14.12 - Fall 05
Solutions for Problem Set #2

Question 1 (30)

By inspection, it can easily be seen that there is no pure strategy Nash Equilibrium of the game. Whenever a cell is considered, it is always possible to find a deviation by at least one of the players. Thus we conclude that there is no pure strategy Nash Equilibrium.

In order to find all the possible mixed strategy Nash equilibria, we have to clear the game from the strictly dominated strategies since we know from the lectures and recitations that in equilibrium, none of the strictly dominated strategies are given a positive probability.

Let's try to see if there is any strictly dominated strategies:
For player 1 B is a best-response (BR) to L. Similarly C is a BR to M and A is a BR to R. Therefore there is no strictly dominated strategy for player 1 at the **first round** of reasoning.

For player 2 playing L is a BR to A. Similarly M is a BR to B and L is a BR to C. Therefore the only possible candidate for strictly dominated strategy is R. Let's see if there is any possible combination of L and M which dominates R. Calling the probability assigned to L as p, the combination of L and M has to satisfy **all** of the following inequalities.

$$\begin{aligned} p*1+(1-p)*0 &> 0 \\ p*1+(1-p)*2 &> 1 \\ p*2+(1-p)*1 &> 1 \end{aligned}$$

These inequalities can be reduced to:

$$p>0,\ 1>p,\ p>0$$

respectively. As a result, any combination of L and M with $0<p<1$ is doing better than playing R. Therefore R is a strictly dominated strategy for player 2 at the first round of reasoning (Do not think that they are playing sequentially. They are just doing the reasoning before playing the **simultaneous** game). Since player 1 knows that player 2 will not play R, he eliminates his own strategy A since in the reduced game (which excludes the strategy R of player 2) A is strictly dominated by both B or C. Now we are left with the following game

	L	M
B	2,1	1,2
C	1,2	2,1

Let's say player 1 plays B with probability μ and player 2 plays L with probability λ. Then the mixed equilibrium should satisfy the following equations

$$\begin{aligned} \mu*1+(1-\mu)*2 &= \mu*2+(1-\mu)*1 \\ \lambda*2+(1-\lambda)*1 &= \lambda*1+(1-\lambda)*2 \end{aligned}$$

Therefore the solution is

$$\mu=\lambda=1/2.$$

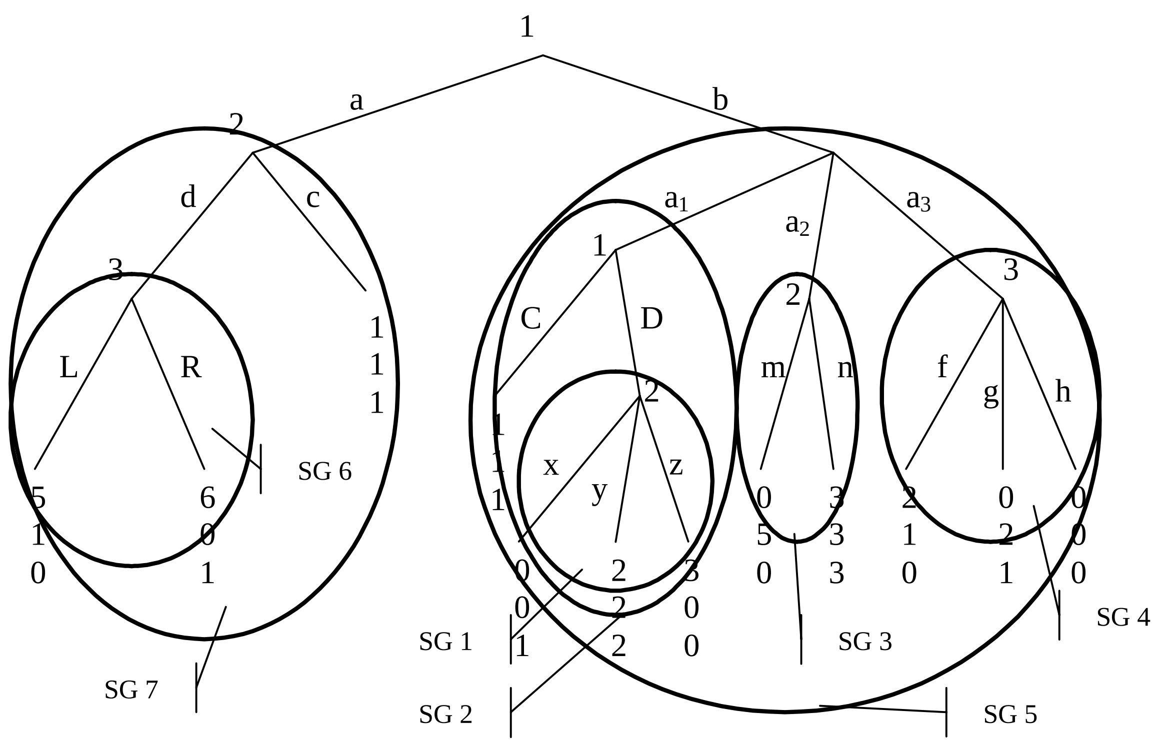
This implies that the only Nash equilibrium is

$$NE = \left[\frac{1}{2}B + \frac{1}{2}C,\ \frac{1}{2}L + \frac{1}{2}M\right]$$

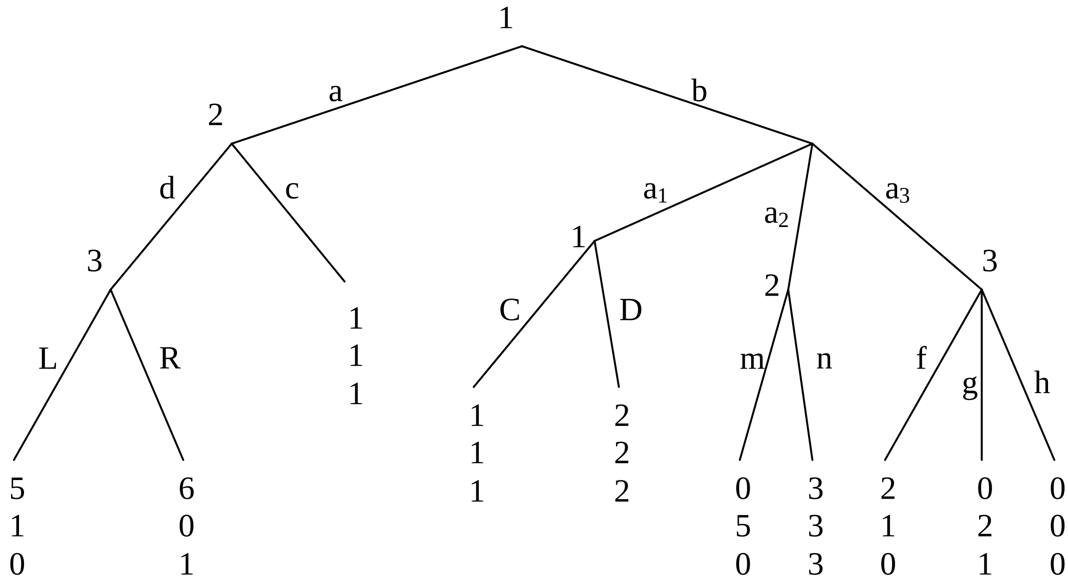
Remember that a Nash equilibrium is the equilibrium strategies **-NOT** the payoffs.

About grading: It was enough to give just one example for p. The details here were only for instructive purposes.

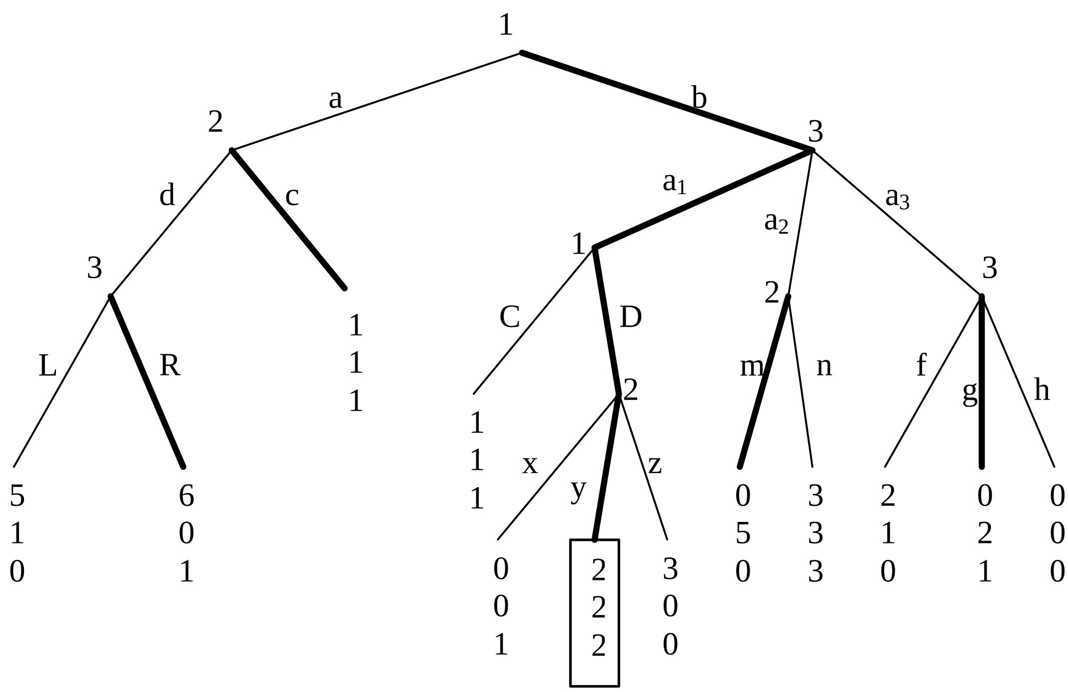
Question 2 (30)



Starting from the smallest subgame (SG 1), we are solving the game backwards. All the proper subgames are taken into a circle in the above picture. Once we solve a SG, that particular subgame is replaced with the sequentially rational decision's payoff. Just to illustrate the substitution process, after solving the first subgame, the tree looks like



Here is the sequentially rational decisions at each node.



As a result, the unique subgame perfect equilibrium of the above game is

$$SPE = (bD,\ cmy,\ a_1Rg)$$

with the equlibrium payoff

$$(2, 2, 2)\,.$$

Question 3 (40)

A disclaimer: as was asked by the question, this solution shows who is eliminated for each combination of surviving contestants, but it does not fully specify strategies. Keep in mind that the part of a strategy that gives someone's behavior in the second round can potentially depend not only on who was eliminated in the first round but also on who voted for whom. Also note that this solution assumes that you cannot vote for yourself, although allowing such actions would not change the results in an interesting way.

There are two rounds to examine, and let's begin by looking at the second round. Let's assume that the players remaining are i, j, and k. The outcome will be that the player with the highest value out of the three remaining is voted off. To see this, let's first establish a preliminary claim: the worst outcome for player i is when he is eliminated, and the best outcome is when the player with the highest value out of players j and k are eliminated. The proof of the first part or the claim is obvious. To see the second part, let's use l to denote the player who was eliminated in the first round and note that

$$\frac{v_i}{v_i+v_j}(v_i+v_j+v_k+v_l+v_i+v_j+v_k+v_i+v_j) > \frac{v_i}{v_i+v_k}(v_i+v_j+v_k+v_l+v_i+v_j+v_k+v_i+v_k)$$

if and only if

$$(v_i+v_k)(v_i+v_j+v_k+v_l+v_i+v_j+v_k+v_i+v_j) > (v_i+v_j)(v_i+v_j+v_k+v_l+v_i+v_j+v_k+v_i+v_k)$$

if and only if

$$v_iv_j+v_k(v_i+v_j+v_k+v_l+v_i+v_j+v_k+v_i+v_j) > v_iv_k+v_j(v_i+v_j+v_k+v_l+v_i+v_j+v_k+v_i+v_k)$$

if and only if

$$v_iv_j+v_kv_i+v_kv_j+v_k(v_i+v_j+v_k+v_l+v_i+v_j+v_k) > v_iv_k+v_jv_i+v_jv_k+v_j(v_i+v_j+v_k+v_l+v_i+v_j+v_k)$$

if and only if

$$v_k(v_i+v_j+v_k+v_l+v_i+v_j+v_k) > v_j(v_i+v_j+v_k+v_l+v_i+v_j+v_k)$$

if and only if

$$v_k > v_j.$$

So the point is that a player gets a higher payoff from entering the lottery against a lower-valued opponent compared to a higher-valued opponent. However, due to the possibility of strategic voting, we shouldn't quite stop here.

Let's look at the case where the players to survive the first round are players 1, 2, and 3. (The other cases are where players 1, 2, and 4 survive; where players 1, 3, and 4 survive; and where players 2, 3, and 4 survive. The arguments showing that the highest-value contestant is eliminated in those cases is the same as in the case where players 1, 2, and 3 remain. All you need to do is change indices.) Player 3 is the last to move. There are four cases to consider for player 3:

1. Players 1 and 2 have both voted for player 3.
2. There is one vote for 1 and one for 2.
3. There is one vote for 1 and one for 3.
4. There is one vote for 2 and one for 3.

In case 1, it doesn't matter what player 3 does. In case 2, player 3 should vote off player 1 (this follows immediately from the claim above). In case 3, player 3 should vote off player 1 because doing so guarantees the best outcome for player 3 (entering the end lottery against player 2), whereas voting for player 2 instead means only a 1/3 chance of the best outcome but a 1/3 chance of the less-preferred outcome of being in the end lottery against player 1 and a 1/3 chance of being eliminated. Case 4 would not come about. Case 4 would mean that player 1 voted for player 2 and player 2 voted for player 3. But if player 1 voted for player 2, player 2 should vote for player 1 because doing so would guarantee the best possible outcome for player 2 (because player 3 would then vote off player 1); voting for player 3 instead would mean (depending on how player 3 votes) either one vote for each of the three contestants (a less-preferred option for player 2) or else that player 2 is voted off by player 3 (again a less-preferred option for player 2).

Now let's go backwards and look at player 2. There are two cases to consider:

1. Player 1 voted for player 2.
2. Player 1 voted for player 3.

Case 1 was actually examined above, and it was established that player 2 should vote for player 1 under this scenario. In case 2, player 2 should also vote for player 1 because doing so would guarantee the best possible outcome for player 2 (because player 3 will then vote off player 1, and player 2 will thus enter the final lottery against player 3), whereas voting for player 3 would mean entering the lottery against player 1 (less-preferred to entering the lottery against player 3).

If we go even further backward and look at player 1, we see that it doesn't matter who player 1 votes for because he'll be eliminated anyway.

Now let's look at the first round. The lowest two surviving contestants are the ones who will eventually enter the final lottery at the very end of the game. It turns out that, even at this point, a player who survives the first round and is one of the contestants who will enter the end lottery would prefer the other player who survives the first round and enters the end lottery to be a lower-value contestant rather than a higher-value one. To see this, let's put ourselves in the shoes of player i and see who we would prefer out of players j and k to survive the first round and eventually play the lottery against. (If players j and k both survive to the second round, the work showing it's better to play the final lottery against a lower-valuation player is the same as above. The work to follow applies to the case that one of those players is eliminated in the first round.) We have

$$\frac{v_i}{v_i+v_j}(v_i+v_j+v_k+v_l+v_i+v_j+v_l+v_i+v_j) > \frac{v_i}{v_i+v_k}(v_i+v_j+v_k+v_l+v_i+v_l+v_k+v_i+v_k)$$

if and only if

$$(v_i+v_k)(v_i+v_j+v_k+v_l+v_i+v_j+v_l+v_i+v_j) > (v_i+v_j)(v_i+v_j+v_k+v_l+v_i+v_l+v_k+v_i+v_k)$$

if and only if

$$2v_iv_j+v_k(v_i+v_j+v_k+v_l+v_i+v_j+v_l+v_i+v_j) > 2v_iv_k+v_j(v_i+v_j+v_k+v_l+v_i+v_l+v_k+v_i+v_k)$$

if and only if

$$2v_iv_j+2v_kv_j+v_k(v_i+v_j+v_k+v_l+v_i+v_l+v_i) > 2v_iv_k+2v_jv_k+v_j(v_i+v_j+v_k+v_l+v_i+v_l+v_i)$$

if and only if

$$-2v_iv_k+v_k(v_i+v_j+v_k+v_l+v_i+v_l+v_i) > -2v_iv_j+v_j(v_i+v_j+v_k+v_l+v_i+v_l+v_i)$$

if and only if

$$v_k(v_i+v_j+v_k+v_l+v_i+v_l-v_i) > v_j(v_i+v_j+v_k+v_l+v_i+v_l-v_i)$$

if and only if

$$v_k > v_j$$

Note that players 3 and 4, given the choice of eliminating each other in the first round or eliminating player 2 in the first round, would thus rather have player 2 be eliminated in the first round and play the end lottery against each other rather than against player 2. But also note that these players would prefer to keep player 1 in the game until the second round rather than player 2 because if players 3 and 4 know that they'll play the lottery against each other in the end anyway, they'd like to raise the size of the pot by keeping player 1 into the second round rather than player 2. Also note that player 1 is indifferent between any possible outcome of the first round because there is no way he will enter the end lottery anyway. We can thus vary player 1's action to see what kinds of equilibria we get that are consistent with sequential rationality. Also remember that the interests of players 3 and 4 are aligned in that they would like to play the lottery against each other in the end and eliminate player 2 in the first round; their two votes can guarantee at least a tie of this option.

If player 1 votes to eliminate player 2, then players 3 and 4 can be assured of getting their best possible outcome and it doesn't matter what player 2 does anyway. If player 1 votes to eliminate player 3, then player 2 should also vote to eliminate player 3 (and then players 3 and 4 should vote to eliminate player 2) because otherwise players 3 and 4 will be able to achieve the necessary number of votes to eliminate player 2 for sure; by providing the second vote for player 3, player 2 will thus have a 50% chance of surviving to the next round. Similarly, if player 1 votes to eliminate player 4, then player 2 should also vote to eliminate player 4 (and players 3 and 4 should then vote to eliminate player 2).

Thus, in the end, there are a couple different possible outcomes of the game:

1. Player 2 is eliminated in the first round, and players 3 and 4 play the end lottery against each other.
2. There is a tie between players 2 and 3 in the first round, and the one who survives the random tiebreaker plays the final lottery against player 4.
3. There is a tie between players 2 and 4 in the first round, and the one who survives the random tiebreaker plays the final lottery against player 3.

But note that, no matter who is eliminated in the first round, the contestant with the highest value among the three remaining will be eliminated in the second round.