22.51 - Interaction of Radiation with Matter

Home Work Set No. 3

1. Find the stationary states of Schrödinger equation for a one-dimensional simple harmonic oscillator subjected to a potential, $V(x) = \frac{1}{2} m\omega_0^2 x^2$, by explicitly solving the differential equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2} m\omega_0^2 x^2 \psi = E\psi . \qquad (1)$$

a) Show that the normalized energy eigen functions are

$$\psi_n(x) = \sqrt{\frac{\beta}{\sqrt{\pi}\,2^n n!}}\, e^{-\xi^2/2} H_n(\xi), \qquad \xi = \sqrt{\frac{m\omega_0}{\hbar}}\, x = \beta x \qquad (2)$$

where $H_n(\xi)$ is the Hermite polynomial of the nth order,

b) and the corresponding energy eigenvalues are

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_0 \;, \;\; n=0, 1, 2,... \qquad (3)$$

c) Obtain the explicit form of $H_n(\xi)$ for the first three orders.

d) Use the ground state wave function to calculate the expectation values of the kinetic and potential energies.

e) Show by an explicit calculation that the ground state is a minimum uncertainty state.

2. Show that the momentum space wave function $\phi_p(q) \equiv \langle q|p\rangle$ satisfies a differential equation

$$\frac{d\phi_p}{dq} = \frac{i}{\hbar} p\phi_p \quad . \qquad (4)$$

It therefore has the form $\phi_p(q) = Ae^{\frac{i}{\hbar}pq}$ (5)

The momentum space wave function is often normalized in such a way that

$$\int_{-\infty}^{\infty} dq\phi^*_{p'}(q)\phi_p(q) = \delta(p' - p) \tag{6}$$

Show that in this case the normalized function takes a form

$$\phi_p(q) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}pq} \tag{7}$$

Starting from $\langle p'|\hat{A}|p''\rangle$, by insertion of unity and using Eq.7, show that

$$\langle p'|\hat{A}|p''\rangle = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dq' \int_{-\infty}^{\infty} dq'' e^{\frac{i}{\hbar}(p''q''-p'q')}\langle q'|\hat{A}|q''\rangle \ . \tag{8}$$

By taking in eq. 8, $\hat{A} = \hat{q}$, show that it can be written as

$$\langle p'|\hat{q}|p''\rangle = i\hbar\frac{d}{dp'}\delta(p' - p''). \tag{9}$$

By induction, then show the general result

$$\langle p'|F(q)|p''\rangle = F(i\hbar\frac{d}{dp'})\langle p'|p''\rangle \tag{10}$$

How would you extend this proof to show

$$\langle p|\hat{q}|\psi\rangle = i\hbar\frac{d}{dp}\langle p|\psi\rangle. \tag{11}$$

3. Given that the Lagrangian of a relativistic point particle of a rest mass m in an electromagnetic field $(\vec{A}, \phi)$ is

$$L = -mc^2\sqrt{1-\beta^2} + \frac{e}{c}\vec{A}\bullet\vec{v} - e\phi \tag{12}$$

a) Show that the Lagrangian gives the right equation of motion for the charge particle in the E-M field.

b) Construct by the standard procedure the Hamiltonian. Note that the Hamiltonian has to be expressed in terms of the conjugate momentum.

c) Give the non-relativistic limit of this Hamiltonian.

4. (a) Show by detailed calculations that the Lagrangian density,

$$\mathrm{L} = \frac{1}{8\pi}\left(E^2 - B^2\right) + \frac{1}{c}\vec{j}\cdot\vec{A} - \rho\phi \qquad (13)$$

when substituting into the continuum Lagrange's equations,

$$\sum_{\nu=1}^{4}\frac{\partial}{\partial x_\nu}\left[\frac{\partial \mathrm{L}}{\partial\left(\partial A_\mu / \partial x_\nu\right)}\right] - \frac{\partial \mathrm{L}}{\partial A_\mu} = 0 \qquad (\mu = 1,2,3,4) \qquad (14)$$

results in the two inhomogeneous Maxwell's equations.

(b) Derive the corresponding field Hamiltonian from the first term of the Lagrangian density.

5. Since the x- and y-components of an angular momentum satisfies the following comutation relation

$$\left[\hat{\mathrm{J}}_x, \hat{\mathrm{J}}_y\right] = i\hat{\mathrm{J}}_z \qquad (15)$$

their expectation values should satisfy an uncertainty relation. Given that the system is in an eigen state of an angular momentum, |j,m>, you are asked to derive such an uncertain relation.

Define $\Delta\mathrm{J}_x = \sqrt{\langle \mathrm{J}_x^2\rangle - \langle \mathrm{J}_x\rangle^2}$, $\Delta\mathrm{J}_y = \sqrt{\langle \mathrm{J}_y^2\rangle - \langle \mathrm{J}_y\rangle^2}$, calculate $\langle \mathrm{J}_x^2\rangle$ and $\langle \mathrm{J}_x\rangle^2$ and hence $\Delta\mathrm{J}_x$ and also $\Delta\mathrm{J}_y$. Show that the following uncertainty relation is valid.

$$\Delta\mathrm{J}_x\,\Delta\mathrm{J}_y = \frac{\hbar^2}{2}\left[j(j+1) - m^2\right]. \qquad (16)$$

Discuss the physical meaning of this equation.

6. We shall study some special properties of a two-dimensional isotropic harmonic oscillator defined by the following Hamiltonian

$$H = \left(\frac{p_1^2}{2m} + \frac{1}{2}m\omega^2 q_1^2\right) + \left(\frac{p_2^2}{2m} + \frac{1}{2}m\omega^2 q_2^2\right). \qquad (17)$$

Introduce two sets of creation and annihilation operators satisfying the commutation relations, $\left[\hat{a}_s, \hat{a}_p^+\right] = \delta_{sp}$, where s, p = 1, 2, show that the Hamiltonian can be transformed into a form

$$H = \hbar\omega\left(\hat{a}_1^+\hat{a}_1 + \hat{a}_2^+\hat{a}_2 + 1\right). \tag{18}$$

Next, introduce another set of operators defined as

$$\begin{aligned} \hat{A}_+ &= \frac{1}{\sqrt{2}}\left(\hat{a}_1 - i\hat{a}_2\right), \quad \hat{A}_- = \frac{1}{\sqrt{2}}\left(\hat{a}_1 + i\hat{a}_2\right) \\ \hat{A}_+^+ &= \frac{1}{\sqrt{2}}\left(\hat{a}_1^+ + i\hat{a}_2^+\right), \quad \hat{A}_-^+ = \frac{1}{\sqrt{2}}\left(\hat{a}_1^+ - i\hat{a}_2^+\right) \end{aligned} \tag{19}$$

(a) Show that the following set of commutation relations hold,

$$\left[\hat{A}_+, \hat{A}_-\right] = \left[\hat{A}_+^+, \hat{A}_-^+\right] = \left[\hat{A}_-, \hat{A}_+^+\right] = 0 \tag{20}$$

$$\left[\hat{A}_+, \hat{A}_+^+\right] = 1 = \left[\hat{A}_-, \hat{A}_-^+\right]. \tag{21}$$

Thus $\hat{A}_+$ and $\hat{A}_+^+$ are the destruction and creation operators of quanta of type +
and $\hat{A}_-$ and $\hat{A}_-^+$ are the destruction and creation operators of quanta of type - .

(b) Introduce a pair of operators: $\hat{N}_+ = \hat{A}_+^+\hat{A}_+$ and $\hat{N}_- = \hat{A}_-^+\hat{A}_-$. Show that $\hat{N}_+$ and $\hat{N}_-$ are Hermitian operators representing the numbers of + quanta
and – quanta respectively.

(c) We can thus introduce a set of state vectors, $\left|n_+ n_-\right\rangle$ such that

$$\begin{aligned} \hat{N}_+\left|n_+ n_-\right\rangle &= n_+\left|n_+ n_-\right\rangle \\ \hat{N}_-\left|n_+ n_-\right\rangle &= n_-\left|n_+ n_-\right\rangle \end{aligned} \quad . \tag{22}$$

Define two new operators:

$$\begin{aligned} \hat{N} &= \hat{N}_+ + \hat{N}_- = \hat{a}_1^+\hat{a}_1 + \hat{a}_2^+\hat{a}_2 \\ \hat{L} &= \hat{N}_+ - \hat{N}_- = i\left(\hat{a}_1\hat{a}_2^+ - \hat{a}_1^+\hat{a}_2\right) \end{aligned} \quad . \tag{23}$$

Show that

$$\left[\hat{N}, \hat{L}\right] = \left[\hat{N}, H\right] = \left[\hat{L}, H\right] = 0, \tag{24}$$

so that $\hat{N}$, $\hat{L}$ and H form a mutually commuting set of operators such that

$$
\begin{aligned}
&H|n_+n_-\rangle = \hbar\omega(n_+ + n_- + 1)|n_+n_-\rangle \\
&\hat{L}|n_+n_-\rangle = (n_+ - n_-)|n_+n_-\rangle
\end{aligned}
\quad . \tag{25}
$$

(d) Show further that the following set of commutation relations exist

$$
\left[\hat{A}_\pm, \hat{L}\right] = \pm\hat{A}_\pm \ , \ \left[\hat{A}_\pm^+, \hat{L}\right] = \mp\hat{A}_\pm^+ \tag{26}
$$

what are the physical meaning of these commutation relations?