Convergence of Fourier Series and Fejer's Theorem

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Abstract

This paper will address the Fourier Series of functions with arbitrary period $2a$. We will derive forms of the Dirichlet and Fejer kernel's, and eventually use these to prove a form of Fejer's theorem generalized to functions of arbitrary period. We also discuss one specific example of a Fourier Series, apply Fejer's theorem to see that it converges to the correct function.

1. Introduction

We assume the reader is familiar with Fourier Series. So, we merely state, without justification, the form of the Fourier series associated with a function $f(x)$ which is $2a$ periodic.

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{a} + b_n sin\frac{n\pi x}{a},$$

where the constants a_n and b_n are

$$a_n = \frac{1}{a}\int_{-a}^{a} f(t)\cos\frac{n\pi t}{a}dt, \quad b_n = \frac{1}{a}\int_{-a}^{a} f(x) sin\frac{n\pi t}{a}dt.$$

We will notate the Nth partial sum of a Fourier series as

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\frac{n\pi x}{a} + b_n \sin\frac{n\pi x}{a}.$$

We will also use $\sigma_N(x)$, the arithmetic mean of the partial sums up to N, defined as follows.

$$\sigma_N(x) = \frac{1}{N+1}\sum_{n=0}^{N} S_n(x)$$

Additionally, in this paper, the following trigonometric identities will be assumed without proof.

$$(1) \quad \cos(x)\cos(y) = \frac{1}{2}(\cos(x-y) + \cos(x+y)),$$

$$(2) \quad \sin(x)\sin(y) = \frac{1}{2}(\cos(x-y) - \cos(x+y)),$$

$$(3) \quad \sin(x)\cos(y) = \frac{1}{2}(\sin(x+y) - \sin(y-x)),$$

$$(4) \quad \sin^2(x) = \frac{1}{2}\left(1 - \cos(2x)\right).$$

2. The Dirichlet Kernel

We substitute the definitions of a_n and b_n into $S_N(x)$ to give

$$S_N(x) = \frac{1}{a}\int_{-a}^{a}\frac{f(t)}{2}dt$$

$$+\sum_{n=1}^{N}\left(\frac{1}{a}\int_{-a}^{a} f(t)\cos\frac{n\pi t}{a}\cos\frac{n\pi x}{a}dt + \frac{1}{a}\int_{-a}^{a} f(t)\sin\frac{n\pi t}{a}\sin\frac{n\pi x}{a}dt\right)$$

We then turn the sum of integrals into the integral of a sum, and apply (1) and (2) in the Introduction. After canceling terms, we have

$$S_N(x) = \frac{1}{a}\int_{-a}^{a} f(t)\left(\frac{1}{2} + \sum_{n=1}^{N}\cos\frac{n\pi(t-x)}{a}\right)dt.$$

Then, we make the substitution $u = t - x$. Since $f(x)$ has period $2a$, this substitution doesn't require us to change the limits of integration. We then have

$$S_N(x) = \frac{1}{a}\int_{-a}^{a} f(x+u)\left(\frac{1}{2} + \sum_{n=1}^{N}\cos\frac{n\pi u}{a}\right)dt.$$

Now, note that we may take

$$2\sin\frac{\pi u}{2a}\left(\frac{1}{2} + \sum_{n=1}^{N}\cos\frac{n\pi u}{a}\right) = \sin\frac{\pi u}{2a} + \sum_{n=1}^{N} 2sin\frac{\pi u}{2a}\cos\frac{n\pi u}{a}.$$

Applying the third trigonometric identity in the Introduction, and canceling terms, we arrive at

$$\frac{1}{2} + \sum_{n=1} N\cos\frac{n\pi u}{a} = \frac{\sin\frac{(N+1/2)\pi u}{a}}{2\sin\frac{\pi u}{2a}}.$$

Substituting this result into the previous integral equation gives what is commonly referred to as *Dirichlet's Integral*.

Dirichlet's Integral 1 $S_N(x) = \frac{1}{2a}\int_{-a}^{a} f(x+u)D_N(u)du.$

Here, D_N is *Dirichlet's Kernel*.

Dirichlet's Kernel 1 $D_N(u) = \frac{\sin\frac{(N+1/2)\pi u}{a}}{\sin\frac{\pi u}{2a}}.$

We will now prove one important property of the Dirichlet Kernel, to be used later. Using the summation form of D_N, we write

$$\int_{-a}^{a} D_N(x)dx = \int_{-a}^{a}\left(1 + 2\sum_{n=1}^{N}\cos\frac{n\pi x}{a}\right)dx.$$

Again, we change the integral of a sum to a sum of integrals and evaluate.

$$\int_{-a}^{a} D_N(x)dx = 2a + 2\sum_{n=1} N\frac{a}{\pi n}(\sin(n\pi) + \sin(-n\pi))$$

The right hand term evaluates to zero, and we have

$$(5) \quad \frac{1}{2a}\int_{-a}^{a} D_N(x)dx = 1$$

3. The Fejer Kernel

We define *Fejer's Kernel* $K_N(x)$ in the following manner.

Fejer's Kernel 1 $K_N(x) = \frac{1}{N+1}\sum_{n=0}^{N} D_n(x)$

However, we desire a closed form expression for Fejer's Kernel. Toward this end, we multiply the kernel by $2sin^2\frac{\pi x}{2a}$, which, after applying the definition of D_n, gives

$$2\sin^2\frac{\pi x}{2a}K_N(x) = \frac{1}{N+1}\sum_{n=0}^{N} 2\sin\frac{(n+1/2)\pi x}{a}\sin\frac{\pi x}{2a}.$$

Again applying (2) and canceling terms, we get

$$2\sin^2\frac{\pi x}{2a}K_N(x) = \frac{1}{N+1}\left(1-\cos\frac{(N+1)\pi x}{a}\right).$$

Finally, applying (4) and solving for $K_N(x)$ yields

$$K_N(x) = \frac{1}{N+1}\frac{1-\cos\frac{(N+1)\pi x}{a}}{1-\cos\frac{\pi x}{2a}}.$$

We wish to prove four properties of Fejer's Kernel. Firstly, that

$$(6)\quad K_N(x) \geq 0.$$

This fact is obvious if we apply (4) to both cosine terms, giving

$$K_N(x) = \frac{1}{N+1}\left(\frac{\sin\frac{(N+1)\pi x}{2a}}{\sin\frac{\pi x}{2a}}\right)^2.$$

For the second property, we use the definition of K_N to write

$$\int_{-a}^{a} K_N(x) = \frac{1}{N+1}\sum_{n=0}^{N}\int_{-a} aD_N(x)dx.$$

Using (5) to evaluate the right hand integral to $2a$ for each n immediately shows that

$$(7)\quad \frac{1}{2a}\int_{-a}^{a} K_N(x)dx = 1.$$

For the third property, we let $0 < \delta \leq |x| \leq a$. This implies that $\cos\frac{\pi\delta}{a} \geq \cos\frac{\pi x}{a}$. This, in turn, implies that

$$K_N(x) = \frac{1}{N+1}\frac{1-\cos\frac{(N+1)\pi x}{a}}{1-\cos\frac{\pi x}{2a}} \leq \frac{1}{N+1}\frac{1-\cos\frac{(N+1)\pi * x}{a}}{1-\cos\frac{\pi\delta}{2a}}.$$

Additionally, it is clear that $1 - \cos\frac{(N+1)\pi x}{a} \leq 2$. So, we arrive at

$$(8) \quad K_N(x) \leq \frac{1}{N+1} \frac{2}{1 - \cos\frac{\pi\delta}{a}}$$

whenever $0 < \delta \leq |x| \leq a$.

The last property is what is called Fejer's Integral. Recalling the definition of $\sigma_N(x)$ and Dirichlet's Integral, we may write

$$\sigma_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} S_n(x) = \frac{1}{N+1} \sum_{n=0}^{N} \frac{1}{2a} \int_{-a}^{a} f(x+u) D_n(u) du.$$

Again taking the sum inside the integral, we find

$$\sigma_N(x) = \frac{1}{2a} \int_{-a}^{a} f(x+u) \left(\frac{1}{N+1} \sum_{n=0}^{N} D_n(u) \right) du.$$

Remembering the definition of $K_N(x)$ at the start of this section, we arrive at Fejer's Integral.

$$(9) \quad \sigma_N(x) = \frac{1}{2a} \int_{-a}^{a} f(x+u) K_N(u) du.$$

4. Convergence by Arithmetic Means

Before proving Fejer's Theorem, we will give a brief proof of the following fact:

Convergence Theorem 1 *If the limit of $\{S_n\}$ exists, then $lim_{n\to\infty} S_n = lim_{n\to\infty} \sigma_n$, where S_n and σ_n have the same definitions as in the introduction.*

Having a proof of this theorem will make Fejer's Theorem much more meaningful.

Proof: Suppose the sequence in question converges to S. That is, $lim_{n\to\infty} S_n = S$. Then, there exists an integer N such that for every $n \geq N$, $|S - S_n| < \frac{\epsilon}{2}$. Then, we may write

$$S - \sigma_n = \frac{(n+1)S}{n+1} - \frac{1}{n+1} \sum_{k=0}^{n} S_k.$$

We let $n \geq N$. We then take S inside the sum and divide the sum into two parts, giving

$$S - \sigma_n = \frac{1}{n+1} \sum_{k=0}^{N} (S - S_k) + \frac{1}{n+1} \sum_{k=N+1}^{n} (S - S_k).$$

We may of course apply the triangle inequality over both sums to give absolute values. Since $N+1 \geq 1$, the right hand sum is less than $(n+1)\frac{\epsilon}{2}$. So we have

$$|S - \sigma_n| < \frac{1}{n+1}\sum_{k=0}^{N} |S - S_k| + \frac{\epsilon}{2}.$$

As N is a constant, we may let n grow large until the left hand sum is less than $\frac{\epsilon}{2}$. So we then have

$$|S - \sigma_n| < \epsilon$$

for all $n \geq M \geq N$, where M is some constant integer. Therefore, σ_n converges to S.

5. Fejer's Theorem

We now come to Fejer's Theorem, which is stated below.

Fejer's Theorem 1 *If f is a real valued, continuous function with period $2a$, then $\sigma_n(x)$ converges uniformly to $f(x)$.*

Proof: Remembering (7) and (9), we write the difference between $\sigma_N(x)$ and $f(x)$ as

$$\sigma_N(x) - f(x) = \frac{1}{2a}\int_{-a}^{a} f(x+u)K_N(u)du - f(x)\frac{1}{2a}\int_{-a}^{a} K_N(u)du.$$

We may combine these integrals, and apply the triangle inequality to get absolute values on both sides. We write $K_N(u)$ outside the absolute value since we have already shown that it is always greater than or equal to zero.

$$|\sigma_N(x) - f(x)| \leq \frac{1}{2a}\int_{-a}^{a} |f(x+u) - f(x)|K_N(u)du.$$

Note that $[-a, a]$ is compact, and f is continuous on that interval. Therefore, f is uniformly continuous on that interval. Since f is uniformly continuous, there exists a $\delta > 0$ (which does not depend on x) such that $|x - y| < \delta$ implies $|f(y) - f(x)| < \frac{\epsilon}{2}$. Without loss of generality, we take $\delta < a$. Additionally, since f is continuous and periodic, it must be bounded above and below. We let $M = sup|f(x)|$. We then have that $|f(x+u) - f(x)| \leq 2M$. Keeping all of this in mind, we divide the above integral into three parts: $-a$ to $-\delta$, $-\delta$ to δ, and δ to a. Taking advantage of the continuity of f in the middle term gives

$$|\sigma_N - f(x)| < \frac{1}{2a}\left(\int_{-a}^{-\delta} 2MK_N(u)du + \frac{\epsilon}{2}\int_{-\delta}^{\delta} K_N(u)du + \int_{a}^{\delta} 2MK_N(u)du\right).$$

The combination of both (6) and (7) requires that the middle term be less than or equal to $2a\frac{\epsilon}{2}$. Using this and (8) on the outer integrals gives

$$|\sigma_N(x) - f(x)| < \frac{2M}{2a(N+1)} \left(\int_{-a}^{-\delta} \frac{2}{1-\cos\frac{\pi\delta}{a}} du + \int_{\delta}^{a} \frac{2}{1-\cos\frac{\pi\delta}{a}} du \right) + \frac{\epsilon}{2}.$$

Evaluating both integrals, we now arrive at

$$|\sigma_N(x) - f(x)| < \frac{4M(a-\delta)}{a(N+1)\left(1-\cos\frac{\pi\delta}{a}\right)} + \frac{\epsilon}{2}.$$

Everything on the right hand side is constant except for N. Thus, for some sufficiently large A,

$$|\sigma_N(x) - f(x)| < \epsilon$$

whenever $N \geq A$.

Note that nowhere is there a dependence on x in the right hand side, as M depends only on δ, which doesn't depend on x. This detail is of great importance, because it places bounds on the difference between σ_N and f everywhere. One can imagine the nightmare in engineering applications if the bound on the error of a Fourier approximation varied from point to point. Such a scenario would significantly decrease the usefulness of Fourier Series.

This concludes our proof of Fejer's Theorem.

6. The Triangle Wave

To conclude, we discuss an explicit example of a function to which the methods of Fourier Series are applicable. The function is the so called triangle wave, defined as follows:

$$T(x) = 2|x - \frac{3}{2}| - 1 \quad for \quad \frac{1}{2} \leq x \leq \frac{5}{2}, \qquad T(x+2) = T(x).$$

This is an odd function of period 2. Straightforward application of the definitions given in the Introduction to this paper shows that the Nth partial sum of the Fourier Series of $T(x)$ is

$$S_N = \frac{8}{\pi^2} \sum_{n=0}^{N} \frac{\sin\frac{n\pi}{2}}{n^2} \sin n\pi x.$$

Noting that for even n, $\sin\frac{n\pi}{2} = 0$, and that for odd n, $\sin\frac{n\pi}{2} = (-1)^{\frac{n-1}{2}}$, we achieve the following form for the partial sums:

$$S_N = \frac{8}{\pi^2} \sum_{n=1,3,5,..}^{N} \frac{(-1)^{\frac{n-1}{2}}}{n^2} \sin n\pi x.$$

Included below are three graphs of $S_3(x)$, $S_5(x)$, and $S_9(x)$, respectively.

Note that the sequence of partial sums clearly converges by the Weierstrass M-test. We simply note that $|(-1)^{\frac{n-1}{2}} \sin n\pi x| \leq 1$. Then, we simply need that $\sum_{n=1,3,..}^{\infty} n^{-2}$ converges, which it most certainly does. Note that this not only proves that the partial sums converge, but that they converge *uniformly*.

Applying the formula for $\sigma_N(x)$ from the introduction to this case, we get

$$\sigma_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} \frac{8}{\pi^2} \sum_{k=1,3,..}^{n} \frac{(-1)^{\frac{k-1}{2}}}{k^2} \sin k\pi x.$$

Noting that the nth partial sum appears $N+1-n$ times in $\sigma_N(x)$, we see that

$$\sigma_N(x) = \frac{8}{\pi^2(N+1)} \sum_{n=1,3,..}^{N} \frac{N+1-n}{n^2} (-1)^{\frac{n-1}{2}} \sin n\pi x.$$

It is not so immediately clear that this sequence converges as N goes to infinity. However, the theorem proved in Section 4 guarantess that it does. Since $\{S_N(x)\}$ converges, $\{\sigma_N(x)\}$ must as well. Furthermore, they converge to the same thing.

Fejer's Theorem, as proved in Section 5, guarantees that

$$\lim_{n\to\infty} \sigma_N(x) = T(x).$$

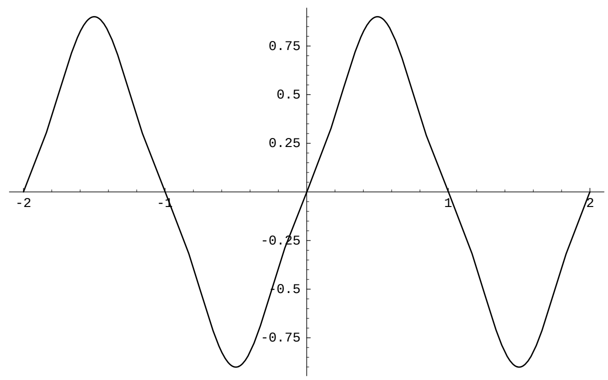
In fact, Fejer's Theorem implies even more: that this convergence is uniform. If we again make use of the theorem from Section 4, we see that

$$\lim_{n\to\infty} S_N(x) = T(x)$$

as well. In this case, this convergence is also uniform, as shown by the Weierstrass M-test above.

Included below, for those who desire visual clarification, are three graphs: $S_3(x)$, $S_5(x)$, and $S_9(x)$, respectively. Note that each successive graph looks more similar to $T(x)$ than the previous one.

This concludes our discussion of the triangle wave, and the paper.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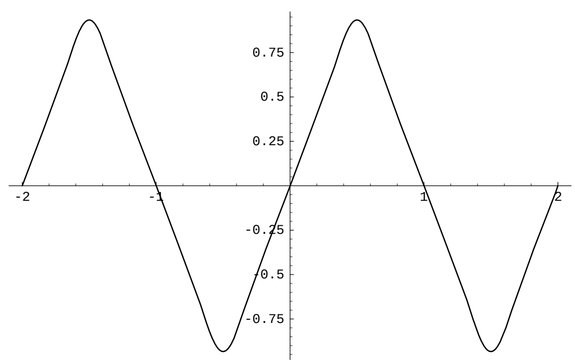
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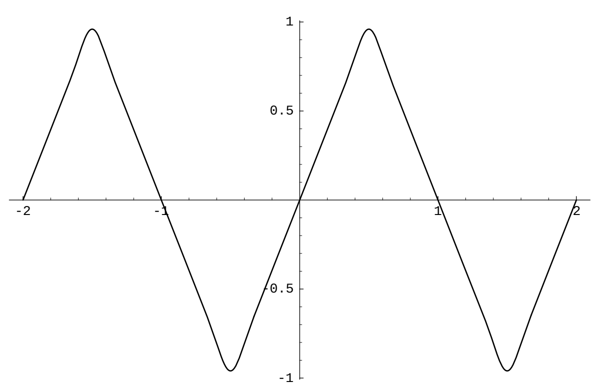
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