22.51 Interaction of Radiation With Matter

Quiz No. 1 Close Book 120 minutes

1. Consider a linear chain of point particles connected by springs of a spring constant K. There are two kinds of particles in the chain: all the even sites (2s) are occupied by a mass M and all the odd sites (2s+1) by a mass m. The distance between the two sites is a. Denote the displacements of the point masses from their equilibrium positions by x_{2s} and x_{2s+1} respectively. This is a model for a one-dimensional diatomic lattice.

(a)(5%)Write down the Lagrangian of the system and derive the equations of motion for the two mass points respectively.

(b)(10%)Solve the equations of motion by assuming solutions of the forms:

$$\begin{aligned} x_{2s} &= A\exp(-i\omega t)\exp\left[i2saq\right] \\ x_{2s+1} &= B\exp(-i\omega t)\exp\left[i(2s+1)aq\right] \end{aligned} \qquad (1)$$

where q is the wave vector of a mode of vibration.

From the condition that the resultant equations have a non-vanishing solution for the amplitudes A and B, derive the ω vs q relation (dispersion relation).

(c)(10%)Plot the dispersion relation for the q-range, $-\frac{\pi}{2a} < q < \frac{\pi}{2a}$, and show that there are an acoustic (ω_a) and an optic (ω_o) branches.

Show that for q = 0, $\omega_a(0) = 0$, and $\omega_o(0) = \left[2K\left(\frac{1}{m} + \frac{1}{M}\right)\right]^{1/2}$,

and for, $q = \pm\frac{\pi}{2a}$, $\omega_a = \left(\frac{2K}{M}\right)^{1/2}$ and $\omega_o = \left(\frac{2K}{m}\right)^{1/2}$, assuming that M > m.

2. A particle of a charge e and mass m is moving in a central potential field $\phi(r)$ and subjecting to an external EM field specified by a vector potential $\vec{A}(\vec{r})$. The Lagrangian of the system is given by:

$$L = \frac{1}{2}mv^2 - e\phi(r) + \frac{e}{c}\vec{v} \bullet \vec{A}(\vec{r}) \qquad (2)$$

(a) (10%) Show first that this Lagrangian gives a correct equation of motion.

(b) (10%) Derive the generalized momenta conjugate to x, y and z. Show that the Hamiltonian corresponding to the given Lagrangian is:

$$H = \frac{1}{2m}(\vec{p} - \frac{e}{c}\vec{A})^2 + e\phi \qquad (3)$$

(c) (5%) Show that in the Coulomb gauge, the Hamiltonian can be reduced to the following form

$$H = \frac{p^2}{2m} - \frac{e}{mc}\vec{A}\cdot\vec{p} + \frac{e^2}{2mc}A^2 + e\phi \qquad (4)$$

3. Given that the ground state wave function of a hydrogen atom is given by:

$$\psi_0(r) = \frac{1}{\sqrt{\pi a_1^3}} e^{-r/a_1} \quad \text{where} \quad a_1 = \frac{\hbar^2}{me^2} = \text{Bohr radius} \qquad (5)$$

(a)(5%) Show by a direct calculation that

$$\overline{V}(r) = -e^2\left(\overline{\frac{1}{r}}\right) = -\frac{e^2}{a_1} \quad . \qquad (6)$$

(b)(10%) Find out the ground state energy by using the Virial theorem. What is the most probable distance from the nucleus one expects to find an electron in a K-shell?

4. You are asked to construct the ground state wave function of a harmonic oscillator in the momentum space (not the usual positional space), and then use it to derive the uncertainty relation for that state. For this purpose, you are given the following information.
The ground state of the harmonic oscillator is denoted by $|0\rangle$, an eigen ket of the number operator with the occupation quantum number equal to zero. The annihilation and creation operators for these states are given by

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} + \frac{i}{m\omega}\hat{p}\right) \quad \text{and} \quad \hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} - \frac{i}{m\omega}\hat{p}\right). \qquad (7)$$

In momentum space, the $\hat{q}$-operator can be shown to be represented by a differential operator:

$$\langle p|\hat{q}|0\rangle = i\hbar\frac{d}{dp}\langle p|0\rangle. \tag{8}$$

You need to use the following integration formulae for your calculations:

$$\int_{-\infty}^{+\infty} dp\ e^{-\alpha p^2} = \sqrt{\frac{\pi}{\alpha}}, \qquad \int_{-\infty}^{+\infty} dp\ p^2 e^{-\alpha p^2} = -\frac{\partial}{\partial\alpha}\int_{-\infty}^{+\infty} dp\ e^{-\alpha p^2}. \tag{9}$$

(a) (10%) Construct the normalized ground state wave function in momentum space $\phi_0(p) \equiv \langle p|0\rangle$

(b) (10%) Derive the uncertainty relation between the position and momentum in the ground state. You are to use the momentum space wave function derived in question (a) for this computation.

(c) (5%) Give the physical meaning of the momentum space wave function.

5. You are asked to construct the three normalized eigen functions, $\langle\theta,\varphi|\ell m\rangle \equiv Y_{\ell m}(\theta,\varphi)$, of the orbital angular momentum of magnitude $1\hbar$, namely, $\ell = 1,\ m = 1, 0, -1$, explicitly using the ladder operators:

$$\hat{L}_+ = \hbar e^{i\varphi}\left(i\cot\theta\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial\theta}\right), \quad \hat{L}_- = \hbar e^{-i\varphi}\left(i\cot\theta\frac{\partial}{\partial\varphi} - \frac{\partial}{\partial\theta}\right). \tag{10}$$

(a) (10%) Use the fact that $\hat{L}_+ Y_{11}(\theta,\varphi) = 0$, determine the function $Y_{11}(\theta,\varphi)$ up to a constant. Use the normalization condition to determine the constant.

(b) (10%) Use the lowering operator $\hat{L}_-$ to obtain the functions $Y_{10}(\theta,\varphi)$ and $Y_{1\bar{1}}(\theta,\varphi)$..