A Final Report of Research on
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SUMMARY

This final report describes research on stochastic and adaptive systems by faculty and students of the Decision and Control Sciences Group of the M.I.T. Laboratory for Information and Decision Systems (formerly Electronic Systems Laboratory) with support provided by the United States Air Force Office of Scientific Research under Grant AFOSR 77-3281B. The Grant Monitor was Charles L. Nefzger, Major, USAF. The time period covered by this report is February 1, 1978 to January 31, 1979.

Substantial progress is reported in the areas of nonlinear filtering, parameter-adaptive control, reliable control system design, and reduced order compensators for singular stochastic control problems.
1. INTRODUCTION

During the grant period of February 1978 - January 31, 1979 of AFOSR Grant 77-3281B, work has progressed on the following main subject areas:

(i) Non-linear Filtering
(ii) Stochastic Control
(iii) Adaptive Control
(iv) Reliable and Robust Control System Designs
(v) Singular Stochastic Control and Dual Compensator Structures
(vi) Nonlinear Control Law Synthesis
(vii) Control Laws Implemented by Finite-state Sequential Machines.

The work was carried out under the joint direction of Professors M. Athans and S.K. Mitter. They were assisted by Professor T. Johnson, Dr. Mark Davis (visiting from Imperial College and supported by the grant), Dr. R. Singh (visitor), Mr. D. Birdwell (fellowship student), Mr. R. Ku, Mr. D. Ocone (fellowship student), Mr. P. Parikh and Mr. D. Wimpey.
2. NONLINEAR FILTERING

During this grant period Professor Mitter has continued his work on non-linear filtering. In this work he was assisted by Dr. Mark Davis (visiting from Imperial College and supported by the current grant) and Mr. D. Ocone (fellowship student).

We have concentrated on the following aspects of the non-linear filtering problems:

(i) Robust Non-linear Filtering
(ii) Filtering for Systems with Multiplicative Noise
(iii) Bounds on Non-linear Filtering
(iv) Perturbation Theory for Non-linear Filtering.

Robust Non-linear Filtering

During the last few years there has been intense activity in the modelling of stochastic dynamical systems. A systematic approach to these questions was given by McShane [1].

Subsequently Balakrishnan (see [2] and references cited therein), and Sussman [3] have explored different aspects of this problem. The basic question has to do with the fact that modelling of noise processes as Ito-processes is apparently not the right thing to do in the implementation of filters. The reason is that physical noise processes have finite bandwidth and the formula for computing conditional expectations using the Ito-model (infinite bandwidth process) when applied to physical processes will not give the correct answer. Balakrishnan and Mitter [4] have argued that a white noise model as proposed by Segal and Gross is much more appropriate when non-linear filters are to be used in real physical situations. Work on this is progressing. Using certain ideas of Clark we are able to give a robust version of the likelihood-
ratio formula where the filter turns out to be a smooth function of the physical data. These ideas are also related to the innovations problem. Here we have been able to obtain positive results for a significant class of non-linear filtering problems.

**Filtering for Systems with Multiplicative Noise**

The goal of this research is to determine explicitly the non-linear filter for state and observation processes which are linear but where the noise enters multiplicatively. This class of models turns out to be quite general since the statistical description of some more general non-linear models can be approximated arbitrarily closely by models of this type. Closed form solutions for a special class of multiplicative noise models have been obtained [5]. Work is now progressing in trying to determine the best polynomial filter for the more general class of linear systems with multiplicative noise. We would like to mention that the class of systems we are considering is probably the only one for which the best linear filter can be computed as a finite dimensional filter.

**Bounds on Non-linear Filtering**

In the doctoral thesis of Galdos, written under the direction of Professor Mitter and supported by a previous grant of the AFOSR, a formula for lower bound of the filtering error was given. This is an a-priori bound which was obtained using information-theoretic ideas. The basic problem with this bound is that it is very difficult to evaluate. Work is now progressing to see if the bound can be evaluated in the asymptotic sense. It appears that the evaluation of the asymptotic error can be solved in certain situations by solving an eigenvalue problem for a certain elliptic partial differential operator. This work will be reported later.
Perturbation Theory for Non-linear Filtering

In recent work [4] Professor Mitter has shown that the conditional density equations for the Kalman filtering problem has a precise analogy to the free quantum field. This correspondence sheds new light even on the Kalman Filtering Problem and indicates how a probabilistic interpretation can be given to the fast algorithms of Kailath and others for the Kalman filter.

This has suggested how perturbation theory can be done systematically for non-linear filtering problems. In particular Mr. D. Ocone (in a forthcoming Ph.D. thesis) has shown how convergent Volterra expansions can be obtained to express conditional expectations of functions of the state process.

REFERENCES (Section 2)


3. ADAPTIVE CONTROL

During the past year, the completed Ph.D. thesis of R. Ku [1] under the supervision of Professor Athans has demonstrated that even in the case of scalar linear systems, if both the state equation and the measurement equation contain white parameters, the resulting optimal control problem, with respect to quadratic performance criteria, admits no closed form analytical solution. Rather it leads to a complex two-point boundary value problem. Numerical methods have been developed to solve this nonlinear two-point-boundary value problem which demonstrates that the uncertainty threshold principle still exists, and that the threshold for the existence of infinite-horizon optimal decision rules is decreased when noisy measurements of the state are made under both multiplicative and additive white noise. It should be self-evident that the lack of analytic solutions to the problem will be present in the multivariable case.

In spite of these analytical difficulties it was decided to study further the multivariable version of the problem. A study was initiated by Mr. P. Parikh and Professor Athans in which the stochastic optimal solution was derived for Linear-Quadratic regulator and tracking problems under the assumption of exact state measurements. General purpose computer subroutines have been developed by Mr. Parikh and Mr. Carrig under the supervision of Professor Athans, and extensive digital computer simulation of a simple two state two input discrete time example have been carried out.

Simultaneously with this effort, we have developed a set of sensitivity equations for the stochastic control problem. The basic idea is that in a stochastic optimal control problem the impact of the variability of a certain system parameter (or sets of parameters) can be only assessed through the use
of a specific performance index. Thus, we are investigating the effect of the variance of a given parameter upon the optimal cost-to-go. Note that for certain objective functions the variance of a given parameter may be dominant, while for other objective functions the variance of this uncertain parameter may not be as significant. Simulation results are showing this effect quite clearly. If the uncertain parameters are characterized by small standard deviations, the response of the system is relatively insensitive to the selection of the weighting matrices in a tracking linear-quadratic context. However, as the parameter variances increase one sees significant deviations in the time trajectories of the control and state variables, thus illustrating the point that the effects of parameter uncertainty cannot be decoupled from the overall system objective. The first documentation of these results will be in the S.M. thesis of P. Parikh scheduled for completion in April 1979.

Since September 1978 Professor Athans and Professor Kendrick (visiting from the University of Texas) have been collaborating in the area of adaptive control so as to understand better the nature of the probing terms and of the caution terms in the dual control adaptive algorithm. Extensive simulations have been carried out and more are necessary to understand fully under what circumstances active learning adaptive control leads to a superior performance as compared to passive learning using (a) the certainty equivalence method and (b) the open loop feedback optimal method. A simple two state discrete-time example, representing a highly aggregated model of the U.S. economy is used for simulation purposes. This research is scheduled for completion in May 1979.

Professor Athans, Professor Kendrick, and Mr. Dersin are also collaborating in the preparation of a paper that also deals with the dual control algorithm,
and in particular with improved definitions of the probing and caution terms and with the relationship of the dual method equations to those obtained under the assumption that the system parameters are white (uncorrelated in time). A paper documenting the results of this study will become available in May 1979.
4. RELIABLE AND ROBUST CONTROL SYSTEM DESIGNS

Mr. J.D. Birdwell, Professor M. Athans, and Dr. D.A. Castanon have continued during the past year their investigations in the area of stochastic control with special emphasis on developing a method of approach and theoretical framework which advances the state of the art in the design of reliable multivariable control systems, with special focus on actuator failures and necessary actuator redundancy levels.

The mathematical model consists of a linear time invariant discrete time dynamical system. Configuration changes in the system dynamics, (such as actuator failures, repairs, introduction of a back-up actuator) are governed by a Markov chain that includes transition probabilities from one configuration state to another. The performance index is a standard quadratic cost functional, over an infinite time interval.

If the dynamic system contains either process white noise and/or noisy measurements of the state, then the stochastic optimal control problem reduces, in general, to a dual problem, and no analytical or efficient algorithmic solution is possible. Thus, the results are obtained under the assumption of full state variable measurements, and in the absence of additive process white noise.

Under the above assumptions, the optimal stochastic control solution can be obtained. The actual system configuration, i.e. failure condition, can be deduced with an one-step delay. The calculation of the optimal control law requires the solution of a set of highly coupled Riccati-like matrix difference equations; if these converge (as the terminal time goes to infinity) one has a reliable design with switching feedback gains, and, if they diverge*, the design is unreliable and the system cannot be stabilized unless more reliable actuators or more redundant actuators are employed. For reliable designs, the

* (This concept is similar to that described as the Uncertainty Threshold Principle.)
feedback system requires a switching gain solution; that is, whenever a system change is detected, the feedback gains must be reconfigured. On the other hand, the necessary reconfiguration gains can be precomputed, from the off-line solutions of the Riccati-like matrix difference equations.

Through the use of the matrix discrete minimum principle, a suboptimal solution was also obtained. In this approach, one wishes to know whether or not it is possible to stabilize the system with a constant feedback gain, which does not change even if the system changes. Once more this can be deduced from another set of coupled Riccati-like matrix difference equations. If they diverge as the terminal time goes to infinity, then a constant gain implementation is unreliable, because it cannot stabilize the system. If, on the other hand, there exists an asymptotic solution to this set of Riccati-like equations then a reliable control system without feedback reconfiguration can be obtained. The implementation requires constant gain state variable feedback, and the feedback gain can be calculated off-line.

In summary, these results can be used for off-line studies relating the open loop dynamics, required performance, actuator mean time to failure, and functional or identical actuator redundancy, with and without feedback gain reconfiguration strategies.

Documentation of these results can be found in Birdwell's thesis [2] and a paper [3].

Robustness Research

Additional results in the robustness area have not been obtained as yet. Our plans are to concentrate upon the effect of modelling errors in the robustness area, through the use of stochastic singular perturbation theory.
References [4] and [5] related to robustness have appeared in the literature during this reporting period. Their contents have been discussed in previous interim reports.

REFERENCES (Sections 3 and 4)


5. SINGULAR STOCHASTIC CONTROL AND DUAL COMPENSATOR STRUCTURES

The main outlines of a theory of reduced-order compensator design are now in place. As previously described, we consider the case of a linear time-invariant system subject to additive Gaussian white noise disturbances and measurement errors; the problem is to design a time-invariant compensator of specified order which minimizes asymptotically the sum of variances of a finite number of linear combinations of state and control variables, assuming that a set of stabilizing gains exists. A key to understanding this problem is the dual singular cases arising when there is zero plant noise or observation noise, as indicated in [8].

In [7], the conditions for zero-order compensation, or output-feedback are given. Together, these special cases suggest the solution of the general problem: the optimal gains are computed by determining two aggregations of the plant dynamics: a "control aggregation" and a "filtering aggregation". These two problems are coupled, but in a tractable way. Together, the solutions of these two problems determine the optimal reduced-order compensator gains. These results are currently being tested by simulation; a detailed technical report will follow.
6. NONLINEAR CONTROL LAW SYNTHESIS

Dr. R.N.P. Singh, a Visiting Scientist at M.I.T., and Professor Johnson have studied the problem of developing functional expansion techniques for the solution of the Hamilton-Jacobi-Bellman equation which characterizes optimal nonlinear feedback laws ([2], [4]). Functional expansion techniques differ from series expansion techniques in that at each iteration the entire functional form of the solution may be modified; thus there is a potential for faster convergence of numerical methods of this type (this was recognized by Bellman many years ago). We have looked at a certain class of nonlinear problems where advantage is to be gained by taking a quadratic initial guess for the optimal cost-to-go; this has the advantage of simplifying later terms in the series expansion, facilitating a recursive computation of the optimal feedback law. We are continuing to investigate the possibility of further analytical results which may be obtained using the method of characteristics.
7. CONTROL LAWS IMPLEMENTED BY FINITE-STATE SEQUENTIAL MACHINES

This avenue of research has proved so promising that a separate proposal has been submitted for its continuation. During the period of the current grant, we have investigated mappings from continuous to discrete variables as well as certain methods for the analysis of feedback systems containing continuous and discrete parts.*

The representation of mappings between continuous domain and discrete range is considered to be a crucial issue in the design and analysis of hybrid systems. A knowledge of the general structure and properties of such mappings could lead to a synthesis procedure for synchronous encoders and decoders. In the sampled-data case, we have investigated the notion of an "acceptor" [1], which may be viewed as a continuous/discrete mapping realizable by a continuous finite-dimensional system followed by a threshold device. The function of an acceptor is to distinguish one class of input sequences from another. A very broad class of (single-input, single-output) acceptors admits a shift realization, wherein the classification is decided as a discontinuous memoryless function of the stored past input value. The advantage of the acceptor formulation is that the action of the device can be specified in advance; it is a generalization of the notion of acceptor from automata theory. We have also considered the possibilities of designing continuous-time encoders; in [6] it is shown that one can devise an asynchronous coder which is continuous. This suggests that asynchronous coding may preserve information content, whereas in general synchronous coding cannot.

Several initial ideas have been developed in the area of control of hybrid systems. We have extended the results of [3] to show that, at least in some

* We refer to continuity of the state-transition map with respect to the state variables, not with respect to time.
cases, a closed-loop hybrid system admits a state-space realization in terms of transition times, continuous system states and discrete system states. With this realization, the state may be propagated directly from one transition time to the next, without the need for "simulation" of the continuous system. This realization also bears a closer resemblance to the models used in queueing and jump process theories. In the sampled-data case, D.N. Wimpey has pursued the possibility of designing finite-state compensators by means of exact finite-state descriptions the input-output properties of a continuous plant viewed through a discontinuous input and output device. Existing results from automata theory may then be applied to design a finite-state regulator for a finite-state system. A report is in preparation. S.N. Jones [5] has proposed a procedure for design of stochastic finite-state controllers for uncertain discontinuous systems.

REFERENCES (Sections 5, 6 and 7)


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