22.51 - Interaction of Radiation with Matter

Home Work Set No. 2

1. Given that the Hamiltonian of a one-dimensional harmonic oscillator is

$$H = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{x}^2 = \hat{T} + \hat{V} \qquad (1)$$

(a) Use the transformation,

$$\hat{x} = \sqrt{\frac{\hbar}{2\omega}}(a + a^+), \qquad \hat{p} = i\sqrt{\frac{\hbar\omega}{2}}(a^+ - a) \qquad (2)$$

where the annihilation and creation operators satisfies a relation $[a, a^+] = 1$, to express the kinetic and potential energy operators in terms of the annihilation and creation operators.

(b) Since the position and momentum operators do not commute, there is an uncertainty relation between these two observables. Show that for the exited state $| n >$ of the harmonic oscillator, the uncertainty relation between the position and momentum takes the equality

$$\Delta x \Delta p = \hbar(n + \frac{1}{2}) \qquad (3)$$

(c) Work out the commutation relation between the potential energy and the kinetic energy operators. Give the uncertainty relation between them. If the oscillator is in its n-th exited state, what is the expectation values of these two energies?

2. If a pair of Hermitian operators, $\hat{A}$ and $\hat{B}$, satisfy the following commutation relation:

$$\left[\hat{A}, \hat{B}\right] = i\hat{C} \qquad (4)$$

(a) Prove that the operator $\hat{C}$ is also Hermitian.

(b) Prove mathematically that the following inequality is valid:

$$(\Delta A)^2(\Delta B)^2 \geq \left(\frac{1}{2}\langle \hat{C}\rangle\right)^2 \tag{5}$$

where $\langle \hat{C}\rangle = \langle \psi|\hat{C}|\psi\rangle$ and $(\Delta A)^2 = \left\langle \left(\hat{A} - \langle \hat{A}\rangle\right)^2\right\rangle$.

For the proof you can look up any quantum mechanics books. But you have to put it in the Dirac notation.

3. Starting from the fundamental commutation relation between $\hat{q}$ and $\hat{p}$, prove a pair of operator relations:

$$[\hat{q}, F(\hat{p},\hat{q})] = i\hbar\frac{\partial F}{\partial \hat{p}} \tag{6}$$

$$[\hat{p}, F(\hat{p},\hat{q})] = -i\hbar\frac{\partial F}{\partial \hat{q}} \tag{7}$$

$$\exp(i\xi\hat{p}/\hbar)\hat{q}\exp(-i\xi\hat{p}/\hbar) = \hat{q} + \xi \tag{8}$$

$$\exp(i\xi\hat{p}/\hbar)F(\hat{q})\exp(-i\xi\hat{p}/\hbar) = F(\hat{q} + \xi) \tag{9}$$

$$\exp(i\xi\hat{q}/\hbar)\hat{p}\exp(-i\xi\hat{q}/\hbar) = \hat{p} - \xi \tag{10}$$

$$\exp(i\xi\hat{q}/\hbar)F(\hat{p})\exp(-i\xi\hat{q}/\hbar) = F(\hat{p} - \xi) \tag{11}$$

where ξ is a c-number and F(q) and F(p) are regular functions.

4. Let us study a system of two identical harmonic oscillators interacting with an interacting energy proportional to the product of the displacements from the equilibrium positions. The Hamiltonian is

$$H(x_1, x_2) = \left(\frac{p_1^2}{2m} + \frac{1}{2}m\omega^2 x_1^2\right) + \left(\frac{p_2^2}{2m} + \frac{1}{2}m\omega^2 x_2^2\right) + \lambda x_1 x_2 \tag{12}$$

(a) Show by transformation to a pair of new variables

$$\xi_1 = \sqrt{\frac{m\omega}{\hbar}}x_1 \text{ and } \xi_2 = \sqrt{\frac{m\omega}{\hbar}}x_2 \tag{13}$$

that the Hamiltonian can be put into the form:

$$H(\xi_1,\xi_2)=\frac{1}{2}\hbar\omega\left[\left(\xi_1^2-\frac{\partial^2}{\partial\xi_1^2}\right)+\left(\xi_2^2-\frac{\partial^2}{\partial\xi_2^2}\right)\right]+\frac{\lambda\hbar}{m\omega}\xi_1\xi_2 \tag{14}$$

(b) Show that by transforming to another pair of new coordinates

$$\eta_1=\sqrt{\frac{\omega_1}{2\omega}}(\xi_1+\xi_2),\ \eta_2=\sqrt{\frac{\omega_1}{2\omega}}(\xi_1-\xi_2),$$
$$\omega_1^2=\omega^2+\frac{\lambda}{m},\ \omega_2^2=\omega^2-\frac{\lambda}{m} \tag{15}$$

the Hamiltonian can finally be transformed into a diagonal form

$$H(\eta_1,\eta_2)=\frac{1}{2}\hbar\omega_1\left(\eta_1^2-\frac{\partial^2}{\partial\eta_1^2}\right)+\frac{1}{2}\hbar\omega_2\left(\eta_2^2-\frac{\partial^2}{\partial\eta_2^2}\right) \tag{16}$$

(c) Find the eigen values of this new Hamiltonian.

5. Let us study the classical analog of the above problem. The Lagrangian is

$$L=\frac{1}{2}\left(\dot{x}^2+\dot{y}^2\right)-\frac{1}{2}\omega_0^2\left(x^2+y^2\right)+\lambda xy \quad . \tag{17}$$

(a) Write down the equations of motion for each degree of freedom.

(b) Let $x(t)=Ae^{i\omega t}$ and $y(t)=Be^{i\omega t}$. Substitute these quantities into equations of motion to obtain the characteristic equation for ω.

(c) The solution of the characteristic equation should give two roots, say ω_1 and ω_2. These are the two eigen frequencies of the normal modes.

(d) Show that for $\omega=\omega_1$, $A=B$; and for $\omega=\omega_2$, $A=-B$. What are motions of the oscillators these correspond to?

(e) The above results suggest that the two normal coordinates Q_1 and Q_2 can be obtained by a transformation

$$x=\frac{Q_1+Q_2}{\sqrt{2}},\ y=\frac{Q_1-Q_2}{\sqrt{2}} \quad . \tag{18}$$

Express the Lagrangian in terms of the normal modes.