6.003 Homework 4

*Please do the following problems by **Wednesday, March 3, 2010**. You need not submit your answers: they will NOT be graded. Solutions will be posted.*

Problems

1. Z transforms

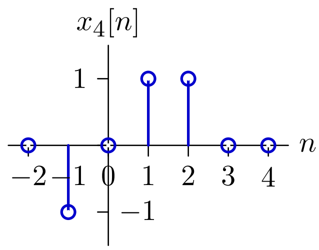
Determine the Z transform (including the region of convergence) for each of the following signals:

a. $x_1[n] = \left(\frac{1}{2}\right)^n u[n-3]$

b. $x_2[n] = (1+n)\left(\frac{1}{3}\right)^n u[n]$

c. $x_3[n] = n\left(\frac{1}{2}\right)^{|n|}$

d.



2. Inverse Z transforms

Determine and sketch all possible signals with Z transforms of the following forms. For each signal, indicate the associated region of convergence.

a. $X_1(z) = \dfrac{1}{z-1}$

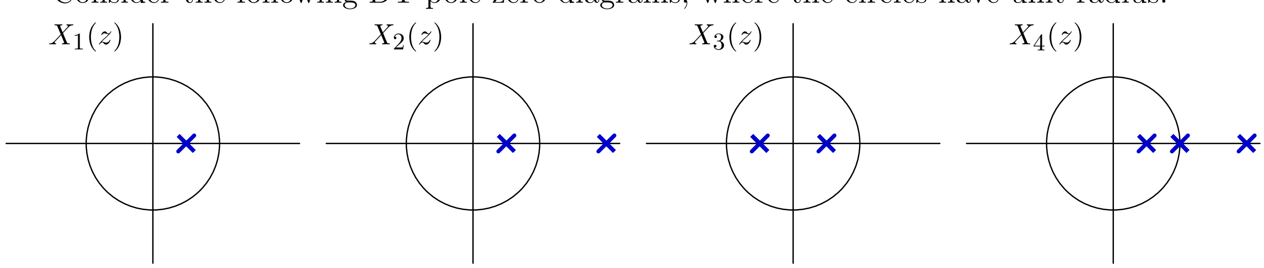
b. $X_2(z) = \dfrac{1}{z(z-1)^2}$

c. $X_3(z) = \dfrac{1}{z^2+z+1}$

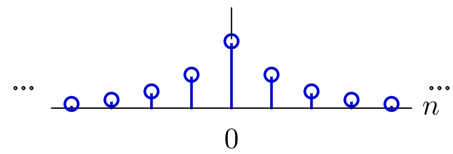
d. $X_4(z) = \left(\dfrac{1-z^2}{z}\right)^2$

3. More symmetries

Consider the following DT pole-zero diagrams, where the circles have unit radius.



a. Which if any of the pole-zero plots could represent the Z transform of the following DT signal?



b. Which if any of the pole-zero diagrams could represent a system that is stable?

c. Which if any of the pole-zero diagrams could represent a system that is causal?

d. Which if any of the pole-zero diagrams could represent a system that is both causal and stable?

4. Z transform

Let $X(z)$ represent the Z transform of $x[n]$, and let $r_0 < |z| < r_1$ represent its region of convergence (ROC).

Let $x[n]$ be represented as the sum of even and odd parts

$$x[n] = x_e[n] + x_o[n]$$

where $x_e[n] = x_e[-n]$ and $x_o[n] = -x_o[-n]$.

a. Under what conditions does the Z transform of $x_e[n]$ exist?

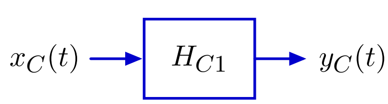
b. Assuming the conditions given in part a, find an expression for the Z transform of $x_e[n]$, including its region of convergence.

5. DT approximation of a CT system

Let H_{C1} represent a **causal** CT system that is described by

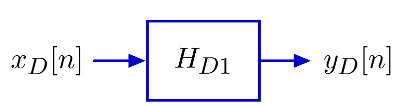
$$\dot{y}_C(t) + 3y_C(t) = x_C(t)$$

where $x_C(t)$ represents the input signal and $y_C(t)$ represents the output signal.



a. Determine the pole(s) of H_{C1}.

Your task is to design a **causal** DT system H_{D1} to approximate the behavior of H_{C1}.



Let $x_D[n] = x_C(nT)$ and $y_D[n] = y_C(nT)$ where T is a constant that represents the time between samples. Then approximate the derivative as

$$\frac{dy_C(t)}{dt} = \frac{y_C(t+T) - y_C(t)}{T}.$$

b. Determine an expression for the pole(s) of H_{D1}.

c. Determine the range of values of T for which H_{D1} is stable.

Now consider a second-order **causal** CT system H_{C2}, which is described by

$$\ddot{y}_C(t) + 100 y_C(t) = x_C(t)\,.$$

d. Determine the pole(s) of H_{C2}.

Design a **causal** DT system H_{D2} to approximate the behavior of H_{C2}. Approximate derivatives as before:

$$\dot{y_C}(t) = \frac{dy_C(t)}{dt} = \frac{y_C(t+T) - y_C(t)}{T} \quad \text{and}$$

$$\frac{d^2 y_C(t)}{dt^2} = \frac{\dot{y_C}(t+T) - \dot{y_C}(t)}{T}\,.$$

e. Determine an expression for the pole(s) of H_{D2}.

f. Determine the range of values of T for which H_{D2} stable.

6. Avoiding excitation

Consider the system described by the following difference equation:

$$y[n] = x[n] + \frac{5}{2} y[n-1] - y[n-2]\,.$$

Find an input $x[n]$ such that the output $y[n]$ is proportional to $\left(\frac{1}{2}\right)^n$ for large values of n. Try to minimize the number of non-zero samples in $x[n]$.

7. Periodic system

Consider this variant of the Fibonacci system:

$$y[n] = y[n-1] - y[n-2] + x[n]$$

where $x[n]$ represents the input and $y[n]$ represents the output.

a. Compute the unit-sample response and show that it is periodic. What is the period?

b. Determine the poles of the system.

c. Decompose the system functional into partial fractions, and use the result to determine a closed-form expression for $h[n]$, the unit-sample response.

8. Growth

Here is a system of difference equations:

$$a[n] = \frac{1}{3}a[n-1] + x[n],$$

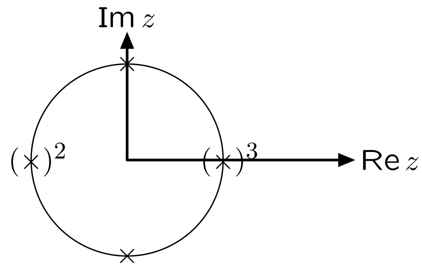
$$\frac{3}{2}c[n] = c[n-1] + x[n],$$

$$y[n] = 2a[n] + 3c[n]$$

in which $x[n]$ represents the input and $y[n]$ represents the output. Estimate, to within 0.01%, the ratio $h[2009]/h[2007]$ where $h[n]$ represents the unit-sample response of the system.

9. Lots of poles

All of the poles of a system fall on the unit circle, as shown in the following plot, where the '2' and '3' means that the adjacent pole, marked with parentheses, is a repeated pole of order 2 or 3 respectively.



Which of the following choices represents the order of growth of this system's unit-sample response for large n? Give the letter of your choice plus the information requested.

a. $y[n]$ is periodic. If you choose this option, determine the period.

b. $y[n] \sim An^k$ (where A is a constant). If you choose this option, determine k.

c. $y[n] \sim Az^n$ (where A is a constant). If you choose this option, determine z.

d. None of the above. If you choose this option, determine a closed-form asymptotic expression for $y[n]$.

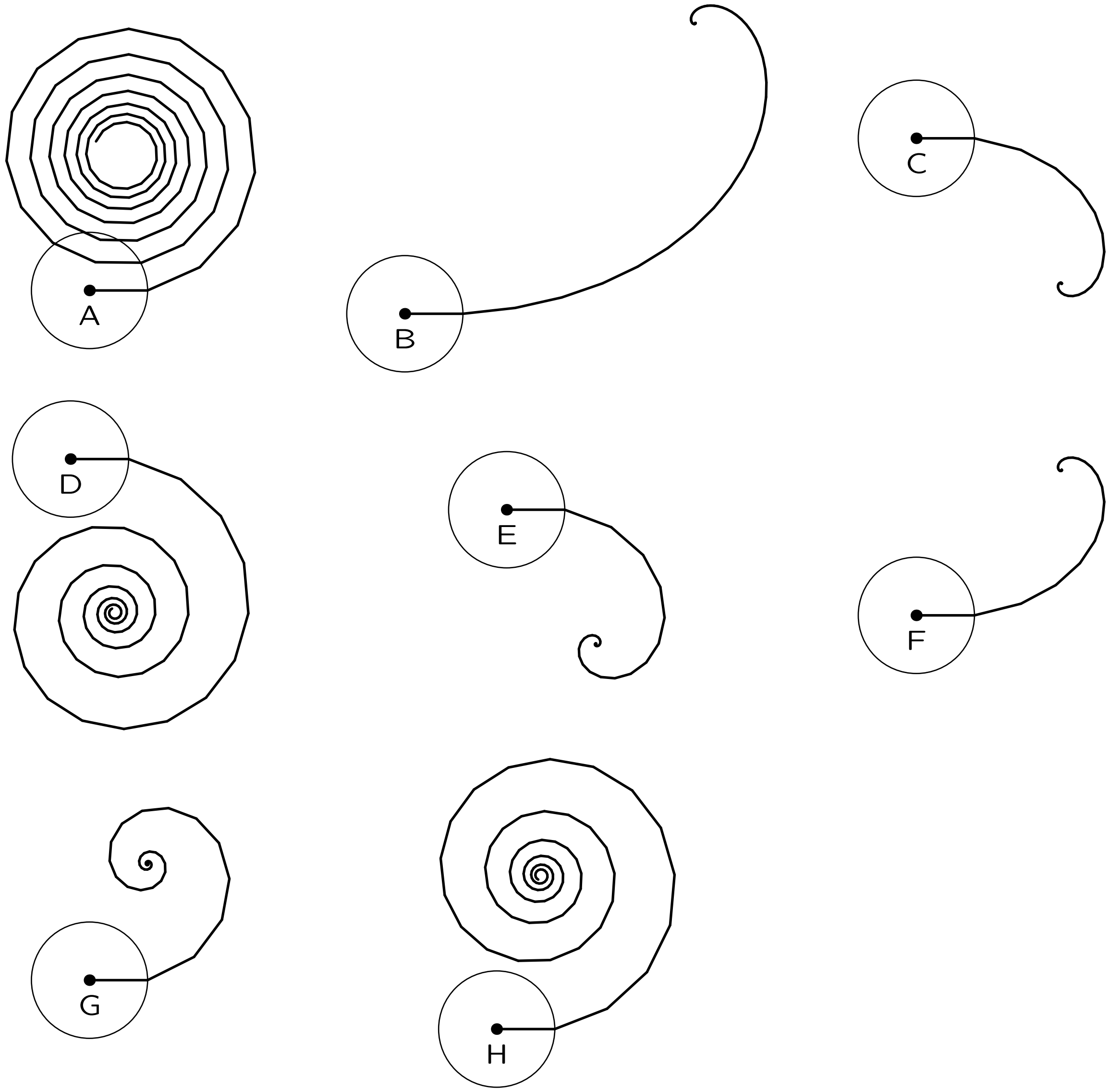
10. Complex Sum

Each diagram below shows the unit circle in the complex plane, with the origin labeled with a dot.

Each diagram illustrates the sum

$$S = \sum_{n=0}^{100} \alpha^n.$$

Determine the diagram for which $\alpha = 0.8 + 0.2j$.


A
B
C
D
E
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