14.12 Game Theory
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Solution to Homework 6

Answer to Problem 1

(a) Suppose player j bids $b_j(v_j) = a + cv_j$. Let $u_i(v_i, v_j, b_i)$ be player i's payoff when player j is playing this strategy, and player i has a valuation v_i and bids b_i. Then

$$\begin{aligned} u_i(v_i, v_j, b_i) &= (v_i - b_i) \text{ if } v_j < \frac{b_i - a}{c} \\ &= \frac{1}{2}(v_i - b_i) + \frac{1}{2}(a + cv_j) \text{ if } v_j = \frac{b_i - a}{c} \\ &= (a + cv_j) \text{ if } v_j > \frac{b_i - a}{c} \end{aligned}$$

Therefore, expected payoff to player i from bidding b_i when his valuation is v_i equals $\int_0^1 u_i(v_i, v, b_i)\, dF(v)$
where $F(v)$ is the cummulative distribution function for player j's valuation v_j. Since v_j has a uniform distribution over $[0, 1]$, $F(v) = v$. Then the expected payoff can be written as

$$\begin{aligned} & \int_0^{\frac{b_i - a}{c}} (v_i - b_i)\, dv + \int_{\frac{b_i - a}{c}}^1 (a + cv)\, dv \\ = \; & (v_i - b_i)\left(\frac{b_i - a}{c}\right) + \left[av + \frac{c(v)^2}{2}\right]_{\frac{b_i - a}{c}}^1 \\ = \; & (v_i - b_i)\left(\frac{b_i - a}{c}\right) + a + \frac{c}{2} - \left(\frac{b_i - a}{c}\right)\left[a + \frac{c}{2}\left(\frac{b_i - a}{c}\right)\right] \\ = \; & (v_i - b_i)\left(\frac{b_i - a}{c}\right) + a + \frac{c}{2} - \left(\frac{a + b_i}{2}\right)\left(\frac{b_i - a}{c}\right) \\ = \; & \frac{1}{2}\left(\frac{b_i - a}{c}\right)(2v_i - 3b_i - a) + a + \frac{c}{2} \end{aligned}$$

Setting the first-order condition with respect to b_i equal to zero, we obtain

$$-3(b_i - a) + (2v_i - 3b_i - a) = 0$$

$$\Longrightarrow b_i = \frac{1}{3}(v_i + a)$$

Therefore, if player j bids $b_j(v_j) = a + cv_j$, then player i's best response is to bid $b_i(v_i) = \frac{a}{3} + \frac{1}{3}v_i$. In a symmetric Bayesian Nash equilibrium, $b_j(v) \equiv b_i(v)$.

$$\implies \frac{a}{3} + \frac{v}{3} \equiv a + cv$$

$$\implies c = \frac{1}{3}, a = 0$$

Therefore, both players 1 and 2 playing $b(v) = \frac{1}{3}v$ is a symmetric, linear Bayesian Nash equilibrium.

(b) Suppose player j bids $b(v_j)$, where $b(.)$ is a strictly increasing, differentiable function. Let $u_i(v_i, v_j, b_i)$ be player i's payoff when player j is playing this strategy, and player i has a valuation v_i and bids b_i. Then

$$\begin{aligned} u_i(v_i, v_j, b_i) &= (v_i - b_i) \text{ if } v_j < b^{-1}(b_i) \\ &= \frac{1}{2}(v_i - b_i) + \frac{1}{2}b(v_j) \text{ if } v_j = b^{-1}(b_i) \\ &= b(v_j) \text{ if } v_j > b^{-1}(b_i) \end{aligned}$$

Following the reasoning in part (a), the expected payoff to player i equals

$$\begin{aligned} & \int_0^{b^{-1}(b_i)} (v_i - b_i)\, dv + \int_{b^{-1}(b_i)}^1 b(v)\, dv \\ = \; & (v_i - b_i)\, b^{-1}(b_i) + [h(v_j)]_{b^{-1}(b_i)}^1 \\ = \; & (v_i - b_i)\, b^{-1}(b_i) + h(1) - h\left(b^{-1}(b_i)\right) \end{aligned}$$

where $h(v)$ is defined such that $h'(v) = b(v)$.

Setting the first-order condition with respect to b_i equal to zero, and writing b_i^* for the best reply, we obtain

$$-b^{-1}(b_i^*) + (v_i - b_i^*)\frac{db^{-1}(b_i^*)}{db_i^*} - h'\left(b^{-1}(b_i^*)\right)\frac{db^{-1}(b_i^*)}{db_i} = 0$$

$$\implies -b^{-1}(b_i^*) + \frac{1}{b'\left(b^{-1}(b_i^*)\right)}\left[v_i - b_i^* - b\left(b^{-1}(b_i^*)\right)\right] = 0$$

For a symmetric equilibrium, we must have $b_i^* = b(v_i) \implies b^{-1}(b_i^*) = v_i$. Hence,

$$-v_i + \frac{1}{b'(v_i)}\left[v_i - 2b(v_i)\right] = 0$$

$$\implies b'(v_i) + \frac{2b(v_i)}{v_i} = 1 \tag{1}$$

(c) For notational simplicity, replace v_i with v in (1). Multiplying throughout (1) with v^2, we obtain

$$b'(v)\,v^2 + 2b(v)\,v = v^2$$

$$\begin{aligned}
\implies \quad & \frac{d}{dv}\left[b(v)\,v^2\right] = v^2 \\
\implies \quad & b(v)\,v^2 = \int v^2 dv \\
\implies \quad & b(v)\,v^2 = \frac{v^3}{3} + c \\
\implies \quad & b(v) = \frac{v}{3} + \frac{c}{v^2}
\end{aligned}$$

For $b(v)$ to be strictly increasing for $v \in [0,1]$, we require $c = 0$. Therefore,

$$b(v) = \frac{v}{3}$$

Answer to Problem 2

(a) To compute a symmetric Bayesian Nash equilibrium, we compute the best response function for player i when all other individuals $j \neq i$ adopt a strategy $b_j(v_j) = a + cv_j$.

If player i bids b_i when her valuation is v_i, then her expected payoff equals

$$[v_i - b_i] \Pr\left(\max_{j \neq i} b_j(v_j) < b_i\right)$$

Note that we can ignore the case where $\max_{j \neq i} b_j(v_j) = b_i$ as this occurs with zero probability. Now,

$$\begin{aligned}
\Pr\left(\max_{j \neq i} b_j(v_j) < b_i\right) &= \Pr\left(\max_{j \neq i} a + cv_j < b_i\right) \\
&= \Pr\left(\max_{j \neq i} v_j < \frac{b_i - a}{c}\right) \\
&= \left(\frac{b_i - a}{100c}\right)^{n-1}
\end{aligned}$$

Therefore, we can write player i's expected payoff from bidding b_i as

$$[v_i - b_i]\left(\frac{b_i - a}{100c}\right)^{n-1}$$

which is maximised at $b_i = \frac{a}{n} + \left(\frac{n-1}{n}\right) v_i$.

Therefore, in a symmetric equilibrium, we must have

$$\frac{a}{n} + \left(\frac{n-1}{n}\right) v_i \equiv a + cv_i$$

$$\implies a = 0, b = \frac{n-1}{n}$$

Therefore, the strategy profile $b_i(v_i) = \left(\frac{n-1}{n}\right) v_i$, $i = 1..n$ is a symmetric, linear Bayesian Nash equilibrium.

(b) The equlibrium payoff to a player with valuation v_i equals

$$[v_i - b_i(v_i)] \Pr\left(\max_{j \neq i} b_j(v_j) < b_i(v_i)\right)$$

$$\begin{aligned}
&= \left[v_i - \left(\frac{n-1}{n}\right) v_i\right] \Pr\left(\max_{j \neq i} \left(\frac{n-1}{n}\right) v_j < \left(\frac{n-1}{n}\right) v_i\right) \\
&= \frac{v_i}{n} \Pr\left(\max_{j \neq i} v_j < v_i\right) \\
&= \frac{v_i}{n} \left(\frac{v_i}{100}\right)^{n-1} \\
&= \frac{(v_i)^n}{n(100)^{n-1}}
\end{aligned}$$

(c) If n=80, then the expected payoff to a player with valuation v equals $\frac{(v)^{80}}{80(100)^{79}}$. This is the cost that a player with valuation v would be willing to incur to play the game. This expression is increasing in v. Therefore, the higher is one's valuation, the 'luckier' the player is. And the difference in expected payoff between the 'luckiest' and 'least lucky' player equals

$$\begin{aligned}
&\frac{100^{80}}{80 \times 100^{79}} - 0 \\
&= \frac{100}{80}
\end{aligned}$$

Answer to Problem 3

Throughout this question, we will refer to XC's strategies as (X,Y) where X = Action taken by XC if t_{XC} = good and Y =Action taken by XC if t_{XC} = bad, with X, Y =A (Advertisement), NA (No Advertisement). Likewise, we will refer to the consumer's strategy as (x, y) where x = consumer's action if he observes A and y = consumer's action if he observes NA, with x, y = B (Buy), D (Don't).

a) See graph attachment.

b) Separating Equilibria: Suppose that (A, NA) is the strategy adopted by XC. In this case, (B, D) is a best response by the representative consumer and the beliefs are $\{\mu^A_{Good} = 1, \mu^{NA}_{Good} = 0\}$.

Let's check XC's best response to (B, D). Clearly, if t_{XC} = good he still prefers A since $R > c$. If t_{XC} =bad, XC prefers D since $r - c < 0$. Therefore, the equilibria is given by { (A, NA), (B, D)} , $\{\mu^A_{Good} = 1, \mu^{NA}_{Good} = 0\}$

c) Pooling Equilibria: Suppose both types of player XC play NA. The representative consumer will learn nothing from the signal and its beliefs about types remain unchanged.$\{\mu^{NA}_{Good} = 0.6, \mu^{NA}_{Good} = 0.4\}$. Then, when the representative consumer observes no advertisement, its expected payoff from buying is more than its expected payoff from don't buying: $(0.6)(1) + (0.4)(-1) = 0.2 > 0$. Therefore, the representative consumer buys.

We still need to specify the consumer's beliefs if it were to observe the signal A. Note that it is not possible to do so using Bayes' rule as neither type sends this signal in equilibrium. If $\mu^A_{Good} > 0.5$ he prefers to buy, and if $\mu^A_{Good} < 0.5$ he does not want to buy. In this case, XC will not want to deviate whatever are the beliefs of the consumer. The good type gets R if he does not deviate which is bigger than what he gets if he deviates, $R - c$ or $-c$. (depending on the consumer decision when he observes A). The bad type also does not want to deviate since r is bigger than $r - c$ or $-c$. Therefore, the equilibria are given by { (NA, NA), (B, B)}, $\{\mu^A_{Good} > 0.5, \mu^{NA}_{Good} = 0.6\}$ and { (NA, NA), (D, B)}, $\{\mu^A_{Good} < 0.5, \mu^{NA}_{Good} = 0.6\}$.

d) In this case the separating equilibria in b) is no longer an equilibria since the bad type would want to deviate and pretend he is a good type by advertising and getting $r - c$ instead of 0.

The pooling equilibria in c) is still an equilibria. But we have also another pooling equilibria where both players decide to advertise (A,A). In this case, the consumer will learn nothing when he observes advertisement and his beliefs about types remain unchanged.$\{\mu^A_{Good} = 0.6, \mu^A_{Good} = 0.4\}$. Therefore, his expected payoff from B is more than its expected payoff from D, $(0.6)(1) + (0.4)(-1) = 0.2 > 0$. Thus, he will decide to buy if observes advertisement. We cannot apply the Bayes' rule consumer's beliefs if it were to observe the signal NA because neither type sends this signal in equilibrium. If $\mu^{NA}_{Good} > 0.5$ he buys, and if $\mu^{NA}_{Good} < 0.5$ he does not want to buy. We can find the range of beliefs for which neither type will deviate from playing A. In order for XC to not deviate, we need the consumer to not buy if it receives signal NA . Thus we need that $\mu^{NA}_{Good} < 0.5$. Therefore, the equilibria is given by { (A,A), (B, NB)} and $\{\mu^A_{Good} = 0.6, \mu^{NA}_{Good} < 0.5\}$.

Answer to Problem 4 (Gibbons, ex. 4.10)
The strategy of player 2 is to buy if $v_2 \geqslant p$ and sell if $v_2 < p$
As for player 1:
$\underset{p}{Max}\ psPr(v_2 > p) + [v_1 - p * (1 - s)]\, Pr(v_2 < p)$
or, equivalently,
$\underset{p}{Max}\ ps(1 - p) + [v_1 - p(1 - s)]\, p$

First-order condition then yields:
$p(v1) = \frac{v_1+s}{2}$
It is also obvious that player 2 strategy is optimal.