22.51 Interaction of Radiation With Matter

Quiz No. 2 Close Book 120 minutes

1. One can consider the light emitted by a gas laser operating below the threshold to be a single mode of optical radiation of frequency ω, in thermal equilibrium with the laser cavity at temperature T, to a good approximation. Suppose one illuminates a single-photon detector (a photo-multiplier tube with a point photo-cathode) with this laser. One counts the number of photons per second repeatedly, and gets a series of counts n_1, n_2, n_3, … One can then compute the average counts <n> and plot out the photo-count distribution function, P(n) = Probability of getting n counts in a measurement.

5% (a) Write down the density operator of this beam of light.

5% (b) compute <n>.

5% (c) Give the form of P(n).

2. Given the following spin-dependent bound scattering lengths of hydrogen and deuterium:

H^1	$b_+ = 1.08 \times 10^{-12}$ cm	$b_- = -4.74 \times 10^{-12}$ cm
D^2	$b_+ = 0.95 \times 10^{-12}$ cm	$b_- = 0.10 \times 10^{-12}$ cm

10 % (a) Calculate the coherent and incoherent cross sections of each isotope.

5 % (b) Explain the reason why the coherent cross section of hydrogen is small but the incoherent cross section is large.

3. In the class, we talked about the fact that when the target nucleus has a spin $\hbar\hat{I}$, the scattering length which characterizes the neutron-nucleus interaction is spin-dependent, having a value $\hat{b}_+$ when the total spin angular momentum of the neutron-nuclear system is I+1/2, and $\hat{b}_-$, when the total spin is I-1/2. Furthermore, if one uses a polarized incident neutron beam, the scattering of neutrons from the nucleus will result in certain probability of having a neutron spin-flip transition. Suppose we have a target nucleus which is hydrogen. For the purpose of calculating the spin-flip probability, it is convenient to introduce the scattering length operator of the proton as:

$$\hat{b}_H = A_H + B_H \hat{\sigma} \cdot \hat{I}$$

(A) (10%) Show that for this case, the appropriate expressions for A and B are:

$$A_H = \frac{3}{4}b_+ + \frac{1}{4}b_- \text{ and } B_H = \frac{1}{2}(b_+ - b_-)$$

(B)(5%) Show that a coherent neutron scattering does not involve a spin flip transition.

(C)(10%) Show that for a polarized incident beam (say spin "up"), 2/3 of the scattering results in the spin flip (namely up-down scattering) and 1/3 results in non-spin flip scattering.

4. In a standard NMR apparatus, one sets up a DC magnetic field of $B_0\hat{e}_z$ in the z-direction and then place an RF coil in it, which supplies an alternating field $2\hat{e}_x B_1 \cos\omega t$, along the x-direction. The sample is put inside the RF coil. Let us call, $\omega_0 = \gamma B_0$, the resonance frequency, where γ is the gyro-magnetic ratio of the nucleus under observation. The NMR signal is observed as a resonance absorption in the coil when the RF frequency ω is swept across the resonance frequency ω_0.

(A) (5%) Show that the Hamiltonian of a spin $\vec{I}$ in the combined DC and RF fields can be written as (putting $\omega_1 = \gamma B_1$):

$$H = H_0 + H_1(t) = -\hbar\omega_0 \hat{I}_z - 2\hbar\omega_1 \hat{I}_x \cos\omega t \quad (1)$$

Explain the reason why the RF field has to be perpendicular to the DC field. What will happen if the RF coil is pointing in the z-direction?

An important method in MNR is to use an RF pulse to manipulate the direction of a spin or the magnetization of the sample and subsequently observing how it relaxes in the DC magnetic field. You are asked to explain this technique quantum mechanically. In this context, answer the following questions.

(B) (10%) In order to proceed with the discussion of how to manipulate the spin with the RF field, let us write

$$H_1(t) = -2\hbar\omega_1 \hat{I}_x \cos\omega t = -\hbar\omega_1\left(\hat{I}_x \cos\omega t - \hat{I}_y \sin\omega t\right) - \hbar\omega_1\left(\hat{I}_x \cos\omega t + \hat{I}_y \sin\omega t\right) \quad (2)$$

and neglect the second term on the right hand side. What is the justification for this approximation? One then rewrite it as:

$$H_1(t) = -\hbar\omega_1\left(\hat{I}_x \cos\omega t - \hat{I}_y \sin\omega t\right) = -\hbar\omega_1 e^{i\omega t\hat{I}_z}\hat{I}_x e^{-i\omega t\hat{I}_z} \qquad (3)$$

Justify the last equality at least for the limiting case when $\omega t \to 0$. If you can prove it in the general case for an arbitrary ωt, you will get a bonus of 5 more points.

(C) (5%) Solve the time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi_s(t)\rangle = (H_0 + H_1(t))|\psi_s(t)\rangle = \left(-\hbar\omega_0\hat{I}_z - \hbar\omega_1 e^{i\omega t\hat{I}_z}\hat{I}_x e^{-i\omega t\hat{I}_z}\right)|\psi_s(t)\rangle \qquad (4)$$

by making a rotating frame transformation (i.e. a transformation to the interaction picture)

$$|\psi_s(t)\rangle = e^{i\omega t\hat{I}_z}|\psi_R(t)\rangle\,. \qquad (5)$$

and show that Eq. 4 in the rotating frame reduces to

$$i\hbar\frac{\partial}{\partial t}|\psi_R(t)\rangle = -\hbar\left[(\omega_0 - \omega)\hat{I}_z + \omega_1\hat{I}_x\right]|\psi_R(t)\rangle \qquad (6)$$

and hence, if one choose the RF frequency in such a way that $\omega = \omega_0$, then the solution of Eq. 6 is

$$|\psi_R(t)\rangle = \exp\left(i\omega_1 t\hat{I}_x\right)|\psi_R(0)\rangle\,. \qquad (7)$$

What is the physical meaning of the rotating frame? What advantage does one get to view the spin dynamics in the rotating frame?

(D) (5%) Use the solution given in Eq. 7 to calculate the expectation values of the z-component of the magnetic moment operator $\hat{\mu} = \gamma\hbar\hat{\vec{I}}$. Show that

$$\langle\hat{\mu}_z(t)\rangle_R \equiv \langle\psi_R(t)|\hat{\mu}_z|\psi_R(t)\rangle = \langle\hat{\mu}_z(0)\rangle_R \cos\omega_1 t - \langle\hat{\mu}_y(0)\rangle_R \sin\omega_1 t \qquad (8)$$

so that if at t = 0, the magnetic moment is pointing along z-direction, at a time such that $t_{\pi/2} = \dfrac{\pi}{2\omega_1}$ later, it will be lying on the x-y plane. An RF pulse of this duration is thus called a $\pi/2$ pulse. Can you use this technique to tilt the spin in –z direction? You may need to use the identity

$$e^{-i\omega_1 t\hat{I}_x}\hat{I}_z e^{i\omega_1 t\hat{I}_x} = \hat{I}_z\cos\omega_1 t - \hat{I}_y\sin\omega_1 t \tag{9}$$

5. Photo-electric effect is a first order radiation interaction process in which a photon of $(\vec{k},\lambda)$ is absorbed by an atom in the ground state $|A\rangle$ and an electron in the ground state is excited to the continuum state $|B\rangle$ of the atom. Consider a simplified case of a hydrogen-like atom having an atomic number Z. The ground state wave function of a hydrogen-like atom is given by

$$\psi_A(r) = \frac{1}{\sqrt{\pi a^3}}e^{-\frac{r}{a}}, \text{ where the Bohr radius } a = \frac{\hbar^2}{me^2 Z} \tag{10}$$

and the final continuum state wave function is approximated by a plane wave

$$\psi_B(\vec{r}) = \frac{1}{L^{3/2}}e^{i\vec{q}\cdot\vec{r}}, \tag{11}$$

where $\vec{p} = \hbar\vec{q}$ is the momentum of the ejected electron. Answer the following questions.

(a) 10% Show that the differential cross-section of the photo-electric effect in the Born approximation is proportional to square of a matrix element (where $\vec{Q} = \vec{k} - \vec{q}$),

$$\frac{d\sigma}{d\Omega} \propto \left|\int e^{i\vec{Q}\cdot\vec{r}}\hat{\varepsilon}_{k\lambda}\cdot\nabla\psi_A(r)d^3r\right|^2, \tag{12}$$

and the matrix element can be simplified to

$$\int e^{i\vec{Q}\cdot\vec{r}}\hat{\varepsilon}_{k\lambda}\cdot\nabla\psi_A(r)d^3r = i(\hat{\varepsilon}_{k\lambda}\cdot\vec{q})\int_0^\infty dr 4\pi r^2\frac{\sin Qr}{Qr}\psi_A(r). \tag{13}$$

(b) 10% Evaluate the Fourier integral on the right-hand side with the help of the formula

$$\int_0^{\infty} dx 4\pi x^2 \frac{\sin xy}{xy} e^{-x} = \frac{8\pi}{\left(1+y^2\right)^2}, \tag{14}$$

and show that the value of the differential cross-section is dominated by the contributions from K-shell electrons and is proportional to the 5th power of Z, the famous signature of the photo-electric effect..