14.12 Game Theory Lecture Notes
Lectures 15-18

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1 Static Games with Incomplete Information

So far we have focused on games in which any piece of information that is known by any player is known by all the players (and indeed common knowledge). Such games are called the games of complete information. Informational concerns do not play any role in such games. In real life, players always have some private information that is not known by other parties. For example, we can hardly know other players' preferences and beliefs as well as they do. Informational concerns play a central role in players' decision making in such strategic environments. In the rest of the course, we will focus on such informational issues. We will consider cases in which a party may have some information that is not known by some other party. Such games are called games of incomplete information or asymmetric information. The informational asymmetries are modeled by Nature's moves. Some players can distinguish certain moves of nature while some others cannot. Consider the following simple example, where a firm is contemplating the hiring of a worker, without knowing how able the worker is.

Example 1 *Consider the game in Figure 1. There are a Firm and a Worker. Worker can be of High ability, in which case he would like to Work when he is hired, or of Low ability, in which case he would rather Shirk. Firm would want to Hire the worker that will work but not the worker that will shirk. Worker knows his ability level. Firm does not know whether the worker is of high ability or low ability. Firm believes that the worker is of high ability with probability* p *and low ability with probability* $1-p$*. Most importantly, the firm knows that the worker knows his own ability level. To model this situation, we let Nature choose between High and Low, with probabilities* p *and* $1-p$*, respectively. We then let the worker observe the choice of Nature, but we do not let the firm observe Nature's choice.*

A player's private information is called his "type". For instance, in the above example Worker has two types: High and Low. Since Firm does not have any private information,

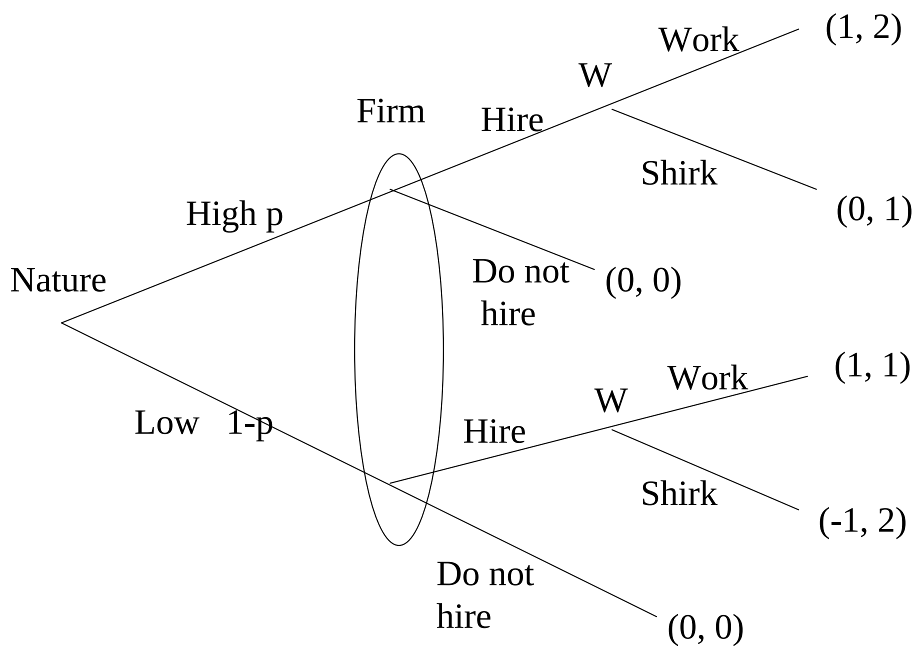


Figure 1:

Firm has only one type. As in the above example, incomplete information is modeled via imperfect-information games where Nature chooses each player's type and privately informs him. These games are called *incomplete-information game* or *Bayesian game*.

Formally, a static game with incomplete information is as follows. First, Nature chooses some $t = (t_1, t_2, \ldots, t_n) \in T$, where each $t \in T$ is selected with probability $p(t)$. Here, $t_i \in T_i$ is the type of player $i \in N = \{1, 2, \ldots, n\}$. Then, each player observes his own type, but not the others'. Finally, players simultaneously choose their actions, each player knowing his own type. We write $a = (a_1, a_2, \ldots, a_2) \in A$ for any list of actions taken by all the players, where $a_i \in A_i$ is the action taken by player i. The game is denoted by (N, T, A, p).

As usual, a strategy of a player determines which action he will take at each information set of his. Here, information sets are identified with types $t_i \in T_i$. Hence, a strategy of a player i is a function $s_i : T_i \to A_i$, mapping his types to his actions. For instance, in the example above, Worker has four strategies: (Work,Work)—meaning that he will work regardless of whether he is of high or low ability, (Work, Shirk)—meaning that he will work if he is of high ability and shirk if he is of low ability, (Shirk, Work), and (Shirk, Shirk).

A Bayesian Nash equilibrium is a Nash equilibrium of a Bayesian game. For instance, for $p > 1/2$, a Bayesian Nash equilibrium of the game in Example 1 is (Hire, (Work,Shirk)). That is, the firm hires the worker, and the worker works if he is of high ability and shirks otherwise. There is also another Nash equilibrium, where the worker chooses to Shirk regardless of his type, and the firm doesn't hire him.

Players' types may be "correlated", meaning that a player "updates" his beliefs about the other players' type when he learns his own type. Since he knows his own type when he takes his action, he maximizes his expected utility with respect to his updated beliefs. We assume that he updates his beliefs using Bayes' Rule.

Bayes' Rule Let A and B be two events, then probability that A occurs conditional on B occurring is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)},$$

where $P(A \cap B)$ is the probability that A and B occur simultaneously, and $P(B)$ is the (unconditional) probability that B occurs.

In static games of incomplete information, the application of Bayes' Rule will often be trivial, but as we move to study dynamic games of incomplete information, the importance of Bayes' Rule will increase.

Let $p_i(t'_{-i}|t_i)$ denote i's belief that the types of all other players is $t'_{-i} = (t'_1, t'_2, ..., t'_{i-1}, t'_{i+1}, \ldots, t'_n)$ given that his type is $t_{i.}$ [We may need to use Bayes' Rule if types across players are 'correlated'. But if they are independent, then life is simpler. In that case, players do not update their beliefs.]

We can now define Bayesian Nash Equilibrium. A strategy profile $s^* = (s^*_1, ..., s^*_n)$ is a *Bayesian Nash Equilibrium* in an n-person static game of incomplete information if and only if for each player i and type $t_i \in T_i$,

$$s^*_i(t_i) \in \arg\max_{a_i} \sum_{t_{-i}} u_i \left[s^*_1(t_1), ..., s^*_{i-1}(t_{i-1}), a_i, s^*_{i+1}(t_{i+1}), ..., s^*_n(t_n), (t_i, t_{-i}) \right] \times p_i(t_{-i}|t_i),$$

where u_i is the utility of player i, and a_i denotes his action. That is, for each player i, each possible type t_i chooses and action that is *optimal* given the *conditional beliefs* $p_i(t_{-i}|t_i)$ of that type against the other players' strategies. Notice that the utility function u_i of player i depends both players' actions and types.[1] Notice also that a Bayesian Nash equilibrium is a

[1]Utility function u_i does not depend the whole of strategies $s_1, \ldots, s_n$, but the expected value of u_i possibly does.

Nash equilibrium of a Bayesian game with the additional property that each type plays a best reply.[2]

Example 2 *Let's check that the strategy profile (Hire, (Work,Shirk)) is a Bayesian Nash equilibrium in the game of Example 1 whenever* $p > 1/2$. *Given the strategy (Work, Shirk) of the worker, the expected utility of the firm from hiring is*

$$\begin{aligned} & u_F(High, Hire, Work)\Pr(High) + u_F(Low, Hire, Shirk)\Pr(Low) \\ = \ & 1 \cdot p + (-1)(1-p) = 2p - 1. \end{aligned}$$

Given the strategy (Work, Shirk) of the worker, the expected utility of the firm from not hiring is

$$u_F(High,\ Do\ not\ Hire, Work)\Pr(High) + u_F(Low,\ Do\ not\ Hire, Shirk)\Pr(Low) = 0.$$

When $p > 1/2$, *we have* $2p - 1 > 0$, *and hence strategy Hire maximizes the firm's expected payoff. For the worker, we need to check optimality for each type separately. For the High type, we have*

$$u_W(High, Hire, Work) = 2 > 1 = u_W(High, Hire, Shirk),$$

as desired. For the Low type, we again have

$$u_W(Low, Hire, Shirk) = 2 > 1 = u_W(Low, Hire, Work).$$

As an exercise, check that (Do not Hire, (Shirk,Shirk)) is also a Bayesian Nash equilibrium. In the following example, both players have private information.

Example 3 *Consider the payoff matrix*

	L	R
X	θ, γ	$1, 2$
Y	$-1, \gamma$	$\theta, 0$

where $\theta \in \{0, 2\}$ *and* $\gamma \in \{1, 3\}$. *Each player knows his own payoff, i.e., player 1 knows the value of* θ, *and player 2 knows the value of* γ. *Independent of the value of* θ, *player 1 finds the both values of* γ *equally likely. Similarly, independent of the value of* γ, *player 2 finds both values of* θ *equally likely. In this game, each player has two types. Player 1 has types* 0 *and* 2, *while player 2 has types* 1 *and* 3. *All type profiles are equally likely, i.e.,*

[2] This property is necessarily satisfied in any Nash equilibrium if all types occur with positive probability.

$p(0,1) = p(0,3) = p(2,1) = p(2,3) = 1/4$. *Towards computing the Bayesian Nash equilibria, note that for type $\theta = 0$ of player 1, action X strictly dominates action Y, i.e., independent of what he thinks about what player 2 plays, it is a best response for player 1 to play X when his type is* 0. *Therefore, for any Bayesian Nash equilibrium s^*, we have*

$$s_1^*(0) = X.$$

Similarly, action L is strictly dominant for type $\gamma = 3$ of player 2, and hence for any Bayesian Nash equilibrium s^*, we have

$$s_2^*(3) = L.$$

We also need to determine the actions of type $\theta = 2$ of player 1 and type $\gamma = 1$ of player 2. Consider type $\theta = 2$ of player 1. For this type, action X is a best response iff probability that player 2 plays L is at least 1/4.[3] *But for type $\theta = 2$, in equilibrium, the probability of L is at least 1/2. To see this, let p be the probability that type $\gamma = 1$ plays L in a possibly mixed strategy equilibrium. Then,*

$$\Pr(s_2 = L) = \Pr(\gamma = 3) \cdot 1 + \Pr(\gamma = 1)\, p = 1/2 + p/2 \geq 1/2.$$

Hence, in any Bayesian Nas equilibrium s^*, type $\theta = 2$ of player 1 plays X, i.e.,

$$s_1^*(2) = X.$$

We finally need to determine the equilibrium action of type $\gamma = 1$. In any equilibrium, for tyepe $\gamma = 1$, the payoff from L is $\gamma = 1$. His expected payoff from R is

$$\begin{aligned} & u_2(s_1^*(0), R, 1)\Pr(\theta = 0) + u_2(s_1^*(2), R, 1)\Pr(\theta = 2) \\ = \quad & u_2(X, R, 1)\Pr(\theta = 0) + u_2(X, R, 1)\Pr(\theta = 2) = u_2(X, R, 1) = 2 > 1. \end{aligned}$$

Hence, in any Bayesian Nash equilibrium s^*, type $\gamma = 1$ plays R, i.e.,

$$s_2^*(1) = R.$$

We have just shown that there exists a unique Bayesian Nash equilibrium s^*, where $s_1^*(0) = s_1^*(2) = X$, $s_2^*(1) = R$, and $s_2^*(3) = L$.

[3] For $\theta = 2$, the expected payoff from X is

$$u_1(X, L, 2)\Pr(s_2 = L) + u_1(X, R, 2)(1 - \Pr(s_2 = L)) = 2\Pr(s_2 = L) + (1 - \Pr(s_2 = L)) = \Pr(s_2 = L) + 1,$$

while the payoff from Y is

$$u_1(Y, L, 2)\Pr(s_2 = L) + u_1(Y, R, 2)(1 - \Pr(s_2 = L)) = -\Pr(s_2 = L) + 2(1 - \Pr(s_2 = L)) = 2 - 3\Pr(s_2 = L).$$

The expected payoff $\Pr(s_2 = L) + 1$ from X is higher than that of Y (i.e. $2 - 3\Pr(s_2 = L)$) iff $\Pr(s_2 = L) > 1/4$.

The following are very sketchy notes about the applications in Gibbons.

Example: Cournot with Incomplete Information.

$P(Q) = a - Q$

$Q = q_1 + q_2$

$c_1(q_1) = cq_1$

Both firms Risk-Neutral

Firm 2's types (private information)

$c_2(q_2) = c_H q_2$ with probability θ

$c_L q_2$ with probability $1 - \theta$

common knowledge among players.

How to find the Bayesian Nash Equilibrium?

Firm 2 has two possible types; and different actions will be chosen for the two different types.

$\{q_2(c_L), q_2(c_H)\}$

Suppose firm 2 is type high.

$\Longrightarrow$

$$\max_{q_2}(P - c_H)q_2 = [a - q_1 - q_2 - c_H]\, q_2$$

given the action of player q_1.

$$\Longrightarrow q_2(c_H) = \frac{a - q_1 - c_H}{2} \qquad (*)$$

Similarly suppose firm 2 is low type:

$$\max_{q_2} [a - q_1 - q_2 - c_H]\, q_2$$

$$q_2(c_L) = \frac{a - q_1 - c_H}{2} \qquad (**)$$

Important Remark: The same level of q_1 in both cases. Why??

Firm 1's problem

$$\max_{q_1} \theta\, [a - q_1 - q_2(c_H) - c]\, q_1 + (1 - \theta)\, [a - q_1 - q_2(c_L) - c]\, q_1$$

$$q_1 = \frac{\theta\left[a - q_2(c_H) - c\right] + (1-\theta)\left[a - q_2(c_L) - c\right]}{2} \qquad (***)$$

Solve *, **, and *** for $q_1, q_2(c_L), q_2(c_H)$

$$q_2^*(c_H) = \frac{a - 2c_H + c}{3} + \frac{(1-\theta)(c_H - c_L)}{6}$$

$$q_2^*(c_L) = \frac{a - 2c_L + c}{3} + \frac{\theta(c_H - c_L)}{6}$$

$$q_1^* = \frac{a - 2c + \theta c_H + (1-\theta)c_L}{3}$$

Harsaryi's Justification for Mixed Strategies

	O	F
O	$2+t_1, 1$	$0, 0$
F	$0, 0$	$1, 2+t_2$

t_1, t_2 private information of players.

t_1, t_2 are independent draws from uniform distribution over $[0, X]$.

Harsanyi shows that as $X \longrightarrow 0$ (as uncertainty disappears), we converge to a mixed strategy equilibrium where 1 plays 0 with probability 2/3 and 2 plays F with probability 2/3. See Gibbons for details.

Auctions

Two bidders for a unique good.

v_i : valuation of bidder i.

Let us assume that v_i's are drawn independently from a uniform distribution over $[0, 1]$. v_i is player i's private information. The game takes the form of both bidders submitting a bid, then the highest bidder wins and pays her bid.

Let b_i be player i's bid.

$$v_i(b_1, b_2, v_1, v_2) = \begin{cases} v_i - b_i \text{ if } b_i > b_j \\ \frac{v_i - b_i}{2} \text{ if } b_i = b_j \\ 0 \text{ if } b_i < b_j \end{cases}$$

$$\max_{b_i}(v_i - b_i)Prob\{b_i > b_j(v_j)|\text{given beliefs of player i}) + \frac{1}{2}(v_i - b_i)Prob\{b_i = b_j(v_j)|...)$$

$\frac{1}{2}(v_i - b_i)Prob\{b_i = b_j(v_j)|...) = 0$ since a continuum of possibilities.

Let us first conjecture the form of the equilibrium: Conjecture: Symmetric and linear equilibrium

$$b = a + cv.$$

Then

$$\max_{b_i}(v_i - b_i)Prob\{b_i \geq a + cv_j\} =$$
$$(v_i - b_i)Prob\{v_j \leq \frac{b_i - a}{c}\} = (v_i - b_i) \cdot \frac{(b_i - a)}{c}$$

FOC:

$$\begin{aligned} b_i &= \frac{v_i + a}{2} \qquad \text{if} \quad v_i \geq a \\ &= a \qquad \text{if} \quad v_i < a \end{aligned} \tag{1}$$

A linear strategy is BR to a linear strategy only if $a = 0$

$$\Longrightarrow b_i = \frac{1}{2}v_i$$

$$b_i = \frac{1}{2}v_j$$

Double Auction

Seller names P_s

Buyer names P_b

$P_b < P_s$ no trade

$P_b \geq P_s$ trade at $p = \frac{P_b + P_s}{2}$

Valuations again private information.

V_b uniform over $(0, 1)$

V_s uniform over $(0, 1)$ and independent from V_b

Strategies $P_b(V_b)$ $P_s(V_s)$

The buyer maximizes

$$\max_{P_b}\left[V_b - \frac{P_b + E\{P_s(V_s)|P_b \geq P_s(V_s)\}}{2}\right] \times Prob\{P_b \geq P_s(V_s)\}$$

where $E(P_s(V_s)|P_b \geq P_s(V_s)$ expected seller bid *conditional* on P_b being greater than $P_s(V_s)$.

Similarly, the seller maximizes

$$\max_{P_s} [P_s + E\{P_b(V_b)|P_b(V_b) \geq P_s] - V_s] \times Prob\{P_b(V_b) \geq P_s\}$$

Equilibrium $\quad P_s(V_j)BR \quad$ to $\quad P_b(V_b)$

$P_b(V_b)BR \quad$ to $\quad P_s(V_s)$

Bayesian Nash Equilibria?

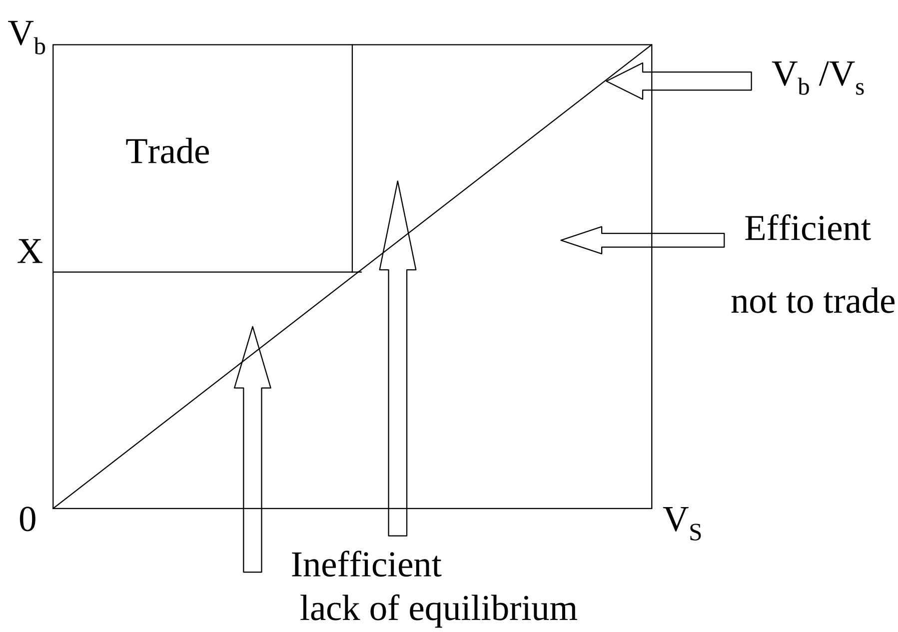
There are many: Let us construct some examples

1. Seller $P_s = X$ if $V_s \leq X$

$P_b = X$ if $V_b \geq X$

An equilibrium with "fixed" price.

Why is this an equilibrium? because given $P_s = X$ if $V_s \leq X$, the buyer does not want to trade with $V_b < X$ and with $V_b > X, P_b = X$ is optimal.



See Gibbons for other equilibria.