22.51 - Interaction of Radiation with Matter

Home Work Set No. 1

1. Starting from the definition of angular momentum: $\vec{L} = \vec{r} \times \vec{p}$, evaluate the following Poisson brackets.

$$\{L_x, L_y\}, \{L_y, L_z\}, \{L_z, L_x\}, \{L_x, L^2\}, \{L_y, L^2\}, \{L_z, L^2\}$$
$$\{L_x, \vec{p}\}, \{L_x, p^2\}, \{L_x, \vec{p}^{\,n}\}, \{L_x, \vec{r}\}, \{L_x, r^2\}, \{L_x, V(r)\}.$$

From the Hamiltonian of a central field problem: $\mathrm{H} = \frac{1}{2m} p^2 + V(r)$, where the potential function V(r) is a function of the scalar distance r, show that

$$\{L_x, H\} = \{L_y, H\} = \{L_z, H\} = 0$$

$$\{L^2, H\} = 0\,.$$

Determine the complete set of the constants of motion of a particle moving under the gravitational field.

2. Consider a symmetrical linear triatomic molecule ABA lying along the x-direction. We number the atoms from right to left by 1, 2 and 3. Find out the normal coordinates for the following two cases:

a) Longitudinal vibrations, i.e. vibrations along the x-direction. Denoting the displacements from the equilibrium positions by x_1, x_2 and x_3 respectively, the Lagrangian is

$$L = \frac{1}{2} m_A (\dot{x}_1^2 + \dot{x}_3^2) + \frac{1}{2} m_B \dot{x}_2^2 - \frac{1}{2} k_1 \left[(x_1 - x_2)^2 + (x_3 - x_2)^2 \right]\,.$$

Find the normal coordinates Qa and Qs, corresponding to the anti-symmetric and symmetric vibrations, their frequencies and draw the corresponding displacements. Note that you have to use a condition that the center of mass of the molecule is not moving during the vibrations.

b) Transverse vibration, i.e. vibration along the y-direction. The Lagrangian is

$$L = \frac{1}{2} m_A (\dot{y}_1^2 + \dot{y}_3^2) + \frac{1}{2} m_B \dot{y}_2^2 - \frac{1}{2} k_2 \ell^2 \delta^2$$

where $\delta = \frac{1}{\ell}\left[(y_1 - y_2) + (y_3 - y_2)\right]$ is the deviation of the angle ABA from the value π.

This variable is the most physically relevant quantity of the vibration and can be chosen as the normal coordinate. Express the Lagrangian in terms of this generalized coordinate. For this purpose you need to use conditions that the center of mass is stationary and the angular momentum around it is zero.

3. The virial theorem is a general properties of a wide class of mechanical systems, periodic or non-periodic. It is a theorem which is statistical in nature.

Consider a general system of mass points with position vectors $\vec{r}_i$ and subjected to forces (external or internal) $\vec{F}_i$. Consider a quantity G of the system, $G = \sum_i \vec{p}_i \cdot \vec{r}_i$, where the summation is over all the particles, show that

$$\frac{dG}{dt} = 2T + \sum_i \vec{F}_i \cdot \vec{r}_i .$$

Consider the time average of both sides of the equation by integrating over it from t = 0 to t = τ and then divide by τ. Make an argument that as $\tau \to \infty$, the left hand side of the equation tends to zero, and one proves the theorem (denoting the time average by a bar)

$$\overline{2T} + \overline{\sum_i \vec{F}_i \cdot \vec{r}_i} = 0 .$$

Show from this theorem that for a system of particles interacting among themselves by an inverse square law forces, the time average of the kinetic energy is equal to negative of the one half of the average potential energy. In the case of particles interacting by a harmonic (quadratic) potential, what is the relation between these average energies?

4. A damped harmonic oscillator is a mass point m, tied to a fixed spring, with a spring constant k, moving in a medium having a friction constant f. In one dimension, the equation of motion is

$$m\frac{d^2x}{dt^2} = -kx - f\frac{dx}{dt} \quad \text{or} \quad \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega_0^2 x = 0 .$$

where $2\lambda = f/m,\ \omega_0^2 = k/m$

Show that the general solution of the above equation can be written as:

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

where r_1 and r_2 are the roots of the quadratic equation.

$$r^2 + 2\lambda r + \omega_0^2 = 0$$

Show that the solution can be classified into three types.

Case (i)

If $\lambda<\omega_0$, namely, small damping,

$x(t)=ae^{-\lambda t}\cos(\omega t+\alpha)$,

where a and α are real and $\omega = \sqrt{\omega_0^2 - \lambda^2}$,

This is a damped oscillation. Show that in this case, the energy of the oscillator decays as

$$E(t) = E_0 e^{-2\lambda t}.$$

Case (ii)

$\lambda = \omega_0$, $r = -\lambda$, is a double root, and the solution can be written as

$x(t)=(c_1 + c_2 t)e^{-\lambda t}$

What kind of motion is this?.

Case (iii)

If $\lambda>\omega_0$, namely large damping, the solution is

$$x(t)=c_1 \exp\left[-\lambda t - \sqrt{\lambda^2 - \omega_0^2}\,t\right] + c_2 \exp\left[-\lambda t + \sqrt{\lambda^2 - \omega_0^2}\,t\right].$$

This is a monotonically decaying motion.

5. Consider a linear chain of point particles connected by springs of a spring constant K. There are two kinds of particles in the chain: all the even sites (2s) are occupied by a mass M and all the odd sites (2s+1) by a mass m. The distance between the two sites is a. Denote the displacements of the point masses from their equilibrium positions by x_{2s} and x_{2s+1} respectively. This is a model for a one-dimensional diatomic lattice.

(a) Write down the Lagrangian of the system and derive the equations of motion for the two mass points.

(b) Solve the equations by assuming solutions of the forms:

$$x_{2s} = A\exp(-i\omega t)\exp[i2saq]$$
$$x_{2s+1} = B\exp(-i\omega t)\exp[i(2s+1)aq]$$

where q is the wave vector of a mode of vibration.

(c) From the condition that the secular equation has a non-vanishing solution for the amplitudes A and B,

derive the dispersion relation:

$$\omega^2 = K\left(\frac{1}{m}+\frac{1}{M}\right) \pm K\left[\left(\frac{1}{m}+\frac{1}{M}\right)^2 - \frac{4\sin^2 qa}{mM}\right]^{1/2}$$

Plot ω vs q (the dispersion relation) for the q-range, $-\frac{\pi}{2a} < q < \frac{\pi}{2a}$, and show that there are an acoustic (ω_a) and an optic (ω_o) branches. There is a band gap between the two branches, which will be larger for the larger mass ratio M/m.

(d) Show that for q = 0, $\omega_o(0) = \left[2K\left(\frac{1}{m}+\frac{1}{M}\right)\right]^{1/2}$ and $\omega_a(0) = 0$, and for $q = \pm\dfrac{\pi}{2a}$, $\omega_o = \left(\dfrac{2K}{m}\right)^{1/2}$ and $\omega_a = \left(\dfrac{2K}{M}\right)^{1/2}$, assuming that M > m.

(e) Plot A/B as a function of q and show that, for the acoustic branch, A = B and for the optic branch,

-MA = mB, at q = 0. Discuss the nature of vibration in the two branches.