14.12 Game Theory (Fall 2003)-Midterm II
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Instructions: This is an open book exam, you can use any written material. You have 1 hour and 20 minutes. The weight of each question is indicated next to it. Good luck!

1. Consider an infinite horizon bargaining game with three players $N = \{1, 2, 3\}$. In the beginning of each period $t \geq 1$, one of the three players is randomly selected to make an offer: Player 1 is selected with probability $\frac{1}{2}$, each one of Players 2 and 3 is selected with probability $\frac{1}{4}$. The selected Player i offers a division of the cake (x_t, y_t, z_t) where $x_t, y_t, z_t \geq 0$ and $x_t + y_t + z_t = 1$ (x_t denotes player 1's share y_t denotes player 2's share and z_t denotes player 3's share). The two other Players j and k observe i's offer (x_t, y_t, z_t), then j and k simultaneously decide whether to accept or reject this offer. If both j and k accept then the division is carried out, if at least one of them reject then the offer is rejected and they proceed to period $t + 1$. Players maximize the discounted sum of their expected payoffs and have the common discount factor $\delta \in (0, 1)$. The selection of who will make an offer is i.i.d. across periods.

   (a) (20pts) Conjecture a SPE strategy profile. Write down formally the strategy profile and verify that it is indeed an SPE by using the single deviation property. How is the cake divided as $\delta \to 1$?

   (b) (5pts. Difficult, for extra credit. Do not spend time on this only unless you are done with all other questions.) Show that the SPE is not unique.

2. Consider following 3×3 stage game G:

	L	M	R
U	5,6	2,2	2,3
M	6,3	3,4	0,3
D	2,1	1,0	0,1

   (a) (5pts) What are the SPE of $G(T)$ when $T < \infty$.

   (b) (10pts) What are the set feasible payoff vectors and the payoff vectors that can be obtained in SPE of $G(\infty)$ by applying the Folk theorem with Nash threats.

   (c) (10pts) Give the SPE trigger strategy profile that yields the payoff vector $(5, 6)$. What is the minimum δ for these strategies to be SPE?

(d) (20pts) What are the minmax payoffs of the players in G? Does the Folk theorem with Nash threats give an SPE of $G(\infty)$ where the payoff is (6,3)? Can you construct an SPE of $G(\infty)$ where the payoff is (6,3) using "Carrot and Stick" type strategies? Indicate the minimum δ for which the strategy profile you construct is an SPE.

3. Consider the following auction of a single object to two bidders $N = \{1, 2\}$. Bidders valuations of the object are i.i.d. uniformly distributed on the interval $[0, 1]$. Each bidder i simultaneously submits a bid $b_i \geq 0$. The highest bidder wins the object and pays the average of the two bids. If the two players have the same bid, then they both win the object with equal probability. Therefore the payoff to player i with type v_i when the bids are b_i and b_j $(i \neq j)$ is:

$$u_i(b_i, b_j; v_i) = \begin{cases} v_i - \frac{b_i+b_j}{2} & \text{if } b_i > b_j, \\ \frac{1}{2}\left(v_i - \frac{b_i+b_j}{2}\right) & \text{if } b_i = b_j, \\ 0 & \text{if } b_i < b_j. \end{cases}$$

(a) (25pts. Symmetric linear BNE) Find a BNE where the bid of player i with valuation v_i can be expressed as:

$$b_i(v_i) = cv_i + a$$

for some $c > 0$ and $a \geq 0$ that do not depend on i.

(b) (10pts. Symmetric BNE in strictly increasing and differentiable strategies) Show that any BNE where player i with valuation v_i bids:

$$b_i(v_i) = b(v_i)$$

for some differentiable function $b : [0, 1] \to \mathbb{R}_+$ with $b'(\cdot) > 0$ that does not depend on i, must be the same as the linear BNE you have identified in (a).