14.12 Game Theory
Fall 2004

Answers for Midterm 1, Fall 2004

1. Consider the following game:

	A	B	C
a	$3,0$	$0,3$	$0,x$
b	$0,3$	$3,0$	$0,x$
c	$x,0$	$x,0$	x,x

(a) The question does not ask to find all Nash equilibria. One way is to find all the pure-strategy Nash equilibria and then use the symmetry of the game to guess a symmetric mixed-strategy Nash equilibria and check that it is indeed a Nash equilibria. The Nash equilibria of that game are

$$(c, C)$$
$$(\frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c; \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C)$$
$$(\frac{1}{2}a + \frac{1}{2}b; \frac{1}{2}A + \frac{1}{2}B)$$

In the solutions, we find all the equilibria starting with the pure-strategy ones. The best responses of player 1 are

$$\begin{aligned} BR_1(A) &= \{a\} \\ BR_1(B) &= \{b\} \\ BR_1(C) &= \{c\} \end{aligned}$$

The best responses of player 2 are

$$\begin{aligned} BR_2(a) &= \{B\} \\ BR_2(b) &= \{A\} \\ BR_2(c) &= \{C\} \end{aligned}$$

Hence the only pure-strategy Nash equilibrium is

$$(c, C)$$

In a mixed-strategy equilibrium, player 2 is playing A with probability p, B with probability q and C with probability $1 - p - q$. The expected payoffs of player 1 are

$$\begin{aligned} EU_1(a) &= 3p \\ EU_1(b) &= 3q \\ EU_1(c) &= 1 \end{aligned}$$

Hence his best responses are

$$\begin{aligned}
p &> \frac{1}{3} \text{ and } p > q \Longrightarrow a \\
q &> \frac{1}{3} \text{ and } q > p \Longrightarrow b \\
\frac{1}{3} &> p \text{ and } \frac{1}{3} > q \Longrightarrow c \\
p &= q > \frac{1}{3} \Longrightarrow \{a, b\} \\
p &= \frac{1}{3} > q \Longrightarrow \{a, c\} \\
q &= \frac{1}{3} > p \Longrightarrow \{b, c\} \\
p &= q = \frac{1}{3} \Longrightarrow \{a, b, c\}
\end{aligned}$$

By symmetry, if player 1 is playing a with probability r, b with probability s and c with probability $1 - r - s$, the best responses of player 2 are

$$\begin{aligned}
r &> \frac{1}{3} \text{ and } r > s \Longrightarrow B \\
s &> \frac{1}{3} \text{ and } r > s \Longrightarrow A \\
\frac{1}{3} &> r \text{ and } \frac{1}{3} > s \Longrightarrow C \\
r &= s > \frac{1}{3} \Longrightarrow \{A, B\} \\
r &= \frac{1}{3} > s \Longrightarrow \{B, C\} \\
s &= \frac{1}{3} > r \Longrightarrow \{A, C\} \\
r &= s = \frac{1}{3} \Longrightarrow \{A, B, C\}
\end{aligned}$$

- Let's first look at an equilibrium where player 2 is mixing among $\{A, B, C\}$ with positive probabilities. In order for this strategy to be a best response it must be the case that player 1's probabilities are $r = s = \frac{1}{3}$. Player 1 is playing his three strategies with a positive probability which is a best response to $p = q = \frac{1}{3}$. We conclude that $(\frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c; \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C)$ is a Nash equilibrium.
- Then look at an equilibrium where player 2 is mixing among $\{A, B\}$. This means that $p + q = 1$ and the best responses of player 1 are

$$\begin{aligned}
p &> \frac{1}{2} \underset{1}{\Longrightarrow} a \underset{2}{\Longrightarrow} B \text{ cannot be an equilibrium} \\
p &= \frac{1}{2} \underset{1}{\Longrightarrow} \{a, b\} \\
p &< \frac{1}{2} \underset{1}{\Longrightarrow} b \underset{2}{\Longrightarrow} A \text{ cannot be an equilibrium}
\end{aligned}$$

  Player 1 is thus mixing among $\{a, b\}$.And using the symmetry, this can be the case in equilibrium if and only if $r = s = \frac{1}{2}$. We conclude that $(\frac{1}{2}a + \frac{1}{2}b; \frac{1}{2}A + \frac{1}{2}B)$ is a Nash equilibrium.
- What about an equilibrium where player 2 is mixing between $\{A, C\}$, i.e. $p \in (0, 1)$ and $q = 0$? $\{A, C\}$ is a best response only if $s = \frac{1}{3}$, b is played with a positive probability. And b is a best response iff $q \geq \frac{1}{3}$ which contradicts that we are looking for an equilibrium with $q = 0$. By symmetry no equilibrium exists with player 2 mixing between $\{B, C\}$.

(b) As $BR_1(C) = \{c\}$ and $BR_2(c) = \{C\}$, (c, C) remains a Nash equilibrium when $x = 2$. What about the mixed-strategy Nash equilibria

- If player 2 plays $\frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C$

$$\begin{aligned} EU_1(a) &= 1 \\ EU_1(b) &= 1 \\ EU_1(c) &= 2 \end{aligned}$$

Player 1's best response is c. Hence $(\frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c; \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C)$ is not a Nash equilibrium.

- If player 2 plays $\frac{1}{2}A + \frac{1}{2}B$

$$\begin{aligned} EU_1(a) &= \frac{3}{2} \\ EU_1(b) &= \frac{3}{2} \\ EU_1(c) &= 2 \end{aligned}$$

Player 1's best response is c. Hence $(\frac{1}{2}a + \frac{1}{2}b; \frac{1}{2}A + \frac{1}{2}B)$ is not a Nash equilibrium.

2. (a) There are two subgames in this game, including the whole game itself. The normal-form representation of the final subgame is as follows:

$P_1 \backslash P_2$	a	b	c
L	(3,1)	(0,0)	(2,2)
R	(0,0)	(1,3)	(2,2)

It is easy to check that the two pure-stratgey NE of this game are (L,c) and (R,b). The game also has a mixed-strategy NE that is given by (1/3 L + 2/3 R, c) but we will concentrate only on the pure-strategy subgame-perfect equilibria.

For the subgame-perfect equilibrium outcome, when we look at Player_1's decision at the first node, the only outcomes we need to consider from the final subgame are the NE that we found above. Player_1 gets a payoff of 2 if he plays X at the start of the game, no matter what the outcome is in the final subgame (of course, the final subgame will not actually be reached if Player1 plays X at the first stage). If he chooses E at the first node, he gets a payoff of 2 if the outcome of the final subgame is (L,c) and a payoff of 1 if the outcome of the final subgame is (R,b). So the pure-strategy subgame perfect equilibria of this game are: {(XR, b), (XL, c), (EL, c)}.

(b) The representation of the game in strategic-form is as follows:

$P_1 \backslash P_2$	a	b	c
EL	3,1	0,0	2,2
ER	0,0	1,3	2,2
XL	2,2	2,2	2,2
XR	2,2	2,2	2,2

(c) The pure-strategy NE are {(XL,b), (XR,b), (EL,c), (XL,c), (XR,c)}. Of these, from part (a), (XL,b) and (XR,c) are not subgame-perfect.

3. Lets refer to the two players as P1 and P2 and without loss of generality, lets assume that P1 moves first and makes an offer (x, y) to P2. Clearly, in equilibrium, we must have $y = 1 - x$.

(a) P2 will only accept a division that leaves him at least as well-off as his next-best option, which is a utility of 0.

This means:

$$\begin{aligned} u_2 &= (1-x) - \alpha x \geq 0 \\ \text{or,} \;\; x &\leq \frac{1}{1+\alpha} \end{aligned}$$

Since P1's utility is strictly increasing in x, in equilibrium $x = \frac{1}{1+\alpha}$ and $y = \frac{\alpha}{1+\alpha}$.

(b) For the backward-induction outcome, we start with the final subgame where P1 or P2 each have a probability of 1/2 of making the offer to the other player. Whichever player makes the offer must behave exactly as in part (a). The expected utilities of each player from this second-stage is:

$$\begin{aligned} EU &= 1/2 * 0 + 1/2 * (\frac{1}{1+\alpha} - \alpha * (\frac{\alpha}{1+\alpha})) \\ &= \frac{1-\alpha}{2} \end{aligned}$$

So in order to make P2 accept the offer in the first-stage, P1 must leave P2 with a utility in the first stage that is at least as high as his expected utility in the second-stage.

This means:

$$(1-x) - \alpha x \geq \frac{1-\alpha}{2}$$

$$\text{or, } x \leq 1/2$$

Since P1's utility is strictly increasing in x, P1 will offer exactly 1/2 to P2 to have him accept the first-stage offer.

The answer is not complete unless we check P1's own utility from having P2 accept in the first-stage or allowing the game to move to the second-stage.

When P1 offers 1/2 in the first-stage, his utility is $\frac{1-\alpha}{2}$. This is the same as his expected utility from the second-stage.

So we will have two types of equilibria in this game. In the first type of equilibrium, P1 offers (x, y) with $x = y = 1/2$ to P2 in the first-stage, in which case P2 will accept his offer in the first-stage and P1 and P2 both earn $\frac{1-\alpha}{2}$. The second type of equilibria will have P1 offering any (x, y) such that $x > 1/2$ and letting the game move to the second-stage, where both players earn $\frac{1-\alpha}{2}$ in expectation.

(c) When $u_2 = (1-x) - \alpha x \geq 0$, P2 will accept any offer between 0 & 1 so P1 maximizes his utility by offering 0 to P2.

When the game is a two-period game as in part (b), the player making the offer in the second stage will keep the entire dollar for himself. The utility to a player when he keeps the dollar is 1 and the utility to a player is α, if he is offered 0 by the other player. It follows that to leave P2 indifferent between accepting or rejecting the offer in the first-stage, P1 must offer $(x, 1-x)$ such that:

$$x + \alpha(1-x) \geq \frac{1+\alpha}{2}$$

$$\text{or, } x \leq 1/2$$

Like before, since P1's utility is strictly increasing in x, P1 will offer exactly 1/2 to P2 to have him accept the first-stage offer.

Once again, the answer is not complete unless we check P1's own utility from having P2 accept his offer in the first-stage or allowing the game to move to the second-stage.

When P1 offers 1/2 in the first-stage, his utility is $\frac{1+\alpha}{2}$. Again, this is equal to his expected utility from the second-stage.

So, like in part (b), we will have two types of equilibria in this game. In the first type of equilibrium, P1 offers (x, y) with $x = y = 1/2$ to P2 in the first-stage, in which case P2 will accept his offer in the first-stage and P1 and P2 both earn $\frac{1+\alpha}{2}$. The second type of equilibria will have P1 offering any (x, y) such that $x > 1/2$ and letting the game move to the second-stage, where both players earn $\frac{1+\alpha}{2}$ in expectation.

4. (a) Note that in this game, given any bid b_j from Player$_j$, there are two ways for Player$_i$ to earn a positive payoff. The first way is to bid a bid $b_i > b_j$ and win $100 - b_i$ and the second way is to tie with Player$_j$'s bid and earn $\frac{100-b_j}{2}$. Thus, the best response (BR) of Player$_i$ to a bid b_j from Player$_j$, is to bid

$$b_i = b_j + 1, \qquad \text{if } 100 - b_j - 1 > \frac{100 - b_j}{2} \tag{1}$$

$$b_i = b_j + 1 \text{ OR } b_j, \qquad \text{if } 100 - b_j - 1 = \frac{100 - b_j}{2} \tag{2}$$

(1) is satisfied whenever $b_j < 98$. When $b_j = 98$, Player$_i$'s best response is either 98 or 99 since they both yield a utility of 1.

Now we can proceed with the iterative process to eliminate strictly dominated strategies.

Round 1: BR$_i$ $\{0, 1, 2, ..., 99\} = \{1, 2, 3, ..., 99\}$

If we assume that Player$_1$ is rational, he will never bid 0 as this is never a best reponse to any bid b_2 from Player$_2$. So we eliminate 0.

Round 2: BR$_i$ $\{1, 2, 3, ..., 99\} = \{2, 3, 4, ..., 99\}$

If we assume that Player$_2$ knows that Player$_1$ is rational, Player$_2$ knows that Player$_1$ will never play 0. A bid of 1 is a best response from Player$_2$ only if Player$_1$ bids 0. If we assume that Player$_2$ is rational, he will never play 1. So we eliminate 1.

Round 3: BR$_i$ $\{2, 3, 4, ..., 99\} = \{3, 4, 5, ..., 99\}$

If we assume that Player$_1$ knows that Player$_2$ knows that Player$_1$ is rational and also that Player$_1$ knows that Player$_2$ is rational, Player$_1$ knows that Player$_2$ will never play 1. Since a bid of 2 by Player$_1$ is a best reponse only to a bid of 1 by Player$_2$, Player$_1$ will never bid 2. So we eliminate 2.

In this way, we can iteratively eliminate all strictly dominated strategies for both player and the only set of rationalizable strategies for both players is $b_i = \{98, 99\}$, $i = 1, 2$. Note that from (1) and (2), a bid of 98 is a best response to both 97 and 98 so that we cannot eliminate it.

(b) The Nash equilibrium in pure strategies will involve the only rationalizable strategies left after the elimination in part (a): $\{b_i^*, b_j^*\} = [\{99, 99\}, \{98, 98\}]$