14.12 Game Theory (Fall 2003)-Midterm II Solutions

1. Consider an infinite horizon bargaining game with three players $N = \{1, 2, 3\}$. In each period t, one of the players is randomly selected to make an offer: Player 1 is selected with probability $\frac{1}{2}$, each one of Players 2 and 3 is selected with probability $\frac{1}{4}$. The selected Player i offers a division of the cake (x_t, y_t, z_t) where $x_t, y_t, z_t \geq 0$ and $x_t + y_t + z_t = 1$ (x_t denotes player 1's share y_t denotes player 2's share and z_t denotes player 3's share). The two other Players j and k observe i's offer (x_t, y_t, z_t), then j and k *simultaneously* accept or reject this offer. If both j and k accept then the division is carried out, if at least one of them rejects then the offer is rejected and they proceed to period $t + 1$.

   Players maximize discounted expected payoffs and have the common discount factor $\delta \in (0, 1)$. If no offer is ever accepted, then each player receives a payoff of zero. The selection of who makes an offer is i.i.d. across periods.

   (a) (15pts) Conjecture an SPE where (as usual) players accept any division where they receive at least δ times their continuation payoff. Write down formally the strategy profile and verify that it is indeed an SPE by using the single deviation property.

   Solution: Denote V_1, V_2, V_3 the payoffs to players in the beginning of each period, *before* the uncertainty about who is making the offer is realized. Clearly the game is stationary, so these continuation payoffs are the same in the beginning of each period.

   Now with probability $\frac{1}{2}$ player 1 will be selected and he will make offer $(1 - \delta V_2 - \delta V_3, \delta V_2, \delta V_3)$ (why?), that will be immediately accepted. If either player 2 or player 3 will get to make an offer (which happens with probability $\frac{1}{4}$ each), player 1 will be offered δV_1 and will accept. Overall, the payoff to player 1 is $V_1 = \frac{1}{2}(1 - \delta V_2 - \delta V_3) + \frac{1}{4}\delta V_1 + \frac{1}{4}\delta V_1$. Similar considerations for players 2 and 3 result in $V_2 = \frac{1}{2}\delta V_2 + \frac{1}{4}(1 - \delta V_1 - \delta V_3) + \frac{1}{4}\delta V_2$ and $V_3 = \frac{1}{2}\delta V_3 + \frac{1}{4}\delta V_3 + \frac{1}{4}(1 - \delta V_1 - \delta V_2)$. This gives a system of three equations with three unknowns, which is easily solved (especially if you rely on symmetry to conjecture $V_2 = V_3$). The answer is $V_1 = \frac{1}{2}, V_2 = V_3 = \frac{1}{4}$, and the offers are calculated accordingly (for instance, if player 1 gets to make an offer, he will offer $(1 - \frac{\delta}{2}, \frac{\delta}{4}, \frac{\delta}{4})$). That constructed strategies are indeed an SPE follows almost immediately from construction.

   (b) (10pts) For any division (x, y, z), construct an SPE where the cake is divided according to (x, y, z) in the first period, no matter who makes the offer.

Solution: The construction below relies heavily on that there are more than two players and that they accept or reject the offer simultaneously. In particular this implies that none of the players never accepting *any* offer is an equilibrium (if my partner is going to reject anyway, I can do nothing by deviating and accepting). Consequently, the following is an SPE: whoever gets to make an offer in the first period offers (x, y, z) and the others accept. If anyone deviates in the first period (either the offer is different from (x, y, z) or someone rejects it) then never accept any offer whatsoever.

2. Consider following 3×3 stage game G:

	L	M	R
U	5,6	2,2	2,3
M	6,3	3,4	0,3
D	2,1	1,0	0,1

In the following restrict attention to only pure strategies.

(a) (5pts) What are the SPE of $G(T)$ when $T < \infty$.

Solution: The unique Nash equilibrium of the stage game is the strategy profile (M, M). Therefore, the unique SPE of the finitely repeated game is to play (M, M) at every stage.

(b) (10pts) What are the set feasible payoff vectors and the payoff vectors that can be obtained in SPE of $G(\infty)$ by applying the Folk theorem with Nash threats.

Solution:

(c) (10pts) Give the SPE trigger strategy profile that yields the payoff vector $(5, 6)$. What is the minimum δ for these strategies to be SPE?

Solution: Consider the following strategy profile: the players are to play (U, L) if the play has been (U, L) at every stage in the past. Otherwise, the players should play (M, M). To establish that this constitutes an SPE, we apply the single deviation principle: in the 'punishment' mode, clearly neither player would want to deviate from playing (M, M) as this is a Nash equilibrium of the stage game. It remains to be checked that neither player wants to deviate from playing (U, L) given the punishment such a deviation would trigger. Player 2, on the other hand, has no reason to deviate from the play (U, L) as L is his best response to player 1 playing U. If player 1 deviates to M (it is clearly not beneficial for her to deviate to D) then she increases her payoff by 1 in the current period but loses 2 in every period thereafter as a result of the punishment. Therefore, it is not in her interest to deviate if

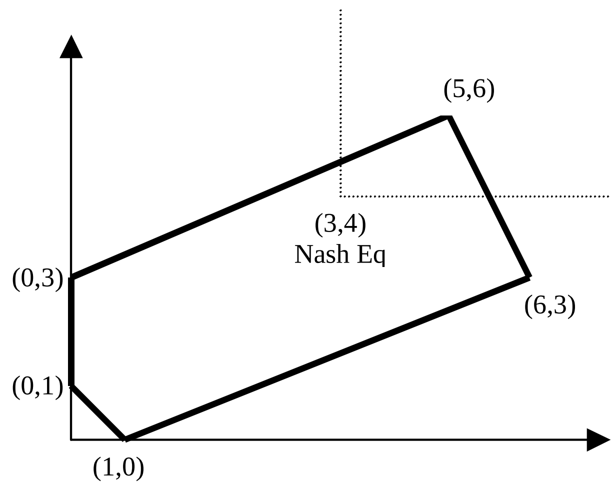

Figure 1:

$1 \geq 2\delta + 2\delta^2 + ... = \frac{2\delta}{1-\delta}$ (where we are discounting future payoffs by δ)
$\Rightarrow \delta \geq \frac{1}{3}$
So the given strategy profile constitutes an SPE with Nash threats for $\delta \geq \frac{1}{3}$

(d) (20pts) Does the Folk theorem with Nash threats give an SPE of $G(\infty)$ with payoff (6,3)? Is there an SPE of $G(\infty)$ with payoff (6,3)? If so indicate the minimum δ for which the strategy profile you construct is an SPE. Same question for the payoff vector (0,3) instead of (6,3).

Solution: Using Nash threats, it is not possible to sustain an equilibrium where the per-period payoff is lower than the Nash equilibrium payoffs of the stage game. Since the Nash equilibrium payoffs are (3,4), it is not possible to construct an SPE where the players receive (6,3) using Nash threats alone.

The minmax payoffs are (2,1). Therefore it is possible to construct an SPE of $G(\infty)$ that yields payoffs (6,3). Here is an example:

The game can be in two phases, C and P. In phase C, the prescribed play is (M, L). In phase P, the prescribed play is (D, M). If either player deviates in phase C, then they are to switch to phase P. If either player deviates in phase P, they remain in phase P. If no deviation occurs in phase C, they remain in phase C. If no deviation occurs in phase P, they switch to phase C. Finally, the game begins in phase C.

We apply the single deviation principle to check if this can be an SPE. In phase C, player two can gain a maximum of 1 point by deviating to M but it would mean going to phase P for one period, which would cost player two 3 points relative to what she would have obtained if the play had remained in phase C. Therefore, it is not in the interest of player 2 to deviate if $1 \leq$

$\delta(3-0) \Rightarrow \delta \geq \frac{1}{3}$. On the other hand, player one cannot gain from deviation in phase 1.

In phase P, player one gains a maximum of 2 points by deviating to M. However, this would mean remaining in phase P an additional period before going back to phase C, and this would cost player one 5 points. Therefore he would not want to deviate if $2 \leq \delta(6-1) \Rightarrow \delta \geq \frac{2}{5}$. In phase P, player two gains 1 point by deviating to L or R but loses 3 points from having to remain in phase C an additional period. Therefore, she would not deviate if $1 \leq \delta(3-0) \Rightarrow \delta \geq \frac{1}{3}$.

Taking all these results together, neither player has an incentive to deviate when $\delta \geq \frac{2}{5}$.

We have thus established that the strategy profile described here constitutes an SPE for $\delta \geq \frac{2}{5}$.

It is not possible to construct an SPE with payoffs (0,3) as 0 is less than player 1's minmax payoff.

3. Consider the following incomplete information version of the Bertrand duopoly game with differentiated products. There are two firms $N = \{1, 2\}$ each with zero marginal cost. Each firm i can either be of High-type with probability θ or Low-type with probability $1-\theta$ (the types of the firms are independently drawn from the same distribution). The type of each firm i is private information of i. After learning his type, each firm i simultaneously determines a price $p_i \geq 0$. The demand for firm i's product is given by

$$\max\{0, 1 - p_i + a^H p_j\}$$

if firm i is of High-type, and by

$$\max\{0, 1 - p_i + a^L p_j\}$$

if firm i is of Low-type, where $i \neq j$ and $a^H > a^L > 0$. Firms maximize expected profits.

(a) (5pts) Formulate the Bayesian game (actions, types, beliefs, and payoffs) corresponding to the above situation. What are the strategies in this game?

Solution: there are two players and for each player two possible types. Player $i = 1, 2$ of type $T = L, H$ has beliefs $(\theta, 1-\theta)$ about the distribution of the other player's type and choses action p_i^T among positive real numbers. If player i is of type $T = L, H$ and player j is of type $S = L, H$, then player i's payoff are her profits: $\pi_i^T \left(p_i^T, p_j^S\right) = p_i^T \left(1 - p_i^T + a^T p_j^S\right)$. A strategy of player i is a pair of prices $\left(p_i^L, p_i^H\right)$ that he can set conditional on each one of his types.

(b) (25pts) Find the BNE. (Assume that θ, a^H, and a^L satisfy conditions for an interior solution.)

Solution: a BNE of this game must be a set of strategies for each player of each possible type such that given the player's strategies and beliefs, none wants to deviate. To compute such equilibria, we look at fixed points of the best response graphs of each player-type. Player i if she is of type T does not know her competitor's type, but she has beliefs about the distribution of her competitor's types. When choosing her best reponse, she chooses the price p_i^T that will maximize her expected profits:

$$\begin{aligned} p_i^T\left(p_j^L, p_j^H\right) &= \arg\max\left\{\theta p_i^T\left(1-p_i^T+a^T p_j^H\right)+(1-\theta)\, p_i^T\left(1-p_i^T+a^T p_j^L\right)\right\} \\ p_i^T\left(p_j^L, p_j^H\right) &= \frac{1+a^T\left(\theta p_j^H+(1-\theta)\, p_j^L\right)}{2} \end{aligned}$$

The BNE, if it is an interior solution, must be a fixed point of the best response graph (of all those 4 different players/types). From the symetry of the game, we can already guess that $p_i^T = p_j^T = p^T$ for $T = H, L$. The conditions for a BNE are:

$$\begin{aligned} p^H &= \frac{1+a^H\left(\theta p^H+(1-\theta)\, p^L\right)}{2} \\ p^L &= \frac{1+a^L\left(\theta p^H+(1-\theta)\, p^L\right)}{2} \\ \text{or } p_1^H &= p_2^H = p^H = \frac{2+(1-\theta)\left(a^H-a^L\right)}{2\left(2-(1-\theta)\, a^L-\theta a^H\right)}, p_1^L = p_2^L = p^L = \frac{2-\theta\left(a^H-a^L\right)}{2\left(2-\theta a^H-(1-\theta)\, a^L\right)} \end{aligned}$$

The BNE consists of the strategy profile $\left(p_1^H, p_1^L; p_2^H, p_2^L\right)$ given above, based on beliefs $(\theta, 1-\theta)$.