14.12 Game Theory
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Homework 1

Due on 9/28/2005 (in class)

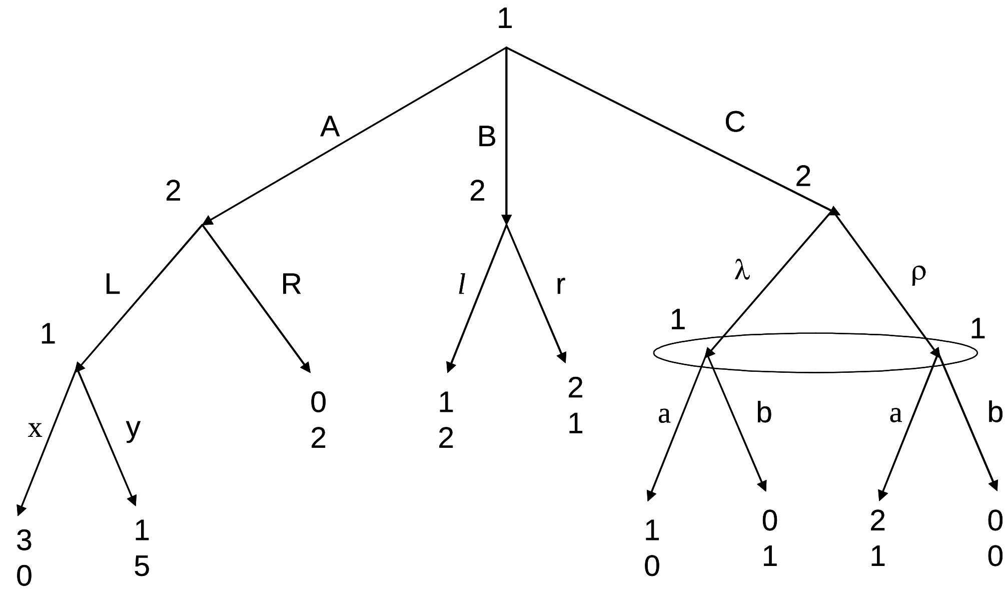
1. Alice is visiting New York. She calls her friend Bob, who lives in New Haven, from a pay phone, and they decide that Bob comes to New York and they meet at the train station. Before they can clarify which train station, Bob's cell phone's battery dies, and they can no longer communicate. Unfortunately, there are two trains from New Haven to New York: Amtrak, which arrives at the Penn Station, and MetroLiner, which arrives at the Grand Central, both arriving at noon. Clearly, Bob may take Amtrak, MetroLiner, or give up and stay home. Alice may check either of the stations (but not both). It is commonly known among them that they are both expected utility maximizers and that the following about their preferences are true. They are indifferent between Bob taking Amtrak while Alice checking the Grand Central and Bob taking MetroLiner while Alice waiting at the Penn Station. That is, missing each other is equally bad. Alice is also indifferent between where they meet. For Alice, waiting at the Penn Station while Bob stays home is as bad as missing each other. But she would feel worse if she waits at the Grand Central and Bob stays home. In particular, Alice's preferences are such that, if she assigns probabilities p, q, r, to Amtrak, MetroLiner, and staying home, respectively, then she would prefer the Penn Station to the Grand Central if and only if $p > q - r/2$. If Bob stays home, he does not care which station Alice checks. He prefers Amtrak to MetroLiner if and only if the probability of Alice waiting at the Penn Station is greater than 1/3. He prefers Amtrak to staying home if the probability of Alice waiting at the Penn Station is greater than 2/3.

   (a) Write a normal-form game that represents the above situation, including the parties' preferences on lotteries.

   (b) Find another pair of payoff functions that represents the same situation.

   (c) Find all the possible outcomes given that it is common knowledge that both players are expected utility maximizers with the above preferences.

2. Consider the following two-player game in extensive form.



(a) Write this game in normal form.

(b) Find all the rationalizable strategies.

(c) Find all Nash equilibria in pure strategies.

(d) Iteratively eliminate all the weakly dominated strategies.

3. In a college there are n students. They are simultaneously sending data over the college's data network. Let $x_i \geq 0$ be the size data sent by student i. Each student i chooses x_i himself or herself. The speed of network is inversely proportional to the total size of the data, so that it takes $x_i t(x_1, \ldots, x_n)$ minutes to send the message where

$$t(x_1, \ldots, x_n) = x_1 + \cdots + x_n.$$

The payoff of student i is

$$x_i - x_i t(x_1, \ldots, x_n).$$

(a) Compute the Nash equilibrium of this game.

(b) Suppose that we introduce a new program so that we pay each student a fixed amount M to each student and charge p for each unit of data he sends. His payoff will be

$$M + x_i - x_i t(x_1, \ldots, x_n) - p x_i.$$

1. Compute the Nash equilibrium under this program.
2. Suppose that we want to make sure that in equilibrium we must break even; i.e. $M = p(\bar{x}_1(M,p) + \cdots + \bar{x}_n(M,p))/n$, where $\bar{x}_n(M,p)$ is the data size of student i in equilibrium under this program. Compute the values M and p that maximize the equilibrium payoff of the students.
3. Briefly discuss your findings.

4. The mechanism in this problem, designed by game theorists David Gale and Lloyd Shapley, is used in many applications, such as matching students with schools or matching potential medical residents with hospitals. We have n candidates and n positions (employers). Each candidate i has a strict preference among the positions $(p_{i1}, p_{i2}, \ldots, p_{in})$, listed in the order of from the best position to the worst position, and each position j has a strict preference over candidates $(c_{i1}, c_{i2}, \ldots, c_{in})$, listed again in the order of from the best to the worst. We know the preferences of the positions, but we need to elicit the preferences of the candidates in order to match them in an efficient way. Each candidate simultaneously submits a ranking $(\hat{p}_{i1}, \hat{p}_{i2}, \ldots, \hat{p}_{in})$ of the positions, which may be possibly different from his true ranking. Then, we use the following mechanism to match them with the positions. We first assign each candidate i to his best declared position, $\hat{p}_{i1}$, as an *applicant* to that position. For each position j, we select the candidate that is find best by j among the applicants to j as its *fiancé* and reject all the other candidates from j. If there are any rejected candidates, then in the next round, we assign each such candidate i to his next best declared position, $\hat{p}_{i2}$, as an applicant to that position. We also consider the fiancé of a position from the previous round as an applicant to that position. We choose the candidate that is found best by j among the applicants to j (including its fiancé from the previous round) as its new fiancé and reject all the other candidates from j. We continue in this manner until there is no rejected candidate. (It has been shown that the process stops at n rounds or less.) The process stops when each position has only one applicant, and we match the applicants to their respective positions. Consider the game played by the candidates, where strategies are submitted rankings.

   (a) Take $n = 3$ and show that submitting the true ranking $(p_{i1}, \ldots, p_{in})$ is a dominant strategy for each candidate i.

   (b) (Bonus) Show that submitting the true ranking is a dominant strategy for each n.