14.12 Game Theory
Fall 2005

Answers to Midterm 1, Fall 2005

Answer to Problem 1

a) Player 1 has the same payoff function in both games, so player 1 trivially has the same preference relation over lotteries with strategy profiles as their outcomes.

What about player 2? In other words, are the payoffs for player 2 in the game on the right a nonnegative affine transformation of the payoffs in the game on the left? In yet other words, do there exist $a \geqslant 0$ and b with $0a + b = 0$, $1a + b = 1$, $4a + b = 3$, and $2a + b = 2$? You can see that we'd need $a = 1$ and $b = 0$ in order to satisfy the first two equations, but this does not satisfy the third equation. So there is no such transformation, and player 2 does not have the same preference relation over lotteries with strategy profiles as their outcomes.

b) Are the payoffs for player 1 in the game on the right a nonnegative affine transformation of the payoffs in the game on the left? In other words, do there exist $a \geqslant 0$ and b with $0a+b = 1$, $6a+b = 4$, $2a+b = 2$, $4a+b = 3$, $4a+b = 3$, and $2a + b = 2$? Yes, you can solve the equations and see that $a = 1/2$ and $b = 1$ are such an a and b.

Are the payoffs for player 2 in the game on the right a nonnegative affine transformation of the payoffs in the game on the left? In other words, do there exist $a \geqslant 0$ and b with $1a+b = 0$, $4a+b = 1$, $4a+b = 1$, $7a+b = 2$, $-2a+b = -1$, and $1a + b = 0$? Yes, you can solve the equations and see that $a = 1/3$ and $b = -1/3$ are such an a and b.

So yes, both players have the same preference relation on lotteries with strategy profiles as their outcomes.

Answer to Problem 2

For player 2 playing M strictly dominates playing L. As a rational player, player 2 will never play L. Knowing that player 2 is rational, player 1 will never play B since A strictly dominates B in the remaining game. Then we are left with the following game:

	M	R
A	4,1	1,0
C	2,0	2,2

In this reduced game, the pure Nash equilibria are obvious: (A, M) and (C, R).

Now, let's look at the mixed equlibrium. Let $P(A) = p$ and $P(C) = 1 - p$ for player 1 and $P(M) = q$ and $P(R) = 1 - q$ for player 2. Then the conditions that these probabilities have to satisfy are

$$\begin{aligned} 4 * q + 1 * (1 - q) &= 2 \\ 1 * p + 0 * (1 - p) &= 0 * p + 2 * (1 - p) \end{aligned}$$

As a result we get

$$\begin{aligned} q &= 1/3 \\ p &= 2/3 \end{aligned}$$

Then the mixed strategy Nash equlibrium is

$$\left(\frac{2}{3}A + \frac{1}{3}C,\ \frac{1}{3}M + \frac{2}{3}R\right)$$

and the pure strategy NE are

$$(A, M) \text{ and } (C, R).$$

Answer to Problem 3

(a) The backwards induction outcome is as below. We first eliminate action y for player 2, by assuming that player 2 is sequentially rational and hence will not play y, which is conditionally dominated by x. We also eliminate action r for player 1, assuming that player 1 is sequentially rational. This is because r is conditionally dominated by l. Second, assuming that player 2 is sequentially rational and that player 2 knows that player 1 is sequentially rational, we eliminate b and c. This is because, knowing that player 1 is sequentially rational, player 2 would know that 1 will not play r, and hence b would lead to payoff of 0, and that by playing c would lead to a payoff of 1. Being sequentially rational she must play a. Finally, assuming that (i) player 1 is sequentially rational, (ii) player 1 knows that player 2 is sequentially rational, and (iii) player 1 knows that player 2 knows that player 1 is sequentially rational, we eliminate L. This is because (ii) and (iii) lead player 1 to conclude that 2 will play a and x, and thus by (i) he plays R.

(b) Player 1 has 4 strategies while player 2 has 6 (named by the actions to be chosen).

	ax	*ay*	*bx*	*by*	*cx*	*cy*
Ll	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1
Lr	0, 1	0, 1	0, 1	0, 1	0, 1	0, 1
Rl	1, 2	1, 2	2, 0	2, 0	1, 1	1, 0
Rr	1, 2	1, 2	−1, 4	−1, 4	1, 1	1, 0

(c) Compute the set of all rationalizable strategies.

First, Ll and Lr are strictly dominated by strategy Rl. Assuming that player 1 is rational, we conclude that he would not play Ll and Lr. We eliminate Ll and Lr, so the game is reduced to

	ax	ay	bx	by	cx	cy
Rl	$1,2$	$1,2$	$2,0$	$2,0$	$1,1$	$1,0$
Rr	$1,2$	$1,2$	$-1,4$	$-1,4$	$1,1$	$1,0$

Now for player 2 cx and cy are strictly dominated by ax. Hence, assuming that (i) player 2 is rational, and that (ii) player 2 knows that player 1 is rational, we eliminate cx and cy. This is because, by (ii), 2 knows that 1 will not play Ll and Lr, and hence by (i) she would not play cx and cy. The game is reduced to

	ax	ay	bx	by
Rl	$1,2$	$1,2$	$2,0$	$2,0$
Rr	$1,2$	$1,2$	$-1,4$	$-1,4$

There is no strictly dominated strategy in the remaining game. Therefore, all the remaining strategies are rationalizable.

Answer to Problem 4

Denote by y the winner of the first round of voting (either the bill or the amendment). In the second round of voting between 0.6 and y, the Moderates will vote for whichever is closer to 0.5; the Democrats will vote for the higher tax rate and the Republicans will vote for the lower tax rate. Since the Democrats and the Republicans will always back different proposals, the winner will be whichever is also backed by the moderates. We can denote the winner of the second round of voting by $f(y)$ defined as follows:

$$\begin{aligned} f(y) &= 0.6 \text{ if } y > 0.6 \text{ or } y < 0.4 \\ &= y \text{ if } y \in [0.4, 0.6] \end{aligned}$$

For the first round of voting, the moderates will choose whichever of x_1 and x_2 will cause the outcome of the second round of voting to be closer to 0.5; i.e. they choose $\arg\min_{x\in\{x_1,x_2\}} |0.5 - f(x)|$. Similarly, the Democrats will choose $\arg\max_{x\in\{x_1,x_2\}} f(x)$. And the Republicans will choose $\arg\min_{x\in\{x_1,x_2\}} f(x)$. Again, since the Democrats and the Republicans will always back different proposals in the first round, the winner will be whichever is also backed by the moderates; therefore $y = \arg\min_{x\in\{x_1,x_2\}} |0.5 - f(x)|$. Then, given x_1, the optimal choice of x_2 for the Democrats is $\min\{0.6, 0.5 + |0.5 - x_1|\}$; i.e. if x_1 is larger than is 0.4, they would choose x_2 to be as large as possible while being closer to 0.5 than is x_1 (so that the moderates back x_2); otherwise they choose 0.6. Then, if the Republicans choose x_1 to be smaller than 0.5 in introducing the bill, the Democrats will introduce an amendment with a higher tax rate, that the moderates back in both rounds. So the best that the Republicans can do is to choose $x_1 = 0.5$.

Thus, a full description of the strategies of the three parties in equilibrium are as follows: Republicans choose $x_1 = 0.5$; in the first round, they vote for $\arg\min_{x\in\{x_1,x_2\}} f(x)$, and in the second round they vote for $\min\{0.6, y\}$; Democrats choose $x_2 = \min\{0.6, 0.5 + |0.5 - x_1|\}$; in the first round, they vote for $\arg\max_{x\in\{x_1,x_2\}} f(x)$ and in the second round they vote for $\max\{0.6, y\}$; as for the moderates, in the first round they vote for $\arg\min_{x\in\{x_1,x_2\}} |0.5 - f(x)|$, and in the second round they vote for $\arg\min_{z\in\{0.6,y\}} |0.5 - z|$.