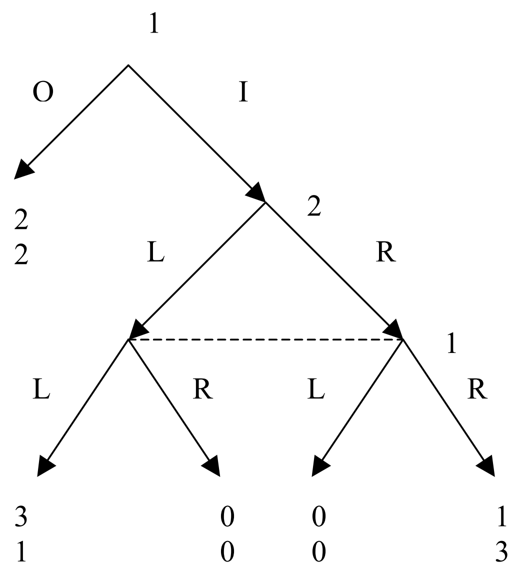
**14.12 Game Theory – Final**

**Instructions.** This is an open book exam; you can use any written material. You have 2 hours 50 minutes. Each question is 20 points. Good luck!

1. Consider the following extensive form game.




   (a) Find the normal form representation of this game.
   **A:**

| | L | R |
|---|---|---|
| OL | 2,2 | 2,2 |
| OR | 2,2 | 2,2 |
| IL | 3,1 | 0,0 |
| IR | 0,0 | 1,3 |

   (b) Find all rationalizable pure strategies.

| | L | R |
|---|---|---|
| OL | 2,2 | 2,2 |
| OR | 2,2 | 2,2 |
| IL | 3,1 | 0,0 |

   (c) Find all pure strategy Nash equilibria.

| | L | R |
|---|---|---|
| OL | **2,2** | **2,2** |
| OR | **2,2** | **2,2** |
| IL | **3,1** | 0,0 |

(d) Which strategies are consistent with all of the following assumptions?

  (i) 1 is rational.
  (ii) 2 is sequentially rational.
  (iii) at the node she moves, 2 knows (i).
  (iv) 1 knows (ii) and (iii).

  **ANSWER:** By (i) 1 does not play IR. Hence, by (iii), at the node she moves, 2 knows that 1 does not play IR, hence he knows IL. Then, by (ii), 2 must play L. Therefore, by (i) and (iv), 1 must play IL. The answer is (IL,L).

2. This question is about a milkman and a customer. At any day, with the given order,

- Milkman puts $m \in [0,1]$ liter of milk and $1-m$ liter of water in a container and closes the container, incurring cost $cm$ for some $c > 0$;
- Customer, without knowing $m$, decides on whether or not to buy the liquid at some price $p$. If she buys, her payoff is $vm - p$ and the milkman's payoff is $p - cm$. If she does not buy, she gets 0, and the milkman gets $-cm$. If she buys, then she learns $m$.

(a) Assume that this is repeated for 100 days, and each player tries to maximize the sum of his or her stage payoffs. Find all subgame-perfect equilibria of this game.

  **ANSWER:** The stage game has a unique Nash equilibrium, in which $m = 0$ and the customer does not buy. Therefore, this finitely repeated game has a unique subgame-perfect equilibrium, in which the stage equilibrium is repeated.

(b) Now assume that this is repeated infinitely many times and each player tries to maximize the discounted sum of his or her stage payoffs, where discount rate is $\delta \in (0,1)$. What is the range of prices $p$ for which there exists a subgame perfect equilibrium such that, everyday, the milkman chooses $m = 1$, and the customer buys on the path of equilibrium play?

  **ANSWER:** The milkman can guarantee himself 0 by always choosing $m = 0$. Hence, his continuation value at any history must be at least 0. Hence, in the worst equilibrium, if he deviates customer should not buy milk forever, giving the milkman exactly 0 as the continuation value. Hence, the SPE we are looking for is *the milkman always chooses m=1 and the customer buys until anyone deviates, and the milkman chooses m=0 and the customer does not buy thereafter.* If the milkman does not deviate, his continuation value will be

$$V = \frac{p-c}{1-\delta}.$$

  The best deviation for him (at any history on the path of equilibrium play) is to choose $m = 0$ (and not being able to sell thereafter). In that case, he will get

$$V_d = p + \delta 0 = p.$$

  In order this to be an equilibrium, we must have $V \geq V_d$; i.e.,

$$\frac{p-c}{1-\delta} \geq p,$$

i.e.,

$$p \geq \frac{c}{\delta}.$$

In order that the customer buy on the equilibrium path, we must also have $p \leq v$. Therefore,

$$v \geq p \geq \frac{c}{\delta}.$$

3. For the game in question 2.a, assume that with probability 0.001, milkman strongly believes that there is some entity who knows what the milkman does and will punish him severely on the day 101 for each day the milkman dilutes the milk (by choosing $m < 1$). Call this type irrational. Assume that this is common knowledge. For some $v > p > c$, find a perfect Bayesian equilibrium of this game. [If you find it easier, take the customers at different dates different, but assume that each customer knows whatever the previous customers knew.]

   **ANSWER: [It is very difficult to give a rigorous answer to this question, so you would get a big partial grade for an informal answer that shows that you understand the reputation from an incomplete-information point of view.]** Irrational type always sets $m = 1$. Since he will be detected whenever he sets $m < 1$ and the customer buys, the rational type will set either $m = 1$ or $m = 0$. We are looking for an equilibrium in which early in the relation the rational milkman will always set $m = 1$ and the customer will always buy, but near the end of the relation the rational milkman will mix between $m = 1$ and $m = 0$, and the customer will mix between buy and not buy.

   In this equilibrium, if the milkman sets $m < 1$ or the customer does not buy at any $t$, then the rational milkman sets $m = 0$ at each $s > t$. In that case, if in addition the costumer buys at some dates in the period $\{t+1, t+2, \ldots, s-1\}$ and if the milkman chooses $m = 1$ at each of those days, then the costumer will assign probability 1 to that the milkman is irrational and buy the milk at $s$; otherwise, he will not buy the milk. On the path of such play, if the milkman sets $m < 1$ or the customer does not buy at any $t$, then the rational milkman sets $m = 0$ and the costumer does not buy at each $s > t$. In order this to be an equilibrium, the probability $\mu_t$ that the milkman is irrational at such history must satisfy

$$\mu_t (100 - t)(v - p) - (1 - \mu_t) p \leq 0,$$

   where the first term is the expected benefit from experimenting (if the milkman happens to be irrational) and the second term is the cost (if he is rational). That is,

$$\mu_t \leq \frac{p}{p + (100 - t)(v - p)}.$$

   Now we determine what happens if the milkman has always been setting $m = 1$, and the customer has been buying. In the last date, the rational type will set $m = 0$, and the rational type will set $m = 1$; hence, the buyer will buy iff

$$\mu_{100} (v - p) - (1 - \mu_{100}) p \geq 0,$$

i.e.,

$$\mu_{100} \geq \frac{p}{v}.$$

Since we want him to mix, we set

$$\mu_{100} = \frac{p}{v}.$$

We derive $\mu_t$ for previous dates using the Bayes' rule and the indifference condition necessary for the customer's mixing. Let's write $\alpha_t$ for the probability that the rational milkman sets $m = 1$ at $t$, and $a_t = \mu_t + (1 - \mu_t)\,\alpha_t$ for the total probability that $m = 1$ at date $t$. Since the customer will be indifferent between buying and not buying at $t+1$, his expected payoff at $t+1$ will be 0. Hence, his expected payoff from buying at $t$ is

$$a_t\,(v - p) + (1 - a_t)\,(-p)\,.$$

For indifference, this must be equal to zero, thus

$$a_t = \frac{p}{v}.$$

On the other hand, by Bayes' rule,

$$\mu_{t+1} = \frac{\mu_t}{a_t}.$$

Therefore,

$$\mu_t = a_t \mu_{t+1} = \frac{p}{v}\mu_{t+1}.$$

That is,

$$\begin{aligned}
\mu_{100} &= \frac{p}{v} \\
\mu_{99} &= \left(\frac{p}{v}\right)^2 \\
\mu_{98} &= \left(\frac{p}{v}\right)^3 \\
&\vdots
\end{aligned}$$

Note that

$$a_t = \frac{p}{v} = \mu_t + (1 - \mu_t)\,\alpha_t \Rightarrow \alpha_t = \frac{\frac{p}{v} - \mu_t}{1 - \mu_t}.$$

Assume that $(p/v)^{100} < 0.001$. Then, we will have a date $t^*$ such that

$$\left(\frac{p}{v}\right)^{101-t^*} < 0.001 < \left(\frac{p}{v}\right)^{100-t^*}.$$

At each date $t > t^*$, we will have $\mu_t = (p/v)^{101-t}$ and the players will mix so that $a_t = \frac{p}{v}$. At each date $t < t^*$, the milkman will set $m = 1$ and the customer will buy. At date $t^*$, the rational milkman will mix so that

$$\left(\frac{p}{v}\right)^{100-t^*} = \mu_{t^*+1} = \frac{\mu_{t^*}}{a_{t^*}} = \frac{0.001}{a_{t^*}},$$

hence

$$a_{t^*} = \frac{0.001}{\left(\frac{p}{v}\right)^{100-t^*}}.$$

Note that $a_{t^*} > p/v$, hence the customer will certainly buy at $t^*$.

Let's write $\beta_t$ for the probability that the customer will buy at day $t$. In the day 99, if the rational milkman sets $m = 1$, he will get

$$U = \beta_{99} (p - c) + \beta_{99}\beta_{100}p + (1 - \beta_{99}) (-c),$$

where the first term is the profit from selling at day 99, the second term is the profit from day 100 (when he will set $m = 0$), and the last term is the loss if the customer does not buy at day 99. If he sets $m = 0$, he will get $\beta_{99}p$ (from the sale at 99, and will get zero thereafter). Hence, he will set $m = 1$ iff

$$\beta_{99} (p - c) + \beta_{99}\beta_{100}p + (1 - \beta_{99}) (-c) \geq \beta_{99}p$$

i.e.,

$$\beta_{99}\beta_{100} \geq \frac{c}{p}.$$

We are looking for an indifference, hence we set

$$\beta_{99}\beta_{100} = \frac{c}{p}.$$

Similarly, at day 98 the rational milkman will set $m = 1$ iff

$$\beta_{98} (p - c) + \beta_{98}\beta_{99}p + (1 - \beta_{98}) (-c) \geq \beta_{98}p,$$

where the second term is due to the fact that at date 99 he will be indifferent between choosing $m = 0$ and $m = 1$. For indifference, we set

$$\beta_{98}\beta_{99} = \frac{c}{p}.$$

We will continue on like this as long as we need the milkman to mix. That is, we will have

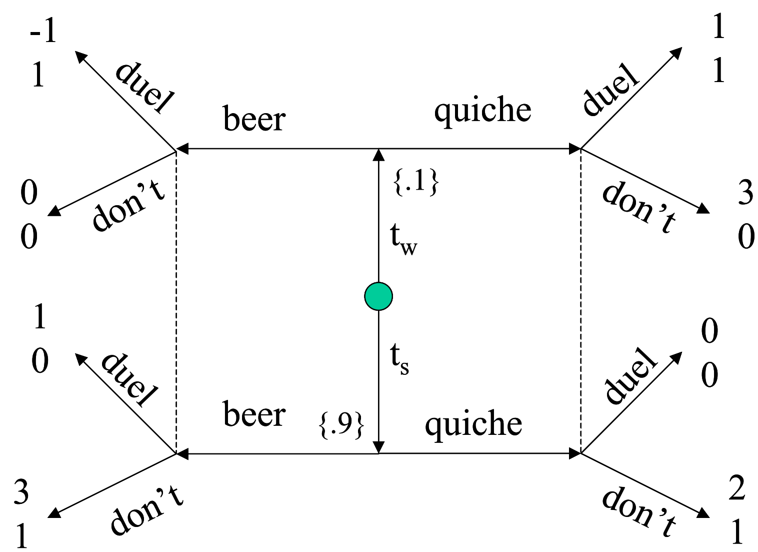
$$\begin{aligned}
\beta_{t^*}\beta_{t^*+1} &= \frac{c}{p}, \\
\beta_{t^*+1}\beta_{t^*+2} &= \frac{c}{p}, \\
&\vdots \\
\beta_{98}\beta_{99} &= \frac{c}{p}, \\
\beta_{99}\beta_{100} &= \frac{c}{p}.
\end{aligned}$$

As we noted before, $\beta_{t^*} = 1$. Hence, $\beta_{t^*+1} = \frac{c}{p}$. Hence, $\beta_{t^*+2} = 1$, ... That is,

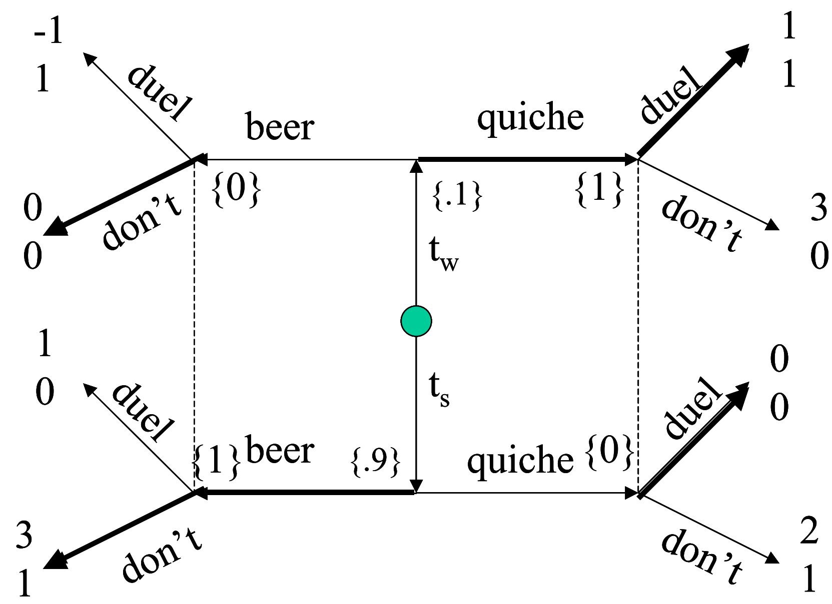
$$\begin{aligned}
\beta_{t^*} &= 1, \\
\beta_{t^*+1} &= \frac{c}{p}, \\
\beta_{t^*+2} &= 1, \\
\beta_{t^*+3} &= \frac{c}{p}, \\
&\vdots
\end{aligned}$$

**Bonus:** [10 points] Discuss what would happen if the irrational type were known to dilute the milk by accident with some small but positive probability.

4. Find a perfect Bayesian equilibrium of the following game.




**ANSWER:**




5. A risk-neutral entrepreneur has a project that requires \$100,000 as an investment, and will yield \$300,000 with probability 1/2, \$0 with probability 1/2. There are two types of entrepreneurs: rich who has a wealth of \$1,000,000, and poor who has \$0. For some reason, the wealthy entrepreneur cannot use his wealth as an investment towards this project. There is also a bank that can lend money with interest rate $\pi$. That is, if the entrepreneur borrows \$100,000 to invest, after the project is completed he will pay back $\$100,000\,(1+\pi)$ — if he has that much money. If his wealth is less than this amount at the end of the project, he will pay all he has. The order of the events is as follows:

   - First, bank posts $\pi$.
   - Then, entrepreneur decides whether to borrow (\$100,000) and invest.
   - Then, uncertainty is resolved.

   (a) Compute the subgame perfect equilibrium for the case when the wealth is common knowledge.

   **ANSWER:** The rich entrepreneur is always going to pay back the loan in full amount, hence his expected payoff from investing (as a change from not investing) is

$$(0.5)(300,000) - 100,000\,(1+\pi)\,.$$

   Hence, he will invest iff this amount is non-negative, i.e.,

$$\pi \leq 1/2.$$

   Thus, the bank will set the interest rate at

$$\pi_R = 1/2.$$

The poor entrepreneur is going to pay back the loan only when the project succeeds. Hence, his expected payoff from investing is

$$(0.5)(300,000 - 100,000\,(1+\pi)).$$

He will invest iff this amount is non-negative, i.e.,

$$\pi \leq 2.$$

Thus, the bank will set the interest rate at

$$\pi_P = 2.$$

(b) Now assume that the bank does not know the wealth of the entrepreneur. The probability that the entrepreneur is rich is 1/4. Compute the perfect Bayesian equilibrium.

**ANSWER:** As in part (a), the rich type will invest iff $\pi \leq \pi_R = .5$, and the poor type will invest iff $\pi \leq \pi_P = 2$. Now, if $\pi \leq \pi_R$, the bank's payoff is

$$\begin{aligned} U(\pi) &= \frac{1}{4}100,000\,(1+\pi) + \frac{3}{4}\left[\frac{1}{2}100,000\,(1+\pi) + \frac{1}{2}0\right] - 100,000 \\ &= \frac{5}{8}100,000\,(1+\pi) - 100,000 \\ &\leq \frac{5}{8}100,000\,(1+\pi_R) - 100,000 \\ &= \frac{5}{8}100,000\,(1+1/2) - 100,000 = -\frac{1}{16}100,000 < 0. \end{aligned}$$

If $\pi_R < \pi \leq \pi_P$, the bank's payoff is

$$\begin{aligned} U(\pi) &= \frac{3}{4}\left[\frac{1}{2}100,000\,(1+\pi) + \frac{1}{2}0 - 100,000\right] \\ &= \frac{3}{8}100,000\,(\pi - 1)\,, \end{aligned}$$

which is maximized at $\pi_P$, yielding $\frac{3}{8}100,000$. If $\pi > \pi_P$, $U(\pi) = 0$. Hence, the bank will choose $\pi = \pi_P$.