of the compact bosons and their right moving superpartners, which transform as target space tangent vectors. The action (taking a flat dilaton background) for such an NLSM is (now labeling the compact dimensions $i, j = 1, \ldots 6$),

$$S = \frac{1}{2\pi\alpha'} \int d^2z \left[G_{i\bar{j}}(X) \partial X^{(i} \bar{\partial} X^{\bar{j})} + B_{i\bar{j}}(X) \partial X^{[i} \bar{\partial} X^{\bar{j}]} \right] + i G_{i\bar{j}}(X) \bar{\psi}^{\bar{j}}_{+} D_{\bar{z}} \psi^{i}_{+} \tag{2.24}$$

where ψ^i_+ are the right moving fermions and $D\psi \equiv \partial\psi + \partial X^j \Gamma^i_{jk} \psi^k$ is the pullback to the worldsheet of the target space covariant derivative. $(0,2)$ SUSY requires that G be hermitian and Kahler. Actually, vanishing of the $U(1)_R$ anomaly for the fermions (the $U(1)_R$ symmetry is part of the $(0,2)$ algebra) requires that the first Chern class of M vanishes, i.e. that the manifold is topologically Calabi-Yau. Again, one can calculate the β function for G perturbatively and one finds again that at lowest order we must have $\beta^G_{i\bar{j}} = R_{i\bar{j}} = 0$, i.e. to lowest order in α' the metric is the CY metric. This is the same requirement from the supergravity analysis, but here we see that the metric will be corrected at higher order in worldsheet perturbation theory. The fact that we have $(0,2)$ SUSY means that we can do whatever we want with the left moving fermions, as long as their CFT has the correct central charge. We can keep them free or we can fiber them over the right moving NLSM in some non-trivial way (this involves adding a covariant derivative term to (2.24) with left moving fermions in some general bundle).

Again, we should remember not to take such an unenlightened, purely geometric view of compactification. Any SCFT's with the correct symmetries and central charges will do.

1.2.7 Constructing Worldsheet Theories With Non-Trivial H

In Chapter 2 we will review $(0,2)$ gauged linear sigma models (GLSMs), 2d gauge theories that are thought to flow to $(0,2)$ NLSMs (which themselves are thought to flow to $(0,2)$ SCFTs). It will turn out that these gauge theories will have a moduli space of vacua that is precisely M, the target space of the low energy NLSM. In order for the GLSM to be consistent and nonanomalous we will again find the requirement that M be a CY manifold.

This, of course, suggests how to make non CY's with H flux–make the GLSM anomalous. In particular we will introduce a quantum gauge anomaly. One might think that this would be disastrous but we add another theory, a classically anomalous gauged WZW model, so that the total gauge anomaly vanishes. The effect will be to fiber the WZW model over the gauge theory moduli space and to create a total space with non-vanishing $d\tilde{H}$ that automatically satisfies (2.23). In the case where the WZW model is a theory of two free, periodic bosons, the vacua we find will reduce to the recently discovered T^2 fibrations over K3 discovered by Fu and Yau ([8, 9]).

1.3 Gauge Gravity Duality and Condensed Matter Physics

1.3.1 Basics of the Correspondence

The holographic principle, gauge gravity duality, or the generalized AdS/CFT conjecture (it goes by many names) is the following remarkable statement–some strongly coupled quantum theories are secretly theories of gravity (perhaps coupled to matter) in some number of extra dimensions. The quickest most heuristic argument for this statement goes as follows: gravity is *weird*. More specifically, GR theorems from the 1970's have taught us that black holes carry entropy associated with the area of their horizons. This, surprisingly, leads to the conclusion that the maximum entropy of some volume, V, is the entropy of the largest black hole that can fit (think not, just throw some more stuff in and increase the entropy until you form a black hole). Therefore, the entropy of V scales like the area that bounds it, A, which is very strange. We are used to working with local quantum field theories which have degrees of freedom at each point in spacetime. Since the maximum entropy is the log of the number of degrees of freedom, entropy should scale like V, not A. This suggests that gravity has the same number of degrees of freedom as some local quantum field theory in one less dimension. Since we have a pretty good idea of what weakly coupled quantum field theories look like (and since they do not, generally, look like gravity in one dimension more), the corresponding local QFT should be strongly coupled. Indeed, this sort of reasoning would lead one to conjecture that inside *every* theory of gravity there is a strongly coupled field theory in extra dimensions. In fact, this is the strongest version of the conjecture. Though this conjecture is incredibly broad, many specific examples have been investigated.

What is this extra dimension from the view of the strongly coupled QFT? Our experience with field theory tells us that we should regard observables as a function of the scale at which we observe them, μ. We also know that operators and correlators depend on μ through their renormalization group equations (RGEs), which, remarkably, are local in μ. This suggests that it is not very crazy to associate the extra gravitational dimension with the energy scale μ of the field theory, and that the equations of motion in the gravitational theory (sometimes called the bulk) somehow encode the RGE equations of the field theory (called the boundary).

The utility of this conjecture is remarkable. Many of the outstanding problems in theoretical physics stem from our lack of good theoretical tools for strongly coupled quantum systems. Though we can write down the Lagrangian for Quantum Chromodynamics (QCD), a through understanding of hadrons and nuclei remains frustratingly elusive because of strong coupling that sets in at $\mu \sim \Lambda_{QCD}$. Similarly, we can not even effectively solve for electrons in a metal because of strong coupling! Typical electron interaction energies are much larger then free electron energies (such as the width of the conduction band), and so we can't treat the interactions as a small perturbation. It is very fortunate that there is often a weakly coupled description that is basically a free Fermi gas: Fermi liquid theory. However, when the material is not a fermi liquid, there are few good theoretical tools.

One might ask what good the conjecture is if we have replaced the strong coupling problem with the problem of solving a gravitational theory, another great, unsolved challenge of theoretical physics. The answer is that sometimes we get lucky and the theories of gravity corresponding to the local QFTs are *classical*. Classical theories of gravity are "easy." As we will see, calculating a quantity for the strongly coupled quantum theory will reduce in the gravity theory to something relatively simple, like solving a wave equation. Of course, this will be a wave equation on some classical curved background, which we can do by hand in only a few circumstances, but we will often be able to make some analytical statements and make even more progress using a laptop computer.

1.3.2 $\mathcal{N} = 4$ Super Yang Mills and AdS/CFT

We briefly review the most studied example of the correspondence: $\mathcal{N} = 4$ $SU(N)$ Super Yang Mills (a theory similar to QCD) and gravity on the space $AdS_5 \times S^5$. What follows is not a proof, but rather the standard decoupling argument of plausibility.

Consider type IIB string theory on a flat, ten dimensional Minkowski background along with N coincident D3 branes. We have not discussed D-branes, but they are extended objects which source RR flux (here F_5) on which open strings end. So we have closed strings in the bulk and open strings constrained to end on the D-branes. Looking at the low energy effective theory, the closed strings give us the usual type IIB supergravity (2.18) while it turns out that the low energy excitations of the open string are a four dimensional, $\mathcal{N} = 4$ (the maximum SUSY in four dimensions) $U(N)$ super Yang Mills theory living on the branes. At low energies, these two sectors decouple, giving the SYM theory and type IIB SUGRA on flat space.

This system has another description; D3 branes can be viewed as solitonic type II SUGRA objects, with metric

$$\begin{aligned} ds^2 &= f^{-1/2}(-dt^2 + \sum_{i=1}^{4} dx_i^2) + f^{1/2}(dr^2 + r^2 d\Omega_5^2) \\ f &\equiv 1 + \frac{R^4}{r^4} \end{aligned} \tag{3.25}$$

constant dilaton, and five flux $F_5 = (1+*)dt\, dx_1\, dx_2\, dx_3\, df^{-1}$. The SUGRA equations relate R to the ten dimensional Newton constant, which, in turn, can be related to g_s and α', $R^4 = 4\pi g_s(\alpha')^2 N$. There are two types of low energy excitations; at $r \gg R$ there are massless modes in an asymptotically flat region–type IIB supergravity. As $r \ll R$, all modes (including string modes) become redshifted to give another sector of low energy excitations. These two sets of IR excitations completely decouple. The first set gives flat, type IIB supergravity, while the second gives IIB string theory on $AdS_5 \times S^5$ (the small r limit of (3.25)) with constant dilaton and F_5 flux on the S^5.

Since our two descriptions of this system involve a decoupled flat IIB SUGRA background, it is natural to identify the $\mathcal{N} = 4$, $SU(N)$ SYM and IIB string theory on $AdS_5 \times S^5$ as equivalent (the $U(1)$ of the SYM completely decouples; from the SUGRA analysis, it lives on the connecting region between the throat and the asymptotically flat region). Of course this isn't a proof; one description is valid precisely when the other one fails. The description in terms of weakly coupled SYM theory is valid when

$$g_{YM}^2 N \sim g_s N \sim \frac{R^4}{(\alpha')^2} \ll 1 \tag{3.26}$$

whereas the description in terms of a string NLSM is valid when the space is weakly curved with respect to the string scale,

$$\frac{R^4}{(\alpha')^2} \gg 1 \tag{3.27}$$

In addition, though this correspondence is beautiful, we would also like it to be *useful*, that is, we would like to be able to calculate something about strongly coupled SYM theory using *classical* gravity (without string corrections). We would like the space to be weakly curved with respect to the Planck length

$$\frac{R^4}{l_p^4} \sim \frac{R^4}{g_s(\alpha')^2} \sim N \gg 1 \tag{3.28}$$

that is, classical SUGRA computes quantities for $\mathcal{N} = 4$ SYM in the $N \to \infty$ limit.

It should be said that in practical use, one usually dimensionally reduces on the S^5 and thinks of the theory as type IIB SUGRA coupled to various matter in AdS_5. That one can do this is far from clear (the sphere is the same size as the characteristic length of AdS), but it turns out that there is a consistent KK reduction.

1.3.3 Other Examples of the Correspondence

Many other examples of the correspondence have been discovered using similar decoupling arguments. For example, one can change the gauge group of the $\mathcal{N} = 4$ theory by replacing the S^5 with an $\mathbb{RP}^5$. One can orbifold the S^5 to break the boundary field theory to $\mathcal{N} = 2$ or replace the S^5 with some more general Sasaki-Einstein space to get $\mathcal{N} = 1$ theories. One can change these theories in additional ways, such as adding wrapped D5 branes and fractional D3 branes to break the conformal symmetry and beautifully investigate the running coupling, confinement and chiral symmetry breaking from the bulk point of view. The former constitute just a very small listing of known gauge/gravity duals.

In recent years, however, a new attitude has been developing. One constructs *some* classical gravity solution in the bulk. If one takes gauge/gravity duality very seriously, this should correspond to *some* boundary field theory. The recent view has been to postpone identifying what this strongly coupled boundary field theory is and to be content calculating interesting things about it. After all, we're not really interested in $\mathcal{N} = 4$ super yang mills or $\mathcal{N} = 1$ klebanov-witten theory–we are interested in QCD; we want to suss out the qualities that these theories have in common with QCD.

As we will see, the isometries of the gravity solution correspond to the spacetime symmetries of the boundary field theory. Suppose we are interested in some class of field theories with a specified spacetime symmetry group. We can study the boundary theories by constructing bulk gravitational solutions with an identical isometry group. If we are lucky, our investigations will lead us to results that are universal across a large range of such theories.

1.3.4 The Dictionary

Though we have discussed gauge/gravity duality as an equivalence of two theories, we have not specified how the variables of the bulk gravity theory map onto the observables of the QFT. For concreteness, we review how this matching works for $\mathcal{N} = 4$ super yang mills and $AdS_5 \times S^5$. We also briefly review how to calculate two point functions.

Symmetries

The two theories (IIB on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM) enjoy all of the same symmetries (as they should if they are to be the same theory!). For example, they both enjoy a bosonic $SO(4,2) \times SO(6)$ symmetry. The AdS_5 metric,

$$ds^2 = \frac{r^2}{R^2}\sum_{\mu=1}^{4}(dx^\mu)^2 + R^2\frac{dr^2}{r^2} \tag{3.29}$$

is obviously $SO(3,1)$ invariant. It is also invariant under dilations, $x^\mu \longrightarrow \lambda x^\mu$, $r \longrightarrow \lambda^{-1}r$. These actually generate the whole conformal group, $SO(4,2)$. The $SO(6)$ symmetry is just rotations of the S^5, its isometry group. In the field theory, the $SO(3,1)$ subgroup is the usual Lorentz group while dilations are also symmetries of the SYM theory (in particular, the dilation symmetry follows at the quantum level from the vanishing of the β function). These subgroups also get enlarged to the conformal group in the usual way. The $SO(6)$ is a global R-symmetry which does not commute with SUSY.

The fermionic symmetries of these theories also match. Type IIB string theory (and supergravity) in flat space has the maximum number of supercharges allowed, 32. As it turns out, the $AdS_5 \times S^5$ background does not break any of these. Similarly, the SYM theory not only has the 16 supercharges from the $\mathcal{N} = 4$ SUSY, but also 16 additional supercharges under the superconformal group, the SUSY completion of $SO(4,2)$.

Fields and Operators

In the event that both sides of the duality are known, we can make a definite mapping between classical fields in the gravity bulk and operators in the boundary. Again, we will use an example of the mapping for $\mathcal{N} = 4$ SYM.

We can organize operator representations of the superconformal algebra by starting with operators that are so called "superconformal primaries." We get other operators in the representation by acting with Q and P of the SUSY algebra on these special operators. It turns out that they are of the form

$$\mathcal{O}^{i_1 \ldots i_l} \equiv Tr(X^{\{i_1} \ldots X^{i_l\}}) \tag{3.30}$$

where $X^{i=1\ldots6}$ are the scalars in the $\mathcal{N} = 4$ gauge multiplet and are vectors under the $SO(6)$ R-symmetry. On the supergravity side, fields on $AdS_5 \times S^5$ can be expanded in spherical harmonics on the S^5. Using x as coordinates on AdS_5 and y on the S^5 (with

$\sum y^2 = 1$), any field $\Phi(x,y)$ can be expanded as

$$\Phi(x,y) = \sum_l \phi_l(x) Y^l(y) \tag{3.31}$$

with the spherical harmonics $Y^l(y) = T_{i_1 \ldots i_l} y^{i_1} \ldots y^{i_l}$.

The way that the R-symmetry acts in both theories strongly suggests that we identify fields in AdS_5 whose S^5 spherical harmonic is $T_{i_1 \ldots i_l} y^{i_1} \ldots y^{i_l}$ with SYM operators $T_{i_1 \ldots i_l} Tr(X^{\{i_1} \ldots X^{i_l\}})$. This is enough to organize the whole spectrum of supergravity perturbations. Since we know which supergravity fields correspond to superconformal primaries, we can get descendant operators by acting with Q and P. Similarly, we can get the supergravity perturbations that correspond to these operators by acting with the corresponding symmetries in $AdS_5 \times S^5$.

Calculating Correlation Functions

We briefly review the GKPW implementation ([10, 11]) of calculating correlation functions using gauge gravity duality. The prescription is

$$\left\langle e^{-\int \phi_0 \mathcal{O}} \right\rangle_{CFT} = Z_{AdS}[\phi] \tag{3.32}$$

On the left hand side, we are calculating an expectation value with some source ϕ_0, where we should think of ϕ_0 as the field in AdS dual to the operator, $\mathcal{O}$, of the field theory. On the right hand side, we should calculate the partition function of the gravitational theory evaluated in a background ϕ whose $r \to \infty$ boundary value is ϕ_0 (we are glossing over the fact that a UV cutoff is usually needed). In terms of the generator of connected correlation functions,

$$W_{CFT} = -\log \left\langle e^{-\int \phi_0 \mathcal{O}} \right\rangle_{CFT} = \log Z_{AdS}[\phi] \tag{3.33}$$

For a classical gravity theory, we can evaluate this partition function at its saddle,

$$W_{CFT} = S_{grav}[\underline{\phi}] \tag{3.34}$$

where $S_{grav}[\underline{\phi}]$ is the gravity action evaluated at an extremum with the constraint that ϕ have boundary value ϕ_0. Connected correlation functions are found by taking functional derivatives of W, for example, the connected two point function is

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_c = \left.\frac{\delta^2 W}{\delta\phi_0(x_1)\delta\phi_0(x_2)}\right|_{\phi_0=0} \tag{3.35}$$

Without going into details here, the steps for calculating, say, a (Euclidean) two point function are

1. Solve the wave equation in the bulk for ϕ with boundary condition $\phi(\text{bdy}) = \phi_0$. For a scalar, this equation is

$$\frac{1}{\sqrt{g}}\partial_A(\sqrt{g}g^{AB}\partial_B)\phi - m^2\phi = 0 \tag{3.36}$$

   Typically, for Euclidean g, one of the solutions will blow up in the bulk and so we throw it away. The condition at the boundary is enough to fix the solution completely (we will have more to say about Lorentzian correlators in Chapter 4).

2. Evaluate this solution, $\underline{\phi}$, on the action.

3. Take two functional derivatives to get $\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_c$.

The form of this two point function will give a relationship between the scaling dimension ν of the operator (the eigenvalue under dilation), and the mass of the field, m. For scalar fields in AdS_{d+1}, $\nu = \sqrt{(d/2)^2 + m^2R^2}$.

For two point functions, we only need to take two functional derivatives before setting $\phi_0 = 0$, and so we do not need to know the full form of the bulk action. We only need the free action (to know the various masses of the fields, however, we do need the full bulk theory). For three point functions and higher, the detailed form of the bulk interactions becomes important. Once known, one can calculate these correlation functions by solving the wave equation perturbatively in ϕ.

We will have much more to say about the computation of fermion two point functions in Chapter 4.

The Boundary Theory in Different States

The previous discussion concerned computing correlation functions in vacua. Very often, however, we want to compute correlators in an ensemble with finite temperature or charge density. Thus, we need to somehow change the bulk geometry correspondingly. The most naive thing is correct–add a black hole in the AdS geometry. The boundary temperature corresponds to the Hawking temperature of the black hole while the charge density corresponds to its charge. We will have more to say about this in Chapter 4.

Therefore, the correspondence is not strictly between a spacetime and a field theory. Really, the correspondence is between a field theory and an asymptotic, boundary metric (and classical dynamics in this geometry). The different classical excitations of the bulk correspond to evaluating expectation values of the field theory in different states.

1.3.5 Application to Condensed Matter Systems

In the last few years, there have been numerous examples of bulk gravitational solutions that may be possibly relevant to condensed matter physics. This is encouraging and exciting because in the laboratory we can create an infinite number of condensed matter systems, while we are constrained by Mother Nature to be able to observe only a small number (1) of strongly coupled particle theories. Some recent examples of the correspondence are the Schrodinger metric solution ([12, 13, 14]), relevant to field theories with Schrodinger symmetry such as fermions at the unitarity limit, and also Lifshitz metrics, which are dual to Lifshitz theories of tricritical points ([15]).

We will focus on another case, first studied in ([16, 17]), the charged black hole in AdS_4. The finite density of "stuff" breaks the Lorentz symmetry of the boundary theory. In the second of these papers, it was shown that the boundary two point function for fermions takes the form

$$G_{\psi\psi} = \langle \psi(x_1)\psi(x_2) \rangle \underbrace{=}_{kspace} \frac{Z}{\omega - v_f|\vec{k} - \vec{k_f}| + i\Gamma} \tag{3.37}$$

where k_f, v_f and Z can only be computed numerically and $\Gamma \sim \omega^{2\nu}$. The singularity of $G_{\psi\psi}$ at $\omega = 0$ and $k = k_f$ signals a fermi surface while finite ω and $k - k_f$ describe excitations above this surface. These duality constructions give a whole theoretical playground of non-fermi liquids. This gives a remarkable handle on a strong coupling problem that has stalwartly refused analytical attack. In particular, $\nu = 1/2$ are so called "strange metals," i.e. high T_c superconductors. In the holographic picture, this ν has a nice geometric interpretation. The near horizon limit of the AdS_4 black hole is $AdS_2 \times \mathbb{R}^2$. Hence, in the IR, the boundary theory develops a new, two dimensional conformal symmetry under the AdS_2. At zero temperature,

$$\lambda = \sqrt{\frac{m^2 + k_f^2}{6} - \frac{q^2}{12}} \tag{3.38}$$

is related to the scaling dimension, ν of ψ under this IR scaling transformation (q, m are its charge and mass) by $\lambda = \nu - 1/2$.

In Chapter 4, we consider one way of expanding this zoo of duals. We take the above system of fermions minimally coupled to the extremal AdS_4 black hole and add the lowest dimensional irrelevant operator

$$\bar{\psi}(g_m + g_e\Gamma)\Sigma^{\mu\nu}\psi F_{\mu\nu} \tag{3.39}$$

which corresponds to turning on a tree level magnetic and electron dipole moments for the bulk fermion. In Chapter 4 we examine how this perturbation changes the boundary field theory. We find that correlators are still described by (3.37), but that turning on the dipole operator generically changes k_f, the location of the fermi surfaces and also changes ν, the scaling dimension of ψ under the AdS_2 conformal symmetry. We present our numerical results and produce "phase diagrams" of the attainable ν versus m.

Chapter 2

Heterotic Flux Vacua From Hybrid Linear Models

2.1 Introduction

As reviewed in Chapter 1, the prevalent paradigm for string compactifications has been Calabi-Yau manifolds. We know, however, that our universe does not have compact dimensions that are a Calabi-Yau–CY's typically come with moduli, massless fields and associated unobserved long range forces. The answer to this has been to turn on Ramond-Ramond fluxes in type II supergravity to fix these moduli (see, for example [18]). From the SUGRA point of view, this is all very beautiful, but RR fluxes have resisted a stringy, worldsheet understanding [1]. To avoid these complications, we might try to quantize pure NS-NS vacua of Type II. Unfortunately, turning on H-flux generically generates tree-level tadpoles which can only be cancelled by decompactifying or adding orientifolds and other RR objects, so that doesn't solve the problem.

The difficulties of quantizing RR fluxes can be avoided by working in a heterotic duality frame, where a tree-level H-flux can be balanced against a 1-loop anomaly via

[1] Considerable progress has been made in quantizing RR backgrounds by Berkovits and collaborators using the pure spinor and hybrid formalisms (see for example [19]). However, a generally applicable and computationally effective formalism analogous to the GLSM remains elusive. For now, RR vacua remain challenging.

the Green-Schwarz mechanism,

$$dH = \alpha' \left(tr\, R \wedge R \; - Tr\, F \wedge F \right).$$

Turning on H flux means that we are no longer working with a CY (nor, indeed, Kahler [20]) manifold. It also means that we are closer to constructing more realistic vacua–some of the moduli are fixed. For example, in CY's there is an unfixed modulus corresponding to the overall size, under which $g \rightarrow t^2 g$ and $H \rightarrow t^2 H$. If equation (2.1) is satisfied nontrivially, each side scales differently ($R \rightarrow t^0 R$) and so this global conformal mode no longer corresponds to a massless direction. Unfortunately, this supergravity equation is abrasively non-linear, making the construction of concrete non-Calabi-Yau solutions exceedingly challenging.

Considerable progress was made on this problem with the identification of a special class of non-trivial solutions in [21, 8, 9, 22]. These solutions all take the form of T^2-fibrations over a base $K3$, with H-flux along the fibration balancing against the curvature of the bundle so as to satisfy the Bianchi identity above. While these vacua have $c_3\left(\mathcal{V}\right) = 0$, and thus have zero generations at the semi-classical level, they provide interesting toy models of non-trivial heterotic flux compactifications.

These solutions are in supergravity, however, and one should be skeptical of the validity of the solutions because the T^2 radii are stuck at the α' scale. As always, it would be best to have string worldsheet CFTs corresponding to these flux vacua. One well known approach for "constructing" such worldsheet CFTs is that of the Gauged Linear Sigma Model (GLSM [23]). The idea here is to construct a two dimensional, supersymmetric gauge theory in the UV that flows in the IR to some desired superconformal field theory. Though not all of the details of the IR superconformal field theory will be known (we will not, for example, be able to compute a CY metric), nonetheless certain observables of the IR CFT are computable using a weakly coupled gauge theory in the UV. Because we are interested in solutions of the heterotic string, we work with $(0, 2)$ supersymmetric GLSM's.

Using the GLSM, the vacua of were [21, 8, 9, 22] realized in [24] by cancelling a 1-loop gauge anomaly against the classical gauge anomaly of a set of dynamical axions. The basic idea here is as follows. In a GLSM with gauge group $[U(1)]^m$, the NS-NS B field of the resulting NLSM is given by

$$B = \sum_{a=1}^{m} \theta^a H_a \tag{1.1}$$

where θ_a are the theta angles of the gauge theory, and H_a are the generators of 2 cohomology of the resulting NLSM target space. Since $H = dB = 0$, these vacua have no H flux. However, we would have H flux if we could somehow promote the θ angles to dynamical axions, all the while maintaining $(0,2)$ SUSY (naively, equation (1.1) would lead us to believe that $dH = 0$ so that the Bianchi identity is trivially satisfied–this turns out not to be the case as $H = dB$ by itself will turn out to be not a gauge invariant quantity).

This mechanism is the pullback to the worldsheet of the spacetime Green-Schwarz effect. The fact that all such models have zero generations follows[2], in the worldsheet description, from the existence of a pair of free right-moving fermions (the superpartners of the axions, which are coordinates on the T^2 fiber) whose zero modes ensure that all spacetime fermions come in non-chiral pairs. For these and other reasons, it would be interesting to generalize these models beyond the original example of T^2-fibrations over Kähler manifolds.

The goal of this chapter is to construct one such generalization. To introduce H-flux, we again pull back spacetime Green-Schwarz anomaly cancellation to a GLSM for the worldsheet CFT. This time, however, we will not require the worldsheet anomaly to be abelian. So long as we are careful to keep all possible anomalies cancelled, making the gauge group non-abelian boils down to replacing the T^2 fiber with some non-abelian group, G (or, more generally, some coset G/H), a subgroup of which is identified with the gauge group of the GLSM. More precisely, rather than starting with

[2]We thank J. Lapan for discussions on this point.

an anomalous abelian GLSM and canceling the anomaly by coupling the theory to a scalar axion in a gauge-non-invariant fashion, we now start with an anomalous non-abelian gauge theory and cancel the anomaly by coupling to a classically-anomalous gauged WZW model. By suitable choice of coset, we can ensure that there are no free right-moving fermions to force the spacetime spectrum to be chiral – these vacua do not, in general, have generation number zero. The result is a hybrid WZW gauged linear sigma model providing a worldsheet description of a large class of new quasi-geometric heterotic flux vacua which reduces to the original T^2 fibration in the abelian case.

Notably, something very similar was done in a pair of beautiful papers by Johnson *et. al.* [25, 26], who built novel $(0, 2)$ "minimal models" by adding $(0, 2)$-singlet left-moving fermions to gauged WZW models so as to cancel the one-loop anomaly generated by the fermions against the classical anomaly of the WZW model. One of the mysteries of those models was where, on the moduli space of string vacua, they arose; one lesson of this line of work is that they arise on the moduli space of non-Kähler flux-vacua of the heterotic string. A similar strategy was also used in a recent paper by Distler and Sharpe [27], who built WZW-fibered non-linear sigma models over Calabi-Yau 3-folds to realize E_8 bundles over topological CYs which could not be otherwise realized via free fermions.

This chapter is organized as follows. In Section 2 we review $(0, 2)$ GLSM's and in Section 3 gauged WZW models. In Section 4 we couple such WZW models to anomalous gauged linear sigma models to cancel the gauge anomaly of the GLSM. In Section 5 we identify the necessary non-anomalous $U(1)_L$ and $U(1)_R$ symmetries needed for a computation of the spectrum. In Section 6 we discuss how some of our models may be obtained by bosonization and fermionization. We then introduce several explicit examples in Section 7 and conclude in Section 8.

2.2 Brief Review of $(0,2)$ Gauged Linear Sigma Models

In this and the next section, we briefly review the superspace multiplets of $2d$, $(0,2)$ supersymmetry, and also the basics of the Gauged Linear Sigma Model (GLSM). The material presented here is standard, and can be found in, for example [23], [28], [29].

Supersymmetric actions are most readily constructed using superspace. Superspace has two bosonic (light cone) coordinates (x^+, x^-) and also two fermionic coordinates $(\theta^+, \bar{\theta}^+)$. SUSY transformations act on general superfields $A(x,\theta)$ as

$$\delta_\epsilon A(x,\theta) = (\epsilon Q_+ - \bar{\epsilon}\bar{Q}_+)A(x,\theta) \tag{2.2}$$

with the Q operators defined on superspace as

$$\begin{aligned} Q_+ &= \frac{\partial}{\partial\theta^+} + i\bar{\theta}^+\partial_+ \\ \bar{Q}_+ &= -\frac{\partial}{\partial\bar{\theta}^+} - i\theta^+\partial_+ \end{aligned} \tag{2.3}$$

We can define superderivatives

$$\begin{aligned} \mathcal{D}_+ &= \frac{\partial}{\partial\theta^+} - i\bar{\theta}^+\partial_+ \\ \bar{\mathcal{D}}_+ &= -\frac{\partial}{\partial\bar{\theta}^+} + i\theta^+\partial_+ \end{aligned} \tag{2.4}$$

One irreducible representation of $(0,2)$ supersymmetry is the chiral multiplet, defined by $\bar{\mathcal{D}}_+\Phi = 0$. It can be expanded in terms of ordinary fields on spacetime as

$$\Phi(x,\theta) = \phi(x) + \sqrt{2}\theta^+\psi_+(x) - i\theta^+\bar{\theta}^+\partial_+\phi(x) \tag{2.5}$$

A fermi multiplet Γ is defined in exactly the same way, except its lowest component is a fermionic field (the condition that $\bar{\mathcal{D}}_+\Phi = 0$ can be relaxed slightly, giving additional fields, which, in the $(2,2)$ case are the adjoint scalars of the gauge multiplet–we ignore this more general case for now)

$$\Gamma(x,\theta) = \gamma_-(x) - \sqrt{2}\theta^+ F(x) - i\theta^+\bar{\theta}^+\partial_+\gamma_-(x) \tag{2.6}$$

Another multiplet is a real superfield $V = V^\dagger$. Such multiplets can have a vector as their lowest component

$$V_- = A_-(x) - 2i\theta^+\bar{\lambda}_-(x) - 2i\bar{\theta}^+\lambda_-(x) + 2\theta^+\bar{\theta}^+ D(x) \tag{2.7}$$

or a scalar

$$V_+ = C + i\theta^+\gamma_+ + i\bar{\theta}^+\bar{\gamma}_+ + \theta^+\bar{\theta}^+ A_+ \tag{2.8}$$

2.2.1 Gauge Theories in Superspace

We can put these ingredients together to form a $(0,2)$ gauge theory in superspace (also known as a GLSM) with gauge group G. We start by introducing N chiral multiplets $\Phi_{i=1\ldots N}$,

$$\Phi_i = \phi_i + \sqrt{2}\theta^+\psi_{+i} - i\theta^+\bar{\theta}^+\partial_+\phi_i$$

transforming in representations R_i of a symmetry group G, together with M fermi multiplets $\Gamma_{a=1\ldots M}$,

$$\Gamma_a = \gamma_{-a} - \sqrt{2}\theta^+ F_a - i\theta^+\bar{\theta}^+\partial_+\gamma_{-a}$$

transforming in representations R_a of G. Since F_a is auxiliary, in the absence of couplings with other multiplets, the left-moving fermions are on-shell SUSY singlets. This is the full matter sector of the GLSM.

We introduce dynamics by gauging G with a $(0,2)$ vector multiplet, $V_\pm$. In components,

$$\begin{aligned} V_- &= A_- - 2i\theta^+\bar{\lambda}_- - 2i\bar{\theta}^+\lambda_- + 2\theta^+\bar{\theta}^+ D \\ V_+ &= C + i\theta^+\gamma_+ + i\bar{\theta}^+\bar{\gamma}_+ + \theta^+\bar{\theta}^+ A_+ \end{aligned} \tag{2.9}$$

The canonical field strength supermultiplet

$$\begin{aligned} \Upsilon_- &= [e^{V_+}\bar{\mathcal{D}}_+ e^{-V_+}, \nabla_-] \\ &= (-2\lambda_- - iD_-\gamma_+) + 2i\theta^+(D + \frac{i}{2}F_{+-} + ...) + 2i\theta^+\bar{\theta}^+(D_+\lambda_- + ...) \end{aligned}$$

transforms in the adjoint of G, where $(...)$ denotes terms that will shortly be set to zero by a choice of gauge. $\nabla_- = \partial_- + iV_-$ is the left-moving gauge-covariant superderivative.

Supergauge transformations are defined using an adjoint-valued chiral gauge parameter

$$B = b + \sqrt{2}\theta^+\beta_+ - i\theta^+\bar{\theta}^+\partial_+ b, \tag{2.10}$$

The matter fields transform according to their representations while $V_\pm$ transform as,

$$\begin{aligned} V_+ &\rightarrow V_+ + i(B - \bar{B}) - i[V_+, B + \bar{B}] + \dots \\ V_- &\rightarrow V_- + i\partial_-(B - \bar{B}) + i[V_+, B + \bar{B}] + \dots \end{aligned}$$

In components, the variation of V_+ takes the form,

$$\begin{aligned} C &\rightarrow C - 2i\mathrm{Im}\, b - i[C, 2\mathrm{Re}\, b] + \dots \\ \gamma_+ &\rightarrow \gamma_+ + \sqrt{2}\beta_+ - i[\gamma_+, 2\mathrm{Re}\, b] - \sqrt{2}[C, \beta_+] + \dots \\ A_+ &\rightarrow A_+ + 2\partial_+\mathrm{Re}\, b - i[A_+, 2\mathrm{Re}\, b] + \sqrt{2}[\gamma_+, \bar{\beta}_+] + \sqrt{2}[\bar{\gamma}_+, \beta_+] - i[C, \partial_+ 2\mathrm{Im}\, b] + \dots, \end{aligned} \tag{2.11}$$

where the $(\dots)$ terms involve higher order commutators involving $\mathrm{Im}\, b$.

As in four dimensions, we can use our super-gauge invariance to fix the non-dynamical components of V_+ to zero[3], leaving V_+ in the form,

$$V_+ = \theta^+\bar{\theta}^+ A_+.$$

This so-called Wess Zumino (WZ) gauge is particularly intuitive, since it makes manifest that the only propagating degrees of freedom in the vector multiplet are the gauge boson $A_\pm$ and the gaugino, λ_- (which resides in V_-). Notably, since any gauge transformation with $\beta_+ = 0$ and $\mathrm{Im}\, b = 0$ preserves the WZ condition, WZ gauge-fixing preserves a residual unfixed gauge symmetry, $A_+ \longrightarrow A_+ + D_+ a$, where $a = 2\mathrm{Re}\, b$. These are just the usual gauge transformations associated with any gauge theory.

Sadly, the benefits of WZ gauge come at a cost. By fixing some of the components of vector superfield V to zero, we have destroyed manifest supersymmetry. Explicitly, under a SUSY transformation with SUSY parameter ϵ, the vector V_+ transforms out of WZ gauge,

$$V_+ \longrightarrow i\theta^+\bar{\epsilon}A_+ - i\bar{\theta}^+\epsilon A_+ + \theta^+\bar{\theta}^+ A_+ \,. \tag{2.12}$$

We can return to WZ gauge by making a further gauge transformation with gauge parameter,

$$B^{WZ} = \frac{-i}{\sqrt{2}}\left(\theta^+\bar{\epsilon}A_+ - \bar{\theta}^+\epsilon A_+\right). \tag{2.13}$$

It is easy to check that this returns us to WZ gauge. The theory is thus only supersymmetric up to a gauge transformation in WZ gauge. As long as our theory is gauge-invariant, this is a technical detail (we will see, in the sections that follow, that if there is a possible gauge anomaly this technical point becomes paramount).

2.2.2 Gauge Invariant Actions

We illustrate how to construct gauge invariant $(0,2)$ actions. For simplicity, we take $G = U(1)$, but this can (and will be) easily generalized to many $U(1)$'s and to non-abelian groups $U(N)$ (we will take $U(N)$ as opposed to $SU(N)$, because we want

[3] Specifically, taking $b - \bar{b} = iC$ sets $C \longrightarrow 0$, while taking $\beta_+ = -\gamma_+$ subsequently sets $\gamma_+ \longrightarrow 0$.

there to be a central $U(1)$ for which we can write down a Fayet–Iliopoulos parameter). Gauge and SUSY invariant kinetic terms for the chiral superfields are

$$\mathcal{L}_{ch} = -\frac{i}{2}\int d^2\theta\, e^{Q_i V_+}\bar{\Phi}_i \nabla_-(e^{Q_i V_+}\Phi_i) = -|D\phi_i|^2 + i\bar{\psi}_{+i}D_-\psi_{+i} \tag{2.14}$$

with $\nabla_- \equiv \partial_- + iV_-$ and D the usual spacetime covariant derivatives. Similarly, gauge invariant terms for the fermi multiplets are given by

$$\mathcal{L}_{fm} = -\frac{1}{2}\int d^2\theta\, \bar{\Gamma}_a e^{2q_a V_+}\Gamma_a = i\bar{\gamma}_{-a}\partial_+\gamma_{-a} + |F_a|^2 \tag{2.15}$$

We can add in Yukawa interactions and a scalar potential by turning on a superpotential

$$\mathcal{L}_J = \frac{1}{\sqrt{2}}\int d\theta^+\, \Lambda_a J^a(\Phi_i)\,|_{\bar{\theta}^+=0} + \text{h.c.} = (\gamma_{-a}\psi_{+i}\frac{\partial J^a}{\partial\phi_i} + F_a J^a(\phi)) + \text{h.c.} \tag{2.16}$$

A gauge invariant field strength superfield can be obtained (in WZ gauge) as

$$\Upsilon_- \equiv [e^{V_+}\bar{\mathcal{D}}_+ e^{-V_+}, \nabla_-] = -2\lambda_- + 2i\theta^+(D - \frac{i}{2}F_{+-}) + 2i\theta^+\bar{\theta}^+\partial_+\lambda_- \tag{2.17}$$

in terms of which an invariant kinetic term is

$$\mathcal{L}_{gauge} = \int d^2\theta\, \bar{\Upsilon}\Upsilon = \frac{1}{2e^2}F_{+-}^2 + \frac{i}{e^2}\bar{\lambda}_-\partial_+\lambda_- + \frac{1}{2e^2}D^2 \tag{2.18}$$

Another invariant term we can write down is

$$\mathcal{L}_{FI} = \int d\theta^+\, t\Upsilon + \text{h.c.} = -rD + \theta F_{+-} \tag{2.19}$$

with $t \equiv ir + \theta$ the complexified Fayet-Iliopoulos parameter.

F_a and D are auxiliary and can be integrated out. After doing so, the scalar potential of the total Lagrangian, $\mathcal{L}_{ch} + \mathcal{L}_{fm} + \mathcal{L}_J + \mathcal{L}_{gauge} + \mathcal{L}_{FI}$, is

$$V(\phi_i) = \frac{e^2}{2}\left(\sum_i Q_i|\phi_i|^2 - r\right)^2 + \sum_a |J^a|^2 \tag{2.20}$$

2.2.3 From GLSM's to NLSM's

Equation (2.20) describes a manifold of SUSY preserving vacua, $\mathcal{M}$, sometimes called a moduli space. Those modes transverse to the vacuum acquire a mass by the Higgs mechanism $\sim e\sqrt{r}$. In the $e \longrightarrow \infty$ limit, these modes decouple and one can show that the resulting theory is a $(0,2)$ NLSM on $\mathcal{M}$ with some gauge bundle for the left moving fermions, V. For the case where $G = U(1)$, one can calculate the Kahler form and B field for the resulting theory, and one finds $J = r\omega$ and $B = \theta\omega$, where ω is the generator of 2-cohomolgy for $\mathcal{M}$. We are interested in GLSM's that flow to $(0,2)$ superconformal field theories. This means that that the model must have a non-anomalous right moving $U(1)_R$ symmetry and a non-anomalous $U(1)_L$ which is used to implement a chiral GSO projection. In addition, in order for the gauge theory to be sensible at all, the gauge symmetry itself must be non-anomalous. For $(2,2)$ GLSM's this is automatically ensured, since right and left moving fermions necessarily live in the same gauge representation. For $(0,2)$ theories this is no longer the case and vanishing of the gauge anomaly must be imposed by hand. Following [28] we investigate these requirements for a class of theories with chiral multiplets Φ^i and P and Fermi Multiplets Λ^a and Γ with charge assignments as in the below chart.

Field	Gauge Charge
Φ_i	w_i
P	$-m$
Λ_a	n_a
Γ	$-d$

The condition for vanishing gauge anomaly is

$$\sum_i w_i^2 + m^2 = \sum_a n_a^2 + d^2 \tag{2.21}$$

In the presence of a superpotential, the requirement that the

$U(1)_R$ and $U(1)_L$ are nonanomalous typically require

$$\begin{aligned} \sum_i w_i &= m \\ \sum_a n_a &= d \end{aligned} \tag{2.22}$$

Geometrically, these conditions, are $c_2(\mathcal{M}) = c_2(V)$, $c_1(\mathcal{M}) = 0$, and $c_1(V) = 0$ respectively in the resulting NLSM. We recognize the second requirement as the Calabi-Yau condition. We now construct some example moduli spaces.

Example: $\mathcal{M} = \mathcal{O}(-n) \to \mathbb{P}^{n-1}$

We examine the case where there are n Φ_i fields each with charge $+1$, 1 P field with charge $-n$, n Λ_i fields with charge $+1$ and 1 Γ field with charge $-n$ so that the anomaly conditions are automatically satisfied. For now, we take no superpotential, i.e. $J^i = 0$. The vanishing of the scalar potential implies

$$\sum_i^n |\phi_i|^2 - np^2 = r \tag{2.23}$$

Let us examine the case when $r \gg 1$. In this case, not all of the ϕ_i's can be zero, and each fixed value of $< p >$ describes an S^{2n-1} of fixed size. In fact, we need to mod out by the gauge transformations, and so each fixed value of $< p >$ determines a $S^{2n-1}/U(1) = \mathbb{P}^{n-1}$. However, we still have a complex degree of freedom, and the total space is a line bundle over $\mathbb{P}^{n-1}$, $\mathcal{M} = \mathcal{O}(-n) \to \mathbb{P}^{n-1}$.

Now, let us examine the other scenario, $r \ll 1$. In this case, $|p| \neq 0$ which gives a mass to the p field. We can use the gauge freedom to fix the phase of p so that

$$p = \sqrt{\frac{\sum_i |\phi_i|^2 - r}{n}} \tag{2.24}$$

at each point. Therefore it would seem our vacuum manifold is here $\mathbb{C}^n$. In fact, there are a $\mathbb{Z}_n$ of residual that leave (2.24) unchanged so that the vacuum manifold is actually the orbifold $\mathcal{M} = \mathbb{C}^n/\mathbb{Z}_n$. In each of these cases, all of the left moving fermions remain masses and so the gauge bundle is just a $\mathcal{O}(-n) \otimes \mathcal{O}(1)^n$ sitting over $\mathcal{M}$.

At low energies, we expect both of these theories to flow to conformal field theories on the respective $\mathcal{M}$'s. Thus we see an example of what is known as the Calabi-Yau/Landau Ginzburg correspondence-there is a continuous parameter, r which interpolates the two CFT's. In the language of the NLSM's, the $\mathbb{C}^n/\mathbb{Z}_n$ CFT is just the $\mathcal{O}(-n) \longrightarrow \mathbb{P}^{n-1}$ CFT "continued to negative Kahler class." In general, one can interpolate smoothly between topologies (without hitting a singular CFT) although there are codimension 1 points in the space of (r, θ) where genuine phase transitions occur. In the gauge theory, these generally correspond to points where the vacuum is in an unbroken Coulomb phase. In general, the picture looks something like:

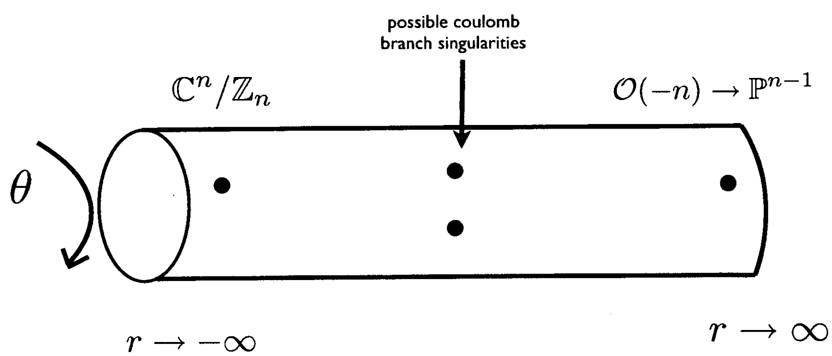

Figure 2-1: A Continuous Family of Conformal Field Theories

Example: Cutting out CY Hypersurfaces in $\mathbb{P}^{n-1}$

In the $r \gg 1$ phase, we can arrange $\mathcal{M}$ to be some compact Calabi-Yau hypersurface sitting inside $\mathbb{P}^{n-1}$ defined by the equation $W(\Phi) = 0$ by adding a superpotential of the form

$$\mathcal{L}_J = \frac{1}{\sqrt{2}} \int d\theta^+ \left(\Gamma W(\Phi) + \Lambda^i P \frac{\partial W}{\partial \Phi_i} \right) + \text{h.c.} \tag{2.25}$$

Vanishing of the scalar potential now also requires

$$\begin{aligned} W(\phi) &= 0 \\ p\frac{\partial W}{\partial \phi_i} &= 0 \end{aligned} \tag{2.26}$$

In general, W will be transverse, meaning that $\frac{\partial W}{\partial \phi_i} = 0$ has no solution except all $\phi_i = 0$. Since this cannot be the case in this phase, the second equation of (2.26) solution forces $p = 0$. This collapses the p fiber (and gives a mass to p), while the first equation of (2.26) cuts out a hypersurface in $\mathbb{P}^{n-1}$. By (2.25) gauge invariance imposes that W is homogenous of degree n which is the condition that a hypersurface in $\mathbb{P}^{n-1}$ be Calabi-Yau. To determine the vector bundle of left moving fermions that sits over this space, we examine the Yukawa couplings in the Lagrangian and find that in order for the fermions to be massless, we must have

$$\sum_i \frac{\partial W}{\partial \phi_i} \gamma_{-i} = 0 \tag{2.27}$$

This equation determines a map $\mathcal{O}(1)^n \to \mathcal{O}(n)$, the kernel of which is the bundle of left moving fermions. In more general models, the left moving fermion bundle is determined by similar, perhaps more complicated sequences of bundle maps.

In the $r \ll 1$ phase, the transversality condition forces all $\phi_i = 0$ in vacua. p gets frozen at the expectation value $p = \sqrt{-r/n}$. The ϕ's remain massless (as do all of the left moving fermions γ_{-i}) and we have a fixed vacuum with a superpotential that has a degenerate critical point at the origin. Such a theory is known as a Landau-Ginzburg theory (and perhaps explains better why the phenomenon introduced in the previous section is known as the Calabi-Yau/Landau Ginzburg correspondence). Actually, we have a $\mathbb{Z}_n$ orbifold of this Landau-Ginzburg theory because of the previously mentioned residual gauge symmetry.

Further Generalizations

The above examples are just a tip of the iceberg in terms of different moduli spaces that can be realized by GLSM's. Our choice of superpotential (2.25) was extremely constrained and by considering different couplings, one can drastically change the structure of the left moving fermion bundle. By involving ϕ's with more general gauge charges, we can engineer hypersurfaces in weighted projected spaces. By including more "P" type fields, we can construct our Calabi-Yau as a complete intersection of several hypersurfaces. By including many copies of $U(1)$, the resulting many FI parameters produce a very rich Calabi-Yau/Landau-Ginzburg phase structure.

Finally, by considering the gauge group $U(N)$, we can construct Calabi-Yau hypersurfaces that sit inside of Grassmannians instead of inside projective space. Very interestingly, Hori and Tong [30] have showed, using some very beautiful strong coupling arguments, that the $r \ll 1$ phase of certain $U(N)$ gauge theories is the Pfaffian Calabi-Yau, which cannot be realized as a complete intersection in a projective space or a Grassmannian.

2.2.4 The Use of GLSM's

Since the "extra dimensions" of string theory are described by a nonlinear sigma model, the ultimate goal of this program is to calculate correlation functions of NLSM's. Calculating these correlation functions directly is difficult to say the least–there are no known 6-d Calabi-Yau metrics in closed form.

The utility of the GLSM is that it reduces to the NLSM in the $e \to \infty$ limit. A fruitful approach might be to calculate correlation functions in the UV using the GLSM.

This story is not so simple, however, as the GLSM is not under calculational control for $e \to \infty$; one wants to use a *weakly* coupled gauge theory instead. Luckily, one can construct field theories related to the GLSM (this process is called "twisting" [31]) whose correlation functions are independent of e. One can then use the twisted, weakly coupled GLSM to compute correlation functions of certain classes of operators in the desired NLSM ([23], [32], [33, 34]).

2.3 Gauged WZW Models

Wess-Zumino-Witten (WZW) models are NLSM's on group manifolds. Gauged WZW models, described in [25] and [26], couple these to gauge fields to realize coset manifolds. A gauged WZW model with $(0,1)$ supersymmetry contains G-valued scalar bosons, $g(x) \in G$, together with right handed Majorana-Weyl superpartners, ψ_+, valued in $\mathfrak{g} = T_G$, the Lie algebra of G. To gauge the WZW model we introduce two vector fields, A^L and A^R, gauging $H_{L,R} \subset G$, where H_L and H_R are generated by left and right multiplication,

$$
\begin{aligned}
g &\to h_L g h_R^{-1} \\
\psi_+ &\to h_R \psi_+ h_R^{-1} \\
A^L &\to h_L d h_L^{-1} + h_L A^L h_L^{-1} \\
A^R &\to h_R d h_R^{-1} + h_R A^R h_R^{-1}.
\end{aligned}
$$

The action of the gauged model is

$$
\begin{aligned}
S = \; & -\frac{k}{4\pi}\int_\Sigma \mathrm{tr}\left[g^{-1}\partial_+ g g^{-1}\partial_- g\right] - i\mathrm{tr}\left[\psi_+ D_-\psi_+\right] \\
& -\frac{ik}{2\pi}\int_\Sigma \mathrm{tr}\left[g^{-1}\partial_+ g A_-^R - A_+^L\partial_- g g^{-1} + iA_-^R g^{-1} A_+^L g + \frac{i}{2}(A_+^L A_-^L + A_+^R A_-^R)\right] \\
& -\frac{ik}{12\pi}\int_V \mathrm{tr}\left[(g^{-1}\partial_i g)(g^{-1}\partial_j g)(g^{-1}\partial_k g)\right]\epsilon^{ijk} \qquad (3.28)
\end{aligned}
$$

where V is a volume bounded by the worldsheet and $D_-\psi_+ = \partial_-\psi_+ - i[A_-^R, \psi_+]$ is the covariant derivative of our right-moving fermions, which take values in the algebra of the coset $G/(H_R \times H_L)$, *i.e.*,

$$\psi_+ \in Lie(G) - Lie(H_R) - Lie(H_L).$$

The model actually has $(0,2)$ supersymmetry if the coset satisfies the following conditions:

- $T_{\mathbb{C}}$, the Lie algebra of $G/(H_R \times H_L)$, has the decomposition $T_{\mathbb{C}} = T_+ \oplus T_-$ of conjugate representations. This is the statement that $G/(H_R \times H_L)$ has a local complex structure.
- $[T_+, T_+] \subset T_+$. and $[T_-, T_-] \subset T_-$. This is the statement that the Nijenhuis tensor vanishes and the complex structure is integrable.
- $\mathrm{tr}(ab) = 0$ if $a, b \in T_+$ or $a, b \in T_-$. This is the statement that there exists a Hermitian $(1,1)$ form on $G/(H_R \times H_L)$.

Under these conditions, the model is invariant under the $(0,2)$ SUSY transformations

$$\begin{aligned}
\delta g &= i\epsilon_1 g \psi_{R-} + i\epsilon_2 g \psi_+ \\
\delta \psi_+ &= \epsilon_1 \Pi_+ (g^{-1} D_+ g - i\psi_+\psi_- - i\psi_-\psi_+) + i\epsilon_2 \psi_+\psi_+ \\
\delta \psi_- &= \epsilon_2 \Pi_- (g^{-1} D_+ g - i\psi_+\psi_- - i\psi_-\psi_+) + i\epsilon_1 \psi_-\psi_- \\
\delta A^L_\pm &= 0 \\
\delta A^R_\pm &= 0
\end{aligned} \tag{3.29}$$

where $\Pi_\pm$ is the projection to $T_\pm$ and $D_\pm g \equiv \partial_\pm g - iA^L_\pm g + igA^R_\pm$. For unitary groups with $g^{-1} = g^\dagger$, consistency of the SUSY transformations requires that $\epsilon_2 = -\bar{\epsilon}_1$.

Finally, and crucially for our later purposes, this action is in fact classically anomalous for a general gauging: under a gauge transformation with left/right gauge parameters α_L, α_R, the action shifts by,

$$\delta S = \frac{k}{4\pi}\left(\mathrm{tr}[\alpha_R\, F^{R'}_{+-}] - \mathrm{tr}[\alpha_L\, F^{L'}_{+-}]\right), \tag{3.30}$$

with F^R the field strength for A^R and F^L the field strength for A^L (the primes indicate that only the dA terms in F appear, *i.e.* only the "consistent" anomaly contributes).

2.3.1 An Example

Let's examine a simple example, the $SU(2)/U(1)$ WZW model, where the $U(1)$ generated by $\sigma_3/2$ has been gauged on the right (*i.e.* with A^R). Since $SU(2)/U(1) \sim \mathbb{P}^1$ is complex and Hermitian (in fact, Kähler), this model should admit a $(0,2)$ supersymmetric extension. As it stands, however, the Lagrangian is not gauge-invariant. To cancel this classical anomaly, we introduce left handed fermions charged under the A^R gauge symmetry; these chiral fermions generate a quantum anomaly which cancels the classical anomaly of the gauged WZW model[4]. The anomaly cancellation condition is then

$$\frac{k}{2} + 1 - Q^2 = 0,$$

where the $k/2$ is the coefficient of the classical anomaly of the WZW model (the $1/2$ is from the normalization of the generators of $SU(2)$), the $+1$ comes from the right handed Weyl fermion in the WZW model with gauge charge +1, and the $-Q^2$ is the contribution from the left handed fermion with charge Q.

Significantly, since left-handed fermions are singlets (on-shell) under right-moving $(0,2)$ supersymmetry, adding them does not spoil $(0,2)$ supersymmetry. These models are known as $(0,2)$ minimal models [26] and have central charge

$$c = \frac{3k}{k+2}\,. \tag{3.31}$$

As an application, we can use these minimal models to build realistic heterotic compactifications with $c = 9$. Condition (3.31) is a very restrictive condition on k – so restrictive, in fact, that the only way to build a vacuum with the correct central charge is to take the tensor product of four theories with $k = 6$ ($Q = 2$) [26]. Of course, one

[4]This specific theory has been used in the construction of worldsheet theories that describe four dimensional heterotic solutions of a black hole of magnetic charge Q, [35].

may generate more possibilities by taking the tensor product of this model with $(2, 2)$ WZW models – these are the so called "doped" models of [26].

2.4 Constructing the Hybrids

As reviewed in the last section, we can cancel the quantum anomaly generated by a set of chiral fermions by coupling in a gauged WZW model, with total anomaly cancellation imposing a single condition relating the charges of the fermions to the level of the WZW model. In this section we will study a natural generalization of this mechanism in which we replace the chiral fermions by a gauged linear sigma model whose fermion content is anomalous. In this more intricate case, vanishing of the net anomaly will again reduce to a set of conditions relating the charges of the matter fields in the gauge theory to the level of the WZW model. Studied semiclassically, the net effect will be to fiber the WZW model non-trivially over the classical target space of the sigma model.

Two points need to be kept in focus. First, neither the gauge theory nor the WZW model is independently invariant under the symmetry we would like to gauge – the gauge theory suffers from a quantum anomaly and the WZW model is classically anomalous. It is only the combination of the two which realizes this symmetry exactly and allows us to gauge. Second, both the gauge theory and the gauged WZW are independently supersymmetric, despite the anomalies. This is obscured when working in WZ gauge, where the SUSY algebra closes only up to a gauge transformation and thus does not close in the presence of a gauge anomaly. However, this is a failure of WZ gauge, not of supersymmetry, and is in any case of no concern so long as we focus on the non-anomalous combination of gauge theory and gauged WZW model.

2.4.1 Gauge Anomalies in the GLSM

Since the $(0,2)$ GLSM contains fermions transforming in chiral representations of the gauge group, it runs the risk of a chiral anomaly which spoils gauge-invariance. Without gauge invariance, negative norm states no longer decouple and unitarity is lost. We must proceed with caution.

In two dimensions the anomaly comes from a diangle diagram with one current insertion and one external gauge boson. The derivation (see for example [36]) of this

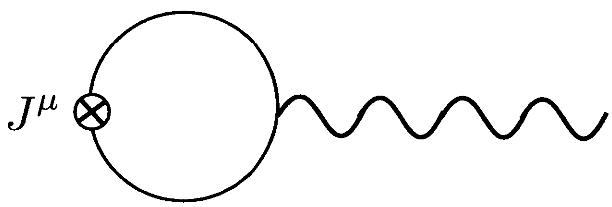

Figure 2-2: An Anomalous Diangle Diagram in 2-d

anomaly proceeds as in four dimensions. In a $U(1)$ gauge theory with N right-handed fermions of charge Q_i and M left-handed fermions of charge q_a, the chiral anomaly of the gauge current, J^μ_G, under variation with gauge parameter α, is

$$\partial_\mu J^\mu_G = \frac{\mathcal{A}}{2\pi}\alpha F_{+-}, \tag{4.32}$$

where the anomaly coefficient is given by $\mathcal{A} = \sum_i Q_i^2 - \sum_a q_a^2$. For a non-abelian theory with semi-simple gauge group, this generalizes to[5]

$$\partial_\mu J^\mu_G = \frac{\mathcal{A}}{4\pi}\mathrm{Tr}\left[\alpha F_{+-}\right] \tag{4.33}$$

with $\mathcal{A}$ again determined by the matter fields and their representations. (Nonabelian anomalies will be discussed in more detail in section 2.7.)

[5]The rather annoying factor of 2 between the abelian and non-abelian anomalies derives from different conventional normalizations of the generators.

In 2-d theories enjoying $(2,2)$ supersymmetry, this anomaly always vanishes, since every right handed fermion lives in a supermultiplet with a left-handed partner, so both transform in the same gauge representation. Said differently, $(2,2)$ supersymmetry only allows for matter that is in a non-chiral representation of the gauge group.

In the $(0,2)$ theories of interest to us, left- and right-chiral fermions live in different representations (chiral and fermi, respectively) of $(0,2)$ supersymmetry, and may thus transform in different representations of the gauge group. We should thus expect a $(0,2)$ supersymmetric extension of this anomaly in our theories. For semi-simple Lie groups, the resultant super anomaly is

$$\frac{\mathcal{A}}{4\pi}\int d\theta^{+}\mathrm{Tr}\left[B\Upsilon\right] + \text{h.c.}\,, \tag{4.34}$$

where B is the gauge parameter and Υ the gauge field strength supermultiplet.

Only models whose chiral anomalies vanish make sense. Nevertheless, let us for the moment soldier on and consider $(0,2)$ models with non-vanishing chiral anomaly. This leads to an important subtlety with WZ gauge, where SUSY is only respected up to a gauge transformation: if the anomalous theory is not invariant under gauge transformations, the theory in WZ gauge would not appear to be supersymmetric. Explicitly, suppose we perform a supersymmetry transformation with parameter ϵ, then apply the WZ-restoring gauge transformation (2.13). The resultant shift in the action is found by evaluating the anomaly on this gauge variation,

$$\frac{\mathcal{A}}{4\pi}\int d\theta^{+}\mathrm{Tr}\left[B^{WZ}\Upsilon\right]. \tag{4.35}$$

Where B^{WZ} is the chiral superfield gauge transformation that returns A_{+} to WZ gauge. Since this is non-vanishing for general Υ, supersymmetry appears broken in WZ gauge. Of course, this is purely an artifact of fixing WZ gauge – if we do not fix WZ gauge, the action is explicitly SUSY-invariant without any additional gauge transformation. Nonetheless, ensuring that this "WZ anomaly" cancels will be a useful check of the gauge-invariance of what follows.

2.4.2 Adding the WZW Theory

In section 2.3, we used the classical anomaly of a gauged WZW model to cancel the quantum chiral anomaly of a charged Weyl fermion. As we have just seen, the quantum anomaly of a general GLSM takes the same form as these earlier anomalies. This suggests a simple way to construct new non-anomalous models by balancing the classical gauge anomaly of a WZW model against the chiral anomaly of a $(0, 2)$ GLSM. As we shall see, the total theory can indeed be made non-anomalous and well defined.

At first glance, there are a number of choices to be made in coupling the GLSM to the WZW model. Explicitly, the GLSM contains a single dynamical vector, A, transforming non-trivially under supersymmetry. The WZW model, by contrast, boasts two non-dynamical vectors, A^L and A^R, which transform trivially under supersymmetry. If our goal is to play the quantum anomaly of one off the classical anomaly of the other, they must be coupled to the same vector. We thus must identify A with either A^L or A^R. In the WZW model, this means promoting one of $A^{L,R}$ to a dynamical vector transforming non-trivially under supersymmetry. So: which do we pick?

Supersymmetry guides our choice. In the GLSM in WZ gauge, A_+ transforms trivially under SUSY while A_- transforms non-trivially, with $\delta A_- = 2i\epsilon\lambda_-$ (2.9). Meanwhile, the vector couplings in the WZW model take the form (3.28),

$$-\frac{ik}{2\pi}\int_\Sigma \mathrm{tr}\left[g^{-1}\partial_+ g A_-^R - A_+^L \partial_- g g^{-1} + iA_-^R g^{-1} A_+^L g + \frac{i}{2}(A_+^L A_-^L + A_+^R A_-^R)\right]$$

If we promote A^R to a dynamical field with the same SUSY variations as A, the WZW action will pick up a term proportional to $g^{-1}\partial_+ g$ under SUSY variation. To preserve SUSY, this must cancel against some other term in the action. Unfortunately, no other vector coupling in the action has a g-dependent SUSY variation. So A^R is out.

By contrast, if we promote A^L to a dynamical field, the variation of the WZW action, while non-zero due to the $A^L_- A^L_+$ term, is at least independent of g and thus has a chance of being cancelled by something in the GLSM action. Explicitly, the action varies by a term proportional to the SUSY parameter, ϵ, the right-chiral boson, A_+, and the level, k, of the WZW model. The component form is easily worked out to be,

$$\delta\mathcal{L}_{WZW} = -\frac{ik}{2\pi}\left(\epsilon\,\mathrm{tr}[A_+\bar{\lambda}_-] + \bar{\epsilon}\,\mathrm{tr}[A_+\lambda_-]\right). \tag{4.36}$$

Worryingly, this does not look like the SUSY variation of any term in the GLSM action, so we again look stuck.

At this point something beautiful happens. Recall that our GLSM in WZ gauge is not in fact supersymmetric, but picks up a non-trivial SUSY-variation due to the anomaly of the WZ-restoring gauge shift. Evaluating this explicitly gives,

$$\delta\mathcal{L}_{gauge} = \frac{\mathcal{A}}{4\pi}2i\left(\epsilon\,\mathrm{Tr}[A_+\bar{\lambda}_-] + \bar{\epsilon}\,\mathrm{Tr}[A_+\lambda_-]\right) \tag{4.37}$$

Note this is just the component form of the supergauge transformation required to return A_+ to WZ gauge after a SUSY transformation

$$\frac{\mathcal{A}}{4\pi}\int d\theta^+\mathrm{Tr}\left[B^{WZ}\Upsilon\right] \tag{4.38}$$

The λ_- part of (4.37) comes from the Υ in (4.38) while $\beta_+ \sim A_+ \in B^{WZ}$ is the supergauge transformation needed to restore WZ gauge after a SUSY transformation. Delightfully, the form of the resulting variation precisely matches the SUSY variation of the WZW model! Thus, requiring that the total variation of the action vanishes then imposes a single condition relating the level, k, to the anomaly, $\mathcal{A}$, of the GLSM,

$$k\,\mathrm{tr}[T^2] = \mathcal{A}\,\mathrm{Tr}[T^2], \tag{4.39}$$

where tr denotes the trace in the WZW model and Tr the trace in the gauge theory, which may involve a different normalization (an endless source of spurious factors of 2).

The beauty of this condition is that its satisfaction ensures not just supersymmetry but also cancellation of the total anomaly. Under a left-gauge variation (3.30) the WZW model picks up a classical variation,

$$\delta\mathcal{L}_{WZW} = -\frac{k}{4\pi}\,\mathrm{tr}[\alpha\, F'_{+-}], \tag{4.40}$$

while the GLSM picks up the chiral anomaly (4.33), so that the total anomaly is,

$$\begin{aligned}\delta\mathcal{L} &= \delta\mathcal{L}_{gauge} + \delta\mathcal{L}_{WZW} \\ &= \frac{\mathcal{A}}{4\pi}\mathrm{Tr}[\alpha\, F'_{+-}] - \frac{k}{4\pi}\,\mathrm{tr}[\alpha\, F'_{+-}].\end{aligned} \tag{4.41}$$

Requiring that the total anomaly vanish thus imposes the same condition as that needed for manifest supersymmetry.

At this point, we have succesfully cancelled the anomaly of our gauge theory by coupling in a classically anomalous WZW model. However, several points deserve further comment. First, we have been cavalier about the role of WZ gauge in the above. As noted, both the WZW model and the gauge theory are independently supersymmetric. However, once we fix to WZ gauge in which SUSY is only a symmetry up to a gauge variation, neither model is manifestly supersymmetric due to the (classical, quantum) anomaly generated by the gauge transformation needed to restore WZ gauge. Thus WZ gauge/SUSY invariance really is nothing other than a measure of anomaly cancellation.

Secondly, while we have discussed in some detail the fate of the vector A_L gauging the left-action, we have not mentioned that of A_R gauging the right-action. In particular, even if we do not promote it to a dynamical vector (which we shan't, as this would upset both gauge-invariance and supersymmetry), so long as we take H_R to be non-trivial, A_R will still couple to an anomalous current. The crucial observation here is that we can always include SUSY-singlet left-moving fermions to cancel this anomaly without altering any of the considerations above. So we have more general models where we can gauge A_R as well.

Finally, as we originally introduced it, our WZW model for the coset $G/(H_R \times H_L)$ contained right handed fermions living in $Lie(G) - Lie(H_R) - Lie(H_L)$ and coupled to the non-dynamical A_L. In our hybrid GLSM, however, A_L is only non-dynamical in the deep IR where the gauge coupling runs strong; at finite energy, the vector field is dynamical and the bosonic field g lives in G/H_R. Correspondingly, ψ_+, the right-handed superpartners of g, begin life valued in $Lie(G) - Lie(H_R)$ in the UV, with the restriction to $Lie(G) - Lie(H_R) - Lie(H_L)$ in the IR coming from their coupling to the gauginos. The full theory is supersymmetric iff G/H_R is a complex manifold with Hermitian metric.

2.5 $U(1)_R$ and $U(1)_L$ Symmetries

The $(0,2)$ superconformal algebra is determined by the stress tensor, $\mathcal{T}_B$, a complex fermionic supercurrent $\mathcal{T}_F$ and a purely holomorphic $U(1)_R$ current $\mathcal{J}_R$. The OPE of $\mathcal{J}_R$ with itself determines the right moving central charge,

$$\mathcal{J}_R(z)\mathcal{J}_R(0) \sim \frac{c_R}{3z^2} + \ldots$$

One of the virtues of the GLSM is that we can often identify a candidate conserved R-current in the UV which flows to purely-right-moving conserved current $\mathcal{J}_R$ in the IR. By 't Hooft anomaly matching and asymptotic freedom, we can (in principle) compute the central charge of the strongly-coupled IR theory by computing weak-coupling OPEs in the UV. A similar story obtains for the left-moving current, $\mathcal{J}_L$.

Our goal in this section is to identify conserved $U(1)_R$ and $U(1)_L$ currents in the UV which flow to purely right/left-moving conserved currents in the IR. In canonical $(0,2)$ GLSMs, it suffices to assign $U(1)_R$ charges to the matter fields compatable with the superpotential such that the R-current $\mathcal{J}_R^\mu$ is non-anomalous, conserved, and orthogonal to all non-anomalous flavor currents. In general, the resulting $\mathcal{J}_R^+$ contains terms that either flow away in the IR or whose divergence is trivial in Q_+ cohomology, so that the on-shell $\mathcal{J}_R^-$ runs to the holomorphic conserved current of the IR superconformal algebra.

For our gauged WZW+GLSM hybrids, identifying the correct R-currents is a little more subtle. Naïvely, the thing to do is assign each field a general R-transformation law, compute the resulting current by varying the action according to this symmetry, and deduce what the R-transformations must be for the resulting current to transform as an R-current. Without loss of generality, we can assign g the charge $QT^{(1)}$, ψ_+ the charge $Q_R T^{(2)}$ and ψ_- the charge $-Q_R T^{(2)}$, where $T^{(1,2)} \in \mathfrak{g}$ specify the embedding of our $U(1)$ in $\mathfrak{g}$, and the Q's are real numbers. We can then try to construct a conserved R-current by varying the action according to this symmetry. However, due of the classical non-gauge invariance of the WZW action, the resulting current, $\mathcal{J}_R$, is in general neither gauge invariant nor holomorphic in the IR. To identify a good R-current, we will need to modify this naïve current to preserve gauge invariance (in a manner very similar to [37]).

For example, let's take an abelian model with $G = U(1) \times U(1)$ (with one of these $U(1)$'s gauged). The right-moving fermions ψ_{+i} in the chiral multiplets Φ_i carry $U(1)_R$ charge q_R^i, the left-moving fermions λ_-^a in the fermi multiplets Γ^a carry charge q_R^a, the gauginos carry charge $+1$, the right-moving fermions ψ_+ in the WZW multiplet carry charge $+1$, and the bosons $\theta^{l=1,2}$ in the WZW model carry shift charge q_l. The corresponding naïve currents are,

$$
\begin{aligned}
\mathcal{J}_R^+ &= \frac{1}{2e^2}\lambda_-\lambda_- + q_a^R\bar{\lambda}_{-a}\lambda_{-a} - \frac{k}{4\pi}q_l\partial_-\theta_l \\
\mathcal{J}_R^- &= q_i^R\bar{\psi}_{+i}\psi_{+i} + \frac{k}{4\pi}\bar{\psi}_+\psi_+ - \frac{k}{4\pi}q_l\partial_+\theta_l + \frac{kq_lN_l}{2\pi}A_+.
\end{aligned}
\tag{5.42}
$$

where N_l is the shift gauge charge of θ_l ($\theta_l \to \theta_l + \alpha N_l$). As expected, these currents are not gauge invariant, nor is it clear that the divergence of $\mathcal{J}_R^+$ is Q-trivial. Happily, it is easy to identify their (classically) gauge invariant cousins as

$$
\begin{aligned}
\widetilde{\mathcal{J}}_R^+ &= \mathcal{J}_R^+ + \frac{kq_l N_l}{4\pi} A_- \\
\widetilde{\mathcal{J}}_R^- &= \mathcal{J}_R^+ - \frac{kq_l N_l}{4\pi} A_+ .
\end{aligned}
\tag{5.43}
$$

The q_l are then chosen such that, if $\mathcal{J}_G$ is the gauge current, the leading term in the $\mathcal{J}_G^+ \widetilde{\mathcal{J}}_R^+$ OPE is equal to that of the $\mathcal{J}_G^- \widetilde{\mathcal{J}}_R^-$ OPE (this is the same as requiring that $\widetilde{\mathcal{J}}_R$ is gauge invariant quantum mechanically). The leading coefficient of the $\widetilde{\mathcal{J}}_R^- \widetilde{\mathcal{J}}_R^-$ OPE minus that of the $\widetilde{\mathcal{J}}_R^+ \widetilde{\mathcal{J}}_R^+$ will give $\hat{c}$, one third of the central charge of the right moving SCFT.

Since e^2 runs strong in the IR, the contributions of the left handed gauginos to $\widetilde{\mathcal{J}}_R^+$ flow away in the IR, while the $U(1)_R$ charged fermi multiplets develop masses. The divergence of the remaining part of $\widetilde{\mathcal{J}}_R^+$ is then,

$$
\begin{aligned}
\partial_+ \widetilde{\mathcal{J}}_R^+ &= \ldots - \frac{kq_l}{4\pi}(\partial_+\partial_-\theta_l - N_l \partial_+ A_-) \\
&= \ldots + \frac{kq_l N_l}{4\pi} F_{+-} \\
&\propto \ldots + \{Q_+, \lambda_-\}
\end{aligned}
\tag{5.44}
$$

where in the second line we have used the θ equation of motion, and in the third we have used the SUSY algebra (for portions of the moduli space with $D = 0$). Since this is Q trivial, we thus expect $\widetilde{J}_R^+$ to flow away completely so that $\widetilde{J}_R^-$ is the holomorphic, right moving R-current in the deep IR.

For non-abelian G, we expect a similar story – the $U(1)_R$ and $U(1)_L$ currents will be a sum of the individual GLSM and WZW currents (for a general WZW model, these will involve the Lie algebra fermions and the Kac-Moody currents), corrected by A dependent terms to preserve gauge invariance.

Before moving on, it is useful to emphasize the apparent latitude we have in building specific examples. Recall that cancellation of the gauge, $U(1)_R$, $U(1)_L$, and mixed

$U(1)_R/U(1)_L$ anomalies of a conventional $(0,2)$ GLSM, plus the requirement that the low energy central charge is an integer, ensures that the target space is Calabi Yau. When anomaly cancellation is ensured by fibering a WZW model over a gauge theory, the ability to assign various $U(1)$ charges to the fibers would seem to free us from the requirement that the base be CY. Of course, in that case the FI parameter of the GLSM runs, and a detailed understanding of the IR CFT requires a more nuanced analysis than the brief discussion above. In the remainder of this note we will focus on the simplest case, in which the base is a CY; it would be interesting to explore the fate of more general examples with Ricci-curved bases.

2.6 An Alternate Construction: Bosonization

WZW models were originally discovered [38] as an answer to the question: "What is the bosonization of an equal number of left and right moving fermions?" For example, N right and left moving Majorana-Weyl fermions may be bozonized into a $k = 1$, $O(N)$ WZW model.

The fibred models discussed above also arise via a combination of bosonization and fermionization. In these models, however, the fermion spectrum is chiral, so we must consider the bosonization and fermionionization of chiral systems (see, for example, [39].) While straightforward, the process is not pretty.

For example, consider the $(0,2)$ CFT given by the tensor product of a free T^2 sigma model (at free-fermion radius) and a non-anomalous abelian GLSM with target space $K3$. The basic strategy is to fermionize the free left-chiral bosons in the T^2 multiplet (this gives a set of free left-moving Weyl fermions) while bosonizing a pair of gauged left-moving fermions in the GLSM (this gives a set of left-gauged chiral bosons which, together with the original free right-chiral bosons, form a left-gauged $U(1) \times U(1)$ WZW model at $k = 2$ [38, 40]). The resulting model is thus the original GLSM coupled to a left-gauged WZW model and free fermions – ie, the T^2 is now fibred over the base, while the fermions are trivial lines. By construction, the contribution to the quantum gauge anomaly of the original left-handed fermions is now generated by the classical anomaly of the gauged WZW model. This is just the left-gauged WZW-fibred GLSM discussed above.

More generally, we can start with a simple $(0,2)$ GLSM with gauge group G_{GLSM} and target space X and tensor on a WZW model for G_{WZW} (perhaps with additional right-gauging by some $H_R \subset G_{WZW}$). Now bosonize some subset of charged left-moving fermions in the GLSM whose contribution to the anomaly lies in $G_{Anom} \subset G_{GLSM} \cap G_{WZW}$ and fermionize the left-chiral bosons of the WZW model such that the final left- and right-chiral bosons form a $G_{WZW}/(G_{Anom} \times H_R)$ WZW model, with the dualized left-moving fermions uncharged under the vector of the GLSM. The quantum anomaly of the fermions is again replaced by the classical anomaly of the WZW model, and the WZW model is now nontrivially fibered over the base GLSM. The result is a hybrid model of precisely the form discussed in this paper. Note, too, that this duality has a more familiar name – it is nothing other than a Narain T-duality of the heterotic string on $X \times (G_{WZW}/H_R)$ [41].

2.7 Some Examples

We now present some basic examples of WZW models fibered over gauged linear sigma models. For simplicity, we will mostly take the base space to be a non-compact projective space or Grassmannian. As usual [23], superpotentials can be turned on to cut out a hypersurface/intersection and compactify the target.

2.7.1 $U(1) \times U(1) \longrightarrow K3$

Consider a $U(1)$ GLSM for $K3$ decorated by some vector bundle, $\mathcal{V} \rightarrow K3$, such that the gauge anomaly $\mathcal{A} = c_2(T_{K3}) - c_2(\mathcal{V})$ is non-zero. As we have seen, we can cancel this anomaly by tensoring in a WZW model with suitable left-$U(1)$ action gauged. The simplest such WZW-fiber we can add while preserving $(0,2)$ supersymmetry is the $G = U(1) \times U(1) \sim T^2$ WZW model. To cancel the anomaly, we gauge this WZW model by a left-acting $U(1)$,

$$H_L = U(1) = \begin{pmatrix} e^{i\alpha N_1} & 0 \\ 0 & e^{i\alpha N_2} \end{pmatrix} . \tag{7.45}$$

The bosonic fiber Lagrangian then takes the form,

$$\begin{aligned} \mathcal{L}_{fiber} &= \frac{k}{4\pi}(\partial_+\theta_l\partial_-\theta_l - 2N_lA_+\partial_-\theta_l + (N_1^2+N_2^2)A_+A_-) \\ &= \frac{k}{4\pi}(D_+\theta_lD_-\theta_l - N_l\theta_lF_{+-}) \end{aligned}$$

where we have chosen bosonic coordinates $g = (e^{i\theta_1}, e^{i\theta_2}) \in G$ such that $D\theta_l \equiv \partial\theta_l - N_lA$, and we have integrated by parts in the second equality. The abelian anomaly is canceled by requiring $k(N_1^2 + N_2^2) = \mathcal{A}$. In terms of the complexified coordinates $\theta = \theta_1 + i\theta_2$ and $\chi = \frac{1}{\sqrt{2}}(\psi_+^1 + i\psi_+^2)$, the SUSY transformations of the WZW fields become,

$$\begin{aligned} \delta\theta &= \sqrt{2}\,\epsilon\,\chi \\ \delta\chi &= -i\frac{\bar{\epsilon}}{\sqrt{2}}(\partial_+\theta - (N_1 + iN_2)A_+). \end{aligned} \tag{7.46}$$

This is nothing but a the torsion linear sigma model of [24], a worldsheet description of heterotic flux vacua first explored in [21, 8, 9, 22] whose semi-classical geometry is a non-Kähler T^2-fibration $T^2 \to X \xrightarrow{\pi} K3$ decorated by a vector bundle $\mathcal{V}_X = \pi^*\mathcal{V}_{K3}$ supported by gauge invariant NS-NS 3-form flux H on the total space X, $H = kN^l(d\theta_l + N_lA) \wedge F$.

The realization of these earlier abelian linear models as special cases of WZW-fibrations clarifies a number of features obscured in the earlier presentation. First, it is now clear why these models arise via bosonization and fermionization – indeed, that is how the original WZW construction arose. Secondly, and importantly, the WZW presentation makes precise one of the suggestive features of the original linear models – namely, the gauge action on the bosonic coordinates on the fiber is chiral, with only the left-action gauged. This plays an important role in the study of "small-radius" phases of the worldsheet theory [42].

Before moving on to a non-abelian example, it is perhaps useful to give a concrete example of a model in which all of the anomalies are explicitly cancelled. We take a particularly simple example – a T^2 fibration over a $K3$ formed by the quartic in $\mathbb{P}^3$. The field content is five chiral multiplets, $\Phi_{i=1,\ldots,4}$, P, and five fermi multiplets, $\Lambda_{a=1,2,3,4}$ and Γ, and one WZW multplet (θ, ψ) which forms a $k = 1$, $U(1) \times U(1)$, WZW model. The various charge assignments are given in the figure (note that the figure excludes the gaugino which is charged $+1$ under $U(1)_R$ and is neutral under $U(1)_L$).

Field	Gauge	$U(1)_R$	$U(1)_L$
Φ_i	1	0	0
P	-4	1	1
$\Lambda_{1,2}$	1	0	-1
$\Lambda_{3,4}$	0	0	-1
Γ	-4	1	0
θ_1	1	0	-1
θ_2	1	0	-1
ψ	0	1	0

Figure 2-3: Charges

To make the target space compact, we also add a superpotential of the form (similar, but more general than (2.25),

$$\sim \int d\theta^+ \left(\Gamma\, W(\phi) + P\,\Lambda_a J^a(\phi)\right) \qquad (7.47)$$

where $W(\phi)$ is a quartic polynomial which cuts out a $K3$ in $\mathbb{P}^3$. The J^a's are cubic and quartic polynomials that ensure transversality and set $p = 0$ in the $r \gg 1$ phase.

Thus, in the usual way, the D terms and F terms conspire to give a $K3$ over which the T^2 is fibered. Notice that the $[U(1)_R]^2$ current computation gives the correct central charge $c = 9$. The model also comes equipped with a gauge bundle, V, from the Fermi multiplets, which, similar to (2.27) are determined by the kernel of the bundle map

$$\mathcal{O}(1)^2 \otimes \mathcal{O}^2 \xrightarrow{J^a} \mathcal{O}(4) \tag{7.48}$$

This was one of the many models studied in [24].

2.7.2 Another Abelian Anomaly: $SU(2)\times U(1) \rightarrow \left[\mathcal{O}(-2) \rightarrow \mathbb{P}^1\right]$

Consider a $U(1)$ GLSM including two chiral multiplets, Φ_i, of charge $+1$, one chiral multiplet, P, of charge -2, and one Fermi multiplet, Γ, of charge -2. The classical higgs branch of this theory is the familiar $\mathcal{O}(-2) \rightarrow \mathbb{P}^1$. Quantum mechanically, this model has a chiral anomaly, so we need to couple in a WZW model.

Instead of adding a $U(1) \times U(1)$ WZW model, let's try fibering over our target a non-abelian WZW model for some group manifold, G, with a $U(1) \subset G$ gauged so as to cancel the abelian anomaly of the GLSM. A particularly simple choice is $G = SU(2) \times U(1)$. Since $SU(2) \times U(1)$ is hyperkahler [43], the WZW model admits $(0,2)$ supersymmetry. In particular, the Lie algebra splits as $T_\pm$ under three inequivalent complex structures. For example, under one of them

$$T_\pm = \{a(1 \pm i\sigma_x) + b(\mp\sigma_z + i\sigma_y)\}. \tag{7.49}$$

To cancel the anomaly of the GLSM, we gauge the left-action of the $U(1)$ factor in WZW model by

$$A_L = NA \tag{7.50}$$

where A is the vector in the GLSM and N is a parameter. Anomaly cancellation then fixes $k = 2$ and $N = 1$. A simple computation confirms that this model is completely non-anomalous. The central charge (over three) of WZW model is [44] $\hat{c} = 2\frac{k+1}{k+2} = 3/2$. The naïve central charge of this model is thus $\hat{c} = 2 + 3/2 = 7/2$. That this naïve counting is indeed correct can be seen by flowing to the Landau-Ginzburg point in the moduli space, $r \to -\infty$. As discussed in [42], the correct description of the theory here is an asymmetric orbifold of a $\mathbb{C}^2$ theory tensored with an $SU(2) \times U(1)$ WZW theory. Since orbifolding by a finite group does not change the central charge [45], the central charge is just the sum of the central charges of the two theories.

2.7.3 Examples With Non-Abelian GLSMs

Field	Gauge
Φ_i	$\square$
P_α	$-q_\alpha$
Λ^m	$\square$
Γ^s	$-d_s$
Σ^σ	$adj.$

Figure 2-4: $U(N_c)$

Let's now start with a $U(N_c)$ gauge theory of the form studied in [30]. These models include N_R chiral multiplets Φ_i transforming in the fundamental, N_P chiral multiplets P_α in the $\det^{-q_\alpha}$ representation, N_Λ Fermi multiplets, Λ^m , in the fundamental and N_Γ Fermi multiplets, Γ^s, in the $\det^{-d_s}$ representation. In addition, we add N_Σ chiral multiplets, Σ_σ, in the adjoint representation. The field content is summarized in Figure 2.

The classical target space is given by the vanishing locus of the D term,

$$D^a_b = e^2 \left(\sum_{i=1}^{N_R} \phi^a_i \phi^\dagger_{bi} - \sum_{\alpha=1}^{N_p} q_\alpha |p_\alpha|^2 \delta^a_b - r\delta^a_b \right) \quad a, b = 1 \dots N_c \,, \tag{7.51}$$

modulo the gauge group, as usual. On the Higgs branch, where $p^\alpha = 0$, the manifold defined by $D = 0$ is the space of N_c planes in $\mathbb{C}^{N_R}$, also known as the Grassmannian $G(N_c, N_R)$. This can be seen as follows–the condition

$$\sum_{i=1}^{N_\phi} \phi^a_i \phi^\dagger_{bi} = r\delta^a_b \tag{7.52}$$

defines a set of N_c orthogonal vectors in $\mathbb{C}^{N_R}$. Applying a $U(N_c)$ transformation will rotate these vectors, but will not change the N_c plane that they span. Therefore modding out by the gauge transformations gives $G(N_c, N_R)$. In the non-anomalous models of [30] , a superpotential restricts the vacuum manifold to be some Calabi-Yau hypersurface of $G(N_c, N_R)$. For our purposes, the non-compact ambient variety suffices, so we will dispense with the superpotential.

To study the anomaly structure of the theory, it is useful to treat the trace and traceless parts of the gauge group separately. For gauge transformations in the $SU(N)$,

$$\partial_\mu j^\mu = \frac{N_R - N_L - 2N_c + 2N_\sigma N_c}{4\pi} Tr(\alpha F'_{+-}) . \tag{7.53}$$

Note that for $SU(N_c)$, $Tr_{Adjoint}(T^a T^b) = 2N_c Tr(T^a T^b)$ so that the factors of N_c in (7.53) come from the gauginos and from the Σ fields. For gauge transformations in the central $U(1)$, on the other hand,

$$\partial_\mu j^\mu = \frac{1}{4\pi}(N_R - N_L + N_c \sum_\alpha q_\alpha^2 - N_c \sum_s d_s^2) \, Tr(\alpha F'_{+-}). \tag{7.54}$$

To cancel these anomalies, we again tensor in and gauge a suitable WZW model. If the non-abelian anomaly is non-trivial, however, the WZW model must also be non-abelian. Let's look at a couple of simple examples.

Non-Abelian Example #1: $SU(2) \times U(1) \to [\oplus_\alpha O(-q_\alpha) \to G(2, N_R)]$

As in a previous example, we start with a $SU(2) \times U(1)$ WZW model, but this time gauge the entire symmetry group (which we identify with the gauge group of the non-abelian GLSM),

$$A_L = N_0 T^0 A^0 + T^a A^a \tag{7.55}$$

where $a = 1, 2, 3$ runs over the $SU(2)$ generators and 0 denotes the central $U(1)$ in both the WZW model and the GLSM. The anomaly is cancelled by requiring

$$\begin{aligned} kN_0^2 &= N_R - N_L + 2\sum_\alpha q_\alpha^2 - 2\sum_s d_s^2 \\ k &= N_R - N_L - 4 + 4N_\sigma . \end{aligned} \tag{7.56}$$

Non-Abelian Example #2: $([U(1)^2]_{k'} \times [SU(2)/U(1)]_k) \to [\oplus_\alpha O(-q_\alpha) \to G(2, N_R)]$

We present a second way of fibering a WZW model over a non-abelian gauge theory–one that utilizes both left and right gauging of the WZW model. Starting with a GLSM of the same form as in the previous example, we now cancel the anomaly by fibering a right-gauged $[SU(2)/U(1)_{right}] \otimes [U(1)^2]$ WZW model.

As above, a $U(1)$ subgroup of the WZW model is left-gauged by the central $U(1)$ of the GLSM, with two integers, $\{N_1, N_2\}$, specifying the embedding of $U(1)$ in $U(1)^2$ such that the abelian anomaly is cancelled. The full $SU(2)$ of the second WZW model is also left-gauged by the $SU(2)$ vector of the GLSM, canceling the non-abelian anomaly. Finally, to cancel the anomaly of $H_R \sim U(1)$, we also add a left-moving fermion with charged Q under the auxiliary $U(1)_{right}$ gauge symmetry of the WZW model. The full anomaly cancellation conditions are thus

$$\begin{aligned} k'(N_1^2 + N_2^2) &= N_R - N_L + 2\sum_\alpha q_\alpha^2 - 2\sum_s d_s^2 \\ k &= N_R - N_L - 4 + 4N_\sigma \\ k &= 2(Q^2 - 1). \end{aligned} \tag{7.57}$$

k and k' refer to the possibly different levels of the tensored WZW models.

2.8 Conclusions

We have shown that anomalies in $(0,2)$ gauged linear sigma models may be cancelled by tensoring them with a suitably gauged WZW model. The resulting gauged WZW+GLSM is manifestly $\mathcal{N} = 2$ supersymmetric and is expected to flow to a non-linear sigma model with NS-NS flux when the mixed gauge-R-anomaly is also vanishing. Along the way we identified a candidate R-current which is in the same Q-cohomology class as the R-current of the twisted SCFT, and is thus expected to flow to the superconformal R-current of the IR SCFT. We also found that these WZW models reduce, in the abelian case, to the "torsion linear sigma models" of [24]; the more general non-abelian case thus provides a natural generalization of these quasi-geometric heterotic flux vacua.

It is straightforward (if tedious) to integrate out the massive vector and matter fields along the semi-classical Higgs branch to construct a one-loop approximation to the geometry and flux of the sigma model to which the gauge theory flows. (We must work at one-loop rather than tree level due to the anomaly.) As in the abelian case studied in detail in [24], the result is again a non-Kähler metric with flux specified by the WZW-fibration and satisfying the Bianchi identity. Moreover, one should be able to construct the analogues of twisted versions of these models and to say something about the low energy fixed point correlation functions using a combination of gauge theory and exact conformal field theory techniques.

Another interesting approach [42], is to flow to a Landau-Ginzberg point of the GLSM, e.g. at $r \rightarrow -\infty$, where the partition function of the full theory reduces to an orbifold of the product of the LG partition function with the WZW partition function. Viewed as a symmetry in either the LG or the WZW theory individually, the orbifold group is anomalous; when the partition functions are taken together, the anomaly cancels. This orbifold CFT is interesting in itself, and its spectrum has recently been computed ([46]). This is the exact conformal field theory avatar of gauge anomaly cancellation in the UV GLSM/WZW hybrid.

Chapter 3

Anisotropic Sigma Models in Three Dimensions

3.1 Introduction

As reviewed in Chapter 1, there has recently been very productive cross fertilization between condensed matter physics and high energy particle physics. For instance, methods in gauge/gravity duality have been developed for field theory duals that may be relevant to condensed matter systems (see for example [12], [13], [14], [16, 17]). These field theory duals break Lorentz invariance and scale space and time anisotropically. In terms of inverse spatial length, the space dimensions scale as $[\vec{x}] = -1$, while time scales as $[t] = -z$, where z is known as the dynamical exponent.

Conversely, techniques in condensed matter theory have affected how we think of high energy theories of physics. Recenetly, Horava wrote down a power counting renormalizable theory of gravity which scales space and time differently ([6], [7]). There has been much interest in this "Horava-Lifshitz gravity," as a model for our own cosmology (see, for example [47, 48, 49]). In [6], the $z = 2$ theory was coupled to Lifshitz scalars to describe a possible quantum theory of membranes (with flat target space metric).

Motivated by this, we explore the one loop quantum properties of $z = 2$ Lifshitz scalars with non-trivial target space metric. We are interested in the one loop requirements for conformal symmetry; since the scalars must be dimensionless, $z = 2$ requires that "space-time," be $2 + 1$ dimensional, i.e. a theory of membranes.

For a theory of strings, the relevant nonlinear sigma model (NLSM) action is

$$S[X] = \int d^2\sigma \, G_{\mu\nu}(X) \, \partial_a X^\mu \partial^a X^\nu \tag{1.1}$$

This action has a geometric global symmetry that corresponds to target space (spacetime) diffeomorphisms; the ∂X's transform as tangent vectors and $G_{\mu\nu}$ transforms as a spacetime metric.

For a theory of membranes with $z = 2$ anisotropic scaling, the requirement that there be no dimensionful parameters suggests a sigma model of the form

$$S[X] = \int dt \, d^2\sigma \, \left[G_{\mu\nu}(X) \, \partial_t X^\mu \partial_t X^\nu - \alpha G_{\mu\nu}(X) \triangle X^\mu \triangle X^\nu \right] \tag{1.2}$$

where α is a dimensionless constant, $\triangle \equiv \partial_a \partial^a$, and $a = \{\sigma_1, \sigma_2\}$ runs over spatial indices. This action, however, no longer has the geometric global symmetry–$\triangle X^\mu$ does not transform as a target space vector. Since we are ultimately interested in a theory whose target space gives something like our own, diffeomorphism invariant spacetime, we'd like to restore this symmetry. The appropriate way to do this at the classical level is to covariantize the action as

$$S[X] = \int dt \, d^2\sigma \, \left[G_{\mu\nu}(X) \, \partial_t X^\mu \partial_t X^\nu - \alpha G_{\mu\nu}(X) (\triangle X^\mu + \Gamma^\mu_{\rho\sigma} \partial_a X^\rho \partial^a X^\sigma)(\triangle X^\nu + \Gamma^\nu_{\alpha\beta} \partial_b X^\alpha \partial^b X^\beta) \right] \tag{1.3}$$

where $G_{\mu\nu}$ transforms as a metric and $\Gamma^\mu_{\alpha\beta}$ as a connection. There is also another possibility, adding the operator

$$-\beta G_{\mu\nu}(X)(\partial_a \partial_b X^\mu + \Gamma^\mu_{\rho\sigma} \partial_a X^\rho \partial_b X^\sigma)(\partial^a \partial^b X^\nu + \Gamma^\nu_{\alpha\beta} \partial^a X^\alpha \partial^b X^\beta) \tag{1.4}$$

Indeed, these three operators could have come with entirely different 2-tensors, say $F_{\mu\nu}, G_{\mu\nu}, H_{\mu\nu}$.

The structure of the paper is as follows–in Section 2 we briefly review the background field method and in Section 3 we review how to compute the beta function for the case of a Lorentz invariant worldsheet. We recover the famous condition for conformal invariance in two dimensions, $\beta_{\mu\nu} \sim R_{\mu\nu} = 0$. In Section 4 we apply similar technology to the anisotropic membrane with identical target space metrics for all three operators and recover the same condition for conformal invariance. Thus the physics, while anisotropic in the worldvolume, recovers isotropy in the target space (at least at one loop). In Section 5 we relax the condition that the metrics be the same and take $F_{\mu\nu} \neq G_{\mu\nu}$ as different metrics for the time and space part of the Lagrangian (we leave the fully general case where there are three different metrics, and Γ is a general tangent bundle connection, to future work). Finally, in Section 6 we close with some comments about coupling the theory to worldvolume gravity going towards a fully dynamical anisotropic theory of membranes.

3.2 The Background Field Method

In this paper, we'll use the background field method for computations. We provide here a brief review; for a comprehensive introduction, see for example [50] and [51]. Suppose we define two generating functionals

$$Z[J] \equiv \int \mathcal{D}\pi\, exp\left[iS[\pi] + iJ\cdot\pi\right] \tag{2.5}$$

and

$$\tilde{Z}[J,X] \equiv \int \mathcal{D}\pi\, exp\left[iS[\pi + X] + iJ\cdot\pi\right] \tag{2.6}$$

(where the $\cdot$ between J and π is shorthand for spacetime integration). We define the generating functionals of connected graphs as $W[J] \equiv -i\ln Z[J]$ and $\tilde{W}[J,X] \equiv$

$-i \ln \tilde{Z}[J, X]$, and their Legendre transforms, the effective actions as

$$\Gamma[\bar{\pi}] = W[J] - J \cdot \bar{\pi} \tag{2.7}$$

and

$$\tilde{\Gamma}[\tilde{\pi}, X] = \tilde{W}[J, X] - J \cdot \tilde{\pi} \tag{2.8}$$

with $\bar{\pi} = \frac{\delta W}{\delta J}$ and $\tilde{\pi} = \frac{\delta \tilde{W}}{\delta J}$. As usual, these effective actions are computed by summing all one particle irreducible (1PI) Feynman diagrams.

The main result of the background field method is

$$\tilde{\Gamma}[\tilde{\pi} = 0, X] = \Gamma[X] \tag{2.9}$$

$\tilde{\Gamma}[\tilde{\pi}, X]$ can be thought of as all 1PI graphs computed with the action shifted by a background classical field, X. Since, at the end of the calculation we set $\tilde{\pi} = 0$, we need only compute graphs with external insertions of the background field.

This is not the only advantage of the background field method. We will be interested in actions that have a global diffeomorphism invariance, such as

$$S[X] = \int d^2\sigma \, G_{\mu\nu}(X) \, \partial_a X^\mu \partial^a X^\nu \tag{2.10}$$

If one were to make a direct attack on this action, one would expand $G_{\mu\nu}(X)$ in normal coordinates $G_{\mu\nu}(X) \approx \eta_{\mu\nu}(X_0) + \ldots$ so that a propagator for X can be defined. In doing so, we would explicitly break the diffeomorphism symmetry; there would be no guarantee that the one loop quantum corrections to the effective action would be geometric and covariant. If, instead, we use the action of a background classical field plus a quantum correction $S[X_0 + \pi]$, we will see that we can define a propagator for π whilst maintaing diffeomorphism symmetry in X_0. Therefore the one loop corrections to the effective action will be covariant with respect to X_0.

3.3 The Isotropic Case

3.3.1 Classical Geometry

We briefly review how the background field calculation works when there is worldvolume Lorentz invariance. We follow a calculation very similar to that in [52] and [53]. The action is

$$S[X] = \int d^2\sigma\, G_{\mu\nu}(X)\, \partial_a X^\mu \partial^a X^\nu \tag{3.11}$$

with $a = \{\sigma_1, \sigma_2\}$. There is a global symmetry

$$\begin{aligned} X^\mu &\rightarrow X^\mu(X) \\ G_{\mu\nu}(X) &\rightarrow G'_{\mu\nu}(X') = \frac{\partial X^\sigma}{\partial X^{\mu\prime}} \frac{\partial X^\tau}{\partial X^{\nu\prime}} G_{\sigma\tau}(X') \end{aligned} \tag{3.12}$$

corresponding to to target space diffeomorphisms. The action is invariant because $v_a^\mu \equiv \partial_a X^\mu$ transforms like a target space vector.

The first step in the background field method is to calculate $S[Y = X_0 + \pi]$. However, because π is the difference between two nearby coordinates, X_0 and Y, it is not covariant. We'd like to write π in terms of covariant objects, such as a vector tangent to the geodesic that connects X_0 and $X_0 + \pi$. If $\lambda^\mu(t)$ is this geodesic, ($\lambda^\mu(0) = X_0^\mu$, $\lambda^\mu(1) = X_0^\mu + \pi^\mu$),

$$\ddot{\lambda}^\mu + \Gamma^\mu_{\nu\sigma} \dot{\lambda}^\nu \dot{\lambda}^\sigma = 0 \tag{3.13}$$

Writing $\dot{\lambda}^\mu(0) = \eta^\mu$, we can Taylor expand $\lambda(t)$. One can use the equation (3.13) to write the higher derivative terms in the expansion in terms of η. The result is

$$\lambda^\mu(t) = X_0^\mu + \eta^\mu t - \frac{1}{2}\Gamma^\mu_{\nu\sigma}\eta^\nu\eta^\sigma t^2 - \frac{1}{3!}\Gamma^\mu_{\nu\sigma\delta}\eta^\nu\eta^\sigma\eta^\delta t^3 + \dots \tag{3.14}$$

where $\Gamma^\mu_{\nu\sigma\delta}$ is the naive covariant derivative on Γ which runs over lower indices only (it is not actually covariant, but is determined using the same rules as covariant derivatives on lower index tensors). Therefore

$$X_0^\mu + \pi^\mu = X_0^\mu + \eta^\mu - \frac{1}{2}\Gamma^\mu_{\nu\sigma}\eta^\nu\eta^\sigma - \frac{1}{3!}\Gamma^\mu_{\nu\sigma\delta}\eta^\nu\eta^\sigma\eta^\delta + \dots \tag{3.15}$$

We can view equation (3.15) as defining a coordinate transformation at the point $X_0^\mu + \pi^\mu$ to new coordinates η^μ. However, equation (3.14) is valid in every coordinate system. We are led to the discovery that in the η^μ coordinates, known as normal coordinates, all Γ's and symmetrized "covariant derivatives" of Γ vanish. We will bar expressions in this special coordinate system. Non-covariant expressions can easily be made covariant in these coordinates. For example, It is easy to show that

$$\partial_\nu \bar{\Gamma}^\mu_{\sigma\rho} = \frac{1}{3}(\bar{R}^\mu{}_{\sigma\nu\rho} + \bar{R}^\mu{}_{\rho\nu\sigma}) \tag{3.16}$$

because the double Γ terms vanish. The utility of normal coordinates is that one can Taylor expand a tensor in normal coordinates and then complete the expansion to build covariant objects. Since both sides then involve tensor expressions, the expansion will be generally covariant and true in all coordinates. For example, for a symmetric 2-tensor

$$\begin{aligned}
\bar{T}_{\mu\nu}(X_0+\pi) &= \bar{T}_{\mu\nu}(X_0) + \partial_\sigma \bar{T}_{\mu\nu}\eta^\sigma + \frac{1}{2}\partial_\sigma\partial_\lambda \bar{T}_{\mu\nu}\eta^\sigma\eta^\lambda + \dots \\
&= \bar{T}_{\mu\nu} + \bar{\nabla}_\sigma \bar{T}_{\mu\nu}\eta^\sigma + \frac{1}{2}(\bar{\nabla}_\sigma\bar{\nabla}_\lambda \bar{T}_{\mu\nu} + \partial_\sigma\bar{\Gamma}^\epsilon_{\mu\lambda}\bar{T}_{\epsilon\nu} + \partial_\sigma\bar{\Gamma}^\epsilon_{\nu\lambda}\bar{T}_{\epsilon\mu})\eta^\sigma\eta^\lambda + \dots \\
&= \bar{T}_{\mu\nu} + \bar{\nabla}_\sigma \bar{T}_{\mu\nu}\eta^\sigma + \frac{1}{2}(\bar{\nabla}_\sigma\bar{\nabla}_\lambda \bar{T}_{\mu\nu} + \frac{1}{3}\bar{R}^\epsilon{}_{\tau\sigma\mu}\bar{T}_{\epsilon\nu} + \frac{1}{3}\bar{R}^\epsilon{}_{\tau\sigma\nu}\bar{T}_{\epsilon\mu})\eta^\sigma\eta^\lambda + \dots
\end{aligned} \tag{3.17}$$

Since both sides are covariant, the Taylor expansion holds regardless of coordinate system. Applying (3.17) to the metric immediately gives

$$G_{\mu\nu}(X_0+\pi) = G_{\mu\nu}(X_0) + \frac{1}{3}R_{\mu\lambda\sigma\nu}(X_0)\eta^\sigma\eta^\lambda + \dots \tag{3.18}$$

We can now expand the rest of (3.11) in normal coordinates

$$\partial_a(X_0^\mu + \pi^\mu) = \partial_a X_0^\mu + \partial_a \eta^\mu - \frac{1}{2}\partial_a X^\nu \partial_\nu \Gamma^\mu_{\lambda\sigma}\eta^\lambda\eta^\sigma + \dots \tag{3.19}$$

where ... are higher derivatives of the Christoffel symbols. Covariantizing this expression,

$$\partial_a(X_0^\mu + \pi^\mu) = \partial_a X_0^\mu + \nabla_a \eta^\mu + \frac{1}{3}R^\mu{}_{\lambda\sigma\nu}\partial_a X^\nu \eta^\lambda \eta^\sigma + \dots \tag{3.20}$$

where ... are terms that involve higher derivatives of the curvature tensor and $\nabla_a\eta^\mu = \partial_a\eta^\mu + \partial_a X^\nu \Gamma^\mu_{\nu\sigma}\eta^\sigma$, the pullback to the worldsheet of the target space covariant derivative. Doing the entire expansion, one finds, covariantly,

$$\begin{aligned}\mathcal{L}(X_0+\pi) &= \mathcal{L}(X_0) + 2G_{\mu\nu}\nabla_a\eta^\mu\partial^a X_0^\nu + G_{\mu\nu}\nabla_a\eta^\mu\nabla^a\eta^\nu \\ &+ R_{\mu\lambda\sigma\nu}\eta^\lambda\eta^\sigma\partial_a X_0^\mu\partial^a X_0^\nu + \frac{4}{3}R_{\mu\lambda\sigma\nu}\eta^\lambda\eta^\sigma\nabla_a\eta^\mu\partial^a X_0^\nu \\ &+ \frac{1}{3}R_{\mu\lambda\sigma\nu}\eta^\lambda\eta^\sigma\nabla_a\eta^\mu\nabla^a\eta^\nu + O(R^2, \nabla R)\end{aligned} \tag{3.21}$$

By choosing X_0 to be a solution to the classical equations of motion, the term linear in η vanishes.

3.3.2 One Loop Beta Function

To do computations with Lagrangian (3.21), we should switch to an orthonormal frame so that we can invert the kinetic term for η. This gives

$$\begin{aligned}\mathcal{L}(X_0+\pi) &= \mathcal{L}(X_0) + \nabla_a\eta^i\nabla^a\eta_i \\ &+ R_{\mu ij\nu}\eta^i\eta^j\partial_a X_0^\mu\partial^a X_0^\nu + \frac{4}{3}R_{jik\nu}\eta^i\eta^k\nabla_a\eta^j\partial^a X_0^\nu \\ &+ \frac{1}{3}R_{ijkl}\eta^k\eta^j\nabla_a\eta^i\nabla^a\eta^l + O(R^2) + O(\nabla R)\end{aligned} \tag{3.22}$$

with $\eta^i \equiv e^i_\mu(X_0)\,\eta^\mu$, and $e^i_\mu(X_0)$ is a vielbein, and all i, j indices have been vielbein rotated. Since we're summing over vacuum bubbles with external field insertions, the

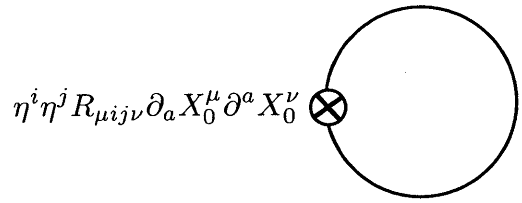

Figure 3-1: Diagram Contributing to the One Loop Beta Function

only contributions at leading order in R to the one loop beta function are vertices with two η's. The $\partial_a \eta^i \partial^a \eta^j$ term gives a propagator

$$\sim \frac{\delta^{ij}}{k^2} \tag{3.23}$$

The contribution of the third term in (3.22) is (see Figure 3-1)

$$\sim R_{\mu\nu} \partial_a X_0^\mu \partial^a X_0^\nu \int \frac{d^2 k}{k^2} \sim R_{\mu\nu} \partial_a X_0^\mu \partial_0^a X_0^\nu \ln\left(\frac{\mu}{\Lambda}\right) \tag{3.24}$$

where μ is an IR cutoff and Λ is a UV cutoff. We take μ to be the same as the renormalization energy scale. The beauty of the background field method is that the effective action must be completely invariant under target space diffeomorphisms and local Lorentz transformations. For example, one might worry that we have not considered all diagrams with two η vertices. The $\nabla_a \eta^i \nabla^a \eta_i$ contains, in addition to the kinetic term for η, contributions to the action of the form

$$\sim \partial_a \eta^i (\omega_\mu)_{ij} \eta^j \partial^a X_0^\mu \tag{3.25}$$

and

$$\sim (\omega_\mu)^i_j (\omega_\nu)_{ik} \eta^j \eta^k \partial_a X_0^\mu \partial^a X_0^\nu \tag{3.26}$$

Since these expressions are not covariant (there are no $\partial\omega$'s around to make them so),

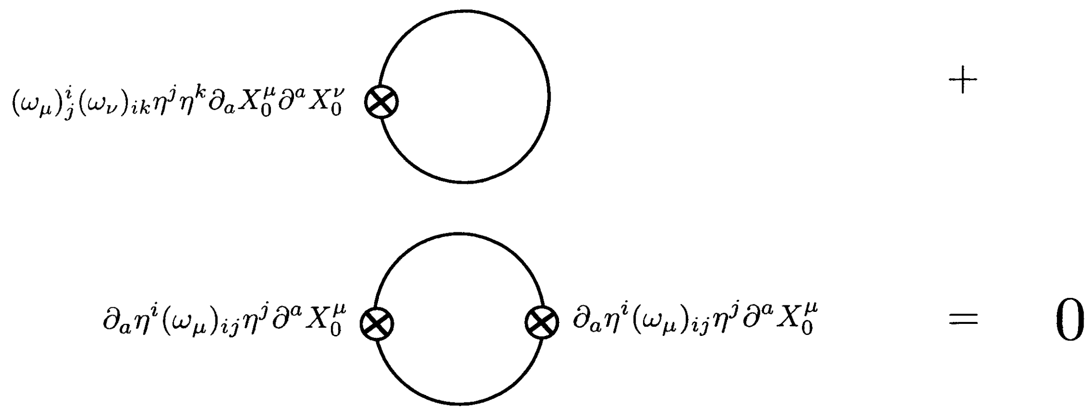

Figure 3-2: Schematic Cancellation of Non-Covariant Terms

their contributions to the effective action must cancel. One can see schematically how this works in Figure 3-2.

Thus, the effective potential to this order (when the appropriate factors of 2π are included) is

$$\Gamma[X_0] = \partial_a X_0^\mu \partial_0^a X_0^\nu (G_{\mu\nu} + \frac{1}{2\pi} R_{\mu\nu} \ln\left(\frac{\mu}{\Lambda}\right)) \tag{3.27}$$

Cancelling the divergence at the renormalization point requires

$$G_{\mu\nu} = G_{\mu\nu}^{ren} + \frac{1}{2\pi} R_{\mu\nu} \ln\left(\frac{\Lambda}{\mu}\right) \tag{3.28}$$

and so

$$\beta_{\mu\nu} = \mu \frac{\partial}{\partial \mu} G_{\mu\nu}^{ren.} = \frac{1}{2\pi} R_{\mu\nu} \tag{3.29}$$

3.4 The Anisotropic NLSM

3.4.1 Classical Geometry

As discussed in the introduction, the appropriate action for a $z = 2$ anisotropic nonlinear sigma model with target space diffeomorphism symmetry is

$$\begin{aligned} S[X] &= \int dt\, d^2\sigma\, \big[G_{\mu\nu}\,\partial_t X^\mu \partial_t X^\nu - \alpha G_{\mu\nu}(\triangle X^\mu + \Gamma^\mu_{\rho\sigma}\partial_a X^\rho \partial^a X^\sigma)(\triangle X^\nu + \Gamma^\nu_{\alpha\beta}\partial_b X^\alpha \partial^b X^\beta) \\ &- \beta G_{\mu\nu}(\partial_a\partial_b X^\mu + \Gamma^\mu_{\rho\sigma}\partial_a X^\rho \partial_b X^\sigma)(\partial^a\partial^b X^\nu + \Gamma^\nu_{\alpha\beta}\partial^a X^\alpha \partial^b X^\beta)\big] \end{aligned} \quad (4.30)$$

(We shall not here address the case when Γ is not the metric connection; later we will address the case when the space and time parts of the Lagangians have different metrics.) $a, b, c, \ldots$ now denote spatial indices and α, β are dimensionless coupling constants. Written more geometrically in terms of $v^\mu_a \equiv \partial_a X^\mu$ and $v^\mu_t \equiv \partial_t X^\mu$ the Lagrangian is

$$\mathcal{L} = G_{\mu\nu}\, v^\mu_t v^\nu_t - \alpha(D_{v_a} v^\mu_a)G_{\mu\nu}(D_{v_b} v^\nu_b) - \beta(D_{v_a} v^\mu_b)G_{\mu\nu}(D_{v_a} v^\nu_b) \quad (4.31)$$

Alternatively, we will also find it useful to decompose the Lagrangian into time and space parts with

$$\begin{aligned} \mathcal{L} &= (\mathcal{L}^{t,\mu\nu}_0 + \mathcal{L}^{s,\mu\nu}_0)G_{\mu\nu} \\ \mathcal{L}^{t,\mu\nu}_0 &= \partial_t X^\mu \partial_t X^\nu \\ \mathcal{L}^{s,\mu\nu}_0 &= -(\partial_a\partial_b X^\mu + \Gamma^\mu_{\rho\sigma}\partial_a X^\rho \partial_b X^\sigma)\mathcal{G}^{abcd}(\partial_c\partial_d X^\nu + \Gamma^\nu_{\alpha\beta}\partial_c X^\alpha \partial_d X^\beta) \end{aligned} \quad (4.32)$$

and

$$\mathcal{G}^{abcd} \equiv \alpha\delta^{ab}\delta^{cd} + \beta\delta^{ac}\delta^{bd} \quad (4.33)$$

Again, we can expand the Lagrangian around a background classical field. For example, we encounter expressions of the following type in normal coordinates

$$\begin{aligned}
\triangle X^\mu + \Gamma^\mu_{\rho\sigma}\partial_a X^\rho \partial^a X^\sigma &= \triangle X^\mu_0 + \partial_a\partial^a\eta^\mu - 2(\partial_\epsilon\Gamma^\mu_{\lambda\sigma})\partial_a X^\epsilon_0 \partial^a \eta^\lambda \eta^\sigma - \frac{1}{2}(\partial_\epsilon \Gamma^\mu_{\lambda\sigma})\triangle X^\epsilon_0 \eta^\lambda\eta^\sigma \\
&+ (\partial_\epsilon\Gamma^\mu_{\rho\sigma}\eta^\epsilon)(\partial_a X^\rho_0 + \partial_a\eta^\rho - \frac{1}{2}\partial_a X^\nu_0 \partial_\nu \Gamma^\rho_{\lambda\epsilon}\eta^\lambda\eta^\epsilon) \\
&\times (\partial_a X^\sigma_0 + \partial_a \eta^\sigma - \frac{1}{2}\partial_a X^\nu_0 \partial_\nu \Gamma^\sigma_{\lambda\epsilon}\eta^\lambda\eta^\epsilon) \qquad (4.34)
\end{aligned}$$

(the other term, with slightly modified worldvolume index contractions has a similar expansion).

After covariantizing and rotating to an orthonormal frame, we find at $O(\eta^2)$ and $O(R)$ (this is all that matters for the one loop β function calculation),

$$\begin{aligned}
\mathcal{L}(X_0 + \pi) &= \mathcal{L}(X_0) + \mathcal{L}_{Lin.} + \nabla_t \eta^i \nabla^t \eta_i - \mathcal{G}^{abcd}(\nabla_a\nabla_b\eta^i)(\nabla_c\nabla_d\eta_i) \\
&+ (\mathcal{L}^{t,\mu\nu}_0(X_0) + \mathcal{L}^{s,\mu\nu}_0(X_0))R_{\mu ij\nu}\eta^i\eta^j \\
&- 4\mathcal{G}^{abcd}(\partial_a\partial_b X^\mu_0 + \Gamma^\mu_{\rho\sigma}\partial_a X^\rho_0 \partial_b X^\sigma_0)\partial_c X^\nu_0 R_{\mu ij\nu}(\nabla_d\eta^j)\eta^i \\
&+ 2\mathcal{G}^{abcd}\partial_c X^\nu_0 \partial_d X^\mu_0 R_{\mu ij\nu}(\nabla_a\nabla_b\eta^i)\eta^j \\
&+ \ldots \qquad (4.35)
\end{aligned}$$

Again $\mathcal{L}_{lin.} = 0$ if we expand around a classical background.

3.4.2 One Loop Beta Function

The terms quadratic in η^i in (4.35) imply a propagator (after Wick rotation, see e.g., [54], [55], [56])

$$\sim \frac{\delta^{ij}}{\omega^2 + (\alpha+\beta)p^4} \qquad (4.36)$$

The terms in the second line combine to give a contribution to renormalize the tree level Lagrangian as

$$\sim -R_{\mu\nu}(\mathcal{L}_0^{t,\mu\nu}(X_0) + \mathcal{L}_0^{s,\mu\nu}(X_0)) \int \frac{d^2p\, d\omega}{\omega^2 + (\alpha+\beta)p^4} \tag{4.37}$$

To determine the integral, we use Schwinger parameters

$$\begin{aligned} \int \frac{d^2p\, d\omega}{\omega^2 + (\alpha+\beta)p^4} &= \int d\gamma\, d\omega\, d^2p\, e^{-\gamma\omega^2} e^{-\gamma(\alpha+\beta)p^4} \\ &= \frac{\pi^2}{2\sqrt{(\alpha+\beta)}} \int_{\frac{1}{\Lambda^2}}^{\frac{1}{\mu^2}} \frac{d\gamma}{\gamma} \\ &\sim \frac{1}{\sqrt{(\alpha+\beta)}} \ln\left(\frac{\Lambda}{\mu}\right) \end{aligned} \tag{4.38}$$

where μ, Λ are again IR and UV cutoffs. So the contribution to the effective action of these terms is

$$\sim -\frac{R_{\mu\nu}}{\sqrt{(\alpha+\beta)}}(\mathcal{L}_0^{t,\mu\nu}(X_0) + \mathcal{L}_0^{s,\mu\nu}(X_0)) \ln\left(\frac{\Lambda}{\mu}\right) \tag{4.39}$$

The contributions of the third line of (4.35) is

$$\sim \int \frac{d^2p\, d\omega\; p^d}{\omega^2 + (\alpha+\beta)p^4} = 0 \tag{4.40}$$

by rotational symmetry of the spatial dimensions (there are additional terms coming from the connection coefficients in ∇, however, these are higher order in R).

The contribution of the fourth line in (4.35) is

$$
\begin{aligned}
&\sim \quad R_{\mu\nu} G^{abcd} \partial_c X_0^\nu \partial_d X_0^\mu \int \frac{d^2p\, d\omega\ p_a p_b}{\omega^2 + (\alpha+\beta)p^4} \\
&\sim \quad (\alpha + \frac{\beta}{2}) R_{\mu\nu} \partial_a X_0^\mu \partial^a X_0^\nu \int d\gamma\, dp\, d\omega\, p^3 e^{-(\alpha+\beta)\gamma p^4} e^{-\alpha\omega^2} \\
&\sim \quad \frac{(\alpha+\frac{\beta}{2})}{(\alpha+\beta)} R_{\mu\nu} \partial_a X_0^\mu \partial^a X_0^\nu \int_{\frac{1}{\Lambda^2}}^\infty d\gamma\, \gamma^{-\frac{3}{2}} \\
&\sim \quad \frac{(\alpha+\frac{\beta}{2})}{(\alpha+\beta)} R_{\mu\nu}\, \partial_a X_0^\mu \partial^a X_0^\nu\, \Lambda
\end{aligned}
\tag{4.41}
$$

with Λ a UV cutoff. In the second line we have used the rotational symmetry of the spatial dimensions. We have generated a relevant operator which vanishes at $R_{\mu\nu} = 0$. We note with curiosity that taking $\beta = -2\alpha$ also makes the operator vanish. It would be nice if there were some underlying symmetry at $\beta = -2\alpha$ that ruled this operator out at higher order in perturbation theory, but we can find none. Presumably this operator gets generated at higher loops even away from $\beta = -2\alpha$. However, when we couple the NLSM to anisotropic gravity, we will argue that there is a symmetry which forbids this operator at different values of α, β.

As in the isotropic case, all contributions to Feynman diagrams from spin connection coefficients hidden in the ∇'s in equation (4.35) are zero. The lowest dimensional covariant operator that they might generate is $Tr(F^2)$, where F is the curvature of the spin connection. This operator occurs at the next order in the curvature expansion.

Thus all operators get renormalized with coefficient $R_{\mu\nu}$, and hence the beta function is the expected $\beta_{\mu\nu} \sim R_{\mu\nu}$ and the one loop conformal point is again at $R_{\mu\nu} = 0$. At one loop, Einstein's equations are required in the target space.

3.5 Extension to the Case of Two Metrics

We now take different metrics for the time and space parts of the Lagrangian, e.g.

$$\mathcal{L} = \mathcal{L}_0^{t,\mu\nu} F_{\mu\nu} + \mathcal{L}_0^{s,\mu\nu} G_{\mu\nu}$$

with $\mathcal{L}_0^{t,\mu\nu}, \mathcal{L}_0^{s,\mu\nu}$ as before. To do the background field expansion of this Lagrangian, we find it easiest to expand in normal coordinates and to use the connection of the space metric G. (This choice is calculationally much easier, as it simplifies the more complicated space sector. If one were to look at the NLSM with three different metrics, this complication would be unavoidable.) Since everything is completely covariant, no target space physics can ultimately depend on this choice. The covariant background field expansion becomes, after rotation with G's vielbein,

$$\begin{aligned}
\mathcal{L}(X_0+\pi) &= \mathcal{L}(X_0) + \mathcal{L}_{Lin.} + \nabla_t\eta^i\nabla^t\eta^j F_{ij} - \mathcal{G}^{abcd}(\nabla_a\nabla_b\eta^i)(\nabla_c\nabla_d\eta_i) \\
&+ \mathcal{L}_0^{t,\mu\nu}(X_0)(R^{\epsilon}{}_{ij\mu}F_{\epsilon\nu} + \frac{1}{2}\nabla_i\nabla_j F_{\mu\nu})\eta^i\eta^j + \mathcal{L}_0^{s,\mu\nu}(X_0)R_{\mu ij\nu}\eta^i\eta^j \\
&- 4\mathcal{G}^{abcd}(\partial_a\partial_b X_0^\mu + \Gamma^\mu_{\rho\sigma}\partial_a X_0^\rho \partial_b X_0^\sigma)\partial_c X_0^\nu R_{\mu ij\nu}(\nabla_d\eta^j)\eta^i \\
&+ 2\mathcal{G}^{abcd}\partial_c X_0^\nu \partial_d X_0^\mu R_{\mu ij\nu}(\nabla_a\nabla_b\eta^i)\eta^j \\
&+ 2\partial_t X_0^\mu(\nabla_i F_{\mu j})(\nabla_t\eta^j)\eta^i \\
&+ \ldots
\end{aligned} \qquad (5.42)$$

The persistence of F_{ij} in the quadratic terms for η gives a strange propagator and we find it difficult to make sense of the necessary Wick rotation. Instead, we expand F perturbatively around G, $F_{\mu\nu} \approx G_{\mu\nu} + \epsilon H_{\mu\nu}$ with ϵ small and calculate everything to first order in ϵ. The propagator is as before, and $\epsilon\nabla_t\eta^i\nabla_t\eta^j H_{ij}$ is treated as a two point interaction. Thus, in addition to the integrals of the previous section, there is always an additional diagram with an insertion of this operator. Again, the fourth line of (5.42) does not contribute at this order, nor does the fifth line (at least when we expand F perturbatively, as it would need two insertions to renormalize the metric).

Schematically, the operator connected to $\mathcal{L}_0^{t,\mu\nu}(X_0)$ gets renormalized as

$$\sim \mathcal{L}_0^{t,\mu\nu}(X_0)(R_{\mu\nu} - \frac{\epsilon}{2}(H_{\mu\sigma}R^{\sigma\tau}{}_{\tau\nu} + H_{\nu\sigma}R^{\sigma\tau}{}_{\tau\mu} + \nabla^2 H_{\mu\nu} + R_{\mu\tau\sigma\nu}H^{\sigma\tau} + R_{\nu\tau\sigma\mu}H^{\sigma\tau}))\ln\left(\frac{\Lambda}{\mu}\right) \quad (5.43)$$

where indices are raised and lowered on H by G. The last two terms of (5.43) come from two point diagrams with an insertion of $\epsilon\nabla_t\eta^i\nabla_t\eta^j H_{ij}$. Similarly $\mathcal{L}_0^{s,\mu\nu}(X_0)$ gets renormalized as

$$\sim \mathcal{L}_0^{s,\mu\nu}(X_0)(R_{\mu\nu} - \frac{\epsilon}{2}(R_{\mu\tau\sigma\nu}H^{\sigma\tau} + R_{\nu\tau\sigma\mu}H^{\sigma\tau}))\ln\left(\frac{\Lambda}{\mu}\right) \quad (5.44)$$

The relevant operator again gets generated, this time with coefficient

$$\sim (\alpha + \frac{\beta}{2})(R_{\mu\nu} - \frac{\epsilon}{2}(R_{\mu\tau\sigma\nu}H^{\sigma\tau} + R_{\nu\tau\sigma\mu}H^{\sigma\tau}))\,\partial_a X_0^\mu \partial^a X_0^\nu\,\Lambda \quad (5.45)$$

which we can again make vanish by taking $\beta = -2\alpha$. Note that equations (5.43) and (5.44) imply

$$\begin{aligned}\beta^G_{\mu\nu} &\sim R_{\mu\nu} - \frac{\epsilon a}{2}(R_{\mu\tau\sigma\nu}H^{\sigma\tau} + R_{\nu\tau\sigma\mu}H^{\sigma\tau}) \\ \beta^H_{\mu\nu} &\sim -\frac{1}{2}(H_{\mu\sigma}R^{\sigma\tau}{}_{\tau\nu} + H_{\nu\sigma}R^{\sigma\tau}{}_{\tau\mu} + \nabla^2 H_{\mu\nu})\end{aligned} \quad (5.46)$$

3.6 Coupling the Model to Worldvolume Gravity

So far, we've just been investigating the one loop properties of $2+1$ nonlinear sigma models with anisotropic worldvolume scaling. In [6], Horava wrote down a putative $2+1$ theory of membranes coupled to anisotropic, Horava-Lifshitz gravity. His theory, however, had flat target space metric. We now wish to write down a Lagrangian which generalizes this to curved target space.

Let us briefly review Horava-Lifshitz gravity in $2+1$ dimensions. The degrees of freedom are a two dimensional metric, g_{ab} and N, and N^a, which are analogous to the lapse and shift vectors in the ADM decomposition of General Relativity [57]. That is, one can think of building a three dimensional metric out of these variables as

$$ds^2 = -N^2dt^2 + g_{ab}(dx^a - N^a dt)(dx^b - N^b dt) \tag{6.47}$$

In, $2+1$ dimensions, the Lagrangian for gravity is

$$\mathcal{L}_{gravity} = \frac{1}{2}\frac{\sqrt{g}}{\kappa^2 N}(\dot{g}_{ab} - \tilde{\nabla}_a N_b - \tilde{\nabla}_b N_a)G^{abcd}(\dot{g}_{cd} - \tilde{\nabla}_c N_d - \tilde{\nabla}_d N_c) \tag{6.48}$$

with κ the Newton's constant, $\tilde{\nabla}$ the metric connection with respect to g_{ab}, and G^{abcd} the "metric on the space of metrics,"

$$G^{abcd} \equiv \left[\frac{1}{2}(g^{ac}g^{bd} + g^{bc}g^{ad}) - \lambda g^{ab}g^{cd}\right] \tag{6.49}$$

λ is a dimensionless coupling constant of the theory. This free field action has an anisotropic global conformal symmetry

$$\begin{aligned} t' &= \lambda^2 t \\ x' &= \lambda x \\ g'_{ab}(t',x') &= g_{ab}(t,x) \\ N'(t',x') &= N(t,x) \\ N_a(t',x') &= \lambda^{-1}N_a(t,x) \end{aligned} \tag{6.50}$$

For generic values of λ, the Lagrangian (6.62) breaks the full 3D diffeomorphism of metric (6.47) to so called "foliation preserving diffeomorphisms," that is, transformations of the form $x' = f(t,x)$, $t' = h(t)$. Under this symmetry, the fields transform as

$$
\begin{aligned}
N(t,x) &= N'(t',x')\frac{dh}{dt} \\
N_a(t,x) &= \frac{dt'}{dt}\frac{\partial x'^d}{\partial x^a}N'_d(t',x') + \frac{\partial x'^c}{\partial t}\frac{\partial x'^d}{\partial x^a}g'_{cd}(t',x') \\
g_{ab}(t,x) &= \frac{\partial x'^c}{\partial x^a}\frac{\partial x'^d}{\partial x^b}g'(t',x')
\end{aligned}
\tag{6.51}
$$

In $2+1$ dimensions, for $\lambda = \frac{1}{2}$, the action also has a local Weyl symmetry

$$
\begin{aligned}
g'(t,x) &= \exp(2\omega(t,x))g(t,x) \\
N'(t,x) &= \exp(2\omega(t,x))N(t,x) \\
N'_a(t,x) &= \exp(2\omega(t,x))N_a(t,x)
\end{aligned}
\tag{6.52}
$$

Indeed, the global part of this weyl symmetry holds for any λ. The anisotropic conformal symmetry (6.50) is a subgroup of (fDiff) $\times$ (global Weyl). For everything subsequent we impose the local version of the Weyl symmetry, $\lambda = 1/2$.

3.6.1 Coupling In The Scalars

We can supplement the gravity action in a way that continues to respect fDiff (for simplicity of exposition, we set all three target space metrics equal to $G_{\mu\nu}$),

$$
\begin{aligned}
\mathcal{L}_{scalar} &= \frac{\sqrt{g}G_{\mu\nu}}{N}(\partial_t X^\mu - N^a\partial_a X^\mu)(\partial_t X^\mu - N^b\partial_b X^\nu) \\
&- \sqrt{g}N\left[\partial_a\partial_b X^\mu - \tilde{\Gamma}^e_{ab}\partial_e X^\mu + \Gamma^\mu_{\rho\sigma}\partial_a X^\rho\partial_b X^\sigma\right] \\
&\times G_{\mu\nu}\mathcal{G}^{abcd} \\
&\times \left[\partial_c\partial_d X^\mu - \tilde{\Gamma}^f_{cd}\partial_f X^\mu + \Gamma^\nu_{\alpha\beta}\partial_c X^\alpha\partial_d X^\beta\right]
\end{aligned}
\tag{6.53}
$$

now with

$$
\mathcal{G}^{abcd} \equiv \left[\alpha g^{ab}g^{cd} + \beta g^{ac}g^{bd}\right] \tag{6.54}
$$

$\tilde{\Gamma}$ is the metric connection of the worldvolume metric, g. For $g_{ab} = \delta_{ab}$, $N = 1$ and $N_a = 0$, this action reduces to the previously discussed anisotropic NLSM. The local Weyl symmetry, under which the scalars are uncharged, continues to hold only if $\beta = 0$, and so we take this to be the case. Note that this Weyl symmetry rules out the generator of the relevant operator that turned up in the NLSM, (4.41) whose covariant form, $N\sqrt{g}g^{ab}\partial_a X^\mu \partial_b X^\nu$ is Weyl non-invariant.

As in string theory, it is worth asking how much of this worldvolume gravity we can gauge away using the fDiff $\times$ Weyl symmetry. $g_{ab}(t,x)$ has three degrees of freedom, whereas the Weyl transformation and the $2d$ diffeomorphsms also have three degrees of freedom, which is enough to set $g_{ab} = \delta_{ab}$. There are residual gauge transformations which do not change this gauge fixing condition. In holomorphic coordinates, $g_{z\bar{z}} = 1/2, g_{zz} = g_{\bar{z}\bar{z}} = 0$. Under a holomorphic transformation combined with a Weyl transformation, $z' = f(t,z)$, $\bar{z}' = \bar{f}(t,\bar{z})$, the diagonal components remain zero and

$$g'(t', z', \bar{z}')_{z\bar{z}} = \exp(2\omega(t,z,\bar{z})) \left| \frac{\partial f(t,z)}{\partial z} \right|^{-2} g_{z\bar{z}} \tag{6.55}$$

So an unfixed gauge symmetry is a holomorphic coordinate transformation (which can now depend on t!) combined with a Weyl transformation of the form $\exp(2\omega(t,z,\bar{z})) = \left|\frac{\partial f(t,z)}{\partial z}\right|^2$. Actually the unfixed gauge symmetry is larger, because we have not fixed time reparametrizations, $t' = h(t)$. Under the total remaining unfixed gauge symmetry, the fields N and N_a transform as

$$\begin{aligned}
t' &= h(t) \\
z' &= f(t,z) \\
\bar{z}' &= \bar{f}(t,\bar{z}) \\
N'(t',z',\bar{z}') &= \left|\frac{\partial f(t,z)}{\partial z}\right|^2 \left(\frac{dh}{dt}\right)^{-1} N(t,z,\bar{z}) \\
N'_{z'}(t',z',\bar{z}') &= \left(\frac{dh}{dt}\right)^{-1} \left[\frac{\partial \bar{f}(t,\bar{z})}{\partial \bar{z}} N_z(t,z,\bar{z}) - \frac{1}{2}\frac{\partial \bar{f}(t,\bar{z})}{\partial t}\right] \\
N'_{\bar{z}'}(t',z',\bar{z}') &= \left(\frac{dh}{dt}\right)^{-1} \left[\frac{\partial f(t,z)}{\partial z} N_{\bar{z}}(t,z,\bar{z}) - \frac{1}{2}\frac{\partial f(t,z)}{\partial t}\right]
\end{aligned} \tag{6.56}$$

Indeed, the above ungauged fixed symmetry for $h(t) = \lambda^2 t$ and $f(t,z) = \lambda z$ is just the global anisotropic conformal symmetry. In conformal coordinates, the gauged fixed action is

$$\begin{aligned}
\mathcal{L} &= \frac{16}{\kappa^2 N}\partial N_z \bar{\partial} N_{\bar{z}} \\
&+ \frac{G_{\mu\nu}}{N}(\partial_t X^\mu - 2N_z\bar{\partial}X^\mu - 2N_{\bar{z}}\partial X^\mu)(\partial_t X^\nu - 2N_z\bar{\partial}X^\nu - 2N_{\bar{z}}\partial X^\nu) \\
&- 4\alpha N \left[\partial\bar{\partial}X^\mu + \Gamma^\mu_{\rho\sigma}\partial X^\rho \bar{\partial} X^\sigma\right] G_{\mu\nu} \left[\partial\bar{\partial}X^\nu + \Gamma^\nu_{\alpha\beta}\partial X^\alpha \bar{\partial} X^\beta\right]
\end{aligned} \tag{6.57}$$

The reader can check that the above action is indeed invariant under (6.56).

3.6.2 A Nambu Goto Form?

We briefly comment on another possible anisotropic membrane action–one that is not based on the sigma model and more closely resembles the Nambu Goto action for strings. A natural starting point for the Nambu Goto action in this case would be something like

$$S = \int d^2\sigma\, dt\ \widetilde{N}\sqrt{\widetilde{G}} \tag{6.58}$$

This is certainly invariant under foliation preserving diffeomorphisms. $\widetilde{G}$ is the pullback to the membrane of the target space metric, while $\widetilde{N}$ is the pullback to the worldsheet of the lapse function, namely

$$\begin{aligned} \widetilde{G}_{ab} &= \partial_a X^\mu \partial_b X_\mu \\ \widetilde{N} &= \sqrt{-(\partial_t X^\mu \partial_t X_\mu - \partial_t X^\mu \partial_a X_\mu \widetilde{G}^{ab} \partial_t X^\nu \partial_b X_\nu)} \end{aligned} \tag{6.59}$$

However, this doesn't respect the anisotropic scaling. Instead, we replace $\widetilde{G}$ with a different metric, a metric on two tensors instead of on vectors

$$F_{(ab)(cd)} \equiv (\partial_a \partial_b X^\mu - \Gamma^e_{ab} \partial_e X^\mu)(\partial_c \partial_d X_\mu - \Gamma^f_{cd} \partial_f X_\mu) \tag{6.60}$$

One can think of F as a 4×4 matrix where each column represents a pair of indices. Then the following action respects foliation preserving diffeomorphisms and scales anisotropically

$$S = \int d^2\sigma\, dt\ \widetilde{N} (\det F)^{\frac{1}{8}} \tag{6.61}$$

Note, that there is also the option of adding the "action for gravity" i.e., adding

$$\mathcal{L}_{gravity} = \frac{1}{2} \frac{\sqrt{\widetilde{G}}}{\kappa^2 \widetilde{N}} (\dot{\widetilde{G}}_{ab} - \tilde{\nabla}_a \widetilde{N}_b - \tilde{\nabla}_b \widetilde{N}_a) \widetilde{G}^{abcd} (\dot{\widetilde{G}}_{cd} - \tilde{\nabla}_c \widetilde{N}_d - \tilde{\nabla}_d \widetilde{N}_c) \tag{6.62}$$

with

$$\widetilde{N}_a \equiv \partial_t X^\mu \partial_a X_\mu \tag{6.63}$$

and

$$G^{abcd} \equiv \left[\frac{1}{2} (\widetilde{G}^{ac} \widetilde{G}^{bd} + \widetilde{G}^{bc} \widetilde{G}^{ad}) - \lambda \widetilde{G}^{ab} \widetilde{G}^{cd} \right] \tag{6.64}$$

3.7 Conclusions

We have constructed the analog of the bosonic NLSM for $z = 2$ anisotropic worldvolumes. We have also calculated the one loop beta function and shown that, incredibly, anisotropy does not change the fact that the target space is forced to be Ricci flat at one loop. We have also shown how to couple this model to worldvolume gravity towards constructing a new, anisotropic theory of membranes.

To properly continue this work, one would need to quantize the theory in the presence of the remaining worldvolume gravity, find the physical modes, carefully gauge fix the Weyl symmetry using ghosts, calculate the critical dimension and calculate the beta functions in the presence of worldvolume gravity, with the most general backgrounds. We leave this work to future research.

While this work was in preparation, a related analysis was of sigma models was done in [58]. Though many calculations are similar, the philosophy and motivation of these authors differ–they work in four dimensions and retain Lorentz invariance.

Chapter 4

Fermi Surfaces and Bulk Dipole Couplings

4.1 Introduction

As described in Chapter 1, the fermionic two point functions of a theory dual to an extremal AdS_4 black hole describes excitations above a fermi surface,

$$G^{\alpha}_{\psi\psi} = \langle\psi(x_1)\psi(x_2)\rangle_\alpha \underbrace{=}_{kspace} \frac{h_1}{k_\perp - \frac{1}{v_f}\omega - h_2 e^{i\gamma(k_f)}\omega^{2\lambda_\alpha(k_f)}} \tag{1.1}$$

with $k_\perp \equiv k - k_f$, and constants $h_1, v_f, \gamma(k_f), \lambda_\alpha(k_f)$. $\lambda_\alpha(k_f)$ is related to the scaling dimension, ν_α, of the k_f^{th} mode of ψ under a low frequency, near horizon AdS_2 symmetry by $\lambda_\alpha(k_f) = \nu_\alpha(k_f) - 1/2$. In this way, the emergent conformal symmetry controls IR properties like dispersion relations and widths.

In this chapter, we consider the existence and location of boundary fermi surfaces in the presence of an additional dimension 5 operator

$$\bar{\psi}(g_m + g_e\Gamma)\Sigma^{\mu\nu}\psi F_{\mu\nu} \tag{1.2}$$

corresponding to magnetic and electric dipole moments for the bulk fermions (Γ is the highest rank Clifford algebra element). Our motivation for this addition is several-fold. First, we would like to investigate the robustness of the previous discoveries. These relied on the canonical dirac action and were insensitive to higher order interactions. However, there exist a large class of higher dimensional quadratic operators that can possibly change these conclusions. This work on dipole couplings is the first attempt to investigate the effects of such operators. Do these higher dimension operators in the bulk drastically alter the existence of fermi surfaces in the boundary? We will find that they do not, but rather the main effects of the dipole couplings are to change the IR AdS_2 scaling dimensions and to change the locations of fermi surfaces in k space (which we will find numerically). As such, we have constructed a much larger parameter space of Non-Fermi liquids for study.

Related to this, we feel that the effects of general higher dimension operators in the bulk deserve further investigation. It is unclear how to interpret bulk Wilsonian RG flow as a boundary effect and we hope that the sustained study of such operators can give us a better understanding of such flow. The dipole moment operators are a natural starting point; they are generic in the sense that they arise from dimensional reduction on a larger space. [1]

Finally, it is useful to investigate generic Green's functions in the boundary that cannot be diagonalized for all k. All previous numerical investigations involved such diagonalizable Green's functions. In the bulk, this corresponds to ability to block diagonalize the Dirac equation. For $g_e \neq 0$, this is no longer the case and we are led to a much more general consideration of 2×2 boundary Green's functions.

---

[1] We thank N. Iqbal for this point.

4.2 Calculating Real Time Spinor Correlation Functions

In chapter 1 we outlined how to calculate two point functions for scalars: solve the wave equation with boundary condition $\phi \to \phi_0$, evaluate the action with this solution, and take two functional derivatives. The wave equation is a second order differential equation and so this ϕ_0 boundary condition does not completely fix the solution; we also need to declare boundary conditions at the (black hole or Poincare) horizon. In Euclidean signature, one linear combination of solutions typically blows up at the horizon and so we fix the solution by requiring regularity. This is not true, however, for real-time correlators, which are typically oscillatory at the horizon. These advanced or retarded real-time correlators can be obtained by the appropriate analytic continuation of the Euclidean ones. We will be interested in the retarded functions; these characterize a system's response to small perturbations. The correct analytic continuation for the retarded function corresponds to choosing the in-going wave solution at the horizon [60].

Actually, there is an easier way to compute retarded two point functions than the one reviewed in the introduction (for a review, see [4]) . At the boundary, a scalar behaves as

$$\phi(r,k) \to r^{d-\nu}\phi_0(k)(1+O\left(\frac{1}{r^2}\right)) + r^{\nu}\phi_1(k)(1+O\left(\frac{1}{r^2}\right)) \qquad (2.3)$$

where $\nu = \sqrt{(d/2)^2 + m^2R^2}$ is the CFT scaling dimension of the corresponding operator, $\mathcal{O}$. The 1-point function in the presence of the source ϕ_0 is given by a sort of functional Hamilton-Jacobi equation

$$\langle \mathcal{O}(k) \rangle_{\phi_0} = \frac{\delta W}{\delta \phi_0(k)} = -\frac{\delta S_{grav}}{\delta \phi_0(k)} = -\left[\lim_{r\to\infty} r^{-d+\nu}\Pi(r,k)\right]_{finite} \qquad (2.4)$$

with

$$\Pi(r,k) \equiv \frac{\partial \mathcal{L}_{grav}}{\partial(\partial_r \phi)} \tag{2.5}$$

The factor of $r^{-d+\nu}$ is needed because we are actually using a UV cutoff and calculating quantities at $r = 1/\epsilon$ with ϵ small. We keep only finite terms; the infinite pieces are UV divergent contact terms that can be removed by proper holographic renormalization. In terms of (2.3),

$$\langle \mathcal{O}(k) \rangle_{\phi_0} = (2\nu - d)\phi_1 \tag{2.6}$$

Linear response theory relates the change in operator vacuum expectation values in the presence of a source ϕ_0 to ϕ_0 and the retarded Green's function,

$$\langle \mathcal{O}(k) \rangle_{\phi_0} = \phi_0(k) G_R(k) \tag{2.7}$$

giving

$$G_R(k) = (2\nu - d)\frac{\phi_1}{\phi_0} \tag{2.8}$$

Therefore, we can read off the two point function from UV data given appropriate boundary conditions at the horizon.

In this chapter, we will follow [60] and use a procedure very similar to the above to calculate spinor correlation functions. For the remainder of the chapter, we will be interested in gauge-gravity duals with a four dimensional bulk. The free spinor action

$$S = \int d^4x \sqrt{-g} i(\bar{\psi}\Gamma^M D_M \psi - m\bar{\psi}\psi) \tag{2.9}$$

results in the curved space Dirac equation

$$\Gamma^M D_M \psi - m\psi = 0 \tag{2.10}$$

with $D_M = \partial_M + (1/4)\omega_{\underline{ab}M}\Gamma^{\underline{ab}}$. ω is the spin connection and M runs over spacetime indices while a, b run over tangent bundle (vielbein) indices. Γ^M's are related to the usual Clifford algebra matrices by a factor of the vielbein, $\Gamma^M = e^M{}_{\underline{a}}\Gamma^{\underline{a}}$. Following the scalar case we require (in a way that will shortly become more clear) $\psi \to \chi_0$ at the boundary and set

$$\left\langle \exp\left[-\int \bar{\chi}_0 \mathcal{O} + \bar{\mathcal{O}}\chi_0\right]\right\rangle_{QFT} = e^{-S_{grav}[\bar{\chi}_0,\chi_0]} \tag{2.11}$$

The most naive application of the GKPW prescription immediately runs into two confusions

1. ψ is a Dirac spinor in 4 dimensions with 4 components and χ_0 is a Dirac spinor in 3 dimensions with 2 components. Our $\psi \to \chi_0$ limit does not really make sense.

2. The Dirac equation is first order. If we fix all of ψ at the boundary, the solution will not, in general, be regular in the interior.

The solution to both of these apparent problems is the same–at the boundary we should only fix half of the components of ψ, which will correspond to the source χ_0. The other components of ψ will be fixed by requiring regularity at the horizon.

Thus, we need to decide which components correspond to the source. Once again, we will associate to the source those components which are largest at the boundary. More specifically, let's work in a basis of gamma matrices with

$$\Gamma^{\underline{r}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \Gamma^{\underline{\mu}} = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^\mu & 0 \end{pmatrix} \quad \psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \tag{2.12}$$

where ψ_+, ψ_- are the chirality eigenvectors of Γ^r, and γ^μ are boundary gamma matrices. This choice makes clear that ψ_+, ψ_- transform as boundary spinors. Near the boundary, the Dirac equation gives

$$\begin{aligned}\psi_+ &\rightarrow A(k)r^{-3/2+m}(1+O\left(\frac{1}{r^2}\right)) + B(k)r^{-5/2-m}(1+O\left(\frac{1}{r^2}\right)) \\ \psi_- &\rightarrow C(k)r^{-5/2+m}(1+O\left(\frac{1}{r^2}\right)) + D(k)r^{-3/2-m}(1+O\left(\frac{1}{r^2}\right)) \quad (2.13)\end{aligned}$$

where A, B, C, D are two component spinors and A, C and B, D are locally related. For now, we focus on $m \geq 0$ (for $m < 0$ we can just switch $A \leftrightarrow D$, $B \leftrightarrow C$). At the boundary, the $A(k)$ term is dominant, and so we set the boundary conditions as

$$A(k) = \chi_0 \quad \text{or} \quad \lim_{r\rightarrow\infty} r^{3/2-m}\psi_+ = \chi_0 \tag{2.14}$$

Analogously to (2.4), we set

$$\langle \bar{\mathcal{O}}(k) \rangle_{\chi_0} = -\left[\lim_{r\rightarrow\infty} r^{m-3/2}\Pi_+(r,k)\right]_{finite} \tag{2.15}$$

Examining (2.13), this gives

$$\Pi_+ = -\sqrt{-g}g_{rr}^{-1/2}\bar{\psi}_- \Rightarrow \langle \bar{\mathcal{O}}(k) \rangle_{\chi_0} = \bar{D} \tag{2.16}$$

If $D(k)$ and $A(k)$ are related by a matrix $S(k)$, $D = SA$ (we should solve the Dirac equation with linearly independent boundary conditions until we can determine all of S), then again linear response theory tells us how to compute the Euclidean correlator,

$$G_E(k) \equiv \langle \mathcal{O}\mathcal{O}^\dagger \rangle = S(k)\gamma^\tau \tag{2.17}$$

Again, we get the retarded Green's function by choosing ingoing boundary conditions at the horizon and by analytically continuing γ^τ,

$$G_R(k) = \langle \mathcal{O}\mathcal{O}^\dagger \rangle = iS(k)\gamma^t \tag{2.18}$$

Actually, for $0 \leq m < 1/2$, all terms in $\psi_\pm$ are normalizable and it is consistent to treat D as the source and A as the response. This is the so-called alternative quantization of [61]. The boundary CFT in alternative quantization differs from that in the usual one by turning on $\mathcal{O}^\dagger\mathcal{O}$ in the CFT lagrangian.

We can calculate ν, the CFT scaling dimension of the boundary operator $\mathcal{O}$ using the following heuristic. Evaluating the boundary CFT at cutoff $r \sim 1/\epsilon$, there is a term in the boundary action

$$\int d^3x \sqrt{\gamma_{\frac{1}{\epsilon}}}\bar{\chi}_0(x, \frac{1}{\epsilon})\mathcal{O}(x, \frac{1}{\epsilon}) \in S_{bdy} \tag{2.19}$$

where γ is the induced metric on the boundary. In terms of some finite renormalized operator,

$$\bar{\chi}_0(x, \frac{1}{\epsilon}) \sim \epsilon^{3/2-m}\bar{\chi}_{0,ren.}(x) \tag{2.20}$$

To make S_{bdy} finite, this suggests that we define $\mathcal{O}(x, \frac{1}{\epsilon}) \sim \epsilon^{3/2+m}\mathcal{O}_{ren.}(x)$ and thus $\nu = 3/2 + m$ for spinors in three dimensions. For alternative quantization, the same reasoning leads to $\nu = 3/2 - m$.

4.3 Fermion Two Point Functions at Finite Density

In the next three sections, we review the work of [17]. We are interested in studying 2-point functions of fermionic operators in a $2 + 1$ dimensional boundary with a finite $U(1)$ charge density. For simplicity, we work at zero temperature.

In the bulk, this ensemble corresponds to studying a black hole in AdS_4 charged under a $U(1)$ gauge field coupled to classical fermions. We will be agnostic about the nature of this classical matter, that is, we will not specify the details of the boundary theory. However, the setup is general enough that it encompasses many known examples of gauge gravity duality (say, $\mathcal{N} = 4$ SYM) and presumably many unknown examples. Thus our study will hopefully be universal over a large range of gauge-gravity duals. We will leave the charge of the fermionic operators, q, general; the actual attainable q are particular to each example.

The bulk action is

$$S_{bulk} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[\mathcal{R} + \frac{6}{R^2} - \frac{R^2}{g_F^2} F_{MN} F^{MN} \right] \tag{3.21}$$

where R is the AdS radius, κ the Newton's constant and g_F the gauge coupling. The solution we are interested in is the charged AdS_4 black hole

$$\begin{aligned} ds^2 &= \frac{r^2}{R^2}(-f dt^2 + d\vec{x}^2) + \frac{R^2}{r^2}\frac{dr^2}{f} \\ f &= 1 + \frac{Q^2}{r^4} - \frac{M}{r^3} \\ A_t &= \mu(1 - \frac{r_0}{r}) \end{aligned} \tag{3.22}$$

with Q, M the black hole charge and mass respectively and $\mu \equiv g_F Q/(R^2 r_0^2)$. r_0 is the outer horizon, i.e. the largest solution to $f(r_0) = 0$. In the boundary, this geometry corresponds to a theory with finite charge density and temperature

$$\begin{aligned} \rho &= \frac{2Q}{\kappa^2 R^2 g_F} \\ T &= \frac{3r_0}{4\pi R^2}\left(1 - \frac{Q^2}{3r_0^4}\right) \end{aligned} \tag{3.23}$$

At extremality, the inner and outer horizons merge into a double zero of f and

$$M = 4\left(\frac{Q}{\sqrt{3}}\right)^{3/2}, \quad Q = \sqrt{3}r_0^2 \Longrightarrow T = 0 \tag{3.24}$$

We will usually work in units with $R = 1$. In addition, in our numerical work, we will often put the horizon at $r_0 = 1$ and set $g_F = 1$.

We want to study the Dirac equation (2.10) in the bulk,

$$\Gamma^\alpha D_\alpha \Psi - m\Psi = 0 \tag{3.25}$$

One can nicely cancel off the spin connection contributions to this equation by defining

$$\Psi = (-gg^{rr})^{\frac{1}{4}} e^{-i\omega t + ik_i x^i} \psi \tag{3.26}$$

with $\vec{x} = (x, y)$ the spatial directions on the boundary. Substituting and rearranging,we get

$$\sqrt{\frac{g_{ii}}{g_{rr}}} (\Gamma^{\underline{r}} \partial_r - m\sqrt{g_{rr}})\psi + iK_\mu \Gamma^{\underline{\mu}} \psi = 0 \tag{3.27}$$

with

$$K_\mu = (-u, k_i) \tag{3.28}$$

and

$$u = \sqrt{\frac{g_{ii}}{-g_{tt}}} \left(\omega + \mu_q (1 - \frac{r_0}{r})\right) \tag{3.29}$$

and $\mu_q = \mu q$. This system of four coupled equations becomes simpler by choosing the k momentum to be entirely in the x direction (which we can do by rotational invariance) and by a canny choice of gamma matrices,

$$\Gamma^{\underline{r}} = \begin{pmatrix} -\sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} \quad \Gamma^{\underline{t}} = \begin{pmatrix} i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{pmatrix} \quad \Gamma^{\underline{x}} = \begin{pmatrix} -\sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}$$
$$\Gamma^{\underline{y}} = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \tag{3.30}$$

Defining $\psi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$ and rearranging gives

$$(\partial_r + m\sqrt{g_{rr}}\sigma^3)\Phi_\alpha = \sqrt{\frac{g_{rr}}{-g_{tt}}} \left(\omega + \mu_q (1 - \frac{r_0}{r})\right) i\sigma^2 \Phi_\alpha + \sqrt{\frac{g_{rr}}{g_{ii}}} k(-1)^\alpha \sigma^1 \Phi_\alpha \tag{3.31}$$

with $\alpha = 1, 2$. This gives two decoupled, real 2×2 equations.

Note that this is a different basis of Γ matrices then the one put forth in the previous section (2.12), useful for calculating two point functions. To calculate, we should relate the two bases. In terms of (3.31), solving the Dirac equation at the boundary gives

$$\Phi_\alpha \sim a_\alpha r^m \begin{pmatrix} 0 \\ 1 \end{pmatrix} + b_\alpha r^{-m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.32}$$

In terms of ψ_+, ψ_- defined in (2.12), this gives

$$\begin{aligned} \psi_+ &\sim A(k) r^m + \ldots \quad , \quad A(k) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\ \psi_- &\sim D(k) r^{-m} + \ldots \quad , \quad D(k) = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \end{aligned} \tag{3.33}$$

The difference in the exponents from (2.13) is due to the r-dependent field redefinition (3.26). In terms of the matrix, S, defined in the previous section

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{3.34}$$

Since the two α equations are decoupled, we can choose independent boundary conditions that do not mix Φ_1 and Φ_2 giving $s_2 = s_3 = 0$ and, from (2.18),

$$G_R = iS\gamma^{t'} = - \begin{pmatrix} b_1/a_1 & 0 \\ 0 & b_2/a_2 \end{pmatrix} \tag{3.35}$$

with

$$\Gamma^{\underline{t}'} = \begin{pmatrix} 0 & \gamma^{t'} \\ \gamma^{t'} & 0 \end{pmatrix} = U\Gamma^t U^{-1} \tag{3.36}$$

where U is the basis change between the Φ_α basis and the chiral basis. For alternative quantization, the source and vev are switched and similar reasoning leads to $\tilde{G}^\alpha_R = -1/G^\alpha_R$.

4.4 Low Frequency Limit of Retarded Greens Functions

We are interested in looking at the low frequency limit (compared to the chemical potential, μ). Naively, we should expand Φ_α in a perturbation series in ω. However, there is a subtlety because the coefficient multiplying ω in (3.31) blows up at the horizon and so we cannot treat ω as a small perturbation there. To deal with this, we split the r-axis into two regions, an inner region (with variable ξ) and an outer region (with r). The inner region has

$$r - r_0 = \frac{\omega R_2^2}{\xi} \quad , \quad \epsilon < \xi < \infty \tag{4.37}$$

and the outer

$$\frac{\omega R_2^2}{\epsilon} < r - r_0 \tag{4.38}$$

with $R_2 \equiv 1/\sqrt{6}$ (really there is a factor also of the AdS radius, R, which we have set to one). The strategy now is to develop the solution as a perturbation series in ω with ξ in the inner region and r in the outer region. Because the distinction between inner/outer involves ω, the inner region equation no longer blows up in the $\omega \longrightarrow 0$ limit and the perturbation series between the two regions is reshuffled.

Let us examine the lowest order solution in the inner region by taking the limit $\omega \longrightarrow 0, \epsilon \longrightarrow 0, \omega R_2^2/\epsilon \longrightarrow 0$. Writing the Dirac equation in this limit gives

$$(-\partial_\xi + \frac{mR_2}{\xi}\sigma_3)\Phi_\alpha = (\omega + \frac{qe_3}{\xi})i\sigma_2\Phi_\alpha + \frac{R_2}{\xi}(-1)^\alpha \frac{k}{r_0}\sigma_1\Phi_\alpha \tag{4.39}$$

with $e_3 \equiv g_F/\sqrt{12}$. This is precisely the Dirac equation for a spinor in $AdS_2 \times \mathbb{R}^2$, $\psi_\alpha = (-gg^{\xi\xi})^{1/4}\Phi_\alpha$, where $\xi = 0$ is the boundary. This near horizon AdS_2 geometry has radius R_2 and constant electric field e_3. In fact, we can think of this as a Dirac spinor on just AdS_2 where the k term is a T-violating mass. We are interested in matching to the outer region by taking $\xi \to 0$. Here, the equation becomes

$$-\xi\partial_\xi\Phi_\alpha = U\Phi_\alpha \tag{4.40}$$

with

$$U = \begin{pmatrix} -mR_2 & e_3q + (-1)^\alpha \frac{kR_2}{r_0} \\ -e_3q + (-1)^\alpha \frac{kR_2}{r_0} & mR_2 \end{pmatrix} \tag{4.41}$$

This matrix has eigenvalues $\pm\lambda_\alpha$ with

$$\lambda_\alpha = \frac{1}{R_2}\sqrt{\frac{k^2R_2^4}{r_0^2} + m^2R_2^4 - q^2e_3^2R_2^2} \tag{4.42}$$

implying that at the boundary the solution is

$$\Phi_\alpha^{I(0)} = A(\omega)v_{+\alpha}\xi^{-\lambda_\alpha} + B(\omega)v_{-\alpha}\xi^{\lambda_\alpha} \tag{4.43}$$

with

$$v_{\pm\alpha} = \begin{pmatrix} mR_2 \mp \lambda_\alpha \\ e_3q - (-1)^\alpha \frac{kR_2}{r_0} \end{pmatrix} \tag{4.44}$$

The AdS_2 retarded Green's function, in the presence of a constant E field, is, generalizing (3.35), B/A. Again, we must set infalling boundary conditions at the horizon. Thus, we can normalize the outer solution as

$$\Phi_\alpha^{I(0)} = v_{+\alpha}\xi^{-\lambda_\alpha} + G_R^\alpha(\omega)v_{-\alpha}\xi^{\lambda_\alpha} \tag{4.45}$$

The AdS_2 retarded Green's function is (see [62])

$$G_R^\alpha(\omega) = e^{-i\pi\lambda_\alpha}\frac{\Gamma(-2\lambda_\alpha)\Gamma(1+\lambda_\alpha - iqe_3)}{\Gamma(2\lambda_\alpha)\Gamma(1-\lambda_\alpha - iqe_3)} \times \frac{(m + i(-1)^\alpha\frac{k}{r_0})R_2 - iqe_3 - \lambda_\alpha}{(m + i(-1)^\alpha\frac{k}{r_0})R_2 - iqe_3 + \lambda_\alpha}(2\omega)^{2\lambda_\alpha} \tag{4.46}$$

Going through the same kind of analysis as in (2.19), λ_α is related to the scaling dimension of the AdS_2 operator, δ_α by $\delta_\alpha = \lambda_\alpha + 1/2$. Note also, that by momentum conservation in $\mathbb{R}^2$, operators with different k do not mix.

Now, we look at the outer region equations. Here, we can safely set $\omega = 0$ to get the lowest order solution. Again, we get

$$\Phi_\alpha^{O(0)} \sim v_{\pm\alpha} \left(\frac{R_2^2}{r - r_0} \right)^{\mp\lambda_\alpha} \tag{4.47}$$

Matching, we conclude

$$\Phi_\alpha^{O(0)} = v_{+\alpha} \left(\frac{R_2^2}{r - r_0} \right)^{-\lambda_\alpha} + G_R^\alpha(\omega) v_{-\alpha} \left(\frac{R_2^2}{r - r_0} \right)^{+\lambda_\alpha} \tag{4.48}$$

Now, in the outer region, we can perturbatively expand the linearly independent solutions

$$\eta_{\pm\alpha} = \eta_{\pm\alpha}^{(0)} + \omega \eta_{\pm\alpha}^{(1)} + \dots \tag{4.49}$$

where we have already solved for $\eta_{\pm\alpha}^{(0)}$. The higher orders can be obtained by solving the dirac equation and requiring that the solution has no piece proportional to the lower order ones. Thus, the matching is entirely determined by the lowest order and we conclude

$$\Phi_\alpha^O = \eta_{+\alpha} + G_R^\alpha(\omega) \eta_{-\alpha} \tag{4.50}$$

To know $\eta_{\pm\alpha}$ we must solve the dirac equation everywhere–we must have all the UV data. $G_R^\alpha(\omega)$ is determined entirely, however, by the IR. At the boundary of AdS_4, we know (from (3.32))

$$\eta_{\pm\alpha}^{(n)} \sim a_{\pm\alpha}^{(n)} r^m \begin{pmatrix} 0 \\ 1 \end{pmatrix} + b_{\pm\alpha}^{(n)} r^{-m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.51}$$

giving the full greens function perturbatively as (3.35)

$$G_R^\alpha(\omega, k) = \frac{b_{+\alpha}^{(0)} + \omega b_{+\alpha}^{(1)} + O(\omega^2) + G_R^\alpha(\omega)(b_{-\alpha}^{(0)} + \omega b_{-\alpha}^{(1)} + O(\omega^2))}{a_{+\alpha}^{(0)} + \omega a_{+\alpha}^{(1)} + O(\omega^2) + G_R^\alpha(\omega)(a_{-\alpha}^{(0)} + \omega a_{-\alpha}^{(1)} + O(\omega^2))} \tag{4.52}$$

We hope the distinction between $G^{\alpha}_{R}(\omega, k)$ (the full AdS_4 greens function) and $\mathcal{G}^{\alpha}_{R}(\omega)$ (that of of the near horizon AdS_2 geometry) is not too confusing.

We note that in [17], it was found that the bosonic low frequency two point function has the same form. However, whenever there is a "fermi surface" for these bosons (see below), there is always an instability.

4.5 Fermi Surfaces

Let us suppose that there exist certain k_f where $a^{(0)}_{+\alpha}(k_f) = 0$. This will generically only happen for real λ_α. For small $k_\perp = k - k_f$ and small ω, the Greens function (4.52) can be written

$$G^{\alpha}_{R}(\omega, k) \approx \frac{h_1}{k_\perp - \frac{1}{v_f}\omega - h_2 e^{i\gamma(k_f)}\omega^{2\lambda_\alpha(k_f)}} \tag{5.53}$$

with

$$\begin{aligned} v_f &\equiv -\frac{\partial_k a^{(0)}_{+\alpha}(k_f)}{a^{(1)}_{+\alpha}(k_f)} \qquad & h_1 &\equiv \frac{b^{(0)}_{+\alpha}(k_f)}{\partial_k a^{(0)}_{+\alpha}(k_f)} \\ h_2 &\equiv -|c(k_f)|\frac{a^{(0)}_{-\alpha}(k_f)}{\partial_k a^{(0)}_{+\alpha}(k_f)} \qquad & \mathcal{G}^{\alpha}_{R}(\omega) &\equiv |c(k)|e^{i\gamma(k)}\omega^{2\lambda_\alpha} \end{aligned} \tag{5.54}$$

This greens function has a pole in the complex ω plane

$$\begin{aligned} \omega_c &= \omega_*(k) - i\Gamma(k) \\ &= \left(\frac{k_\perp}{h_2}\right)^{\frac{1}{2\lambda_\alpha(k_f)}} e^{-i\frac{\gamma(k_f)}{2\lambda_\alpha(k_f)}} \quad , \quad \lambda_\alpha(k_f) < \frac{1}{2} \\ &= v_f k_\perp - h_2 v_f e^{i\gamma(k_f)}(v_f k_\perp)^{2\lambda_\alpha(k_f)} \quad , \quad \lambda_\alpha(k_f) > \frac{1}{2} \end{aligned} \tag{5.55}$$

We interpret the $\omega = 0$, $k_\perp = 0$ singularity as a fermi surface and the finite ω polls as particle-like excitations above this fermi surface. Since, in general, it is not true that $\Gamma \sim \omega^2$, we are looking at systems not described by Fermi Liquid Theory, i.e. Non-

Fermi liquids. Looking at (5.55), the excitations have dispersion relation $\omega_*(k) \propto k_\perp^z$ with

$$\begin{aligned} z &= \frac{1}{2\lambda_\alpha(k_f)} \;,\quad \lambda_\alpha(k_f) < \frac{1}{2} \\ &= 1 \;,\quad \lambda_\alpha(k_f) > \frac{1}{2} \end{aligned} \tag{5.56}$$

and widths $\Gamma(k) \propto k_\perp^\delta$ with

$$\begin{aligned} \delta &= \frac{1}{2\lambda_\alpha(k_f)} \;,\quad \lambda_\alpha(k_f) < \frac{1}{2} \\ &= 2\lambda_\alpha(k_f) \;,\quad \lambda_\alpha(k_f) > \frac{1}{2} \end{aligned} \tag{5.57}$$

For $\lambda_\alpha(k_f) < \frac{1}{2}$, the width and energy are comparable and the excitations are not stable quasi-particles. For $\lambda_\alpha(k_f) > \frac{1}{2}$, as we scale towards the fermi surface, the ratio of lifetime to energy goes to zero and there are such quasi stable excitations. For $\lambda_\alpha(k_f) = 1/2$, both $G_R^\alpha(\omega)$ and $a_{+\alpha}^{(1)}$ have poles which cancel, leaving a log in the greens function. It is

$$G_R^\alpha(\omega, k) \approx \frac{h_1}{k_\perp + c_1\omega + \tilde{c}_1\omega\log\omega} \tag{5.58}$$

with $\tilde{c}_1$ real and c_1 complex. Such a greens function is thought to be relevant for high T_c cuprates and have been dubbed "marginal fermi liquids," in the literature.

Thus, we have written down a Greens function for excitations about a fermi surface. The low energy properties, such as the form of the lifetime and dispersion relation are entirely determined by the scaling dimensions of an emergent near horizon (IR) conformal field theory.

4.6 Turning on Dipole Couplings Couplings

We want to look at the effects of adding an electric or magnetic dipole term in the bulk on the existence and location of fermi surfaces. To do this, we use the Dirac Lagrangian density,

$$\mathcal{L} = i(\bar{\Psi}\Gamma^\alpha D_\alpha \psi - m\bar{\Psi}\Psi) - \bar{\Psi}(g_m + g_e\Gamma)\Sigma^{\mu\nu}\Psi F_{\mu\nu} \tag{6.59}$$

with (in our basis)

$$\Gamma \equiv \Gamma^{\underline{t}}\Gamma^{\underline{r}}\Gamma^{\underline{x}}\Gamma^{\underline{y}} = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} \tag{6.60}$$

The Dirac equation is now

$$\Gamma^\alpha D_\alpha \Psi - m\psi + i(g_m + g_e\Gamma)\Sigma^{\mu\nu}F_{\mu\nu}\Psi = 0 \tag{6.61}$$

Once again, we can cancel the spin part of the covariant derivative by making the definition (3.26). Doing the same sort of manipulations leading up to (3.31) and using $F_{rt} = \mu r_0/r^2$, we get

$$\begin{aligned}(\partial_r + m\sqrt{g_{rr}}\sigma^3)\Phi_\alpha &= \sqrt{\frac{g_{rr}}{-g_{tt}}}\left(\omega + \mu_q(1 - \frac{r_0}{r})\right) i\sigma^2\Phi_\alpha + \sqrt{\frac{g_{rr}}{g_{ii}}}k(-1)^\alpha\sigma^1\Phi_\alpha \\ &+ 2\mu r_0(-g_{tt})^{-\frac{1}{2}}(g_m\sigma^1\Phi_\alpha + g_e\sigma^3\Phi_\beta)\frac{1}{r^2} \end{aligned} \tag{6.62}$$

where again $\alpha = 1, 2$ and $\beta \neq \alpha$ is the other Φ. Note that in this basis the dirac equation is no longer block diagonal, though it is still real. The dipole terms have no effect boundary behavior of this equation.

Because there is mixing with $g_e \neq 0$, the process for extracting the Green's function is slightly more complicated. Equation (3.34) still holds, but we can no longer choose two sets of boundary conditions such that G_R is diagonal. Instead, we use two sets of linearly independent boundary conditions, I, and II. (3.34) becomes

$$\begin{pmatrix} b_1^I & b_1^{II} \\ b_2^I & b_2^{II} \end{pmatrix} = \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} \begin{pmatrix} a_1^I & a_1^{II} \\ a_2^I & a_2^{II} \end{pmatrix} \tag{6.63}$$

or $B = SA$ in matrix notation. The greens function is $G_R(\omega, k) = BA^{-1}$.

4.6.1 Discrete Symmetries

We can discover several discrete symmetries by examining the effect of conjugating the dirac equation (6.62) and the infalling boundary conditions with certain simple matrices, U. For example, when we pick our two sets of infalling boundary conditions in 2×2 blocks, conjugating with the matrix

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\Gamma^{\underline{x}}\Gamma^{\underline{y}} \tag{6.64}$$

switches the sign of k in the Dirac equation and switches the two sets of boundary conditions. We learn that

$$G(\omega, -k) = UG(\omega, k)U \tag{6.65}$$

When $g_e = 0$, we can take a diagonal basis, leading to $G_1(\omega, -k) = G_2(\omega, k)$. For the general mixed case, we note that $\det G(\omega, -k) = \det G(\omega, k)$, so that our graphs of fermi surfaces in the (k, q) plane will be invariant under $k \to -k$. In a similar way, by examining the effect of U on (6.62) and on the boundary conditions, the choice

$$U = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} = \Gamma^{\underline{r}} \implies G(-\omega, -k; -q, -g_m, g_e) = -G^*(\omega, k; q, g_m, g_e) \tag{6.66}$$

This, along with the first discrete symmetry, implies that our fermi surface plots with $g_m = 0$ will be symmetric under $q \to -q$. Finally, the choice

$$U = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} = \Gamma^{\underline{r}}\Gamma^{\underline{t}} \implies G(\omega, -k; -m, -g_m, -g_e) = -\left[G(\omega, k; m, g_m, g_e)\right]^{-1} \tag{6.67}$$

In particular, this implies that alternative quantization is equivalent to taking $(m, g_m, g_e) \to (-m, -g_m, -g_e)$.

4.6.2 The Low Frequency Limit

Again, we develop a perturbation series in ω by splitting the r-axis into inner and outer regions. The lowest order inner region equation is

$$\begin{aligned}(-\partial_\xi + \frac{mR_2}{\xi}\sigma^3)\Phi_\alpha &= \left(\omega + \frac{qe_3}{\xi}\right) i\sigma^2\Phi_\alpha + \frac{R_2}{r_0\xi}k(-1)^\alpha\sigma^1\Phi_\alpha \\ &+ 2\frac{e_3}{\xi}(g_m\sigma^1\Phi_\alpha + g_e\sigma^3\Phi_\beta)\frac{1}{R_2}\end{aligned} \tag{6.68}$$

Near the boundary of AdS_2, we get

$$-\xi\partial_\xi\Phi_\alpha = i\sigma_2 qe_3\Phi_\alpha - R_2\left(m\sigma_3 + \tilde{M}_\alpha\sigma^1\right)\Phi_\alpha + 2g_e\frac{e_3}{R_2}\sigma^3\Phi_\beta \tag{6.69}$$

with

$$\tilde{M}_\alpha \equiv -(-1)^\alpha\frac{k}{r_0} + 2\frac{e_3}{R_2^2}g_m \tag{6.70}$$

Again,

$$-\xi\partial_\xi\psi = U(g_e, g_m)\psi \tag{6.71}$$

and the four eigenvalues of U are

$$\pm\lambda_{1,2} = \pm\frac{1}{R_2}\sqrt{(m^2 + \frac{k^2}{r_0^2})R_2^4 + e_3^2(4(g_e^2 + g_m^2) - q^2R_2^2) \pm 4e_3R_2^2\sqrt{g_m^2\frac{k^2}{r_0^2} + g_e^2(m^2 + \frac{k^2}{r_0^2})}} \tag{6.72}$$

where the $1, 2$ correlates with the $\pm$ in the square root. Thus the dimensions of operators in the IR CFT are significantly changed. In the case of $g_e = g_m = 0$, the usual case of two degenerate eigenvalues obtains.

By making a basis change on (6.68), we can block diagonalize it (though we cannot do so for the full dirac equation)

$$\left(-\partial_\xi - i(\omega + \frac{qe_3}{\xi})\sigma_2 + \frac{\nu_\pm}{\xi}\sigma_1\right)\tilde{\Phi}_{1,2} = 0 \tag{6.73}$$

with

$$\begin{aligned} \nu_- &= \sqrt{\frac{4e_3^2\left(g_e^2 + g_m^2\right)R_2^2 + \left(m^2 + \frac{k^2}{r_0^2}\right)R_2^6 - 4\sqrt{e_3^2\left(g_m^2\frac{k^2}{r_0^2} + g_e^2\left(m^2 + \frac{k^2}{r_0^2}\right)\right)R_2^8}}{R_2^4}} \\ \nu_+ &= \sqrt{\frac{4e_3^2\left(g_e^2 + g_m^2\right)R_2^2 + \left(m^2 + \frac{k^2}{r_0^2}\right)R_2^6 + 4\sqrt{e_3^2\left(g_m^2\frac{k^2}{r_0^2} + g_e^2\left(m^2 + \frac{k^2}{r_0^2}\right)\right)R_2^8}}{R_2^4}} \end{aligned} \tag{6.74}$$

This is exactly the same AdS_2 dirac equation as (4.39), with

$$\begin{aligned} (-1)^\alpha\frac{k}{r_0} &\rightarrow -\frac{\nu_\pm}{R_2} \\ m &\rightarrow 0 \\ \lambda_\alpha &\rightarrow \sqrt{\nu_\pm^2 - q^2e_3^2} = \lambda_{1,2} \end{aligned} \tag{6.75}$$

Thus,

$$G_R^{1,2}(\omega) = e^{-i\pi\lambda_{1,2}} \frac{\Gamma(-2\lambda_{1,2})\Gamma(1+\lambda_{1,2}-iqe_3)}{\Gamma(2\lambda_{1,2})\Gamma(1-\lambda_{1,2}-iqe_3)} \times \frac{-i\nu_\pm - iqe_3 - \lambda_{1,2}}{-i\nu_\pm - iqe_3 + \lambda_{1,2}}(2\omega)^{2\lambda_{1,2}} \tag{6.76}$$

As in (4.48), we can match in the outer region onto either $G_R^1(\omega)$ or $G_R^2(\omega)$, this defines our two boundary conditions. The components in the outer region, however, will generically be mixed 4-spinors. We will have two solutions

$$\begin{aligned} \psi^I &= \eta_+^I + G_R^1(\omega)\eta_-^I \\ \psi^{II} &= \eta_+^{II} + G_R^2(\omega)\eta_-^{II} \end{aligned} \tag{6.77}$$

We can expand A and B (6.63) perturbatively in ω near the boundary. For example,

$$B = B_+^{(0)} + \omega B_+^{(1)} + O(\omega^2) + (B_-^{(0)} + \omega B_-^{(1)} + O(\omega^2))G_R(\omega) \tag{6.78}$$

with

$$G_R(\omega) \equiv \begin{pmatrix} G_R^1(\omega) & 0 \\ 0 & G_R^2(\omega) \end{pmatrix} \tag{6.79}$$

The equation for the low frequency greens function is

$$B_+^{(0)} + \omega B_+^{(1)} + (B_-^{(0)} + \omega B_-^{(1)})G_R(\omega) = G_R(\omega, k)\left[(A_+^{(0)} + \omega A_+^{(1)}) + (A_-^{(0)} + \omega A_-^{(1)})G_R(\omega)\right] \tag{6.80}$$

All previous equations for correlation functions, say (5.53) hold, with a_α's and b_α's replaced by matrices A's and B's, $G_R(\omega)$ replaced by the matrix $G_R(\omega)$ and all denominators replaced by matrix inverses. The fermi surface is now defined by

$$\det\left[A_+^{(0)}(k_f)\right] = 0 \tag{6.81}$$

The dispersion relation and width, the analogues of (5.56) and (5.57), are determined by solving

$$\det\left[(A_+^{(0)}(k_f)+\partial_k A_+^{(0)}(k_f)k_\perp+\omega A_+^{(1)}(k_f)+A_-^{(0)}(k_f)G_R(\omega)\right]=0 \qquad (6.82)$$

Thus, in general, the dispersion relation and width will be controlled by the smallest of λ_1, λ_2.

For simplicity with the numerics, we will find it easiest to deal with nonzero g_m and g_e separately.

4.6.3 $g_m \neq 0$, $g_e = 0$

In this case, there is no need to do any basis changing; the dirac equation is block diagonal. The near horizon equation is (setting $r_0 = 1$)

$$(-\partial_\xi+\frac{mR_2}{\xi}\sigma_3)\Phi_\alpha=(\omega+\frac{qe_3}{\xi})i\sigma_2\Phi_\alpha+\frac{R_2}{\xi}((-1)^\alpha k+\frac{2e_3g_m}{R_2^2})\sigma_1\Phi_\alpha \qquad (6.83)$$

Thus the effect of the magnetic dipole in the near horizon limit is to shift the momentum oppositely in the two blocks. In the matching region (the AdS_2 boundary), the solution goes like

$$\Phi_\alpha \sim \xi^{-\lambda_\alpha}v_{+\alpha}+\xi^{\lambda_\alpha}v_{-\alpha} \qquad (6.84)$$

with

$$\lambda_\alpha=\frac{1}{R_2}\sqrt{(kR_2^2+(-1)^\alpha 2g_m e_3)^2+m^2R_2^4-q^2e_3^2R_2^2} \qquad (6.85)$$

and

$$v_{\pm\alpha}=\begin{pmatrix} mR_2\mp\lambda_\alpha \\ e_3(q-\frac{2g_m}{R_2})-(-1)^\alpha kR_2\end{pmatrix} \qquad (6.86)$$

Matching onto the near horizon region,

$$\Phi_\alpha^O=\eta_{+\alpha}+G_R^\alpha(\omega)\eta_{-\alpha} \qquad (6.87)$$

where the effect of the dipole coupling is to shift k in $G_R^\alpha(\omega)$, and to change the UV data $\eta_\pm$. The AdS_2 greens function is

$$\begin{aligned} G_R^\alpha(\omega) &= e^{-i\pi\lambda_\alpha}\frac{\Gamma(-2\lambda_\alpha)\Gamma(1+\lambda_\alpha-iqe_3)}{\Gamma(2\lambda_\alpha)\Gamma(1-\lambda_\alpha-iqe_3)}\times\frac{(m+in_\alpha)-iqe_3-\lambda_\alpha}{(m+in_\alpha)-iqe_3+\lambda_\alpha}(2\omega)^{2\lambda_\alpha} \\ n_\alpha &= (-1)^\alpha k+\frac{2e_3g_m}{R_2^2} \end{aligned} \tag{6.88}$$

4.6.4 $g_e \neq 0$, $g_m = 0$

In this case, the ($r_0 = 1$) near horizon equation is

$$(-\partial_\xi+\frac{mR_2}{\xi}\sigma_3)\Phi_\alpha = (\omega+\frac{qe_3}{\xi})i\sigma_2\Phi_\alpha+\frac{R_2}{\xi}(-1)^\alpha k\sigma_1\Phi_\alpha+\frac{2e_3}{\xi}g_e\sigma^3\Phi_\beta \tag{6.89}$$

For completeness, the unitary transformation which block diagonalizes this is

$$\begin{aligned} U &= \frac{1}{2\sqrt{2}\sqrt{k^2+m^2+m\sqrt{m^2+k^2}}}\begin{pmatrix} -A_+ & -A_- & iA_- & -iA_+ \\ A_- & -A_+ & iA_+ & iA_- \\ -A_- & -A_+ & -iA_+ & iA_- \\ A_+ & -A_- & -iA_- & -iA_+ \end{pmatrix} \\ A_\pm &= m\pm k+\sqrt{m^2+k^2} \end{aligned} \tag{6.90}$$

giving

$$-\partial_\xi\tilde{\Phi}_\alpha = (\omega+\frac{qe_3}{\xi})i\sigma_2\tilde{\Phi}_\alpha+\frac{R_2}{\xi}((-1)^\alpha\sqrt{m^2+k^2}+\frac{2e_3g_e}{R_2^2})\sigma_1\tilde{\Phi}_\alpha \tag{6.91}$$

In the matching region, the solution again goes like

$$\tilde{\Phi}_\alpha \sim \xi^{-\lambda_\alpha}v_{+\alpha}+\xi^{\lambda_\alpha}v_{-\alpha} \tag{6.92}$$

with

$$\lambda_\alpha = \frac{1}{R_2}\sqrt{(\sqrt{m^2+k^2}R_2^2+(-1)^\alpha 2g_ee_3)^2-q^2e_3^2R_2^2} \tag{6.93}$$

and

$$v_{\pm\alpha} = \begin{pmatrix} \mp\lambda_\alpha \\ e_3(q - \frac{2g_e}{R_2}) - (-1)^\alpha\sqrt{m^2+k^2}R_2 \end{pmatrix} \tag{6.94}$$

The full outer region solution is again

$$\tilde{\Phi}^O_\alpha = \eta_{+\alpha} + G^\alpha_R(\omega)\eta_{-\alpha} \tag{6.95}$$

and

$$\begin{aligned} G^\alpha_R(\omega) &= e^{-i\pi\lambda_\alpha}\frac{\Gamma(-2\lambda_\alpha)\Gamma(1+\lambda_\alpha - iqe_3)}{\Gamma(2\lambda_\alpha)\Gamma(1-\lambda_\alpha - iqe_3)} \times \frac{in_\alpha - iqe_3 - \lambda_\alpha}{in_\alpha - iqe_3 + \lambda_\alpha}(2\omega)^{2\lambda_\alpha} \\ n_\alpha &= (-1)^\alpha\sqrt{m^2+k^2} + \frac{2e_3 g_e}{R_2^2} \end{aligned} \tag{6.96}$$

4.7 Numerical Results

To find fermi surfaces, we look for k_f such that $a_+^{(0)}(k_f) = 0$. By (4.51), this corresponds to $\omega = 0$ solutions to the dirac equation which are normalizable (because of mixing, the process is slightly more complicated for $g_e \neq 0$; we review it below). We implement this procedure by numerically integrating the $\omega = 0$ equation to the boundary and looking for zeros of $a_+^{(0)}$ for some range of k and q .

Such numerical work was done in [17] for $g_m = g_e = 0$. There, it was found that fermi surfaces existed in branches in the (k, q) plane that were basically straight lines jutting out of an oscillatory region (a region where the AdS_2 operator dimensions (4.42) are imaginary and inside which there exist no fermi surfaces). See Figure 4-1 for such a graph with $m = 0.4$. The oscillatory region is shaded green.

4.7.1 $g_m \neq 0$, $g_e = 0$

For $g_m \neq 0$, the above structure is preserved; there are fermi surface branches jutting out of oscillatory regions in the (k, q) plane. By (6.85), turning on g_m keeps intact the

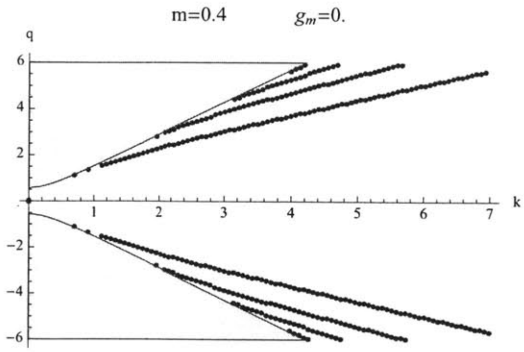

Figure 4-1: Fermi Surfaces $g_e = g_m = 0, m = 0.4$

shape of the oscillatory region, moving it to larger k (it also moves another copy to smaller k, but we focus on $k > 0$ as everything is $k \to -k$ invariant). We note the following qualitative observations

1. As the oscillatory region moves to larger k, it "eats" fermi surfaces in the (k, q) plane. These fermi surfaces branches move to higher and higher q for larger $|g_m|$ (see Figure 4-2).

2. The dipole coupling seems to have the most effect at low q, where it flattens and curves fermi surface branches close to the oscillatory region. Far from the oscillatory regions, the branches asymptote to straight lines.

3. This effect is most pronounced for $m < 0$ (alternative quantization). For m negative enough, local and global maxima and minima can develop in fermi surface branches near the oscillatory region. See Figure 4-3 for an example of how such a minimum develops as m is lowered. Also, in Figure 4-4 we plot results for g_m fixed and $m = -0.4, 0, 0.4$.

In [17], a "phase diagram"was constructed in the (m, q) plane which showed the attainable λ_α's for the primary fermi surface (that with the largest k_f for a given q). Here, we construct a similar phase diagram for $g_m = -2$. Because of various ambiguities and discontinuities, we focus only on the $q > 0$ branch affected most by the dipole coupling. Because the branch gets flattened near the oscillatory region, there are more attainable λ's than with $g_m = 0$ for the same range of m, q. Also, because of #3, when global or local minima occur we must pick what we mean as the primary fermi surface within a branch (note that this differs from $g_m = 0$ where the choice is made between *different* branches). We choose that fermi surface with the largest λ_α. Note, in this case, such a fermi surface actually has smaller k_f.

4.7.2 $g_e \neq 0, g_m = 0$

For $g_e \neq 0$, there is mixing. We change bases in the $\omega = 0$ dirac equation so that (6.83) is the near horizon limit. We then use two different infalling boundary conditions, each corresponding to a distinct AdS_2 dimension. We integrate this out to the boundary, change basis back to the original spin basis and numerically look for zeros of $\det A_+^{(0)} = a_{+1}^{I(0)} a_{+2}^{II(0)} - a_{+1}^{II(0)} a_{+2}^{I(0)}$ (see 6.63). Some observations:

1. Fermi surface branches continue to jut out of an oscillatory region. As $|g_e|$ is increased, the oscillatory regions kiss and move to the right (see figure 4-6). As in the magnetic case, the oscillatory region and "eats" fermi surface branches as it moves to the right.

2. Fermi surface branches are created to the left of the oscillatory region which I will call the "interesting region". As one increases $|g_e|$ more fermi surfaces are created in this region. There can also be local maxima or minima created near the oscillatory region as in the magnetic dipole case.

3. For $g_e \neq 0$ the fermi surfaces are much more gently sloping in the interesting region than for fermi surface branches with similar g_m (and all other constants comparable). For $m = 0$, the fermi surface branches are nearly flat.

4. There seem to be small gaps between the fermi surface branches at $k = 0$. This indicates local maxima or minima at $k = 0$. As one lowers $|m|$, the gaps become larger and larger (see Figure 4-7), although for $m < 0$ there always exists a branch with a large gap.

4.8 Conclusions

We have found that fermi surfaces are robust under the addition of quadratic magnetic and electric dipole operators. Turning on these couplings changes the scaling dimension of the emergent IR AdS_2 symmetry.

It would be interesting and important to have a better analytic understanding of the meaning of the bulk dipole couplings in terms of the boundary field theory. In vacuum, these couplings do not affect the fermion two point function. They do, however, change the structure of the current-fermion-fermion three point function. The full calculation is complicated, but one simple characterization is the following. With $g_m = g_e = 0$, there are no terms in $\langle \bar{\Psi}_\alpha \Psi_\beta J^\mu \rangle$ proportional to second rank clifford algebra elements; turning on the dipole couplings creates such terms. It would be interesting to understand better the physical implications of this.

It was previously found that the existence of fermi surfaces in ordinary quantization was correlated with the existence of the oscillatory region in k space where the AdS_2 scaling dimension becomes imaginary. The oscillatory region corresponds to pair production in the AdS_2 region and so a heuristic interpretation is that this creates a bulk fermi surface leading to a boundary one. However, it appears that this explanation is not necessary; boundary fermi surfaces can appear in alternative quantization without oscillatory regions. However, the alternative quantization is unstable in the RG sense; any small addition of the double trace operator $\mathcal{O}^\dagger \mathcal{O}$ flows the CFT to the that of ordinary quantization. Thus, the fermi surfaces without oscillatory region are also unstable; they flow away, as can be seen most clearly in Figure 5 of [17].

As discussed above, the existence of the oscillatory region implies that the bulk fermi surface has support at the black hole horizon. Naively, backreaction should be suppressed by a factor of $1/N^2$. However, when one integrates the horizon charge density, there is a log divergence which can offset this suppression in the extreme IR; backreaction cannot be ignored [59]. For this reason, it would be interesting to find a deformation of the action which allows fermi surfaces that are RG stable. Unfortunately, our numerical results indicate that the dipole couplings are not such a deformation. Although the couplings change the shape and location of the oscillatory region in the (k, q) plane, we again find that fermi surfaces in ordinary quantization only occur with an oscillatory region.

Finally, we have found that the dipole operators curve fermi surface branches in the (k, q) plane close to the oscillatory region. For certain values of g_m and g_e we can create local maxima and minima of these branches. It would be interesting if we could embed this system into a larger one where q is a tunable parameter. In this context, a local maximum, for example, would represent two fermi surfaces that merge and annihilate as q is continuously increased. We leave such an embedding to future work.

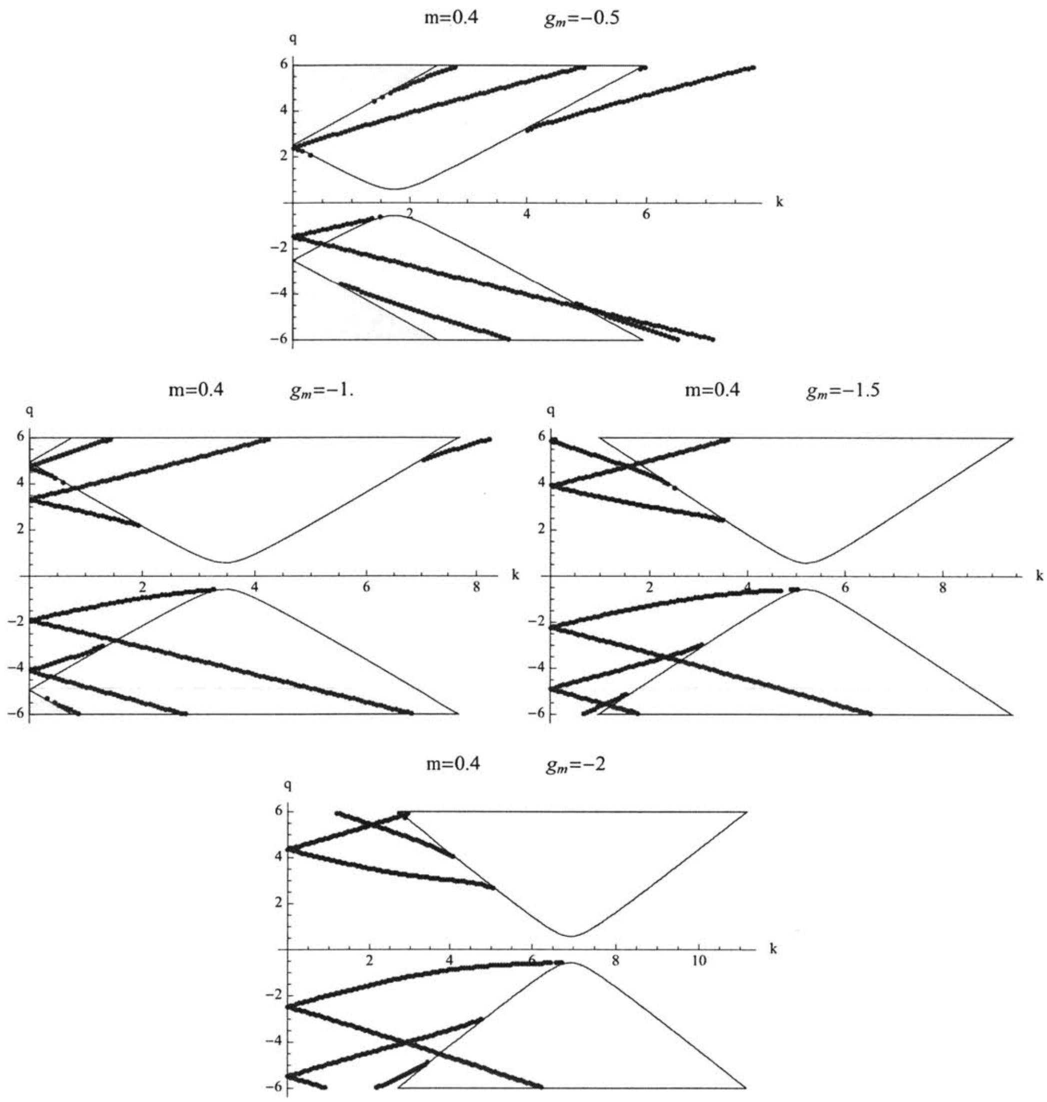

Figure 4-2: Fermi Surfaces for Increased $|g_m|$ with $m = 0.4$

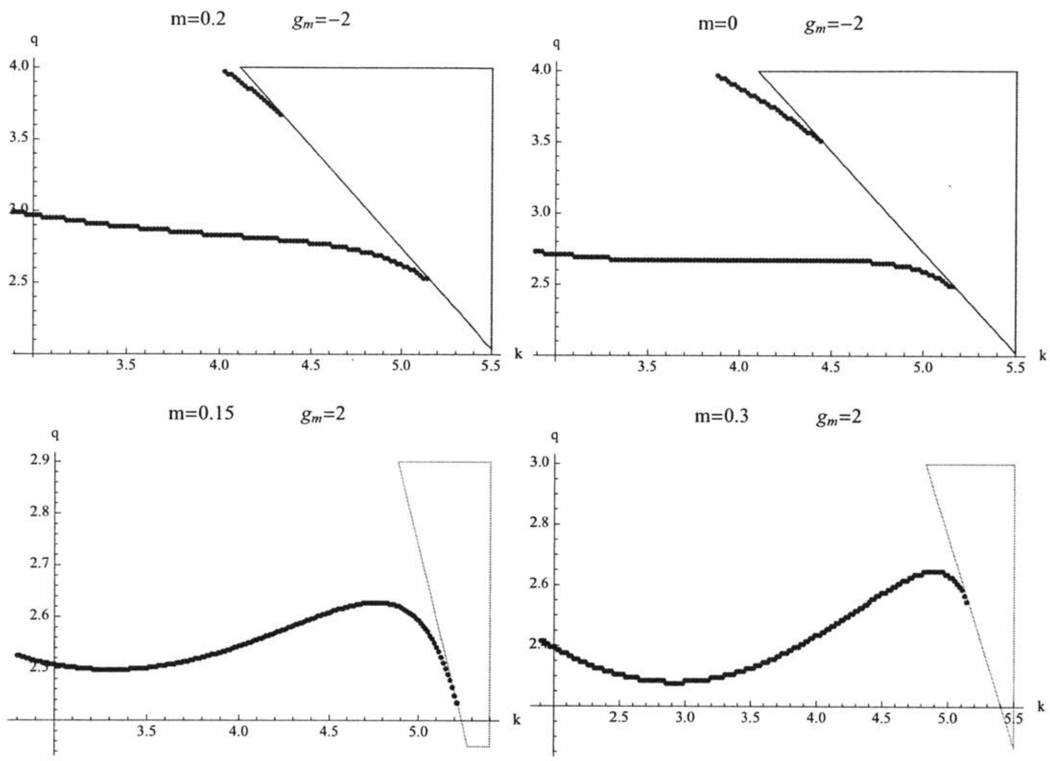

Figure 4-3: Developing local and global minimum for $m < 0$ with $g_m = -2$

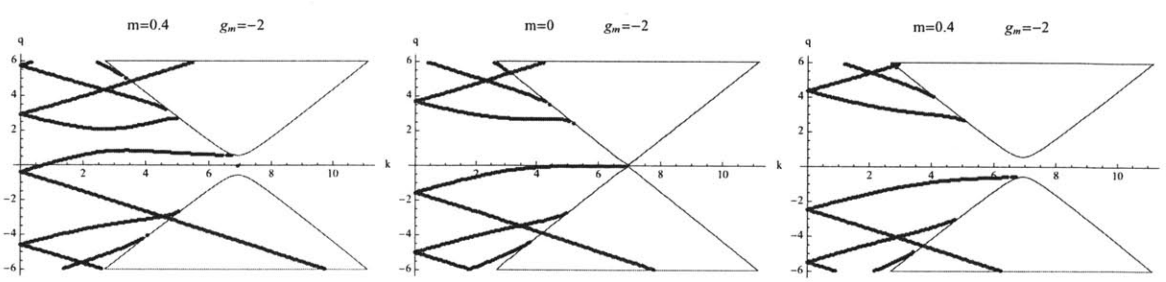

Figure 4-4: Fixed $g_m = -2$ for $m = -0.4, 0, 0.4$

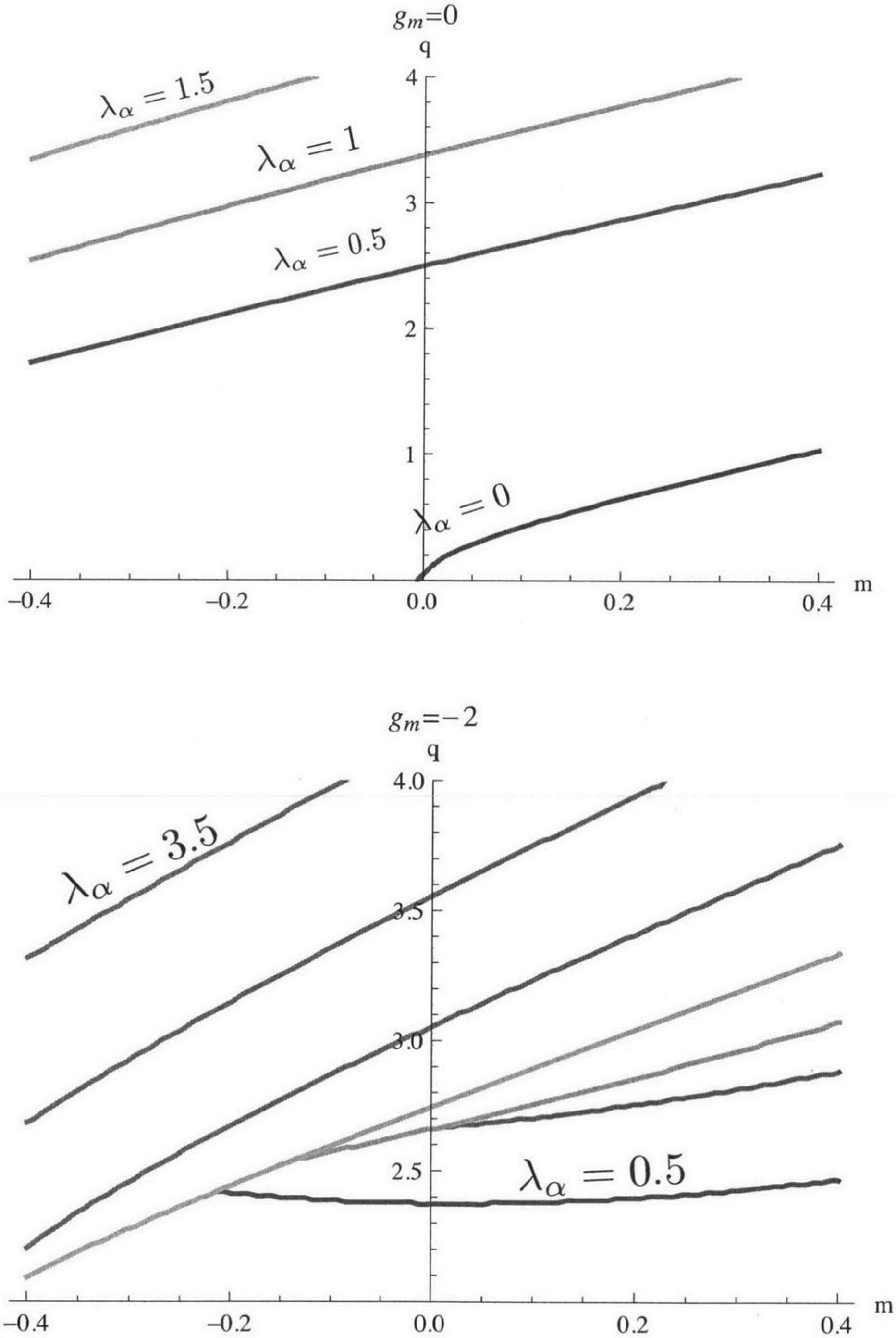

Figure 4-5: "Phase Diagram" for $g_m = -2$

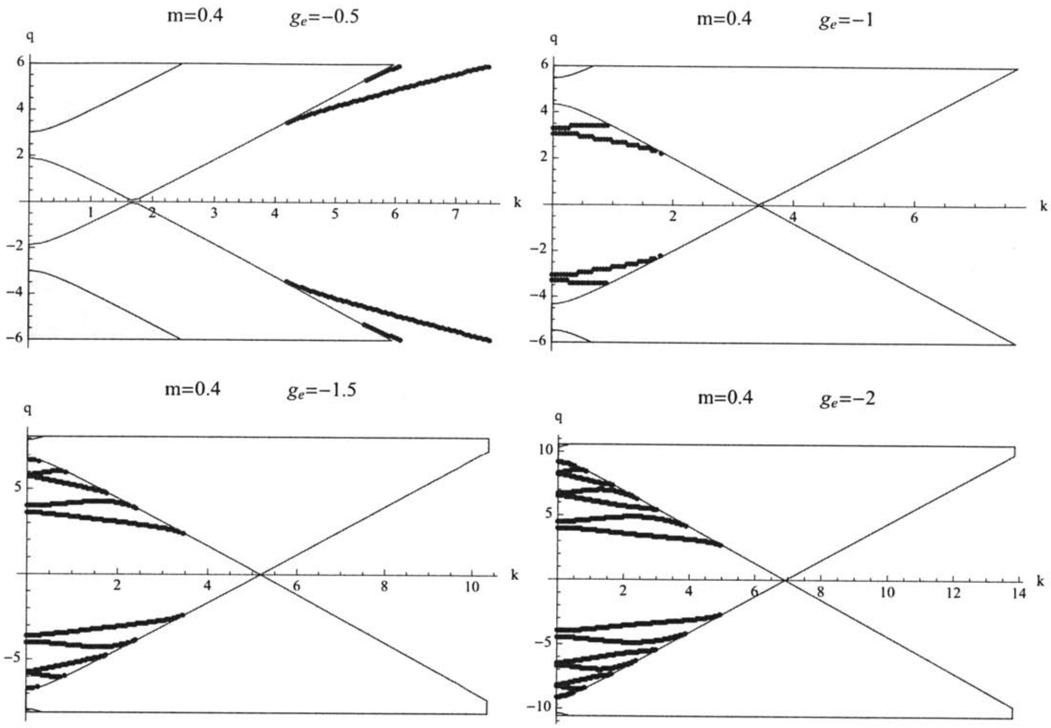

Figure 4-6: Increasing $|g_e|$

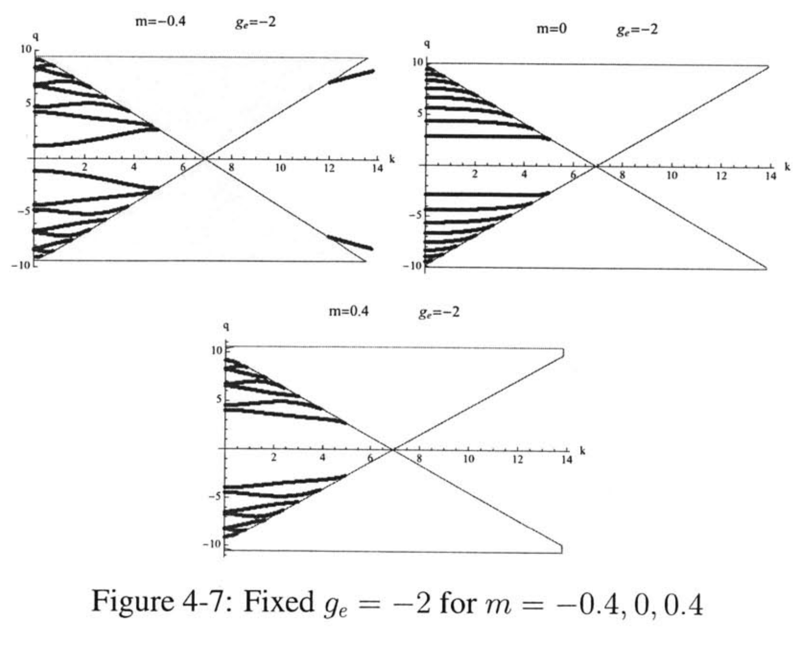

Figure 4-7: Fixed $g_e = -2$ for $m = -0.4, 0, 0.4$

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