14.12 Game Theory – Midterm II - SOLUTIONS
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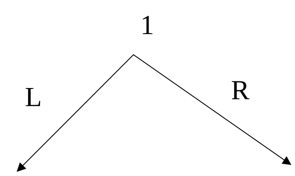
Instructions. This is an open book exam; you can use any written material. You have one hour and 20 minutes. Each question is 35 points. Good luck!

THERE WAS A TYPO IN PROBLEM 3, PART (b), WHICH PASSED UNNOTICED 5 INDEPENDENT AUDITS. [The functional form is supposed to be $x_i = a(1-c_i)^3 + b$ instead of $x_i = a(1-c_i)^2 + b$.] I CAN SEE THE NEGATIVE IMPACT OF THIS ON YOUR PERFORMANCE ON PARTS 3(b) AND 3(c). YOUR EXAM WILL BE GRADED ON THE BASIS OF YOUR WORK UNTIL PROBLEM 3, PART (a). I REARRANGE THE POINTS SO THAT THE TOTAL SCORE WILL BE 100. I WILL GIVE BONUSES FOR YOUR WORK ON PROBLEM 3 PARTS (b) AND (c), WHICH MAY ADD UP TO 25 POINTS. SINCE I DON'T WANT THE BONUSES TO AFFECT THE CURVE, I WILL ADD THE BONUS POINTS (WEIGHTED BY THE WEIGHT OF THE MIDTERM) AFTER THE TRESHOLDS FOR LETTER GRADES ARE DETERMINED — SIMILARLY TO THE QUIZ GRADES.

1. (35).Use a forward induction argument to predict the outcome of the following game, where player 1 first chooses between the two simultaneous-move subgames. (Your reasoning must be sound and based on rationality assumptions that are clearly stated.)



	h	t
H	2,1	1,2
T	1,2	2,1

	g	b
G	3,3	0,0
B	0,0	1,1

Let us assume that player 2 is sequentially rational and has a strong belief in sequentially rationality of player 1. Then, if player 1 plays R at the starting node, then player 2 must rule out the possibility that player 1 intends to play B in following subgame; because this would yield a payoff of at most 1 to player 1 and by choosing L and then $\frac{1}{2}H + \frac{1}{2}T$, she could have had an expected payoff of $1\frac{1}{2}$ (regardless of player 2's strategy). If player 1 does not play B in the subgame, i.e. if player 1 plays G, then player 2 should play g.

Now, let us assume that player 1 is sequentially rational and has a strong belief in the assumptions previously made. Then player 1 knows that by playing R, she would obtain a payoff of 3 which is more than the expected payoff from playing L. Therefore, if player 1 is sequentially rational, she would play R at the starting node.

Thus, if we assume that both players are sequentially rational and that player 2 has a correct strong belief in the sequential rationality of player 1 and that player 1 has a correct strong belief in these preceding assumptions, then we can predict that, in the game above, player 1 first plays R and the two players then play (G, g).

2. Below, there are pairs of stage games and strategy profiles. For each pair, check whether the strategy profile is a subgame-perfect equilibrium of the game in which the stage game is repeated infinitely many times. Each agent tries to maximize the discounted sum of his expected payoffs in the stage game, and the discount rate is $\delta = 0.99$. (Clearly explain your reasoning in each case.)

   (a) (15) **Stage Game:** There are $n > 2$ players. Each player, simultaneously, decides whether to contribute \$1 to a public good production project. The amount of public good produced is $y = (x_1 + \cdots + x_n)/2$, where $x_i \in \{0, 1\}$ is the level of contribution for player i. The payoff of a player i is $y - x_i$.

   Strategy profile: Each player contributes, choosing $x_i = 1$, if and only if the amount of public good produced at each previous date is greater than $n/4$; otherwise each chooses $x_i = 0$. (According to this strategy profile, each player contributes in the first period.)

   Answer: If all the other players are playing according to this strategy, player i has a single deviation which is profitable. He can deviate to not contribute at $t = 0$ and have a higher utility. Indeed whether he contributes or not, the future will not be affected as $y(t = 0) > \frac{n}{4}$ in the two cases.

$$\begin{aligned} \text{if } x_i &= 1,\ y = \frac{n}{2} > \frac{n}{4} \\ \text{if } x_i &= 0,\ y = \frac{n-1}{2} > \frac{n}{4} \end{aligned}$$

   Providing $x_i = 0$ at $t = 0$ gives player i a higher payoff at $t = 0$ than $x_i = 1$ and the same payoff in all the next periods. This strategy profile is not a subgame-perfect equilibrium.

   (b) (15) **Stage Game:** Linear Cournot Duopoly: There are two firms. Simultaneously each firm i supplies $q_i \geq 0$ units of a good, which is sold at price $P = \max\{1 - (q_1 + q_2), 0\}$. The cost is equal to zero.

   Strategy profile: There are two modes: Cartel and Competition. The game starts at Cartel mode. In Cartel mode, each supplies $q_i = 1/4$. In Cartel mode, if each supplies $q_i = 1/4$, they remain in Cartel mode in the next period; otherwise they switch to Competition mode in the next period. In Competition mode, each supplies $q_i = 1/2$. In Competition mode, they automatically switch to Cartel mode in the next period.

   Answer: Look at one deviation of firm i in the Competition mode. In that mode, they automatically switch back to the Cartel mode whatever they do. The firms should be on their best-response curve, otherwise they can reach a higher current profit without changing the future profits. If firm j is playing $q_j = \frac{1}{2}$, the best response of firm i is $q_i = \frac{1}{4}$ as

$$\frac{1}{4} = \arg\max_{q_i} \ q_i(1 - \frac{1}{2} - q_i)$$

Hence firm i has a profitable single deviation which is in the Competition mode, play $q_i = \frac{1}{4}$.

(c) (15) **Stage Game:** Linear Cournot Duopoly of part (b).

Strategy profile: There are two modes: Cartel and Competition. The game starts at Cartel mode. In Cartel mode, each supplies $q_i = 1/4$. In Cartel mode, if each supplies $q_i = 1/4$, they remain in Cartel mode in the next period; otherwise they switch to Competition mode in the next period. In Competition mode, each supplies $q_i = 1/2$. In Competition mode, they switch to Cartel mode in the next period if and only if both supply $q_i = 1/2$; otherwise they remain in Competition mode in the next period, too.

Answer: Let's check that none of the single deviations of firm i is profitable.

1. If period t is in the Cartel mode, by following the strategy firm i gets

$$\frac{1}{8} + \delta\frac{1}{8} + \delta^2\frac{1}{8} + \delta^3\frac{1}{8} + \delta^4\frac{1}{8} + ...$$

as the firms produce $(\frac{1}{4}, \frac{1}{4})$ forever. If firm i deviates, whatever his deviation they switch to the Competition mode in period $t+1$. Hence let's check that the most profitable deviation is not profitable, i.e. that firm i does not want to deviate to $q_i = BR(\frac{1}{4}) = \frac{3}{8}$

$$\frac{3}{8} = \arg\max_{q_i} \;\; q_i(1 - \frac{1}{4} - q_i)$$

In this single deviation the firm i produces $(\frac{3}{8}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, ...)$ and firm j produces $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, ...)$. The firm i gets

$$\frac{9}{64} + \delta 0 + \delta^2\frac{1}{8} + \delta^3\frac{1}{8} + \delta^4\frac{1}{8} + ...$$

Therefore the deviation is not profitable iff

$$\begin{aligned} \frac{1}{8} + \delta\frac{1}{8} &> \frac{9}{64} + \delta 0 \\ \Leftrightarrow \delta &> \frac{1}{8} \end{aligned}$$

2. If period t is in the Competition mode, by following the strategy firm i gets

$$0 + \delta\frac{1}{8} + \delta^2\frac{1}{8} + \delta^3\frac{1}{8} + \delta^4\frac{1}{8} + ...$$

as the firms produce $(\frac{1}{2}, \frac{1}{2})$ at t and $(\frac{1}{4}, \frac{1}{4})$ thereafter. If firm i deviates, whatever his deviation they stay in the Competition mode in period $t+1$. Hence let's check that the most profitable deviation is not profitable, i.e. that firm i does not want to deviate to $q_i = BR(\frac{1}{2}) = \frac{1}{4}$

$$\frac{1}{4} = \arg\max_{q_i} \;\; q_i(1 - \frac{1}{2} - q_i)$$

In this single deviation the firm i produces $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, ...)$ and firm j produces $(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, ...)$. The firm i gets

$$\frac{1}{16} + \delta 0 + \delta^2 \frac{1}{8} + \delta^3 \frac{1}{8} + \delta^4 \frac{1}{8} + ...$$

Therefore the deviation is not profitable iff

$$\begin{aligned} 0 + \delta \frac{1}{8} &> \frac{1}{16} + \delta 0 \\ \Leftrightarrow \delta &> \frac{1}{2} \end{aligned}$$

3. This strategy profile is a subgame-perfect equilibrium.

3. Consider two pharmaceutical companies, who are competing to develop a new drug, called Xenodyne. Simultaneously, each firm i invests x_i amount of money in R&D. The firm that invests more develops the drug first; if they invest equal amounts, then each firm is equally likely to develop the drug first. (The probability that they develop the drug at the same time is zero.) The firm that develops the drug first obtains a patent for the drug and becomes a monopolist in the market for Xenodyne. The other firm ceases to exist, obtaining the payoff of zero, minus its investment in R&D. The monopolist then produces $Q \geq 0$ units of Xenodyne at marginal cost c_i and sells it at price $P = \max\{1 - Q, 0\}$, obtaining payoff of $(P - c_i) Q$, minus its investment in R&D, where Q is chosen by the monopolist. Here, c_i is privately known by firm i, and c_1 and c_2 are independently and identically distributed by uniform distribution on $[0, 1]$.

   (a) (20) Write this game formally as a static Bayesian game.
   ANSWER:

   - Type space: $T_1 = T_2 = [0, 1]$.
   - Action space: $A_1 = A_2 = [0, \infty) \times [0, \infty)^{[0,\infty)\times[0,\infty)}$, where an action is a pair (x_i, Q_i), where Q_i is a function of x_1 and x_2.
   - $$u_i(x_1, Q_1, x_2, Q_2) = \begin{cases} P(Q_i(x_1, x_2)) Q_i(x_1, x_2) - x_i & \text{if } x_i > x_j \\ P(Q_i(x_1, x_2)) Q_i(x_1, x_2)/2 - x_i & \text{if } x_i = x_j \\ -x_i & \text{otherwise.} \end{cases}$$
   - $p_{c_j|c_i}$ is uniform distribution on $[0, 1]$.

   (b) (15) Find a symmetric Bayesian Nash equilibrium of the above game in which each player's investment is of the form $x_i = a(1 - c_i)^3 + b$ for some parameters a and b. [If you can, you may want to solve part (c) first.]
   ANSWER: See part (c).

   (c) (10) Show that the equilibrium in part (b) is the only Bayesian Nash equilibrium in which both firms act sequentially rationally and in which x_i is an increasing, differentiable function of $(1 - c_i)$.

ANSWER: By sequential rationality, a monopolist produces

$$Q_i = 1 - c_i/2$$

in order to maximize its profit, obtaining payoff of

$$(1 - c_i)^2 / 4,$$

minus the investment in R&D. Define new variable

$$\theta_i = 1 - c_i$$

which is also independently and identically distributed with uniform distribution on $[0, 1]$. Let x be the strategy played in a symmetric equilibrium, so that $x_1 = x(\theta_1)$ and $x_2 = x(\theta_2)$. Now, the expected payoff of firm i is

$$E[u_i] = \frac{\theta_i^2}{4} \Pr(x_i > x_j) - x_i.$$

This is because with probability $\Pr(x_i > x_j)$ the firm will become monopolist and get the monopoly profit $\theta_i^2/4$ and will pay the investment cost x_i with probability 1. Since x is increasing, $\Pr(x_i = x_j) = 0$. Now,

$$\Pr(x_i > x_j) = \Pr(x_i > x(\theta_j)) = \Pr\left(\theta_j < x^{-1}(x_i)\right) = x^{-1}(x_i).$$

Hence,

$$E[u_i] = \frac{\theta_i^2}{4} x^{-1}(x_i) - x_i.$$

Therefore, the first-order condition for maximization is

$$0 = \frac{\partial E[u_i]}{\partial x_i} = \frac{\theta_i^2}{4} \frac{1}{x'(\theta_i)} - 1,$$

showing that

$$x'(\theta_i) = \frac{\theta_i^2}{4},$$

and therefore

$$x(\theta_i) = \frac{\theta_i^3}{12} + const,$$

where the $const = 0$, so that $x(0) = 0$.