22.51 -- Interaction of Radiation With Matter

Home Work Set 5

1. Calculation of isotope incoherence.

Suppose there are α = 1,2,3.....n isotopic species in the sample. Denoting the fractional abundance of the α-th species by c_α. Then we have

$$\overline{b} = \sum_{\alpha=1}^{n} c_\alpha b_\alpha \qquad \overline{b^2} = \sum_{\alpha=1}^{n} c_\alpha b_\alpha^2 \qquad (1)$$

(a) Show that for the simplest case of two isotopes, denoting the abundance of the first isotope by c,

$$b_{inc}^2 = c(1-c)(b_1 - b_2)^2 \qquad (2)$$

(b) Take Argon as an example. Argon has two principal isotopes, A^{36} and A^{40} with b_{36} = 2.43 x 10^{-12} cm, b_{40} = 0.193 x 10^{-12} cm. The natural Argon has c_{36} = 0.00337 and c_{40} = 0.996. Calculate the coherent to incoherent cross section ratio,

(c) If you can buy individual isotopes commercially, you can make a sample with the maximum incoherent cross section and hence the minimum coherent cross section. What is the mixing ratio of the isotopes and what is the resultant cross sections?

2. Given an atom with only two stationary states |1> and |2>, having energies $\hbar\omega_1 < \hbar\omega_2$. At time t = 0, the system is in its ground state |1>. A perturbation W, not dependent on time, is switched on at $t = 0^+$. You are asked to calculate the probabilities of finding the system in either states after time t.

Hint: Solve

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = (H + W)|\psi(t)\rangle \qquad (3)$$

by letting

$$|\psi(t)\rangle = c_1(t)e^{-i\omega_1 t}|1> + c_2(t)e^{-i\omega_2 t}|2> \qquad (4)$$

with the initial conditions $c_1(0) = 1$ and $c_2(0) = 0$. You are asked to calculate $|c_1(t)|^2$ & $|c_2(t)|^2$.

3. The Hamiltonian for a one-dimensional harmonic oscillator driven by an external field E in the x-direction is

$$H = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x} - eE\hat{x} \qquad (5)$$

Use the equation of motion in Heisenberg picture to solve for the time dependence of the position operator $\hat{x}(t)$. Show that

$$\left[\hat{x}(t_1), \hat{x}(t_2)\right] = \frac{i\hbar}{m\omega}\sin\omega(t_2 - t_1) \qquad (6)$$

which means that an operator that commutes at the same time needs not commute at different times.

4. Proof of Weyl identity (H. Weyl, Zeit. Physik $\underline{46}$, 1 (1927)).

If two operators $\hat{A}$ and $\hat{B}$ commute with their commutator $\left[\hat{A}, \hat{B}\right]$, namely,

$$\left[\hat{A},\left[\hat{A}, \hat{B}\right]\right] = \left[\hat{B},\left[\hat{A}, \hat{B}\right]\right] = 0 \qquad (7)$$

then

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}}e^{\frac{1}{2}\left[\hat{A},\hat{B}\right]} \qquad (8)$$

(a) Show that the following relation is valid $\left[\hat{B},\hat{A}^n\right] = n\hat{A}^{n-1}\left[\hat{B},\hat{A}\right]$, because of eq. 7.

(b) Then show that the following operator relation holds:

$$\left[\hat{B}, e^{-\hat{A}x}\right] = -xe^{-\hat{A}x}\left[\hat{B}, \hat{A}\right] \qquad (9)$$

(c) Rewrite eq. 9 to obtain

$$e^{\hat{A}x}\hat{B}e^{-\hat{A}x} = \hat{B} + \left[\hat{A},\hat{B}\right]x. \qquad (10)$$

(d) Consider next the function:

$$F(x) = e^{\hat{A}x} e^{\hat{B}x}.$$
(11)

By differentiating F(x) with respect to x, show that the following differential equation is valid:

$$\frac{dF(x)}{dx} = \left(\hat{A} + \hat{B} + x \left[\hat{A}, \hat{B} \right] \right) F(x)$$
(12)

(e) By integrating eq. 12, show finally that the eq. 8 is valid.

5. Show that, for neutrons whose wavelength is large compared to the inter-proton distance of a hydrogen molecule, the total neutron scattering cross-sections for ortho- and para-hydrogens are respectively:

$$\sigma_{ortho} = \frac{4\pi}{9} \left\{ \left(3b_+ + b_- \right)^2 + 2 \left(b_+ - b_- \right)^2 \right\}$$

$$\sigma_{para} = \frac{4\pi}{9} \left\{ \left(3b_+ + b_- \right)^2 \right\}$$
(13)

where b_+ and b_- are the triplet and the singlet scattering lengths respectively for the proton. In an ortho-hydrogen molecule, the two proton spins are parallel (total nuclear spin of the molecule, J = 1) and in a para-hydrogen molecule, the spins are anti-parallel (J = 0).

To work out the important results in Eq. 13, you can consider the scattering length operator, introduced in the class, of the proton as:

$$\hat{b}_H = A + B\hat{\sigma} \cdot \hat{I}$$
(14)

and write down the appropriate expressions for A and B. If the wavelength of neutrons is large compared to the inter-proton distance in the hydrogen molecule, the two scattered neutron waves from the two protons are in phase, so the coherent part of the scattering length operators adds. The scattering length operator for the hydrogen molecule can therefore be written as:

$$\hat{b}_{mol} = 2A + B\hat{\sigma} \cdot \hat{J}$$
(15)

where $\hat{J}$ is the total spin operator of the molecule. It has a value J = 1 for the O-H and J = 0 for the P-H. The total cross-section can then be calculated as:

$$\sigma = 4\pi \left\langle \hat{b}_{mol}^2 \right\rangle$$
(16)

where the average is taken over the unpolarized spin orientations of the incident neutrons. You have finally to correct for the fact that in a hydrogen gas, the molecules are considered to be free rather than bound, as calculated above using a bound scattering length operator.