14.12 Game Theory Lecture Notes
Lectures 17-18

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1 Static Applications with Incomplete Information

These notes are about some economic applications with incomplete information. They are meant to illustrate the common techniques in computing Bayesian Nash equilibria in static games of incomplete information. These applications have been discussed in Gibbons' textbook. The notes illustrate the basic steps in analysis and fills some important details. We will consider three applications. In Cournot duopoly, I will explain how one computes the Bayesian Nash equilibria when there is a continuum of actions but finitely many types. The other two applications will be the first-price auction and double auction. In these applications, there will be a continuum of actions and a continuum of types. In that case, it is not easy to compute all equilibria, and one often considers equilibria with certain functional forms. Here, we will consider (i) symmetric, linear equilibrium, (ii) symmetric but not necessarily linear equilibrium, and (iii) linear but not necessarily symmetric equilibrium. I will explain what symmetry and linearity means when we come there.

1.1 Cournot Duopoly with incomplete information

We have a Cournot duopoly with inverse-demand function

$$P(Q) = a - Q$$

where $Q = q_1 + q_2$. The marginal cost of Firm 1 is $c = 0$, and this is common knowledge. Firm 2's marginal cost c_2 is its own private information. It can take values of

$$\begin{array}{ll} c_H & \text{with probability } \theta \text{, and} \\ c_L & \text{with probability } 1-\theta. \end{array}$$

Each firm maximizes its expected profit.

Here, Firm 1 has just one type, and Firm 2 has two types. Hence, a strategy of Firm 1 is a real number q_1, while a strategy of Firm 2 is two real numbers $q_2(c_H)$ and $q_2(c_L)$, one for when the cost is c_H and one for when the cost is c_L.

Bayesian Nash Equilibrium We now compute a Bayesian Nash equilibrium $(q_1^*, q_2^*(c_H), q_2^*(c_L))$. We consider each type of each firm separately. First consider the high type (c_H) of Firm 2. In equilibrium, that type knows that Firm 1 produces q_1^*. Hence, its production level, $q_2^*(c_H)$, solves the maximization problem

$$\max_{q_2}(P - c_H)q_2 = \max_{q_2}\left[a - q_1^* - q_2 - c_H\right]q_2.$$

Hence,

$$q_2^*(c_H) = \frac{a - q_1^* - c_H}{2} \tag{1}$$

Now consider the low type (c_L) of Firm 2. In equilibrium, that type also knows that Firm 1 produces q_1^*. Hence, its production level, $q_2^*(c_L)$, solves the maximization problem

$$\max_{q_2}\left[a - q_1^* - q_2 - c_L\right]q_2.$$

Hence,

$$q_2^*(c_L) = \frac{a - q_1^* - c_L}{2}. \tag{2}$$

The important point here is that both types consider the same q_1^*, as that is the strategy of Firm 1, whose type is known by both types of Firm 2. Now consider Firm 1. It has one type. Firm 1 knows the strategy of Firm 2, but since it does not know which type of Firm 2 it faces, it does not know the production level of Firm 2. It thinks that the production level of Firm 2 is $q_2^*(c_H)$ with probability θ and $q_2^*(c_L)$ with probability $1-\theta$. Hence, its strategy q_1^* solves the maximization problem

$$\begin{aligned} & \max_{q_1} \theta \left[a - q_1 - q_2^*(c_H)\right] q_1 + (1-\theta) \left[a - q_1 - q_2^*(c_L)\right] q_1 \\ = & \max_{q_1} \left\{a - q_1 - \left[\theta q_2^*(c_H) + (1-\theta) q_2^*(c_L)\right]\right\} q_1. \end{aligned}$$

The equality is due to the fact that the production level q_2 of Firm 2 enters the payoff $[a - q_1 - q_2]\, q_1 = [a - q_1]\, q_1 - q_1 q_2$ of Firm 1 linearly. The term

$$E[q_2] = \theta q_2^*(c_H) + (1-\theta) q_2^*(c_L)$$

is the expected production level of Firm 2. Hence, Firm 1 just plays a best response to the expected production level. (We can do this when and only when the action of the other players affect the payoff of the player linearly.) Therefore,

$$q_1^* = \frac{a - E[q_2]}{2} = \frac{a - \left[\theta q_2^*(c_H) + (1-\theta) q_2^*(c_L)\right]}{2}. \tag{3}$$

To compute the Bayesian Nash equilibrium, we simply need to solve the three linear equations (1), (2), and (3) for $q_1^*, q_2^*(c_L), q_2^*(c_H)$. We write

$$\begin{pmatrix} q_1^* \\ q_2^*(c_H) \\ q_2^*(c_L) \end{pmatrix} = \begin{bmatrix} 2 & \theta & 1-\theta \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}^{-1} \begin{pmatrix} a \\ a - c_H \\ a - c_L \end{pmatrix},$$

yielding

$$\begin{aligned} q_2^*(c_H) &= \frac{a - 2c_H}{3} + \frac{(1-\theta)(c_H - c_L)}{6} \\ q_2^*(c_L) &= \frac{a - 2c_L}{3} - \frac{\theta(c_H - c_L)}{6} \\ q_1^* &= \frac{a + \theta c_H + (1-\theta) c_L}{3}. \end{aligned}$$

1.2 First-price Auction

We have an object, and two bidders want to buy it through an auction. Simultaneously, each bidder i submits a bid $b_i \geq 0$. Then, the highest bidder wins the object and pays her bid. If they bid the same number, then the winner is determined by a coin toss. The value of the object for bidder i is v_i, which is privately known by bidder i. That is,

v_i is the type of bidder i. We assume that v_1 and v_2 are "independently and identically distributed" with uniform distribution over $[0,1]$. This precisely means that knowing her own value v_i, bidder i believes that the other bidder's value v_j is distributed with uniform distribution over $[0,1]$, and the type space of each player is $[0,1]$. Recall that the beliefs of a player about the other player's types may depend on the player's own type. Independence assumes that it doesn't.

Here, the actions are b_i, coming from the action spaces $[0,\infty)$; the types are v_i, coming from the type spaces $[0,1]$; beliefs are uniform distributions over $[0,1]$ for each type. To complete the description of the game, we also need to determine the utility functions. The utility functions are given by

$$u_i(b_1,b_2,v_1,v_2) = \begin{cases} v_i - b_i & \text{if } b_i > b_j, \\ \frac{v_i - b_i}{2} & \text{if } b_i = b_j, \\ 0 & \text{if } b_i < b_j. \end{cases}$$

In a Bayesian Nash equilibrium, each type v_i maximizes the expected payoff

$$E\left[u_i(b_1,b_2,v_1,v_2)|v_i\right] = (v_i - b_i)\Pr\{b_i > b_j(v_j)\} + \frac{1}{2}(v_i - b_i)\Pr\{b_i = b_j(v_j)\} \qquad (4)$$

over b_i. Next, we will compute the Bayesian Nash equilibria. First, we consider a special equilibrium. The technique we will use here is a common technique in computing Bayesian Nash equilibria, and pay close attentions to the steps.

Symmetric, linear equilibrium We will now compute a symmetric, linear equilibrium. Symmetric means that equilibrium action $b_i(v_i)$ of each type v_i is given by

$$b_i(v_i) = b(v_i)$$

for some function b from type space to action space, where b is the same function for all players. Linear means that b is an affine function of v_i, i.e.

$$b_i(v_i) = a + cv_i.$$

To compute symmetric, linear equilibrium, we follow the following steps.

Step 1 *Assume that we have a symmetric linear equilibrium:*

$$\begin{aligned} b_1(v_1) &= a + cv_1 \\ b_2(v_2) &= a + cv_2 \end{aligned}$$

for all types v_1 and v_2 for some constants a and c, that will be determined later. The important thing here is the constants do not depend on the players or their types.

Step 2 *Compute the best reply function of each type.* Fix some type v_i. To compute her best reply, first note that $c > 0$. [This is not obvious; you need to read Gibbons and think about it.] Then, for any fixed value b_i,

$$\Pr\{b_i = b_j(v_j)\} = 0, \tag{5}$$

as b_j is strictly increasing in v_j by Step 1. It is also true that $a \leq b_i(v_i) \leq v_i$. [Again, you need to figure this out!] Hence,

$$\begin{aligned} E[u_i(b_1, b_2, v_1, v_2)|v_i] &= (v_i - b_i)\Pr\{b_i \geq a + cv_j\} \\ &= (v_i - b_i)\Pr\{v_j \leq \frac{b_i - a}{c}\} \\ &= (v_i - b_i) \cdot \frac{b_i - a}{c}. \end{aligned}$$

Here, the first equality is obtained simply by substituting (5) to (4). The second equality is simple algebra, and the third equality is due to the fact that v_j is distributed by uniform distribution on $[0, 1]$. [If you are taking this course, the last step must be obvious to you!] To compute the best reply, we must maximize the last expression over b_i. Taking the derivative and setting equal to zero yields

$$b_i = \frac{v_i + a}{2}. \tag{6}$$

Step 3 *Verify that best -reply functions are indeed affine*, i.e., b_i is of the form $b_i = a + cv_i$. Indeed, we rewrite (6) as

$$b_i = \frac{1}{2}v_i + \frac{a}{2}. \tag{7}$$

We check that both $1/2$ and $a/2$ are constant, i.e., they do not depend on v_i, and they are same for both players.

Step 4 *Compute the constants a and c.* To do this, we observe that in order to have an equilibrium, the best reply b_i in (6) must be equal to $b(v_i)$:

$$b_i = b(v_i),$$

or equivalently

$$\frac{1}{2}v_i + \frac{a}{2} = cv_i + a.$$

This must be an identity, i.e. it must remain true for all values of v_i. Hence, the coefficient of v_i must be equal in both sides:

$$c = \frac{1}{2}.$$

The intercept must be same in both sides, too:

$$a = \frac{a}{2}.$$

Thus,

$$a = 0.$$

This yields the symmetric, linear Bayesian Nash equilibrium:

$$b_i(v_i) = \frac{1}{2}v_i.$$

Any symmetric equilibrium We now compute a symmetric Bayesian Nash equilibrium without assuming that b is linear. We will assume that b is strictly increasing and differentiable.

Step 1 *Assume that we have a Bayesian Nash equilibrium of the form*

$$\begin{aligned} b_1(v_1) &= b(v_1) \\ b_2(v_2) &= b(v_2) \end{aligned}$$

for some increasing, differentiable function b.

Step 2 *Compute the best reply of each type, or compute the first-order condition that must be satisfied by the best reply.* To this end, compute that, given that the other player j is playing according to equilibrium, the expected payoff of playing b_i for type v_i is

$$\begin{aligned} E\left[u_i\left(b_1, b_2, v_1, v_2\right)|v_i\right] &= \left(v_i - b_i\right)\Pr\{b_i \geq b(v_j)\} \\ &= \left(v_i - b_i\right)\Pr\{v_j \leq b^{-1}\left(b_i\right)\} \\ &= \left(v_i - b_i\right)b^{-1}\left(b_i\right), \end{aligned}$$

where b^{-1} is the inverse of b. Here, the first equality holds because b is strictly increasing; the second equality is obtained by again using the fact that b is increasing, and the last equality is by the fact that v_j is uniformly distributed on $[0, 1]$. The first-order condition is obtained by taking the partial derivative of the last expression with respect to b_i and setting it equal to zero. To avoid confusion, let us write b_i^* for the best reply. Then, the first-order condition is

$$-b^{-1}\left(b_i^*\right) + \left(v_i - b_i^*\right)\left.\frac{db^{-1}}{db_i}\right|_{b_i = b_i^*} = 0.$$

Using the formula on the derivative of the inverse function, we re-write this as

$$-b^{-1}\left(b_i^*\right) + \left(v_i - b_i^*\right)\left.\frac{1}{b'\left(v\right)}\right|_{b(v) = b_i^*} = 0. \tag{8}$$

Step 3 *Identify the best reply with the equilibrium action*, towards computing the equilibrium action. That is, set

$$b_i^* = b\left(v_i\right).$$

Substituting this in (8), obtain

$$-v_i + \left(v_i - b\left(v_i\right)\right)\frac{1}{b'\left(v_i\right)} = 0. \tag{9}$$

By simple algebra, we obtain

$$b'\left(v_i\right)v_i + b\left(v_i\right) = v_i$$

and hence

$$\frac{d\left[b\left(v_i\right)v_i\right]}{dv_i} = v_i.$$

Therefore,

$$\begin{aligned} b\left(v_i\right) v_i &= v_i^2/2 + const. \\ b\left(v_i\right) &= v_i/2 + const/v_i. \end{aligned}$$

Since $b(0) \neq \infty$, it must be that $const = 0$. Therefore,

$$b\left(v_i\right) = v_i/2.$$

In this case, we were lucky. In general, one obtains a differential equation as in (9), but the equation is not easily solvable in general. Make sure that you understand the steps until finding the differential equation well.

1.3 Double Auction

We will now consider a "double auction". Although the term refers to an auction, it is actually about a simple bargaining problem. We have a Seller, who owns an object, and a Buyer. They want to trade the object through the following mechanism. Simultaneously, Seller names p_s and Buyer names p_b.

- If $p_b < p_s$, then there is no trade;
- if $p_b \geq p_s$, then they trade at price

$$p = \frac{p_b + p_s}{2}.$$

The value of the object for Seller is v_s and for Buyer is v_b. Each player knows her own valuation privately. We assume that v_s and v_b are independently and identically distributed with uniform distribution on $[0, 1]$. [Recall from the first-price auction what this means.] Then, the payoffs are

$$\begin{aligned} u_b &= \begin{cases} v_b - \frac{p_b+p_s}{2} & \text{if } p_b \geq p_s \\ 0 & \text{otherwise} \end{cases} \\ u_s &= \begin{cases} \frac{p_b+p_s}{2} - v_s & \text{if } p_b \geq p_s \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

We will now compute Bayesian Nash equilibria. In an equilibrium we have a price $p_s(v_s)$ for each type v_s of the seller and a price $p_b(v_b)$ for each type v_b of the buyer. In a Bayesian Nash equilibrium, $p_b(v_b)$ solves the maximization problem

$$\max_{p_b} E\left[v_b - \frac{p_b + p_s(v_s)}{2} : p_b \geq p_s(v_s)\right],$$

and $p_s(v_s)$ solves the maximization problem

$$\max_{p_s} E\left[\frac{p_s + p_b(v_b)}{2} - v_s : p_b(v_b) \geq p_s\right],$$

where $E[x : A]$ is the "integral" of x on set A. (We have $E[x : A] = E[x|A]\Pr(A)$, where $E[x|A]$ is the conditional expectation of x given A. Make sure that you know all these terms!!!)

In this game, there are many Bayesian Nash equilibria. For example, one equilibrium is given by

$$\begin{aligned} p_b &= \begin{cases} X & \text{if } v_b \geq X \\ 0 & \text{otherwise} \end{cases}, \\ p_s &= \begin{cases} X & \text{if } v_s \leq X \\ 1 & \text{otherwise} \end{cases} \end{aligned}$$

for some any fixed number $X \in [0,1]$. We will now consider the Bayesian Nash equilibrium with linear strategies.

Equilibrium with linear strategies We will consider an equilibrium where the strategies are affine functions of valuation, but they are not necessarily symmetric.

Step 1 *Assume that we have an equilibrium with linear strategies:*

$$\begin{aligned} p_b(v_b) &= a_b + c_b v_b \\ p_s(v_s) &= a_s + c_s v_s \end{aligned}$$

for some constants a_b, c_b, a_s, and c_s. We also assume that $c_b > 0$ and $c_s > 0$. [Notice that a and c may be different for buyer and the seller.]

Step 2 *Compute the best responses for all types.* To do this, first note that

$$p_b \geq p_s(v_s) = a_s + c_s v_s \iff v_s \leq \frac{p_b - a_s}{c_s} \tag{10}$$

and

$$p_s \leq p_b(v_b) = a_b + c_b v_b \iff v_b \geq \frac{p_s - a_b}{c_b}. \tag{11}$$

We will now compute the best reply for a type v_b. Given that the seller plays according to the given equilibrium, his expected payoff from playing p_b is

$$\begin{aligned} E\left[u_b\left(p_b, p_s, v_b, v_s\right)|v_b\right] &= E\left[v_b - \frac{p_b + p_s(v_s)}{2} : p_b \geq p_s(v_s)\right] \\ &= \int_0^{\frac{p_b - a_s}{c_s}} \left[v_b - \frac{p_b + p_s(v_s)}{2}\right] dv_s, \end{aligned}$$

where the last equality is obtained by substituting (10). By substituting $p_s(v_s) = a_s + c_s v_s$ in this expression, we obtain

$$E\left[u_b\left(p_b, p_s, v_b, v_s\right)|v_b\right] = \int_0^{\frac{p_b - a_s}{c_s}} \left[v_b - \frac{p_b + a_s + c_s v_s}{2}\right] dv_s.$$

After some simple algebra,[1] this equation becomes

$$E\left[u_b\left(p_b, p_s, v_b, v_s\right)|v_b\right] = \frac{p_b - a_s}{c_s}\left(v_b - \frac{3p_b + a_s}{4}\right).$$

To compute the best reply, we take the derivative of the last expression with respect to p_b and set it equal to zero. This yields

$$\frac{1}{c_s}\left(v_b - \frac{3p_b + a_s}{4}\right) - \frac{3(p_b - a_s)}{4c_s} = 0$$

[1] We can write the integral as

$$\begin{aligned} & \frac{p_b - a_s}{c_s}\left(v_b - \frac{p_b + a_s}{2}\right) - \frac{c_s}{2}\int_0^{\frac{p_b - a_s}{c_s}} v_s dv_s \\ = \; & \frac{p_b - a_s}{c_s}\left(v_b - \frac{p_b + a_s}{2}\right) - \frac{c_s}{4}\left(\frac{p_b - a_s}{c_s}\right)^2 \\ = \; & \frac{p_b - a_s}{c_s}\left(v_b - \frac{p_b + a_s}{2} - \frac{p_b - a_s}{4}\right) \\ = \; & \frac{p_b - a_s}{c_s}\left(v_b - \frac{3p_b + a_s}{4}\right). \end{aligned}$$

Solving for p_b, we obtain

$$p_b = \frac{2}{3}v_b + \frac{1}{3}a_s. \tag{12}$$

Now we compute the best reply of a type v_s. As in before, his expected payoff of playing p_s in equilibrium is

$$\begin{aligned} E\left[u_s\left(p_b, p_s, v_b, v_s\right)|v_s\right] &= E\left[\frac{p_s + p_b(v_b)}{2} - v_s : p_b(v_b) \geq p_s\right] \\ &= \int_{\frac{p_s - a_b}{c_b}}^{1} \left[\frac{p_s + a_b + c_b v_b}{2} - v_s\right] dv_b, \end{aligned}$$

where the last equality is by (11) and $p_b\left(v_b\right) = a_b + c_b v_b$. After some simple algebra,[2] this becomes

$$E\left[u_s\left(p_b, p_s, v_b, v_s\right)|v_s\right] = \left(1 - \frac{p_s - a_b}{c_b}\right)\left[\frac{3p_s + a_b}{4} - v_s + \frac{c_b}{4}\right].$$

Once again, in order to compute the best reply, we take the derivative of the last expression with respect to p_s and set it equal to zero. This yields

$$-\frac{1}{c_b}\left[\frac{3p_s + a_b}{4} - v_s + \frac{1}{4}\right] + \frac{3}{4}\left(1 - \frac{p_s - a_b}{c_b}\right) = 0,$$

or equivalently

$$-\left[\frac{3p_s + a_b}{4} - v_s + \frac{c_b}{4}\right] + \frac{3}{4}\left(c_b - \left(p_s - a_b\right)\right) = 0.$$

Solving for p_s, we obtain

$$\frac{3p_s}{2} = -\frac{a_b}{4} + v_s - \frac{c_b}{4} + \frac{3}{4}\left(c_b + a_b\right) = v_s + \frac{a_b + c_b}{2}.$$

[2] We write the integral as

$$\begin{aligned} & \left(1 - \frac{p_s - a_b}{c_b}\right)\left[\frac{p_s + a_b}{2} - v_s\right] + \frac{c_b}{2}\int_{\frac{p_s - a_b}{c_b}}^{1} v_b dv_b \\ = & \left(1 - \frac{p_s - a_b}{c_b}\right)\left[\frac{p_s + a_b}{2} - v_s\right] + \frac{c_b}{4}\left(1 - \left(\frac{p_s - a_b}{c_b}\right)^2\right) \\ = & \left(1 - \frac{p_s - a_b}{c_b}\right)\left[\frac{p_s + a_b}{2} - v_s + \frac{c_b}{4} + \frac{p_s - a_b}{4}\right] \\ = & \left(1 - \frac{p_s - a_b}{c_b}\right)\left[\frac{3p_s + a_b}{4} - v_s + \frac{c_b}{4}\right] \end{aligned}$$

Therefore,

$$p_s = \frac{2}{3}v_s + \frac{a_b + c_b}{3}. \tag{13}$$

Step 3 *Verify that best replies are of the form that is assumed in Step 1.* Inspecting (12) and (13), one concludes that this is indeed the case. The important point here is to check that in (12) the coefficient 2/3 and the intercept $\frac{1}{3}a_s$ are constants, independent of v_b. Similarly for the coefficient and the intercept in (13).

Step 4 *Compute the constants.* To do this, we identify the coefficients and the intercepts in the best replies with the relevant constants in the functional form in Step 1. Firstly, by (12) and $p_b(v_b) = p_b$, we must have the identity

$$a_b + c_b v_b = \frac{1}{3}a_s + \frac{2}{3}v_b.$$

That is,

$$a_b = \frac{1}{3}a_s \tag{14}$$

and

$$c_b = \frac{2}{3}. \tag{15}$$

Similarly, by (13) and $p_s(v_s) = p_s$, we must have the identity

$$a_s + c_s v_s = \frac{a_b + c_b}{3} + \frac{2}{3}v_s.$$

That is,

$$a_s = \frac{a_b + c_b}{3} \tag{16}$$

and

$$c_s = \frac{2}{3}. \tag{17}$$

Solving (14), (15), (16), and (17), we obtain $a_b = 1/12$ and $a_s = 1/4$.

Therefore, the linear Bayesian Nash equilibrium is given by

$$p_b(v_b) = \frac{2}{3}v_b + \frac{1}{12} \tag{18}$$

$$p_s(v_s) = \frac{2}{3}v_s + \frac{1}{4}. \tag{19}$$

In this equilibrium, we have trade iff

$$p_b(v_b) \geq p_s(v_s)$$

iff

$$\frac{2}{3}v_b + \frac{1}{12} \geq \frac{2}{3}v_s + \frac{1}{4}$$

iff

$$v_b - v_s \geq \frac{3}{2}\left(\frac{1}{4} - \frac{1}{12}\right) = \frac{3}{2}\frac{1}{6} = \frac{1}{4}.$$