Air Injection in Axial Compressors: Modeling, Experimental Validation, and Control of Instabilities

by

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Abstract

This thesis presents verified models for predicting the effects of air injection in high speed axial flow compressors. Using this and other models, new control concepts are studied for reducing the actuation levels required to stabilize compressor instabilities.

A steady air injection model is developed that consists of two parts: a control volume analysis that models the change in mass, momentum, and swirl induced by jet actuators, and a streamline curvature analysis which models the response of the compressor blade rows. The model is validated using measurements of total pressure profiles, total temperature profiles, and speedlines from a single-stage transonic compressor. The results show that the pressure rise associated with air injection in a compressor is primarily due to changes in blade row performance characteristics resulting from changes in flow redistribution within the compressor blade rows. The steady air injection model is used to improve the predictions of an unsteady two-dimensional, linear, compressible stall and surge inception model.

A framework for implementing nonlinear controllers was set up and tested. The control concept consists of appending a robust estimator of the incompressible states to control laws which consider only the incompressible modes. The concept was tested on a single-stage transonic compressor operating at a tip relative Mach number of 1.25. In this compressor, no operating range extension was achieved with constant gain feedback control, but when a robust $H_\infty$ estimator was appended to the constant gain feedback control, the stalling mass was reduced by 9.0%.

An analytical study of methods to reduce air injection levels, either by reducing the number of actuators, or by injecting in a "single-sided" (nominally off) manner was investigated. Extension of the stable operating range with reduced actuator mass flow was demonstrated analytically. A single-sided, sliding mode control scheme was demonstrated analytically to have good operability properties and achieve a larger extension of the stable operating range. Two actuation configurations with fewer actuators were selected based on a parametric study, and their feasibility was demonstrated experimentally.

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NOMENCLATURE

Symbols

\( a \) Speed of sound \((\sqrt{\gamma RT})\)
\( A \) Annulus area
\( B \) Nondimensional surge stability parameter
\( B \) Upstream potential wave coefficient
\( C \) Downstream potential wave coefficient
\( D \) Vortical wave coefficient
\( E \) Entropic wave coefficient
\( j \) \( \sqrt{-1} \)
\( M \) Mach number \((\frac{V}{a})\)
\( n \) Harmonic number
\( P \) Pressure
\( r \) Mean radius
\( R \) Gas constant for air
\( s \) Laplace transform variable
\( t \) Time
\( T \) Temperature
\( U \) Wheel speed
\( V \) Velocity
\( W \) Blade passage relative velocity
\( x \) Axial coordinate
\( z \) Z-transform variable
Greek Symbols

$\alpha_n$  Upstream potential wave propagation coefficient
$\beta$  Relative flow angle
$\beta_n$  Downstream potential wave propagation coefficient
$\chi_n$  Convecting wave coefficient
$\gamma$  Ratio of specific heats
$\lambda$  Eigenvalue
$\sigma$  Eigenvalue real part (growth rate)
$\omega$  Eigenvalue imaginary part (frequency)
$\omega_{\text{loss}}$  Relative total pressure loss coefficient
$\Omega$  Rotor frequency
$\phi$  Flow coefficient ($\frac{\dot{\gamma}}{\dot{\rho}}$)
$\rho$  Density
$\tau_p$  Total pressure loss lag time constant
$\tau_d$  Deviation lag time constant
$\theta$  Tangential coordinate
$\xi$  Blade row stagger angle

Operators, Superscripts, and Subscripts

$\delta(\cdot)$  Small perturbation
$\bar{\cdot}$  Mean value
$\vec{\cdot}$  Vector
$\hat{\cdot}$  Spatial Fourier coefficient
$\dot{\cdot}$  Derivative with respect to time
$\Delta$  Difference operator
$\nabla$  Gradient operator
\(
\frac{D()}{Dt} \quad \text{Total derivative}
\)

\((\cdot)^* \quad \text{Complex conjugate}
\)

\(F(s) \quad \text{Laplace transform of } f(t)
\)

\(F(z) \quad \text{Z-transform of } f(t)
\)

\((\cdot)_r \quad \text{Real part}
\)

\((\cdot)_i \quad \text{Imaginary part}
\)

\((\cdot)' \quad \text{Relative frame value}
\)

\((\cdot)_x \quad \text{Axial component}
\)

\((\cdot)_\theta \quad \text{Tangential component}
\)

\((\cdot)_{tip} \quad \text{Blade tip}
\)

\((\cdot)_k \quad \text{Blade row and inter-blade row gap number}
\)

\((\cdot)_n \quad n^{th} \text{ spatial harmonic value}
\)

\((\cdot)_{t} \quad \text{Stagnation (or total) quantity}
\)

\((\cdot)_{nd} \quad \text{Non-dimensional value}
\)

**Acronyms**

CFD \hspace{1cm} \text{Computational fluid dynamics}

ETFE \hspace{1cm} \text{Empirical transfer function estimate}

MIMO \hspace{1cm} \text{Multi-input, multi-output}

SISO \hspace{1cm} \text{Single-input, single-output}

LQR \hspace{1cm} \text{Linear quadratic regulator}

LQG \hspace{1cm} \text{Linear quadratic Gaussian}

RMS \hspace{1cm} \text{Root mean square}

DFT \hspace{1cm} \text{Discrete Fourier transform}

SFC \hspace{1cm} \text{Spatial Fourier coefficient}

1-D \hspace{1cm} \text{One-dimensional (x direction)}

2-D \hspace{1cm} \text{Two-dimensional (x and } \theta \text{ directions)}

3-D \hspace{1cm} \text{Three-dimensional (r, x, and } \theta \text{ directions)}
**Matrices and Vectors**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>Matrices denoted by bold uppercase characters</td>
</tr>
<tr>
<td>$A^T$</td>
<td>Transpose of $A$</td>
</tr>
<tr>
<td>$a_i$</td>
<td>$i^{th}$ column vector of $A$</td>
</tr>
<tr>
<td>$a_{ij}$</td>
<td>$(i, j)$ entry of $A$</td>
</tr>
<tr>
<td>$I$</td>
<td>Identity matrix</td>
</tr>
<tr>
<td>$O$</td>
<td>Matrix of zeros</td>
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</table>
A gas turbine engine consists of three primary components: a compressor, a combustion chamber, and a turbine. Some engine designers claim that the compressor is the most important component of the gas turbine engine\(^1\). The operating range, performance, and reliability of gas turbine engines are limited by aerodynamic instabilities that occur in the compressor at low mass flow rates. Two of such compressor instabilities are known as rotating stall and surge. A comprehensive review of the flow instabilities present in compression systems is given in reference [55] by Greitzer. Figure 1-1 shows a schematic compressor characteristic generally used to characterize the compressor performance. Compressor speedlines have two limiting mass flow rates, a maximum at which no more mass can go through the compressor and a minimum at which compressor instabilities set in. The maximum mass flow capacity of the compressor is limited by the choking condition at the throat of the compressor flowpath. As the mass flow rate through the compressor is decreased from the maximum possible value at the choking point (in an experiment, the mass flow rate is decreased by closing the throttle), the pressure rise increases. This trend continues until point A, the lowest mass flow rate at which the compressor can operate with axisymmetric flow, is reached. If the mass flow rate is decreased any further beyond point A, an abrupt transition occurs whereby the steady symmetric flow of air through the compressor changes to an unsteady asymmetric flow with a dramatic reduction in pressure rise. This is rotating

\(^1\)In a typical gas turbine engine, an increase in turbine efficiency of 1% corresponds to a 0.3% improvement in compressor efficiency [33].
stall. At this point, one of two things can happen depending on the Greitzer B parameter\(^2\) of the compression system. Either the system can operate with a constant annulus averaged mass flow rate at reduced pressure rise (point B, for compression systems with low B parameter), or the system can operate with large amplitude fluctuations in the annulus averaged mass flow rate and pressure rise i.e., 'surge' (for compression systems with high B parameter). For compression systems with a low B parameter, the compressor can be returned to its axisymmetric flow by increasing the mass flow rate (opening the throttle) to point C. \(A-B-C-D-A\) represents the hysteresis loop associated with this transition between the stable axisymmetric and stable non-axisymmetric operating conditions.

---

\(2\)The Greitzer B parameter is a measure of the compliance relative to the inertia of the compression system.

---

**Figure 1-1:** Schematic compressor characteristic, showing rotating stall condition.

In this chapter, a description of rotating stall and surge, its propagation mechanism including the major stall inception mechanisms observed in experiments, are presented in Section
1.1 The concept of active rotating stall control is presented in Section 1.2. A review of the previous work on active rotating stall and surge control reported in literature is presented in Section 1.3. The motivation, objectives, and contributions of this research are presented in Section 1.4, and the thesis outline is presented in Section 1.5.

1.1 Rotating Stall and Surge

Rotating stall is a two- or three-dimensional non-axisymmetric unsteady flow phenomenon whereby portions of the flow with velocity deficit (known as stall cells) travel around the compressor annulus at 30% to 70% of the rotor speed. Surge is a large amplitude one-dimensional (axisymmetric) flow oscillation in the compressor. The spatial structure of rotating stall and surge in an axial flow compressor, as well as the unsteady pressure transients for both types of instability are illustrated in Figure 1-2.

Figure 1-2: Schematic of rotating stall and surge in axial flow compressors (from [55]).

A physical mechanism for the propagation of a disturbance (or stall cell) from blade to blade around the compressor annulus during rotating stall inception was first presented by Emmons et al. [32]. The classical explanation for stall-cell propagation proposed by Emmons et al. is based on a flow separation phenomenon governed by the angle of attack. Compressor blades will experience flow separation at high angles of attack (which results when there is a reduction in the axial velocity of the air stream entering the compressor). Figure 1-3 is an unwrapped diagram of a compressor annulus showing a row of highly loaded rotor blades. A minor physical irregularity or flow nonuniformity can cause momentary overloading and separation in the middle blade passages. The separated flows create a blockage in the blade passage. This increased blockage within the middle blade passages
(operating under separation) diverts the incoming streamlines to either side of the affected area. This redistribution in the upstream flow field causes the blades above the separated region to experience an increase in angle of attack and separate, and the blades below the separated region to experience a decrease in angle of attack and become less separated. The resulting effect is that the region of separated flow and high blockage propagates relative to the blade row in the direction shown. Over the years, different mechanisms governing the stall inception process have been reported.

Two major types of stall inception have been experimentally identified in axial compressors: modal stall inception and short length-scale stall inception\(^3\). The main differentiating characteristics between these two types of stall inception are the length scale and rotation rate of the perturbations. Modal stall perturbations have a characteristic length of the compressor annulus while the size of the short length-scale disturbances is of the order of the blade spacing (pitch). Short length-scale disturbances initially rotate at higher speeds (\(\approx 70\%\) of the rotor speed) than the fully developed stall cell (which rotate at \(\approx 30 - 50\%\) of the rotor speed). The hydrodynamic theory of compressor stability developed by Moore and Greitzer [91, 92] has been shown to predict the modal stall inception of several compressors. According to the Moore-Greitzer model, the critical parameter characterizing instability for compressors that exhibit modal stall inception is the slope of the total-to-static compressor characteristic. Thus, the modal stall inception model requires the description of the entire compression system instead of the flow structure within the blade passages to pre-

---

\(^3\)Both types of stall inception were observed on NASA Stage 35 albeit under different operating conditions. Short length scale stall inception was observed when a tip-radial distortion screen was introduced in the inlet duct but when air was injected, the stall inception changed to modal [115].
dict instability. Hoying [68] found that the tip clearance flow structure is fundamental to the development of short length-scale stall inception. The instability for short length-scale stall inception was shown to develop directly out of a blade passage structure, and the tip clearance vortex was found to be the critical parameter characterizing this instability. Thus, unlike the modal stall inception model, the stability criterion for short length-scale stall inception is local in nature. A computational model for simulating short wavelength stall inception in multistage compressors was developed by Gong [52]. According to the hydrodynamic theory for modal stall inception by Moore and Greitzer [91, 92], sinusoidal rotating waves will become underdamped near the stall point and these spatial sinusoids should be detectable at small amplitudes before stall inception. This is consistent with the results from stall inception measurements by McDougall et al. [86, 87], Longley [80], and Garnier et al. [45, 46] which showed that modal stall cells originate from small amplitude, long length scale disturbances. On the other hand, Day [22, 24] showed that stall cells representing short length scale disturbances, can originate without any precursors. Day and Camp [23] described the nature and appearance of the modal and short length-scale stall inception phenomena in several low speed compressor configurations, and developed a simple model (criterion) for predicting whether a compressor will exhibit modal or short length scale stall inception. According to Day and Camp, short length-scale stall inception occurs for compressors where the critical incidence at the rotor tip is reached before the neutral stability point of the total-to-static pressure rise characteristic ($\phi_{t}=0$), and modal stall inception occurs for compressors where the neutral stability point of the total-to-static pressure rise characteristic ($\phi_{s}=0$) is reached before the critical incidence is exceeded.

Two types of surge cycles were identified by Greitzer [54]: classic surge and deep surge. Deep surge involves significant flow reversals in the compressor during at least part of the surge cycle whereas classic surge does not involve such flow reversals. Classic surge requires the compressor to recover from stall to uniform forward flow. The critical parameter determining whether (upon the onset of instability) a compression system will settle into a stable rotating stall or enter a surge cycle is the Greitzer B parameter. This parameter can be interpreted as a measure of the ratio of compliance to inertia of the compression system. It also indicates the ratio of two characteristic times: 1) the time required by the compressor to pump sufficient mass to raise the pressure in the plenum from the minimum
pressure sustainable by the compressor to the normal operating pressure, and 2) the time it takes the fluid to go through the compressor. Due to the flow reversals involved in deep surge, a flame can often be seen at the intake and exhaust of an engine as the combustion moves forward and backward in the combustor during engine surge. While rotating stall and surge are separate phenomena, previous research has shown that rotating stall is usually a precursor to surge in many compressors [97].

1.2 The Concept of Active Control

Rotating stall must be avoided in axial flow compressors because it can lead to severe unsteady loading on the compressor blades which can in turn lead to excessive blade vibration, a large drop in the pressure rise delivered by the compressor, a dramatic drop in the compressor efficiency which may cause the burner to overheat, and possibly surge. Surge overstresses the compressor blades, and can blow out the flame in the combustor (this is known as engine flame-out) leading to engine shut-down, and damage. If the compressor for a jet engine that does not surge (low B parameter) settles into a stable rotating stall, the engine may not recover from this condition unless it is shut-down and re-started.

The current approach for avoiding rotating stall is to operate the compressor away from the stall point. This approach requires operating the engine with a stall margin or stability margin so that any transients or disturbances will not drive the system to rotating stall. However, operating the compressor with a stall margin will limit the range and performance of the compressor since it will be operating at a lower pressure rise, possibly with lower efficiency. This is wasteful, and a more modern approach for increasing the useful operating range and thus maximizing the pressure rise and efficiency from the compressor is by active control.

In 1989, Epstein et al. [34] proposed the use of active feedback control techniques to extend the compressor operating range by delaying the onset of rotating stall and surge. This involves the integration of sensors and actuators into the compression system so that the unsteady flow perturbations, which are the precursors of rotating stall and surge, can be measured and subsequently stabilized by applying appropriate feedback. Figure 1-4 shows how the compressor surge line can be moved with active feedback control. Without feedback control, the compressor is operated at point A to maintain a safety margin from
the surge line. With active feedback control, the surge line is moved to the left such that the compressor can be operated at point B. Thus with active feedback control, the compressor can be operated at higher performance levels. Over the years, the validity of this concept has been demonstrated on several research compressors by Ffowcs Williams and Huang [134], Pinsley et al. [100], Day [21], Paduano et al. [96, 98], Haynes et al. [61, 62], Gysling et al. [58, 57, 59], Eveker et al. [36], D’Andrea et al. [19, 20], Behnken et al. [7, 9], van Schalkwyk et al. [124, 126], Vo [128], Protz [101], Freeman et al. [42], and Weigl et al. [130, 131].

Figure 1-4: Schematic compressor characteristic, showing extension in the unstable portion using active feedback control.

A potential application of the performance improvement associated with active feedback control is to reduce the weight of an aircraft engine. The number of compressor stages
required to achieve the same overall pressure rise can be reduced by using active control to extend the operating range of a compressor with steep speedlines. Since the compressor makes up a large percent of an aircraft engine weight, reducing the number of compressor stages will reduce the engine weight thus increasing its thrust to weight ratio. More detailed studies on the performance enhancement of aircraft engines using actively stabilized compressors have been conducted by Seymour [108] and Tow [120]. Seymour [108] used an advanced tactical fighter as the baseline airframe, a standard high-low-high combat profile as the baseline mission, and a mixed flow afterburning turbofan as the baseline engine to show that a 20% increase in the surge margin of a high pressure compressor with steep speedlines resulted in an 11.2% increase in mission radius, an 8.3% decrease in takeoff gross weight, a 7.3% decrease in aircraft operating weight, and an 8.1% decrease in the aircraft’s total wetted area. Tow [120] used a low bypass ratio, mixed flow, afterburning turbofan which is typical of high performance military fighter applications to show that at cruise operating conditions of 35,000 feet and Mach number of 0.85, an additional stall margin of 5% from active control improved the cruise specific fuel consumption by 0.74% to 0.82%.

1.3 Previous Work on Active Control

The stabilization of compression systems by means of active control has been implemented on several research compressors using different actuators to manipulate the compressor flow field. The set of active control experiments reported in literature can be classified into two groups: active control for operating range extension, and active control to improve the operability of the compressor in the presence of disturbances. The difference between these two approaches is illustrated in Figure 1-5. With active control for operating range extension, unstable axisymmetric equilibrium points are linearly stabilized so that the hysteresis occurs at a lower mass flow rate. The objective of active control for operability is not to extend the stable axisymmetric characteristic, but to eliminate the hysteresis loop normally associated with rotating stall, so that the compressor can operate at a point closer to the surge line without being driven to full scale rotating stall by internal or external disturbances. As shown in Figure 1-5, both approaches allow the compressor to be operated at a lower mass flow rate and produce an improvement in performance.
Extending the operating range of axial flow compressors is based on the notion of feedback stabilization, where the unstable axisymmetric equilibrium points are linearly stabilized. The strategy for modal stall control consists of damping the pre-stall harmonics. Since the instability at low mass flow rates is governed by the higher harmonics, multiple sensors are typically needed to detect these higher harmonics, and several actuators may be required to damp them out. Also, since the higher harmonic disturbances rotate at higher rates, high bandwidth actuators will be needed to damp them out. Since the objective of active control for operability is not to extend the axisymmetric flow operating range of the compressor which means that only the lower harmonics are needed to be stabilized, fewer and low bandwidth actuators are required for implementing the necessary one-dimensional actuation schemes.

1.3.1 Active control for operating range extension

The actuation schemes for active and passive surge control reported in literature include movable plenum wall, variable throttle area, bleed valves, and tailored structures; and the actuation schemes for active and passive rotating stall control reported in literature include inlet guide vanes, bleed valves, and air injection.

Active surge control: Ffowcs Williams and Huang [134] used a movable plenum wall (loudspeaker mounted in the compressor plenum), driven by a signal proportional the unsteady plenum pressure, to suppress surge in a centrifugal turbocharger. Pinsley et al. [100] used a variable area throttle, driven by a signal proportional to the unsteady plenum pressure to suppress surge in a centrifugal compressor. A 25% reduction in the surge point mass

Figure 1-5: Schematic showing characteristics for uncontrolled and controlled compressor.
flow was obtained over a significant range of speeds and pressure ratios. Gysling et al. [58] demonstrated the suppression of surge in a centrifugal compressor using a movable plenum wall as tailored structure. By tuning the movable plenum wall with a spring and damper to passively damp the unsteady pressure perturbations in the plenum, the surge line of the centrifugal compressor was shifted roughly 25% for a significant portion of the corrected speed range examined. The effectiveness of different approaches for active compressor surge control was evaluated by Simon et al. [109] using the verified surge model by Greitzer [54]. By considering five actuators (injection in the compressor duct, close-coupled valve control, plenum bleed valve, plenum heat addition, and a movable plenum wall) and four sensors (compressor duct mass flow, plenum pressure, compressor face static pressure, and compressor face total pressure), the effectiveness of different actuator-sensor pairs (a total of twenty pairings) were evaluated with a proportional control law. The close-coupled valve and injector were found to be the most promising actuators for the four sensors studied.

Active control using inlet guide vanes: Paduano et al. [96, 98] used an array of twelve high-response movable inlet guide vanes in the compressor upstream to extend the operating range of a low speed single-stage axial compressor. The circumferential waves of velocity perturbations which precede stall were sensed using a circumferential set of hot wires mounted at an axial location (station) ahead of the rotors, and the harmonic components of these circumferential waves of velocity perturbations were computed. By commanding the actuators (inlet guide vanes) based on the constant gain control law (which entails feeding back each harmonic perturbation modified by a complex gain), an operating range extension of 23% was obtained when the first, second, and third modes were (independently) stabilized. Haynes et al. [61, 62] applied the same modal control scheme to a low speed three-stage axial compressor and obtained an operating range extension of 7.8% by stabilizing the first and second modes. van Schalkwyk et al. [124, 126] also used an array of twelve high-response inlet guide vanes to extend the operating range of a three-stage axial compressor with circumferential inlet distortion by 3.7% using a distributed feedback control law. The control strategy consisted of measuring the entire shape of the velocity perturbations at midspan with an array of sixteen equally spaced hot wires mounted at an axial location ahead of the first stage rotor, and feeding back a rotated and amplified version of the disturbance to the actuators.
Active control using air injection: Day [21] was able to delay the onset of rotating stall in a four-stage compressor using fast-acting air injection valves and two different methods. The control strategy consisted of detecting the emerging stall cells from velocity perturbation measurements using hot wire probes placed at several axial locations throughout the compressor and injecting air with an array of twelve individually controllable valves equally spaced around the circumference near the tips of the first rotor. The valves were either opened or closed, but could not be held at an intermediate position. When all twelve valves were opened at the same time, the maximum amount of air that could be injected was less than 1% of the compressor mass flow at the stall point. In the first method, which resulted in a 4% improvement in stall margin, modal perturbations were effectively damped by adjusting the number of valves open at anytime and by varying the opening times and the supply pressure to the valves. In the second method, which resulted in a 6% improvement in stall margin, emerging stall cells were removed by injecting air in their immediate vicinity using localized pulsing jets. If a stall cell was detected, the central valve in the immediate vicinity of the stall cell was turned on and closed when no stall cells were detected. It should be noted that this control approach is not of the continuous type suggested by Epstein et al. [34] because the actuators could only be switched on or off and could not be modulated between the opened and closed positions.

Gysling [57, 59] was able to extend the operating range of a single stage low speed axial compressor by 10% using an aeromechanical feedback system to modulate the amount of air injected into the compressor face. The control strategy used an array of wall jets upstream of the compressor which were being regulated by locally reacting reed valves. The tuned reed valves responded to flowfield pressure perturbations that precede rotating stall. Based on the success of the passive control experiments by Gysling and Greitzer [59], coupled with a theoretical study by Hendricks and Gysling [64] which showed that the most effective actuation and sensing scheme for stabilizing modal perturbations was to sense the axial velocity and inject high pressure air into the compressor face, electro-mechanical injectors were designed by Diaz [26] for rotating stall stabilization in a low speed compressor. Using an array of twenty-four air injectors designed by Diaz [26], Vo [128] was able to extend the operating range of a three-stage low speed axial compressor by 5.5%. Vo [128] used the same control strategy as Paduano [96, 98] where the first, second, and third harmonics of
the pre-stall velocity perturbations (sensed with an array of sixteen equally spaced hot-wire
probes located upstream of the rotor) were stabilized.

To demonstrate active stabilization of rotating stall and surge in high speed compressors, a
set of high bandwidth electro-mechanical injectors were designed by Berndt [10, 11]. Using
these high bandwidth electro-mechanical injectors, Weigl [130, 131] was able to extend the
operating range of a transonic single stage axial compressor with different control laws at
70% speed and 100% speed. The control strategy was to sense upstream wall static pressure
patterns and feed back the measured signal to an array of twelve separately modulated air
injectors upstream of the rotor based on a control law. At 70% speed, an 11% reduction in
the stalling mass flow was achieved by damping the first and second spatial harmonics of the
pre-stall rotating stall perturbations using a constant gain feedback control law similar to
the ones implemented on low speed compressors by Paduano [96], Haynes [61], and Vo [128].
At 100% speed, a 3.5% reduction in the stalling mass flow was achieved by stabilizing the
multiple modes which comprise the first three harmonic perturbations using a robust $H_\infty$
control law. Spakovsky et al. [115, 116] were able to extend the operating range of the same
single-stage transonic axial flow compressor with inlet distortion using the same actuators,
sensors, and control strategy as Weigl [130]. For the single-stage transonic compressor
operating at 85% speed with an inlet radial distortion, a range extension of 18.5% was
obtained using a robust $H_\infty$ controller to stabilize the first harmonic. For the single-stage
transonic compressor operating at 85% speed with an inlet circumferential distortion, a
range extension of 13.5% was obtained using a constant gain controller to stabilize the first
harmonic perturbations and a range extension of 14.8% was obtained using a robust $H_\infty$
controller to stabilize the first harmonic perturbations.

1.3.2 Active control for operability
Active rotating stall and surge control for operability has been implemented with bleed
valves and air injectors. Based on the Moore-Greitzer three state model for rotating stall and
surge in axial flow compressors described in [91, 92], Liaw and Abed [78] designed a control
law using 1-d bleed valves. The first successful implementation of a bleed valve controller
was reported by Eveker et al. [36]. Freeman et al. [42] were able to achieve successful
rotating stall control experiments by implementing axisymmetric air injection with various
injection configurations. Using pulsed air injection to control the onset of rotating stall, D'Andrea et al. [19, 20] and Behnken et al. [7] were able to slightly extend the operating range and eliminate the hysteresis loop (normally associated with rotating stall) in a low speed axial flow compressor. The control strategy consisted of measuring the unsteady pressure near the rotor face, and using an array of high bandwidth, binary, solenoid-driven valves (pulsed jet injectors) to implement a discrete, nonlinear, pulsed control algorithm. The control algorithm determines the magnitude and phase of the first mode of rotating stall, and controls the air injected in front of the the rotor face. Behnken et al. [9] and Yeung et al. [137, 136] were able to eliminate the hysteresis loop associated with rotating stall and decrease the likelihood of surge in a low speed axial flow compressor by combining the air injection controller in [19, 20, 7] and the surge controller introduced by Badmus et al. [5] and Eveker et al. [36].

1.4 Motivation and Objectives

Most of the previous work on active rotating stall and surge control has been focused on demonstration of the concept of active control. Implementation of active control in industrial applications such as aircraft engines, however, depends on the cost and size of the actuation system. Therefore, the next generation of active control experiments should be focused on reducing the actuation requirements, and implementing control schemes that maximize performance with limited actuation. For example, the implementation of active control using air injectors in an aircraft engine will be limited by the availability of high pressure air and the size of the actuators.

The focus of this research is to develop tools that can be used to meet some of the challenges associated with the cost and size of actuation system. A feasible source of high pressure air is to recirculate air from the downstream stage of a multistage compressor. To minimize efficiency penalties, the amount of air made available for control will be limited. In addition, the air will be supplied at high temperatures. Therefore, one of the goals of this research is to develop active control schemes requiring substantially less mass flow and/or fewer actuators. Another goal of this research is to develop a verified model for steady air injection that can be used to investigate the feasibility of using high temperature air for active rotating stall control in high speed compressors.
An important tool for designing and evaluating control laws for active control studies in rotating stall and surge is a simple reliable model that captures the essential physics of the instabilities. Computational fluid dynamic (CFD) models exist for both modal and short length scale stall inception [119, 99, 118, 52]. These are not low order models to be used for control law design. Escuret and Elder [35] and Badmus et al. [3] have proposed various 1-D models for use in control law design. These models are useful for only a limited class of compressors because they ignore the essential physics of rotating stall, which is a two- or three-dimensional phenomenon. A suitable model of limited dimensionality that has been very useful for control law design in low speed machines is the three state Moore-Greitzer model [91, 92]. The three state Moore-Greitzer model has been extensively used to design and implement (1D) axisymmetric nonlinear control laws and (2D) non-axisymmetric control laws. However, the three state model does not model the second and higher harmonics, which have been shown to interact strongly with the lower harmonics during stall inception. To account for this coupling between the first and higher harmonics, Mansoux et al. [84, 85] developed a high fidelity nonlinear control theoretic formulation of the Moore-Greitzer rotating stall model that is also suitable for control analysis and design. This model will be modified in this research to incorporate the effects of air injection, and used to design nonlinear control laws.

The two-dimensional, linearized stability model of Moore and Greitzer [91, 92] was extended to the compressible flow regime by Bonnaure [13] and Hendricks et al. [63] to describe the stall inception process in high speed compressors. Bonnaure's model was converted to a control-theoretic input-output formulation by Feulner [38, 39]. Hendricks et al. [65] extended Feulner's linear, compressible model to a nonlinear, numerical simulation of rotating stall and surge inception in high speed compressors. Weigl [130] and Fréchette [41] refined and validated Feulner's model on a transonic single stage compressor. However, the current state-of-the-art is able to capture the qualitative behavior of rotating stall and surge inception in the transonic compressor but is unable to accurately capture the exact modal stability and input-output dynamics. One of the goals of this research is to improve the model so that the input-output dynamics can be captured for air injection actuation.

Unsteady disturbances in the compressible model are modeled as small-amplitude harmonic fluctuations about a steady background flow. Thus the steady flow parameters are inputs
to the compressible model. Using an error propagation analysis on a representative 11-stage high speed axial flow compressor, Fréchette [41] found the total uncertainty range on surge margin prediction due to input errors to be 3.6% for a nominal surge margin of 21.2%. The main sources of uncertainty were found to be the steady flow parameters (contributing almost 75% of the uncertainty) and the time lag constant for losses (contributing almost 25% of the uncertainty). Therefore, the first step in improving the compressible model is to develop a model for predicting the steady background flow.

The goals of this research can be summarized as follows:

- Develop control laws that extend the operating range of a transonic compressor with substantially less mass flow and/or fewer actuators.
- Develop and apply verified models of the steady behavior of air injection in high speed axial flow compressors.
- Refine and validate an existing modal stall inception compressible model.

1.5 Thesis Outline

This thesis is organized in two parts: the first part presents the air injection model, experimental verification of the model, and a feasibility study using the model; the second part presents new control concepts for reducing the actuation requirements, and results from control experiments. A description outlining the contents of each chapter follows.

Chapter 2: Experimental Setup
The test facility used for this research is described. Details of the instrumentation used for steady and unsteady measurements are presented. The actuators used are described, the calibration procedure for these actuators is given and the frequency response of the actuators are presented.

Chapter 3: Modeling the Effects of Air Injection in Axial Compressors
This chapter presents models for the effects of steady air injection on the performance of high speed axial flow compressors, and a modified version of the theoretical stall inception model for high speed machines with air injection actuation.
Chapter 4: Experimental Validation of Air Injection Models
In this chapter, models for the steady effects of air injection are validated experimentally using total pressure surveys, total temperature surveys, and speedlines for NASA Stage 35. Forced response measurements of the pre-stall dynamics for NASA Stage 35 are used to validate the theoretical stall inception model.

Chapter 5: Feasibility Study using the Steady Air Injection Model
In this chapter, a feasibility study is performed using the verified steady air injection model. The feasibility study examines the effect of injecting high temperature air into a transonic compressor.

Chapter 6: New Control Concepts for Rotating Stall and Surge Control
Control concepts for reducing the actuation requirements for rotating stall and surge suppression are presented in this chapter. The actuation scheme and control laws for reducing the amount of mass required are presented; the control techniques are evaluated using nonlinear simulations; and the control concept for reducing the number of actuators is presented.

Chapter 7: Experimental Results from Control Experiments
This chapter presents results from control experiments verifying the control concept for reducing the amount of mass required for rotating stall and surge suppression, and results from control experiments validating the proposed actuation configurations for reducing the number of actuators.

Chapter 8: Conclusions
In this chapter, a general overview of the modeling and control aspects of this research are presented. The results and conclusions from experiments and simulations are summarized. Finally, a future course of action is recommended.
CHAPTER 2

EXPERIMENTAL APPARATUS

An overview of the hardware and software used for this research is presented in this chapter. The compressor test rig and instrumentation are described in Section 2.1, the jet actuation system used for implementing the control schemes is described in Section 2.2, and the control data acquisition system is described in Section 2.3.

2.1 Testbed for Control Experiments

In this section, the test facility, design performance parameters of the transonic compressor, and instrumentation layout showing the configuration of sensors and actuators on the test section are described.

2.1.1 Test Facility

The experiments reported in this thesis were conducted in the NASA Lewis Research Center (LeRC) Single-Stage Axial Compressor Test Facility shown in Figure 2-1. This is an open circuit facility for testing advanced compressor stages such as NASA Stages 35, 37, and 38 which are representative of the inlet, middle, and rear stages for an eight-stage 20:1 pressure ratio core compressor. A 2.2 Megawatt DC drive motor drives the compressor in the test section. Air at atmospheric pressure passes through a calibrated thin-plate orifice and upstream plenum which contains a straightening screen to ensure uniform flow in the test section. At the test section, the air goes from the upstream duct through the compressor stage where the pressure is increased to the downstream duct, through the throttle and then exhausted to the atmosphere. The throttle is a sleeve-type valve.
which regulates the amount of air going through the compressor by adjusting the exit area. The geometries of NASA Stages 35, 37, and 38 are summarized in Table 2.1 and their corresponding overall design parameters are listed in Table 2.2. For more details on the design and blade-element performance measurements for these stages, the reader is referred to references [103, 104, 105, 106].

Figure 2-1: Schematic of NASA LeRC compressor test facility from [104].

2.1.2 Instrumentation Layout

The test section was instrumented with jet actuators, steady sensors, and unsteady sensors. A meridional cross-sectional view of the test section showing the axial locations of the actuators, steady sensors, and unsteady sensors is given in Figure 2-2. The actuator, which consists of a servo motor, a sleeve valve and an injector is described in more detail in Section 2.2. The steady flow field sensors consist of hub and casing static pressure sensors, upstream plenum pressure and temperature sensors, downstream total pressure and total temperature rakes, and Kiel-headed total pressure and total temperature area traverse probes for measuring steady profiles. The unsteady sensors consist of high bandwidth wall static pressure sensors which measure pre-stall perturbations for system identification and
control experiments, and unsteady total pressure cobra probe for measuring unsteady total pressure profiles.

**Steady Measurements:** The NASA LeRC test facility steady state compressor performance instrumentation is known as the ESCORT system and is described in detail by Bruckner et al. in reference [15]. The steady measurements made using this ESCORT system include static pressure, static temperature, total pressure, total temperature, and mass flow rate. During compressor operation, the ESCORT system regularly calibrates the steady sensors, and continuously updates and displays the steady measurements. The locations of the steady pressure and temperature sensors throughout the test section including the standard NASA nomenclature used to identify the axial stations tabulated in Table 2.3 of reference [130] are shown in Figure 2-3. For uniform inlet flows where the static pressure is fairly uniform radially, the steady state static pressure at a given axial location was computed by averaging the hub and casing wall static pressure measurements. The steady wall static pressures at Stations B, C, D, F, I, and K were also used as reference pressures for calibrating the unsteady pressure transducers located at those same axial locations. Kiel-headed total pressure and total temperature area traverse probes at Station J are used for measuring steady radial and circumferential profiles. The total pressure and temperature
Table 2.2: Design parameters for NASA Stages 35, 37, and 38 from [103, 104, 105, 106].

<table>
<thead>
<tr>
<th></th>
<th>Rotor 35</th>
<th>Rotor 37</th>
<th>Rotor 38</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total pressure ratio</td>
<td>1.865</td>
<td>2.106</td>
<td>2.105</td>
</tr>
<tr>
<td>Total temperature ratio</td>
<td>1.225</td>
<td>1.270</td>
<td>1.269</td>
</tr>
<tr>
<td>Adiabatic efficiency</td>
<td>0.865</td>
<td>0.877</td>
<td>0.878</td>
</tr>
<tr>
<td>Stage 35</td>
<td>Stage 37</td>
<td>Stage 38</td>
<td></td>
</tr>
<tr>
<td>Total pressure ratio</td>
<td>1.820</td>
<td>2.050</td>
<td>2.050</td>
</tr>
<tr>
<td>Total temperature ratio</td>
<td>1.225</td>
<td>1.270</td>
<td>1.269</td>
</tr>
<tr>
<td>Adiabatic efficiency</td>
<td>0.828</td>
<td>0.842</td>
<td>0.844</td>
</tr>
<tr>
<td>Flow coefficient</td>
<td>0.451</td>
<td>0.453</td>
<td>0.448</td>
</tr>
<tr>
<td>Mass flow rate</td>
<td>20.188 kg/s</td>
<td>20.188 kg/s</td>
<td>20.188 kg/s</td>
</tr>
<tr>
<td>Rotor tip speed</td>
<td>454.456 m/s</td>
<td>454.136 m/s</td>
<td>455.096 m/s</td>
</tr>
<tr>
<td>Rotor rotational frequency</td>
<td>286.5 Hz</td>
<td>286.5 Hz</td>
<td>286.5 Hz</td>
</tr>
</tbody>
</table>

downstream of the compressor was computed by mass averaging the rake measurements at Station K. The uncertainties in the steady compressor performance measurements are reported in Table 2.4 of reference [130]. The orifice mass flow is the amount of air flowing through the plenum and inlet duct upstream of the compressor. The total mass flow through all twelve injectors was measured with a separate venturi meter in the injector air supply duct. The total mass flow through the compressor is the sum of the corrected orifice and actuator mass flows.

Unsteady Measurements: The unsteady measurements consisted of pressure transients measured with high bandwidth Kulite pressure sensors during the system identification and control runs. The axial and circumferential locations on the casing where the Kulite pressure sensors can be placed are summarized in Table 2.5 of reference [130]. Weigl [130] determined that the eight sensors at Station F\(^1\) provided the highest signal to noise ratio near stall and were therefore utilized for system identification and control experiments. Consult reference [130] for more details on the high bandwidth pressure transducers.

\(^1\)The eight wall static Kulite pressure sensors at Station F were located at the following circumferential locations: 15°, 75°, 105°, 165°, 195°, 255°, 285°, and 345°. The [1, 0] mode was found to be the critical mode driving the transonic compressor to instability. According to the Moore-Greitzer model, the mode shape of this [1, 0] mode decays axially away from the compressor. Therefore, the location close to the compressor is the most desirable.
2.2 Jet Actuation System

The actuation scheme for this research (injection of high pressure air into the compressor) was implemented with a jet actuation system which converts an electrical input signal into
a fluid dynamic output (injected mass). The jet actuation system consists of a control unit which was manufactured by Moog Inc. [102] and an array of twelve jet actuators designed by Berndt [10]. Figure 2-4 shows a block diagram representing the various components of the jet actuation system. As shown in Figures 2-2 and 2-3, the actuators are located at 5.97 cm (1.07 rotor chord lengths) upstream of the rotor leading edge (Station E). The twelve actuators are evenly spaced at 30° increments; the first is located at 0° and the twelfth at 330°. Oil free, laboratory compressed air is supplied to the actuators through a large circular plenum, and four 1.27 cm diameter hoses connect each injector to the supply plenum. The details of the assembly, calibration, and operational procedures for the jet actuation system are given in Appendix A.

2.2.1 Electrical Control Unit

The control unit is a 12-channel electronic system that monitors and controls the position of the servo motor and was manufactured by Moog Inc. [102]. It requires a 115VAC/10A and two single-phase 208VAC/30A power supplies to run, and handles both static and dynamic input signals (or commands) which control the servo motor position. When operating in the static mode, each of the twelve actuators can be independently commanded by a potentiometer on the electronics cabinet, and when operating in the dynamic mode (for example during control runs), each of the twelve actuators can be independently commanded.
through a BNC connector on the cabinet.

The motor controller contains a noncontact displacement measuring system which senses the motor position. Kaman Instrumentation Corporation, the manufacturer, describes the sensors as follows [73]:

The Kaman sensing unit uses an inductive technique to determine the position of a conductive target relative to the system sensor as illustrated in Figure 2-5. An AC current flows through the sensor coil, generating an electromagnetic field which radiates out from the sensor. As the conductive target (made from Aluminium) enters this field, the sensor induces a current flow which in turn produces a secondary opposing field, reducing the intensity of the original. This opposing electromagnetic field results in an impedance variation in the sensor coil. The sensor coil makes up one leg of a balanced bridge network. As the target changes position within the sensor field, the bridge network senses impedance changes in the sensor coil and passes the information on to signal conditioning electronics for conversion to an analog voltage. This voltage is directly proportional to target displacement. Since nonconductive materials intervening between the sensor and target have little or no effect on system output, environmental contaminants such as oil, dirt, humidity, and magnetic fields have virtually no effect on system performance.

![Diagram](image)

**Figure 2-5:** Kaman sensing unit from [73].

The sensed position outputs from the motor electronics are low-pass filtered at 1.0 kHz and amplified with a gain of 5 with Preston 830XWB filters. Using an analog feedback circuit, the motor controller outputs a coil current which is proportional to the sum of the position and velocity error signals. The controller gain for each of the servo motor was tuned by Moog Inc. to maximize the bandwidth.
2.2.2 Actuator Components

The jet actuator was designed by Berndt [10, 11], and consists of a servo motor, a sleeve valve, and an injector. A cross-section of the actuator layout showing its components is given in Figure 2-6. The servo motor is the prime mover that opens and closes the valve. It is a linear force electro-mechanical servo motor from Moog Inc. that employs a variable reluctance magnetic flux path to generate force on the motor armature. A valve sleeve which weighs about 30 gram is attached to the motor armature (shaft). The valve sleeve slides over a cylinder which contains a slotted orifice and separates the supply air from the injector, thus regulating the slot orifice area and modulating the choked flow through the orifice. High pressure air enters the valve body and passes through the slotted central cylinder, and exits from the valve center into the injector. The injector imparts momentum to the injectant flow and directs the flow from the valve into the upstream of the compressor. Berndt designed two injection schemes; a sheet injector with fluid dynamic influence limited radially to the outer 15% of the compressor annulus (affecting mainly the tip region of the blade span), and a 3-hole injector with fluid dynamic influence extending radially to the outer 40% of the compressor annulus. The sheet and 3-hole injectors were designed such that they could inject air to the compressor at $-30^\circ$, $-15^\circ$, $0^\circ$, $+15^\circ$, and $+30^\circ$ yaw\textsuperscript{2}. A complete description of the development of the actuator prototype can be found in reference [10].

\textsuperscript{2}The circumferential angles are referenced from the top dead center of the casing. By convention, an angle is positive in the direction of rotor rotation (the rotor angular velocity vector is directed downstream). Both the sheet and 3-hole injectors at different orientations were tested on NASA Stage 35 by Weigl et al. [131] and the sheet injector at $-15^\circ$ yaw was selected for the control applications.
2.2.3 Actuator Calibration

Each of the twelve actuators in the jet actuation system was assembled and calibrated to meet the design specifications of the prototype given in Table 2.3. The design specifications require that the mass flow rate for each actuator varies linearly from 0.02 \text{ kg/s} at the fully closed position to 0.10 \text{ kg/s} at the fully opened position when supplied with 100 \text{ psig} air. To meet each of these flow levels, the vertical position of the valve sleeve relative to the slot opening was set by individually shimming each of the twelve sleeve valves on the servo motor shafts. The radial clearance between the valve sleeve and cylinder is 25 \text{ \mu m}. Berndt designed the valve with such a small clearance in order to reduce frictional damping without producing a large leakage flow when the valve is fully closed. However, with such a close tolerance, special care had to be taken in assembling the valve so that the sleeve
does not rub against the cylinder when it moves while regulating the slot orifice area. A complete description of the centering procedure used for assembling the valves is described in Appendix A.

### Table 2.3: Valve design specifications from [10].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>units</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Supply pressure (gauge)</td>
<td>kPa</td>
<td>700</td>
</tr>
<tr>
<td>Valve command</td>
<td>V</td>
<td>±10V</td>
</tr>
<tr>
<td>Valve stroke</td>
<td>in</td>
<td>0.02</td>
</tr>
<tr>
<td>Radial clearance</td>
<td>μs</td>
<td>25</td>
</tr>
<tr>
<td>Maximum mass flow</td>
<td>kg/s</td>
<td>0.10</td>
</tr>
<tr>
<td>Minimum mass flow</td>
<td>kg/s</td>
<td>0.02</td>
</tr>
<tr>
<td>Orifice flow area</td>
<td>mm²</td>
<td>60</td>
</tr>
<tr>
<td>Bandwidth (-3dB)</td>
<td>Hz</td>
<td>400</td>
</tr>
<tr>
<td>Phase at -3dB</td>
<td>deg</td>
<td>100</td>
</tr>
</tbody>
</table>

The linear electro-mechanical servo motor used to move the valve sleeve employs a variable reluctance magnetic flux path to generate force on the motor armature. The hysteresis due to the induction coils that vary the magnetic field in the flux path for different cavity pressures is shown by the steady state characteristic plots of valve sleeve position versus electrical input command in Figure 2-7 and the coil current versus electrical input command in Figure 2-8. The solid line represents the steady path followed when the input command voltage was varied from -10 V to +10 V, and the dashed line represents the path followed when the input command voltage was varied from +10 V to -10 V. As shown in Figure 2-9a, for a given valve setting, the valve sleeve was observed to move down (closing the slot orifice area) as the air supply pressure (and equivalently the cavity pressure) was increased. The corresponding changes in coil current with increasing cavity pressure is shown in Figure 2-9b. The pressure forces cause the sleeve to move down (i.e., closes the valve) because the valve sleeve is supported by a motor spring which has a spring constant of 1000 lbm/in. The actuators were calibrated such that when the input voltage command was 0.0 V, the valve sleeve was at the mean position of 0.0 inch (corresponding to a coil current of 0.0
A) at the design supply pressure of 100 psig. Figure 2-10a shows the measured steady state characteristics of the actuator at 0%, 50%, and 100% valve opening. The relationship between the mass flow rate and the air supply pressure (or cavity pressure) is linear for most of the operating pressure range because the valve orifice is choked. The measured steady state actuator characteristics at the design supply pressure of 100 psig is shown in Figure 2-10b. When the valve is fully closed, there is a leakage flow through the valve caused by the radial clearance of 25μm between the valve sleeve and cylinder. The relationship between the mass flow rate and the input command voltage is linear from the fully closed position (-10 V input command) to the fully open position (+10 V input command). Therefore, the actuator is a linear mass flow modulator. The total mass flow rates from all twelve actuators operating at design conditions are summarized in Table 2.4 which also lists the percentage of the design compressor mass flow for the fully closed, half open, and fully open valve positions. The steady state valve characteristics for each of the twelve actuators is given by:

\[ m(t) = b x(t) + c \]  

(2.1)

where \( m \) is the total mass flow rate in kg/s going into the injector, \( x \) is the normalized valve sleeve position\(^3\), and the constants \( b \) and \( c \) depend on the air supply pressure. For any actuator operating at design conditions of 100 psig air supply pressure, \( b = 0.0736 \) and \( c = 0.0246 \text{kg/s} \). The calibration constants \( b \) and \( c \) at different supply pressures for each actuator can be obtained from steady valve characteristics (similar to the one in Figure 2-10b) generated from the steady actuator calibration characteristics in Figure 2-10a. The complete set of calibration measurements for all twelve actuators are given in Appendix A.

<table>
<thead>
<tr>
<th>Valve Position</th>
<th>Total Mass Flow Rate (kg/s)</th>
<th>% of Compressor Design Mass Flow</th>
<th>Input Command (V)</th>
<th>Position Sensor (inch)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fully closed</td>
<td>0.295</td>
<td>1.46</td>
<td>-10</td>
<td>0.01</td>
</tr>
<tr>
<td>Half open</td>
<td>0.735</td>
<td>3.64</td>
<td>0</td>
<td>0.0</td>
</tr>
<tr>
<td>Fully open</td>
<td>1.178</td>
<td>5.82</td>
<td>10</td>
<td>-0.01</td>
</tr>
</tbody>
</table>

\(^3\)The valve sleeve position is normalized such that sensor positions of -0.01 in, 0 in, and +0.01 in correspond to 0, 0.5, and 1.0 respectively.
To estimate the flow properties (Mach number, velocity, static pressure, static temperature, total pressure, and total temperature) of the jet coming out of the sheet injector, one of the injectors was instrumented with a static and total pressure probe. The probe was located in the rectangular slot just before the flow shoots out of the injector and into the compressor. The Mach number of the jet from the actuator with the valves at 50% and 100% opening when air was supplied at various pressures are shown in Figure 2-11.
Valve sleeve position hysteresis plots at 0 psig and 100 psig cavity pressure.

(a) Valve sleeve position hysteresis plot at 0 psig cavity pressure

(b) Valve sleeve position hysteresis plot at 100 psig cavity pressure

Figure 2-7: Valve sleeve position hysteresis plots at 0 psig and 100 psig.
Figure 2-8: Coil current hysteresis plots at 0 psig and 100 psig.
Figure 2-9: Effect of injector cavity pressure on the valve sleeve position and motor coil current.
(a) Steady actuator characteristics at 0%, 50%, and 100% valve opening

(b) Steady valve characteristics at 100 psig air supply pressure

Figure 2-10: Actuator steady state characteristics.
Figure 2-11: Injector exit Mach number versus supply pressure for sheet injectors at 50% and 100% valve opening (from [14, 117]).
2.2.4 Actuator Dynamics

The dynamics of the jet actuation system are represented by a transfer function from the input electrical signal (input command voltage) to the fluid dynamic output (injected mass or velocity of jet from the injectors) sensed at the face of the compressor. The overall system dynamics can be broken down into four components: the force motor dynamics, the valve dynamics, the injector dynamics, and the time delay for the fluid to travel from the injector exit to the face of the compressor.

**Electro-Mechanical Force Motor:** The frequency response of the actuator is limited by the first order response time due to induction of the coils that vary the magnetic field in the flux path. As part of the calibration of the jet actuation system, the transfer functions of the twelve servo motors were measured at 0 psig and 100 psig cavity pressures. The transfer function from the input command\(^4\) to the valve sleeve position\(^5\) for one of the servo motors is shown in Figure 2-12. The complete set of servo motor transfer functions for all twelve actuators are given in Appendix A. A model fit consisting of two zeros and three poles was found to sufficiently represent the servo motor dynamics over the frequency range of 0 - 500 Hz (see Figure 2-12) which is the frequency range of interest for the control experiments in this research. The polynomial representation of the model fit to the servo motor dynamics is given by:

\[
\frac{\bar{x}(s)}{\bar{u}(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} \tag{2.2}
\]

where \(\bar{x}\) is the perturbation from the mean of the normalized valve sleeve position, and \(\bar{u}\) is the perturbation from the mean of the normalized input command.

\(^4\)The input command is normalized such that inputs of -10 V, 0 V, and +10 V correspond to 0, 0.5, and 1.0 respectively.

\(^5\)The valve sleeve position is normalized such that sensor positions of -0.01 in, 0 in, and +0.01 in correspond to 0, 0.5, and 1.0 respectively.
Valve: The valve is modeled as an isentropic nozzle. The models in reference [69] for throttling and flow through an isentropic nozzle are used to model the valve. The mass flow rate, $\dot{m}$, from an upstream plenum “u” to a downstream plenum “d” through an isentropic nozzle depends upon the pressure ratio, $P_r = \frac{P_d}{P_u}$, rather than the pressure drop $P_u - P_d$, across the nozzle. The mass flow rate, $\dot{m}$, is given by:

$$\dot{m} = C_d A \frac{P_u}{\sqrt{T_u}} \sqrt{\frac{2\gamma}{R(\gamma - 1)}} \left( \sqrt{P_r^{2/\gamma}} - P_r^{(\gamma+1)/\gamma} \right)$$  \hspace{1cm} (2.3)$$

where $C_d$ is the discharge coefficient, typically $\approx 0.5$, $A$ is the orifice area, $\gamma$ is the ratio of specific heats, $P_u$ is the pressure of the upstream plenum, $T_u$ is the temperature of the upstream plenum, and $P_d$ is the pressure of the downstream plenum. For $P_r \leq P_{r_{\text{crit}}} = \left[ \frac{2}{\gamma+1} \right]^{\gamma/(\gamma-1)}$, the flow is choked, and the mass flow rate is independent of the downstream pressure, $P_d$. To calculate the mass flow rate through the valve: if $P_r > P_{r_{\text{crit}}}$ i.e., subsonic...
flow, then substitute $P_r = \frac{P_r}{P_{\text{so}}}$ into equation 2.3, but if $P_r \leq P_{r_{\text{crit}}}$ i.e., supersonic flow, then substitute $P_r = P_{r_{\text{crit}}}$ into equation 2.3.

If the flow is choked and the air supply stagnation pressure and temperature are constant, the valve position (and equivalently the slot orifice area) will be linearly related to the mass flow rate through the slot orifice into the injector, and because the flow-through time is very small (10 μs), the valve acts quasi-steadily. Therefore, the valve portion of the actuator is modeled as a quasi-steady, linear mass flow regulator. The quasi-steady behavior between the valve sleeve position and the mass flow rate going into the injector is represented by the transfer function:

$$\frac{\dot{m}(s)}{\ddot{x}(s)} = b$$  \hspace{1cm} (2.4)$$

where $\dot{m}$ is the perturbation from the mean of the mass flow rate in kg/s going into the injector, $\ddot{x}$ is the perturbation from the mean of the normalized valve sleeve position, and the constant $b = 0.0736$ for each actuator at the design air supply pressure of 100 psig. Values for the constant $b$ at different supply pressures can be obtained from steady valve characteristics (similar to the one in Figure 2-10b) generated from the steady actuator calibration characteristics in Figure 2-10a.

**Injector:** The fluid dynamic response of the injector portion of the actuator was found to be characterized by a quasi-steady spatial distribution of momentum flux, modified temporally by a first order lag due to the internal fluid dynamics of the injector. Berndt showed that the sheet injector dynamics was captured by a first order Helmholtz resonator (see Appendix A). The transfer function of the injector dynamics is given by:

$$\frac{\ddot{u}_j(s)}{\dot{m}(s)} = \frac{1}{\rho_j A_j \frac{V_p}{a_p}} \frac{1}{T s + 1}$$  \hspace{1cm} (2.5)$$

where $T$ is the time constant given by $T = \frac{V_p}{\rho_j A_j a_p} V_p$ is the volume of the injector plenum, $A_j$ is the injector exit area, $\rho_j$ is the density of the jet from the injector, $\ddot{u}_j$ is the mean velocity of the jet from the injector, and $a_p$ is the speed of sound in the injector plenum. Figure 2-13 shows the sheet injector time constant for different compressor inlet duct mass flow rates with the actuator valve 50% open and air supplied at 100 psig. The injector time
constant depends on the exit jet velocity which is set by the static pressure at the injector exit (see equation A.1 in Appendix A). But the static pressure at the injector exit is set by the flow conditions in the compressor inlet duct. As the compressor inlet duct mass flow rate changes, the inlet duct dynamic head $\frac{1}{2}pu_{in}^2$ will change. Since the inlet duct total pressure $P_t = P + \frac{1}{2}pu_{in}^2$ remains constant, the static pressure will change with mass flow rate. Therefore, the injector time constant changes with the compressor inlet duct mass flow rate because the inlet duct static pressure (and equivalently the static pressure at the injector exit and hence jet exit velocity) changes.

![Figure 2-13: Sheet injector time constant for different compressor inlet duct mass flow rates with actuator valve 50% open and air supplied at 100 psig.](image)

**Convective Time Delay:** The final important dynamic effect of the actuation system is the pure time delay accounting for the convective lag of the fluid jet travelling from the injector to the face of the compressor. This convective time delay is estimated based on the distance between the injectors and the rotor blades, and the mean exit velocity of the jet.
The unsteady velocity at the face of the compressor is given by:

$$\tilde{u}_c(t) = \tilde{u}_j(t - \tau)$$  \hspace{1cm} (2.6)

where $\tilde{u}_c(t)$ is the unsteady velocity at the compressor face, $\tilde{u}_j(t)$ is the unsteady velocity of the jet from the injector, and the convective time delay is the ratio of the distance between the injector and the compressor inlet, $x$, to the mean exit velocity of the jet $\bar{u}_j$ i.e., $\tau = \frac{x}{\bar{u}_j}$. For the flowpath configuration of NASA Stage 35, $x = 0.0597m$. The mean exit jet velocity, $\bar{u}_j$, depends on the type of injector, the level of injection, the air supply pressure and temperature, and the flow conditions at the compressor inlet duct. The mean exit velocity of the jet, $\bar{u}_j$, for different compressor inlet duct mass flow rates was determined using Berndt's actuator model which is summarized in Section A.3. Figure 2-14 shows the convective time delay for different compressor inlet duct mass flow rates with the sheet injector 50% open and air supplied at 100 psig.

![Figure 2-14: Convective time delay for different compressor inlet duct mass flow rates with the sheet injector 50% open and air supplied at 100 psig.](image)
2.3 Data Acquisition and Control

This section presents an overview of the data acquisition, processing and control. This system is separate from the ESCORT system (steady state performance instrumentation system) described in Section 2.1.2. The data acquisition system was used to measure unsteady wall static pressures during stall transients, conduct forced response tests, and implement feedback control laws. The data was sampled at 3.0 kHz which is roughly ten times the design rotor frequency. High bandwidth is necessary to ensure good frequency resolution for spectral analysis and to minimize the effect of controller time delays. The time delay associated with the anti-alias filtering, computer calculations, and zero-order hold was estimated by comparing the measured actuator servo-motor transfer functions with independent analog measurements of the servo motor dynamics. A conservative estimate of the time delay was found to be equal to 1.5 sampling periods (0.5 ms) [129]. A detailed documentation of the hardware used for this research can be found in Section 2.5 of reference [130]. The data acquisition and control software is written in ‘C’ and optimized for speed with a Watcom C/C++ compiler. The software was modified from that of Weigl [129] to implement the nonlinear control laws and the MIMO control laws for the reduced actuation concepts discussed in this thesis. A single program performs sensor calibration, stall transient measurements, forced response testing, and feedback control. The system is capable of implementing spatial Fourier calculations and dynamic compensators with up to 60 states at 3.0 kHz. The data is stored in a circular buffer that is triggered manually with the keyboard. MATLAB is the primary software tool used for data reduction, analysis, and plotting.
This chapter describes the steady and unsteady models for air injection in high speed axial flow compressors. In Section 3.1, the approach that was used to model the effect of steady air injection on the performance of high speed axial flow compressors is presented. The steady air injection model breaks up the effect of air injection into two main components: changing the flow profiles coming into the compressor (this component consists of a mass effect, a momentum effect, and an inlet swirl effect), and the response of the compressor blade rows to variable inlet profiles. The change in the compressor inlet flow profile resulting from steady air injection was modeled using wind tunnel tests and control volume analysis, and the response of the compressor blade rows to the inlet flow profiles are modeled using a streamline curvature method in analysis mode. The steady air injection model is useful because it provides the mean values of the nonuniform steady background flow effects required in unsteady models.

In Section 3.2, the global dynamic (i.e., rotating stall) behavior of high speed compressors, in response to unsteady air injection, is modeled using the 2D compressible rotating
stall inception model described by Bonnare [13]. This is a linearized model in which each flow quantity consists of a steady mean value and a small-amplitude perturbation. Over the years, this compressible model has been modified by Feulner [38], Fréchette [41], and Weigl[130]. However, the state-of-the-art in 2D compressible modeling is only able to capture the qualitative behavior of rotating stall and surge inception in high speed compressors; it is unable to accurately capture the exact modal stability and input-output dynamics. The modifications in Section 3.2 are aimed at providing a more quantitatively accurate model. A review of the current state-of-the-art is presented, followed by modifications to incorporate the effects of air injection. The primary modifications are: (1) the actuator boundary condition is modified to incorporate the changes in mean flow across the jet actuator; (2) total pressure loss is modeled at the blade row trailing edge instead of the leading edge; (3) blade row performance characteristics are made actuation-dependent; (4) a boundary condition that models the change in disturbances in a flowpath with variable cross-sectional areas is added; and (5) a generalized complex acoustic impedance relating the static pressure and axial velocity perturbations is added at the end conditions. The primary application of the compressible rotating stall inception model is for studying the effects of compressor design parameters on stability, and for designing and evaluating feedback control laws.

The steady air injection model is used in Chapter 5 to predict the effect of injecting hot air to a transonic compressor. This application is motivated by the fact that to implement active control in aeroengines using air injection, a feasible source of air is to recirculate air from a downstream stage of a multistage compressor. Such a source will supply air at a high pressure and high temperature. The model can be further used to investigate the effectiveness of air injection in multistage compressors, and to perform a parametric study for determining the optimal stage from which air can be recirculated.

3.1 Steady Air Injection Model

The Steady Air Injection model presented in this section predicts the circumferentially mass averaged spanwise flow profiles at various axial locations along the compressor flowpath, given the performance characteristics for each of its blade rows, when subjected to steady air injection. The model is described here for the experimental setup discussed in Chapter 2, i.e., the jet actuator is placed in the upstream duct of a single stage transonic compressor.
However, it can be extended to multistage compressors with the jet actuator placed at different axial locations. The components of the compressor model consist of a jet actuator in an upstream duct for injecting high pressure air, a rotor, an inter-blade row gap, a stator, and a downstream duct. The model accounts for compressibility effects, swirl sensitivity and radial non-uniformity of the flow profiles resulting from air being injected at various radial locations. The modeling is divided into two components: the jet actuator in a duct and the compressor blade rows. Figure 3-1 shows a schematic overview of the air injection model components for a single stage compressor with a jet actuator in the upstream duct. Three computing stations are shown to identify the two components of the model. The first component accounts for the mixing that takes place in the upstream duct, between the jet from the actuator and the flow in the inlet duct. This model generates a radial flow profile at the compressor face. The second component accounts for the interaction of this radial profile with the compressor. With this approach, profiles measured in a wind tunnel are used to specify the profiles at the injector plane. Wind tunnel measurements at the injector plane instead of the rotor inlet plane are used because the actual compressor flowpath is convergent but the wind tunnel was not. Convection of the profile from the injector plane to the rotor inlet will be accounted for in the streamline curvature code. The flow profile at the rotor inlet will thus be computed based on the interaction between the upstream duct, the blade rows and the downstream duct.
3.1.1 Jet Actuator in Upstream Duct

This component of the injection model corresponds to the control volume bounded by the mixing plane and the injector plane shown in Figure 3-1. The key mechanism for the operation of the jet actuator in a duct is fluid mixing. Wind tunnel tests by Berndt [10] were used to determine the mixed out profiles of the jet actuator. Since the tests were conducted on a rectangular wind tunnel of uniform cross-sectional area and the compressor has a convergent flowpath, the model procedure separates the mixing effect from the nozzle effect. The model assumes that partial mixing first takes place in a uniform upstream duct and the mixed out flow is then convected downstream from the injector plane through the convergent duct. This approach of separating the mixing effect and the nozzle effect was adopted in order to have a modeling approach that will be able to account for changes in the profile shapes that result from interaction with the blade rows and downstream duct.

Compressible control volume analysis was used to analyze the fluid dynamic pumping system
in the upstream duct. By making use of the limited number of profiles that were measured at a single operating condition, this control volume analysis gives a systematic approach for computing the profiles at different operating conditions. The primary stream (jet from actuator) and secondary stream (inlet background flow) are assumed to flow isentropically from known stagnation conditions to the mixing plane shown in Figure 3-1. The primary and secondary velocity profiles are assumed to be uniform at the mixing plane, and the partially mixed flow at the injector plane is assumed to have a velocity profile which can be determined from wind tunnel tests. If the distance between the mixing plane and the injector plane were infinite, so that both the primary and secondary streams were fully mixed out, the velocity profile at the injector plane would be uniform. Thus, as a measure of the nonuniformity of the velocity profile or degree of mixing, a mixing parameter $\beta$, is defined as:

$$\beta = \frac{\int_A \rho_2 V_2^2 dA}{\rho_\infty V_\infty^2 A_\infty}$$

A $\beta$ parameter of unity corresponds to a uniform velocity profile, which will be obtained for fully mixed out streams. $\beta$ is greater than unity for nonuniform velocity profiles resulting from partially mixed out streams. Thus the $\beta$ parameter is a measure of the degree of mixing that has taken place. The flow in the injector is assumed to be subsonic so that its exit static pressure is that in the compressor duct at the mixing plane. This implies that losses occur in the valve, reducing the total pressure in the injector from the high supply values (a supersonic injector would not have its exit static pressure equal to that in the compressor duct, but would have a shock structure reducing the static pressure to that of the compressor duct). The static pressure profile at the injector plane is also assumed to be uniform. Implicit in this assumption is the notion that the streamlines between the mixing plane and the injector plane are straight. Since the Mach number of the flows involved are high, the gases are considered to be ideal compressible gases with constant heat capacities.

**Control Volume Analysis**

The control volume approach provides a powerful method for obtaining the overall changes in flow properties in processes such as the jet pump system which is of interest to us. It consists of mass conservation, momentum conservation and energy conservation. Since this
approach requires the velocity profiles to be known, the 2-dimensional density, axial velocity and tangential velocity profiles at the injector plane are assumed to have the following form:

\[
\begin{align*}
\rho &= \rho_0 + \Gamma p \Phi(r, \theta, z, \theta_j, \dot{m}_j) \\
V_x &= V_{xo} + \Gamma V_x \Phi(r, \theta, z, \theta_j, \dot{m}_j) \\
V_{\theta} &= V_{\theta o} + \Gamma V_{\theta} \Phi(r, \theta, z, \theta_j, \dot{m}_j)
\end{align*}
\]

(3.2)

where \( \Phi(r, \theta, z, \theta_j, \dot{m}_j) \) is a shape function which depends on the distance between the mixing and injector planes, the injector type\(^3\), the mass flow being injected, \( \dot{m}_j \), and the orientation of the jet actuator, \( \theta_j \). Figure 3-2a shows the shape function for the sheet injector at 0° yaw with the valve 100% opened and air being supplied at 100 psig, and Figure 3-2b shows the shape function for the 3-hole injector at 0° yaw with the valve 100% opened and air being supplied at 100 psig. These shape functions are based on wind tunnel measurements from [10].

---

\(^3\)Two type of injectors were designed by Berndt: a "sheet" injector which injects air at the tip, and a "3-hole" injector which injects air over a wider radial span.
Conservation of Mass: The steady state mass conservation law is given by:

\[ \int_A \rho V_{rn} dA = 0 \]  

(3.3)

where \( V_{rn} \) is the outward normal component of velocity relative to the control surface. Applying the mass conservation law to the control surface enclosing the mixing and injector planes shown in Figure 3-1:

\[ \dot{m} = \dot{m}_{act} + \dot{m}_{in} \]

\[ = \rho_{act} V_{x_{act}} A_{act} + \rho_{in} V_{x_{in}} A_{in} \]  

(3.4)

\[ = \int_A \rho V_d dA \]

\[ = \rho_{\infty} V_{x_{\infty}} A_{\infty} \]

where the subscript “in” corresponds to the inlet duct background flows at the mixing plane, “act” corresponds to the jet stream from the actuator at the mixing plane, the integral is performed at the injector plane, and the subscript “\( \infty \)” corresponds to the case when there is complete mixing i.e., the flow properties that will result if the jet stream from the actuator and the inlet background flow were allowed to completely mix out over an infinitely long duct with no losses.

Conservation of Axial Momentum: The steady state axial momentum conservation law is given by:

\[ \int_A \rho V_x V_{rn} dA = \sum F_x \]  

(3.5)

where \( V_x \) is the axial component of velocity, and \( \sum F_x \) represents the axial component of the vector sum of all the forces exerted on the control volume. Applying the axial momentum conservation law to the control surface enclosing the mixing and injector planes shown in Figure 3-1:

\[ \int_A \rho V_x^2 dA - \left[ \rho_{act} V_{x_{act}}^2 A_{act} + \rho_{in} V_{x_{in}}^2 A_{in} \right] = P_{act} A_{act} + P_{in} A_{in} - PA \]  

(3.6)
and applying the axial momentum conservation law to the control surface between the mixing plane and an arbitrary fully mixed out location gives:

$$\rho_{\infty} V_{x_{\infty}} A_{\infty} - \left[ \rho_{\text{act}} V_{x_{\text{act}}} A_{\text{act}} + \rho_{\text{in}} V_{x_{\text{in}}} A_{\text{in}} \right] = P_{\text{act}} A_{\text{act}} + P_{\text{in}} A_{\text{in}} - P_{\infty} A_{\infty}$$

(3.7)

Remember that we are assuming $P_{\text{act}} = P_{\text{in}}$, and the mixing takes place in a uniform area duct $A = A_{\text{act}} + A_{\text{in}}$. This can be easily extended to the case where there is a change in area.

**Conservation of Tangential Momentum:** The steady state tangential momentum conservation law is given by:

$$\int_{A} \rho V_\theta V_r dA = \sum F_\theta$$

(3.8)

where $V_\theta$ is the tangential component of velocity, and $\sum F_\theta$ represents the tangential component of the vector sum of all the forces exerted on the control volume. Applying the tangential momentum conservation law to the control surface enclosing the mixing and injector planes shown in Figure 3-1:

$$\int_{A} \rho V_\theta V_x dA - \left[ \rho_{\text{act}} V_{\text{act}} V_{x_{\text{act}}} A_{\text{act}} + \rho_{\text{in}} V_{\text{in}} V_{x_{\text{in}}} A_{\text{in}} \right] = 0$$

(3.9)

and applying the tangential momentum conservation law to the control surface between the mixing plane and an arbitrary fully mixed out location gives:

$$\rho_{\infty} V_{\theta_{\infty}} V_{x_{\infty}} A_{\infty} - \left[ \rho_{\text{act}} V_{\text{act}} V_{x_{\text{act}}} A_{\text{act}} + \rho_{\text{in}} V_{\text{in}} V_{x_{\text{in}}} A_{\text{in}} \right] = 0$$

(3.10)

**Conservation of Energy:** The steady state energy conservation law is given by:

$$\int_{A} \rho V_r C_p T dA = \dot{Q}$$

(3.11)

where $\dot{Q}$ is the net heat flow into the control volume. Assuming a constant specific heat capacity $C_p$, and no external heat source (i.e., $\dot{Q} = 0$), the energy conservation law applied
to the control surfaces gives:

\[ H_t = \dot{m}_{act}C_pT_{act} + \dot{m}_{in}C_pT_{in} = \int_A \rho V_x C_p T dA = \dot{m}C_pT_{\infty} \quad (3.12) \]

Calibration Procedure

The procedure for determining the calibration constants \( \Gamma_\rho, \Gamma_{V_x}, \) and \( \Gamma_{V_r} \) defined in equation 3.2 for different operating conditions is outlined in this section. This calibration was achieved by numerically solving equations 3.4, 3.6, 3.9, and 3.12 simultaneously. The flow diagram of the numerical procedure for computing the calibration constants is shown in Figure 3-3.

First, a value is selected for \( \Gamma_\rho \) such that the density of the partially mixed flow at the injector plane lies in the interval \( \rho_o \leq \rho \leq (\rho_o + \rho_{act}) \). Second, \( \Gamma_{V_x} \) is determined using...
the mass conservation relation in equation 3.4 and the selected density profile (which is determined using the $\Gamma_\rho$ selected above). Third, the static pressure at the injector plane is determined using the axial momentum conservation relation in equation 3.6, the density profile (i.e., $\Gamma_\rho$), and the axial velocity profile (i.e., $\Gamma_{V_x}$) determined above. Fourth, the tangential velocity at the injector plane is determined using the tangential momentum conservation relation in equation 3.9, the density profile (i.e., $\Gamma_\rho$), and the axial velocity profile (i.e., $\Gamma_{V_x}$). Finally, all the flow parameters determined are used to check whether the energy conservation relation in equation 3.12 is satisfied. If the energy conservation is satisfied, the calibration parameters selected are used to determine the flow profiles at the injector plane. However, if the energy conservation relation is not satisfied, the calibration procedure is repeated with a new selection for $\Gamma_\rho$ until the energy conservation relation is satisfied to within a given tolerance.

Results from Calibration Procedure
In this section, the calibration results predicted by the control volume analysis are compared with wind tunnel measurements taken at the design mass flow rate. Berndt [10] measured the partially mixed profile for both the 3-hole and sheet injectors at various levels of blowing with a wind tunnel mass flow rate of 20 kg/s and the injectors at 0° yaw. Figure 3-4 shows the mass and swirl sensitivities of the momentum flux ratio in the wind tunnel. The momentum flux ratio, MFR, is defined as the ratio of the momentum flux of the partially mixed out profile to the momentum flux of the wind tunnel freestream, and is given by:

$$MFR = \frac{\rho V_x^2}{\rho_{in} V_{x_{in}}^2} \quad (3.13)$$

where $\rho$ and $V_x$ represent the density and axial velocity respectively of the partially mixed out profile, $\rho_{in}$ and $V_{x_{in}}$ represent the density and axial velocity respectively of the wind tunnel freestream. Figure 3-4a shows the model prediction of how the mass averaged momentum flux ratio changes with the wind tunnel mass flow rate for different amount of air injected at design operating conditions. The symbols at 20 kg/s indicate the corresponding values from Berndt's wind tunnel measurements. Figure 3-4b shows the effect of the jet actuator orientation on the momentum flux ratio at a wind tunnel mass flow rate of 20 kg/s. The swirl sensitivity plots are normalized by the corresponding values at 0° yaw.
(a) Mass sensitivity: variation of momentum flux with wind tunnel mass flow rate

(b) Swirl sensitivity: variation of momentum flux with the jet orientation

Figure 3-4: Mass and swirl sensitivities for the sheet injector at different injection levels in the wind tunnel.
A parametric study was performed to investigate the mass and swirl sensitivities of the steady flow field quantities in the upstream duct of NASA Stage 35. The fluid dynamic influence of the sheet injector is limited radially to the outer 15% of the compressor annulus (affecting mainly the tip region of the blade span). To quantify this localized effect of the sheet injectors at the tip, the mass averages presented in this section were performed over the tip 15% radial span. Figure 3-5 shows the mass and swirl sensitivities in the upstream duct of NASA Stage 35 of the mixing parameter defined in equation 3.1. The mass sensitivity plot in Figure 3-5a shows that the degree of mixing increases with the compressor inlet duct mass flow rate. Since the mixing length is finite, the increase in degree of mixing with the compressor inlet duct mass flow rate is due to the fact that the flow parameters in the inlet duct approach the flow parameters of the jet from the actuator. The swirl sensitivity plot in Figure 3-5b shows that the degree of mixing decreases with jet orientation at the design mass flow rate of 20 kg/s. The mass sensitivity plots in Figures 3-6a, 3-7a, and 3-8a show that the fluid in the upstream duct is pumped to higher velocities, higher Mach numbers, and higher total pressures by the jet from the actuator, with the effect of the jet actuator decreasing as the compressor inlet duct flow parameters approach that of the jet from the actuator. These trends are consistent with the observed behavior of other jet pumps [37]. The swirl sensitivity plots in Figures 3-6b, 3-7b, and 3-8b show that for a given compressor inlet duct flow, the velocity, Mach number, and total pressure change with the jet orientation angle. As will be shown in Section 4.2.2, the localized effects in the tip region have a strong effect on the compressor blade row performance. Because of the large disparity in flow properties at the tip region from the hub, the compressor blade row sections will operate at significantly different portions on its performance map. The result of this is that the flow redistribution within the blade passage is changed in order to maintain radial equilibrium. Thus the compressor is very sensitive to the localized effects of the jet actuators. The control volume analysis described in this section is used in Sections 4.2 and 5.1 to compute the flow profiles at the injector plane (see Figure 3-1).

\footnote{A \( \beta \) parameter of unity corresponds to complete mixing where the mixed out profiles are uniform. This will be the case for two fluid streams mixing over an infinite length, or two streams with identical flow properties. An increase in the degree of mixing corresponds to \( \beta \rightarrow 1 \).}
(a) Mass sensitivity: variation of mixing parameter with the compressor inlet duct mass flow rate

(b) Swirl sensitivity: variation of mixing parameter with the jet orientation

Figure 3-5: Mass and swirl sensitivity of mixing parameter for the sheet injector at different injection levels.
(a) Mass sensitivity: variation of velocity with the compressor inlet duct mass flow rate

(b) Swirl sensitivity: variation of velocity with the jet orientation

Figure 3-6: Mass and swirl sensitivity of velocity for the sheet injector at different injection levels.
(a) Mass sensitivity: variation of Mach number with the compressor inlet duct mass flow rate

(b) Swirl sensitivity: variation of Mach number with the jet orientation

Figure 3-7: Mass and swirl sensitivity of Mach number for the sheet injector at different injection levels.
(a) Mass sensitivity: variation of total pressure with the compressor inlet duct mass flow rate

(b) Swirl sensitivity: variation of total pressure with the jet orientation

Figure 3-8: Mass and swirl sensitivity of total pressure for the sheet injector at different injection levels.
3.1.2 Compressor

The compressor comprises the computing region from the injector plane to the exit plane in Figure 3-1, which includes the portion of the upstream duct from the injector plane to the rotor inlet, the rotor, the rotor-stator gap, the stator, and the downstream duct. The objective of the second component of the steady injection model is to predict how the compressor will respond to various radial profiles at the injection plane. The throughflow from the injector plane to the exit plane is modeled using a streamline curvature analysis. The approach adopted for this research is that described in references [25, 16, 74]. This approach has been selected because it emphasizes the physical basis of the streamline curvature analysis, and it applies to a large group of machines with various geometries. Given the flowpath geometry and blade shapes, as well as some information about the blade performance, the throughflow calculations are used to predict the flow. A brief theoretical overview is outlined below.

Overview of Theoretical Background

Despite the fact that the flow in compressors is inherently three-dimensional, the basis for all throughflow calculations is to obtain a solution for an axisymmetric flow. The throughflow is modeled as an axisymmetric flow in which changes in angular momentum and in the fluid's total enthalpy are caused by the rotating and stationary blade rows. This may be regarded as being obtained by circumferentially averaging all flow properties or by solving for the mean flow on a mean blade-to-blade stream surface whose thickness and inclinations are determined by the geometry of the blade rows. The assumption of axial symmetry makes it possible to define a series of meridional streamlines, and the fluid particles are assumed to move through the compressor along the associated surfaces of revolution. The streamline curvature analysis is the most commonly used method for calculating the flow on the meridional surface. Figure 3-9 shows the coordinate system for streamline curvature analysis of the meridional flow. The principle of streamline curvature analysis is to write the equation of motion along lines roughly perpendicular to these stream surfaces (called quasi-orthogonal lines) in terms of the curvature of the surfaces in the meridional plane. The three-dimensional nature of the flow (3-D streamsurfaces) can be accounted for using the views or surfaces shown in Figure 3-9. In the meridional plane, the local tangent to the streamline in the meridional surface is denoted by the unit vector $\vec{n}$, and the normal
to the streamline in the meridional surface is denoted by the unit vector $\hat{r}$. Since the streamlines are not known in advance and are to be determined iteratively as the solution progresses, arbitrary directions called quasi-orthogonals which are roughly perpendicular to the streamlines and do not change with the calculation are chosen in advance. The unit vector $\hat{e}$ denotes the quasi-orthogonal on the mean streamsurface, and the unit vector $\hat{q}$ denotes the projection of $\hat{e}$ onto the meridional plane, thus both $\hat{q}$ and $\hat{e}$ are inclined at an angle $\gamma$ to the $r - \theta$ plane. $\epsilon$ is the angle of inclination to the radial direction of the projection of the quasi-orthogonal $\hat{e}$ onto the $r - \theta$ plane (view along the flow axis) i.e., $\epsilon$ is the local inclination of the mean streamsurface to the meridional plane. This inclination represents the tilting of the streamsurfaces i.e., the twisting and warping of the streamsurfaces as they pass through the blade passage.

![Diagram](image)

**Figure 3-9:** Coordinate system for streamline curvature calculation of meridional flow (from [16]).

For a fluid particle at B, the components of acceleration include the substantive acceleration in the flow direction (meridional direction), the centripetal acceleration produced by the absolute swirl velocity, the centripetal acceleration produced by the flow following a path with radius of curvature, and the circumferential acceleration which is the local blade loading. Applying the momentum equation to the streamsurface in the $\hat{e}$ direction will give an expression equating the static pressure gradient to the acceleration in the $\hat{e}$ direction. The static pressure gradient can be replaced with enthalpy and entropy gradients by using thermodynamic relations. Writing the resulting equation in its conventional form for
gradients in the direction of the quasi-orthogonal in the meridional surface results in:

\[
\frac{1}{2} \frac{\partial V_m^2}{\partial q} = \frac{\partial h_t}{\partial q} - T \frac{\partial s}{\partial q} + V_m \frac{\partial V_m}{\partial m} \sin(\phi + \gamma) + \frac{V_m^2}{r_m} \cos(\phi + \gamma) - \frac{1}{2r^2} \frac{\partial (r^2 V_{\theta}^2)}{\partial q} + \frac{V_m}{r} \frac{\partial (r V_{\theta})}{\partial m} \tan \epsilon
\]

(3.14)

This momentum equation in the meridional plane is sometimes referred to as the radial equilibrium equation and is the basis of all streamline curvature throughflow calculation methods. Consult reference [16] for details of the derivation.

**Aerodynamic Analysis of Compressor Components**

The different components of the compressor can be classified into ducts and blade rows. Ducts comprise the upstream annulus, the inter-blade row gap, and downstream annulus, whereas the bladerows comprise guide vanes, rotors and stators. Streamline curvature analysis of the meridional flow boils down to solving the momentum equation 3.14, which is a differential equation of the meridional velocity throughout the flow regime. Evaluating the right hand side of the equation requires solving for the total enthalpy, entropy, and tangential velocity in the different system components.

**Ducts:** Using Figure 3-1, the ducts in our example will include the regions from the injector plane to the rotor inlet, the rotor exit plane to the stator inlet plane, and the stator exit plane to the exit plane. In the ducts, total enthalpy (thus total temperature) and total pressure are conserved along the streamlines, and as the flow is convected in the ducts, the profiles will change such that angular momentum, \( r V_{\theta} \), remains constant. These three conservation equations, in addition to the flow properties at the inlet of the duct can be used to determine all the flow properties required for solving the equilibrium momentum equation in the meridional plane within a duct.

**Blade Rows:** An actuator disk model is used to model the blade rows. Losses associated with each blade row are convected along meridional streamlines. For a given set of inlet flow conditions, the actuator disk model defines the relative exit flow angle (hence exit flow tangential velocity \( V_{\theta} \)), stagnation temperature, and stagnation pressure at the trailing edge of the blade row. The relative exit flow angle can be obtained from correlations derived
using experimental measurements, or calculated using computational fluid dynamic (CFD) solutions. The change in total enthalpy that occurs along a streamline as it passes through a blade row (rotor or stator) is given by the Euler Turbine Equation:

$$h_{TE} - h_{LE} = U_{TE}V_{θ_{TE}} - U_{LE}V_{θ_{LE}}$$  \hspace{1cm} (3.15)

where $U_{LE} = Ωr_{LE}$, and $U_{TE} = Ωr_{TE}$. An alternative way of looking at the Euler Turbine Equation is that rothalpy, $I_t$ ($I_t = h_t - ΩrV_θ = h_t' - \frac{Ω^2r^2}{2}$) is conserved along meridional streamlines. For stators where $Ω = 0$, the Euler Turbine Equation reduces to the conservation of stagnation enthalpy (or equivalently the conservation of stagnation temperature). The total pressure ratio across a blade row is related to the ideal or isentropic value by:

$$\frac{P_{ITE}}{P_{ILE}} = \left[\frac{P_{ITE}}{P_{ILE}}\right]_{\text{isentropic}} \left\{ 1 - \bar{ω}_r \left[\frac{P_{I^TLE}}{P_{ITE}}\right]_{\text{isentropic}} \left( 1 - \frac{p_{LE}}{p_{ITE}} \right) \right\}$$  \hspace{1cm} (3.16)

where $P_{ITE}$ is the total pressure at the blade row trailing edge, $P_{ILE}$ is the total pressure at the blade row leading edge, $P_{I^TLE}$ is the relative total pressure at the blade row leading edge, $P_{ITE}$ is the relative total pressure at the blade row trailing edge, and $p_{LE}$ is the static pressure at the blade row leading edge. The isentropic total pressure ratios can be obtained from thermodynamic relations of isentropic processes using the inlet and outlet total temperatures, which in turn are obtained from the Euler Turbine Equation. Since the stagnation enthalpy (or stagnation temperature) is conserved across a stator, the isentropic total pressure ratio between the leading edge and trailing edge is equal to unity. Thus the total pressure ratio expression across a stator reduces to:

$$\frac{P_{ITE}}{P_{ILE}} = 1 - \bar{ω}_s \left( 1 - \frac{p_{LE}}{P_{ITE}} \right)$$  \hspace{1cm} (3.17)

Calculating the total pressure at the blade row trailing edge requires correlations for the relative total pressure loss coefficients $ω_r$ and $ω_s$ for the rotor and stator respectively.

**Geometric Input Requirements**

As the name suggests, these are the inputs that can be derived from the geometric information of the compressor usually supplied by the manufacturers. The geometric inputs include
flowpath coordinates, mean streamsurface, and mechanical blockage.

**Flowpath:** The flowpath is the geometry of the annulus endwalls guiding the flow from the inlet plenum to the discharge plenum. The streamline curvature method is extremely sensitive to the shape of the hub and casing. Therefore, great care needs to be taken to make the surfaces used in the calculation smoothly curved in the meridional plane, even if the actual compressor has significant discontinuities of radius or curvature. This smoothing of the hub and casing shape is justified because the real discontinuities are smoothed out by boundary layers.

**Mean Streamsurface:** The streamsurface shape changes from the upstream duct where they start as surfaces of revolution but twist and warp as they pass through the blade rows. The inclination of the streamsurface to the radial direction is represented by $\epsilon$ in Figure 3-9. Normally $\epsilon$ changes but in order to simplify the computations involved without sacrificing the accuracy, a mean streamsurface is defined.

**Mechanical Blockage Factor:** Mechanical blockage, $B_m$, is the proportion of the annulus blocked by the blades. Thus the mechanical blockage is the airfoil tangential thickness and can be calculated from the coordinates of the suction and pressure surfaces at various radial locations. It should be noted that this mechanical blockage factor, $B_m$, is different from tangential blockage factor, $B_\theta$, which is an aerodynamic input and will be defined later.

**Aerodynamic Input Requirements**
The aerodynamic input requirements for this analysis are correlation functions for the relative total pressure loss coefficient, the relative exit flow angle, and the tangential blockage parameter. Leading edge relative Mach number, incidence angle, axial velocity density ratio, Reynolds number, free-stream turbulence parameters, and diffusion factor level are the predominant aerodynamic quantities that have an effect on blade row loss and deviation. In [74], Kerrebrock states that the Reynolds number dependence can be ignored because for most compressor operating conditions, the Reynolds number is high enough that the
cascade performance is insensitive to the actual value.\textsuperscript{5} On the other hand, relative Mach number, incidence angle, and axial velocity density ratio dependence must be accounted for. There are several difficulties associated with the generalization of axial velocity density ratio (AVDR) effects. First, AVDR is a measure of the effective streamtube area and the rate of area variation through the blade row. This will therefore require that the streamlines through the blade rows be known prior to iteratively solving the equilibrium solution of the meridional flow. Secondly, the AVDR effect in higher Mach number cases is naturally related to the shock wave pattern existing in the blade row and is thus difficult to isolate for correlation. Finally, the AVDR effect is completely interdependent and not separable from the general effect of blade row aerodynamic loading level. Therefore, the correlation functions were assumed to be dependent on the relative Mach number, and incidence angle at the inlet of the blade row. A radial dependence is also included to model the spanwise variation of the loss and deviation.

Relative Total Pressure Loss Coefficient Correlation: The relative total pressure loss coefficient is defined as:

\[ \omega_{\text{loss}} = \frac{P_{\text{vTE isen}} - P'_{\text{vTE}}}{P'_{\text{LE}} - P_{\text{LE}}} \]  

(3.18)

where \( P_{\text{vTE isen}} \) is the isentropic relative total pressure at the blade row trailing edge, \( P'_{\text{vTE}} \) is the actual relative total pressure at the blade row trailing edge, \( P'_{\text{LE}} \) is the relative total pressure at the blade row leading edge, and \( P_{\text{LE}} \) is the static pressure at the blade row leading edge. To account for the relative Mach number and relative inlet flow angle dependence, quadratic polynomials in both relative Mach number and relative flow angle at the blade row inlet were used to represent the relative total pressure loss coefficient correlation function at various radial locations. Thus the relative total pressure loss correlation for each blade row is given by the polynomial:

\[ \omega_{\text{loss}}(\dot{m}_j, r, \beta_{\text{LE}}, M'_{\text{LE}}) = a_0 + a_1 \beta_{\text{LE}} + a_2 \beta_{\text{LE}}^2 + a_3 M'_{\text{LE}} + a_4 M'^2_{\text{LE}} + a_5 \beta_{\text{LE}} M'_{\text{LE}} \]

(3.19)

\textsuperscript{5}As stated more precisely by Johnsen et al. in [72], the Reynolds number doesn't affect the loss or deviation of a cascade if it is above a threshold of about \( 2 \times 10^5 \). Below this threshold value, there is a rapid rise in both the loss and deviation.

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where the polynomial coefficients \( a_i = a_i(\hat{m}_j, r) \) for \( i = 0, 1, 2, 3, 4, 5 \) depend on the spanwise location, and the injection configuration which includes the amount of mass injected, the type of injector, and the orientation of the jet actuators. These polynomial coefficients can be determined experimentally or through CFD calculations. Quadratic polynomials were selected to reflect the compressor 'loss bucket' near stall which causes the overall compressor characteristic to change slope. However, this is limited to mass flows near stall because, for higher Mach numbers, the loss bucket has an exponential shape. Therefore, to accommodate the wide range of Mach numbers, piecewise polynomial fits were used. It should be noted that the polynomial representation presents an advantage in that it facilitates the computation of partial derivatives required for the unsteady model.

Relative Exit Flow Angle Correlation: Similar to the correlation function for relative total pressure loss coefficient, piecewise quadratic polynomials in relative Mach number and relative flow angle at the blade row inlet were used to represent the correlation function for the relative exit flow angle or deviation. Thus the relative exit flow angle correlation is given by the polynomial:

\[
\beta_{TE}(\hat{m}_j, r, \beta_{LE}, M'_{LE}) = b_0 + b_1 \beta_{LE} + b_2 \beta_{LE}^2 + b_3 M'_{LE} + b_4 M'^2_{LE} + b_5 \beta_{LE} M'_{LE} 
\]

(3.20)

As before, the polynomial coefficients \( b_i = b_i(\hat{m}_j, r) \) for \( i = 0, 1, 2, 3, 4, 5 \) depend on the spanwise location and the injection configuration, and can be determined from experimental measurements or CFD calculations using least squares fits. The overwhelming sources of inaccuracy in throughflow calculations come from the uncertainties associated with the endwall boundary layer and the prediction of such quantities as the exit flow angle or deviation.

Tangential Blockage Factor: The “throughflow” analysis in compressor design is a two-dimensional axisymmetric calculation describing the spanwise variation of the flow at streamwise locations, both within and between blade rows from the inlet of the compressor to its discharge. The term “tangential blockage” [28] was introduced to account for the nonaxisymmetry (or departure from axisymmetry) in the flow due to blade wakes, corner
stall, tip leakage, and the endwall boundary layers and secondary flows. The essence of the problem is that the nonaxisymmetry of the flow causes a significant difference between the circumferential mass and area averages at any radius. For example, the mass-averaged total pressure is usually higher than the area-averaged total pressure, since the low total pressure region in a wake usually has associated with it a low axial velocity, i.e., a low mass flux. Thus tangential blockage is the parameter that bridges the gap between the assumption of axisymmetry in the analysis and the fact that the flow in an actual compressor is nonaxisymmetric.

The definition of tangential blockage in [28] is based on an incompressible flow regime. As shown below, the definition of tangential blockage has been modified for this research to include the compressible flow regime. Because quantities such as work depend on mass averages, throughflow analyses are generally considered to provide distributions of mass-averaged quantities. However, mass flow is an area-averaged quantity, i.e., the integral of mass flux \( (pV) \) over area. Tangential blockage provides the link between these two types of averages. The mass flow, \( \dot{m} \), through an annulus may be expressed as follows:

\[
\dot{m} = 2\pi \int_{r_h}^{r_t} \rho \bar{V}_x r \, dr = 2\pi \int_{r_h}^{r_t} B_\theta \rho \bar{V}_x^m r \, dr
\]

For an incompressible flow regime, the tangential blockage factor, \( B_\theta \), is defined as the ratio of the circumferentially area-averaged axial flow speed, \( \bar{V}_x^a \), to the axial flow speed based on the circumferentially mass-averaged total and static pressures, flow yaw and pitch angles, \( \bar{V}_x^m \). Extending to the compressible flow regime, the tangential blockage factor can be defined as the ratio of the circumferentially area-averaged mass flux, \( \rho \bar{V}_x^a \), to the circumferentially mass-averaged mass flux, \( \rho \bar{V}_x^m \), and is thus given by:

\[
B_\theta(r) = \frac{\rho \bar{V}_x^a}{\rho \bar{V}_x^m}
\]

It should be noted that the tangential blockage factor, \( B_\theta \), is a function of radius, and when the flow is axisymmetric, the mass and area averages are equal making \( B_\theta \) equal to unity. The importance of the blockage factor in predicting the flow field of multistage compressors using through-flow analysis is demonstrated in references [31, 29, 30].

The tangential blockage is just one approach used to account for the nonaxisymmetric nature
of the flow in the compressor. An alternative approach is to incorporate a spanwise mixing factor into the streamline curvature analysis. Gallimore et al. [44] investigated the cause of spanwise mixing in multistage axial flow compressors directly using an ethylene tracer gas technique in two low-speed, four-stage machines and found the dominant mechanism to be a turbulent type diffusion and not radial convection. Gallimore [43] incorporated the effect of spanwise mixing into an axisymmetric streamline curvature throughflow program, and found that the spanwise variations of exit total pressure in multistage machines were accurately predicted by including the effect of mixing on loss distributions inferred from measurements.

Meridional Flow Solution Procedure
For the steady air injection model, the streamline curvature is in analysis mode. In addition to the geometric inputs of flowpath geometry, mean streamsurface, and mechanical blockage due to blades, this analysis requires as aerodynamic input a row-by-row description of the aerofoil loss, deviation, and tangential blockage. Figure 3-10 shows the equilibrium solution flow chart summarizing the sequence explained below.

1. Initialize Iteration. The starting point is to choose the positions for quasi-orthogonals throughout the compressor flowpath. This is done by first assuming the streamline pattern for the compressor flowpath of interest. Examples of the initial quasi-orthogonals selected for two different compressors are shown in Figure 3-11. For the assumed streamline pattern calculate the streamline slopes and curvatures.

2. Guess Meridional Velocity. Guess the meridional velocity component at each grid point. For the first iteration, a radially uniform meridional velocity component profile is assumed from the first to the last quasi-orthogonal. Define the $T_l$, $P_l$, $V_\theta$, and $s$ profiles for the first quasi-orthogonal.

3. Compute $V_\theta$, $T_l$, and $P_l$. Starting with the flow parameters from the first quasi-orthogonal, compute the tangential velocity, total temperature, total pressure and other relevant flow parameters such as static pressure, static temperature, density, entropy for the subsequent quasi-orthogonals using the blade-to-blade aerodynamic analysis described above.
4. **Evaluate Terms of Momentum Equation.** Using the flow properties determined from the aerodynamic analysis of compressor components, compute the meridional derivatives and other terms on the right-hand side of the equilibrium momentum equation 3.14. In order to be able to evaluate the meridional derivatives throughout the flow field, the streamlines are assumed to have no curvature at the first and last quasi-orthogonals.

5. **Integrate Equilibrium Equation.** To evaluate the meridional velocity profiles, the momentum equation is integrated along the quasi-orthogonals using various numerical integration routines with the previous estimates of the meridional velocity profiles used for the integration constants. For stability reasons, the calculated meridional velocity profiles are then adjusted using a velocity relaxation factor, \( R_v \), according to the following relation:

\[
V_{m_{\text{new}}} = V_{m_{\text{previous}}} + R_v(V_{m_{\text{calculated}}} - V_{m_{\text{previous}}}).
\]

6. **Compute Overall Mass Flow rate.** Using the new meridional velocity profiles at the various quasi-orthogonals, the overall mass flow rate across each computing station is evaluated. A check is then made on the total flow level to see whether it is within a certain tolerance of the flow level at the first quasi-orthogonal. For example, the tolerance criterion used in this research for any given computing station is:

\[
\left| \frac{n_{\text{new}} - n_{\text{ini}}}{n_{\text{ini}}} \right| \leq \frac{1}{1000}.
\]

If the mass flowrate does not satisfy the selected tolerance criterion, the meridional velocity, \( V_m \), is adjusted and the steps 4, 5, and 6 are repeated until the tolerance criterion is satisfied.

7. **Check Continuity between Streamlines.** A continuity check is made on the streamlines, whereby the area between the hub and each streamline must, at all stations, contain the same fraction of the total flow. The streamlines are moved until the continuity condition is satisfied within a tolerance of 0.005. For stability reasons, the streamline movements are made using the grid relaxation factor, \( R_r \), according to the following relation:

\[
\tau_{\text{new}} = \tau_{\text{previous}} + R_r(\tau_{\text{calculated}} - \tau_{\text{previous}}).
\]

8. **Check Convergence Criteria.** Finally, a check is made on the convergence of meridional velocities and streamline locations. The meridional velocity convergence test is satisfied if the meridional velocities at each mesh point is reproduced from one iteration to the next to within 0.01 of the mean velocity or 9.7536 \( m/s \) depending on which is greater, and the streamline convergence test is satisfied if the change in streamline
radius is less than 0.001 of the computing station length.

Figure 3-10: Equilibrium Solution Flow Chart (Q-O stands for quasi-orthogonal).
Flowpath Geometry for NASA Stage 35

(a) Meridional view of NASA Stage 35

Flowpath Geometry of ADLARF 2 Stage Fan

(b) Meridional view of ADLARF 2 Stage military fan from [53]

Figure 3-11: Initial Quasi-orthogonals for NASA Stage 35 and ADLARF 2 Stage Fan.
Summary of Steady Air Injection Model
The steady air injection model consists of two components: the first component uses a control volume analysis and wind tunnel measurements to model the changes in flow profiles entering the compressor, and the second component uses a streamline curvature analysis to model the response of the compressor blade rows to the different inlet spanwise profiles generated by the jet actuator. The control volume analysis requires as inputs, the stagnation properties of flow in the compressor inlet duct and the stagnation properties of the air supplied to the jet actuator, and computes the 2D profiles of the partially mixed out flow at the compressor inlet. The streamline curvature analysis requires both geometric and aerodynamic inputs. The geometric inputs include the meridional flowpath coordinates and geometry of each blade row, and the aerodynamic inputs include the spanwise performance correlations of relative total pressure loss coefficient, the relative exit flow angle, and tangential blockage factor. The outputs of the streamline curvature analysis are the spanwise profiles of the steady flow quantities throughout the compressor. Experimentally measured spanwise profiles and compressor speedlines were used to validate the steady air injection model in Section 4.2. In Chapter 5, the steady air injection model is used to predict the effect of injecting hot air into a single-stage transonic compressor. The steady air injection model is important because it provides the mean steady values required for the unsteady models which form an integral part of compressor stability analysis.

3.2 Compressible Stall Inception Model
The inception of rotating stall and surge in high-speed compressors is modeled by a hydrodynamic stability analysis of the compression system. The 2-D fluid dynamic compressible model was developed by Bonnaure [13] to describe the stall inception process in high speed compressors. This model is an extension of the two-dimensional, linearized stability model of Moore and Greitzer [91, 92] to the compressible flow regime. Hendricks et al. [63] used the model to analyze the onset of rotating stall in high speed axial flow compressors, and to investigate the influence of parameters such as blade tip Mach number and compressor length on compressor stability. Feulner [38, 39] later added actuator and sensor models to the compressible model and developed a state space model suitable for control applications. Further modifications and refinements to the Feulner Model were made by Weigl [130] and
Fréchette [41]. Fréchette also applied the model to the redesign of a high speed multi-stage (11-stage) compressor to increase its stability and in [41], provides an in-depth overview of the model formulation from a fluid dynamics perspective and an energy-based analysis of stall inception in high speed compressors.

The compression system model for a gas turbine engine is shown in Figure 3-12. It consists of an inlet boundary condition, an upstream duct, rotors, stators, inter-blade row gaps, a downstream duct, and an exit boundary condition. The exit boundary condition will model the interaction of the compression system with the combustor and turbine.

![Figure 3-12: Schematic of compression system for compressible model (from [130]).](image)

In the compressible model, the compression system components are classified under three main groups: ducts, blade rows, and boundary conditions. In this research, the models for the ducts and blade rows are not modified. The ducts comprise the upstream duct, the inter-blade row gaps, and the downstream duct; the flow in these ducts is governed by a 2-D wave equation. The blade rows comprise the rotors and stators; the flow in these is governed by a 1-D wave equation. See reference [38] for the details of the fluid quantities that characterize the flow field in the ducts and blade rows. The compressible model is obtained by combining the solutions of the 2-D wave equations in the ducts to the solutions of the 1-D wave equations in the blade rows using the blade row leading and trailing edge boundary conditions. This section presents the modifications made to these and other boundary conditions.

The disturbances between different compression system components are related by bound-
ary conditions. The jet actuator boundary condition relates the disturbances in the two adjacent ducts separated by the jet actuator (modeled as an actuator disk); the leading edge and trailing edge boundary conditions relate the duct and blade row disturbances; and the sudden expansion or contraction boundary condition relates the disturbances between two adjacent ducts of different cross-sectional areas. Interactions between the compression system and its environment are incorporated through the end conditions which consist of the inlet and exit boundary conditions.

3.2.1 Jet Actuator Boundary Condition

From the steady injection model presented in Section 3.1 and the results from the validation experiments presented in Section 4.2, air injection can be modeled as having two main effects: changing the flow profiles entering the compressor, and changing the compressor blade row performance characteristics. The flow changes in the upstream duct associated with air injection are modeled by the jet actuator boundary condition, and the effects on the blade row performance characteristics are incorporated in the blade row boundary conditions.

We redefine the control term as the perturbation from the mean injected mass flow rate normalized by a reference mass flow rate, \( \dot{m}_{\text{ref}} = \rho_o A_o U_o \):

\[
u(s) = \frac{\delta \dot{m}_j}{\rho_o A_o U_o}\]  \hspace{1cm} (3.23)

The jet actuator boundary condition models the flow changes in the duct containing the jet actuator and thus provides a relation between the harmonic disturbance coefficients in the ducts upstream and downstream of the actuator. Changes in the flow coming into the compressor due to air injection are modeled by an actuator disk with mass, momentum, and swirl effects. Thus there will be changes in axial velocity (mass effect), tangential velocity (swirl effect), and total pressure (momentum effect due to increase in dynamic pressure) across the jet actuator disk. Since the mean flow properties change across the actuator, the jet actuator separates the original duct into two ducts with different mean flow properties at the location of the actuator. Changes in duct flow variables across the jet actuator disk are quantified by applying the continuity, axial momentum, tangential momentum and energy conservation equations. These are the same boundary conditions across the jet actuator.
used in references [38, 130]. They have been modified as follows:

**Mass Conservation:** The mass conservation or continuity equation across the jet actuator is:

\[
\frac{\delta m_2}{m_\text{ref}} = \frac{\delta \dot{m}_1}{m_\text{ref}} + u(s)
\]  

(3.24)

and the corresponding perturbation equation in terms of the duct flow quantities is:

\[
\frac{\rho_2 a_2}{\rho U_o} \left[ M_{x_2} \frac{\delta \rho_2}{\rho_2} + \frac{\delta V_{x_2}}{a_2} \right] = \frac{\rho_1 a_1}{\rho U_o} \left[ M_{x_1} \frac{\delta \rho_1}{\rho_1} + \frac{\delta V_{x_1}}{a_1} \right] + u(s)
\]

(3.25)

**Axial Momentum Conservation:** The axial momentum conservation equation is:

\[
P_2 A + \rho_2 V_{x_2}^2 A = P_1 A + \rho_1 V_{x_1}^2 A + \rho_j V_j^2 \cos \theta_j A_j
\]

(3.26)

and the corresponding perturbation equation in terms of duct flow quantities is:

\[
\frac{P_2}{\rho U_o^2} \frac{\delta P_2}{P_2} + \frac{\rho_2 V_{x_2}^2}{\rho_2 U_o^2} \frac{\delta \rho_2}{\rho_2} + \frac{\rho_2 V_{x_2} a_2}{\rho_2 U_o^2} \frac{\delta V_{x_2}}{a_2} = \frac{P_1}{\rho U_o^2} \frac{\delta P_1}{P_1} + \frac{\rho_1 V_{x_1}^2}{\rho_1 U_o^2} \frac{\delta \rho_1}{\rho_1} + \frac{\rho_1 V_1 a_1}{\rho_1 U_o^2} \frac{\delta V_{x_1}}{a_1} + 2 \frac{V_j \cos \theta_j}{U_o} u(s)
\]

(3.27)

This equation is the modification of equation A.36 in reference [38].

**Tangential Momentum Conservation:** The tangential momentum conservation equation is:

\[
\rho_2 V_{x_2} \delta V_{x_2} = \rho_1 \delta V_{x_1} A + \rho_j V_j^2 \sin \theta_j A_j
\]

(3.28)

and the corresponding perturbation equation in terms of the duct flow quantities is:

\[
\frac{\rho_2 V_{x_2}}{\rho U_o^2} \frac{\delta \rho_2}{\rho_2} + \frac{\rho_2 V_{x_2} a_2}{\rho_2 U_o^2} \frac{\delta \rho_2}{\rho_2} + \frac{\rho_2 V_{x_2} a_2}{\rho_2 U_o^2} \frac{\delta V_{x_2}}{a_2} = \frac{\rho_1 V_{x_1}}{\rho U_o^2} \frac{\delta \rho_1}{\rho_1} + \frac{\rho_1 V_{x_1} a_1}{\rho U_o^2} \frac{\delta V_{x_1}}{a_1} + \frac{\rho_1 V_{x_1} a_1}{\rho U_o^2} \frac{\delta V_{x_1}}{a_1} + 2 \frac{V_j \sin \theta_j}{U_o} u(s)
\]

(3.29)

This equation is the modification of equation A.37 in reference [38].
Energy Conservation: The energy conservation across the jet actuator is:

\[ \dot{m}_2 C_p T_{t_2} = \dot{m}_1 C_p T_{t_1} + \dot{m}_3 C_p T_{t_j} \]  

(3.30)

and the corresponding perturbation equation is:

\[
\frac{\rho_2 V_{xz_2} T_{t_2}}{\rho_0 U_0 T_{t_0}} \left[ \frac{\delta \rho_2}{\bar{\rho}_2} + M_{xz_2}^{-1} \frac{\delta V_{xz_2}}{\bar{a}_2} + \frac{\delta T_{t_2}}{T_{t_2}} \right] = \frac{\rho_1 V_{xz_1} T_{t_1}}{\rho_0 U_0 T_{t_0}} \left[ \frac{\delta \rho_1}{\bar{\rho}_1} + M_{xz_1}^{-1} \frac{\delta V_{xz_1}}{\bar{a}_1} + \frac{\delta T_{t_1}}{T_{t_1}} \right] + \frac{T_{t_j}}{T_{t_0}} u(s)
\]

(3.31)

This equation is the modification of equation A.38 in reference [38]. The corresponding perturbation equation for the energy conservation boundary condition in terms of the duct flow quantities is obtained by substituting the perturbation equations for \( \delta T_{t_1} \) and \( \delta T_{t_2} \) into equation 3.31. Combining the perturbation equations for mass conservation, axial momentum conservation, tangential momentum conservation, and energy conservation boundary conditions will give:

\[
\begin{bmatrix}
\frac{\delta P}{P} \\
\frac{\delta \rho}{\rho} \\
\frac{\delta V_x}{a} \\
\frac{\delta V_a}{a}
\end{bmatrix}_{ka,\text{down}} = \begin{bmatrix}
\frac{\delta P}{P} \\
\frac{\delta \rho}{\rho} \\
\frac{\delta V_x}{a} \\
\frac{\delta V_a}{a}
\end{bmatrix}_{ka,\text{up}} + \begin{bmatrix}
b_{ka} \\
b_{ka} \\
b_{ka} \\
b_{ka}
\end{bmatrix}_{ka,\text{up}} u(s)
\]

(3.32)

where \( \mathbf{J}_{ka,\text{up}} \) contains the mean flow properties of the duct upstream of the jet actuator, \( \mathbf{J}_{ka,\text{down}} \) contains the mean flow properties of the duct downstream of the jet actuator, and \( \mathbf{b}_{ka} \) contains the mean flow properties of the jet actuator. This equation is the modification of equation A.40 in reference [38]. The main difference is that in equation 3.32, \( \mathbf{J}_{ka,\text{down}} \neq \mathbf{J}_{ka,\text{up}} \) whereas in equation A.40 of reference [38], \( \mathbf{J}_{ka} = \mathbf{J}_{ka,\text{down}} = \mathbf{J}_{ka,\text{up}} \). The details of the refinement to the actuator model are given in Appendix D. The actuator boundary condition in reference [38] assumes that the steady flow field quantities are the same upstream and downstream of the actuator. The results in Figures 3-6a, 3-7a, and 3-8a show that across the jet actuator, the mass averaged velocity, Mach number, and total pressure respectively change when air is injected. This implies that assuming the same mean flow upstream and downstream of the jet actuator is not accurate. Thus the modified actuator model which incorporates the changes in mean flow across the jet actuator more accurately represent the actuator behavior. Using equation 2.25 in reference [38] to simplify
the modified actuator boundary condition in equation 3.32, the actuator model relating the perturbation coefficients in the ducts upstream and downstream of the actuator is:

\[
\begin{bmatrix}
B \\
C \\
D \\
E
\end{bmatrix}_{ka,\text{downstream}} = A_j(s) \begin{bmatrix}
B \\
C \\
D \\
E
\end{bmatrix}_{ka,\text{upstream}} + b_j(s) \tilde{u}(s)
\]  

(3.33)

where the control term, \(\tilde{u}(s)\), is the corresponding spatial harmonic of the unsteady injection (injected mass flow normalized by a reference mass flow, \(\dot{m}_{\text{ref}} = \rho_o A_o U_o\)), and the matrices \(A_j(s)\) and \(b_j(s)\) are frequency dependent matrices containing mean flow parameters of the ducts upstream and downstream of the actuator. \(B, C, D,\) and \(E\) are the upstream potential wave coefficient, downstream potential wave coefficient, vortical wave coefficient, and entropic wave coefficient respectively in a duct as described in reference [38].

3.2.2 Leading Edge Boundary Conditions

The boundary conditions applied at the blade row leading edge are: mass conservation, rothalpy conservation, and relative total pressure conservation. The equations for the leading edge mass conservation and rothalpy conservation boundary conditions are the same as in references [13, 38]. The relative total pressure boundary condition at the leading edge is modified as follows:

- In references [13, 38], all of the blade row total pressure loss is modeled at the leading edge. Here, this assumption is refined to be consistent with the modeling assumptions made in the steady air injection model described in Section 3.1, where all the total pressure loss is modeled at the blade row trailing edge. Since all the total pressure loss is modeled at the trailing edge, the relative total pressure is conserved at the leading edge. Using “1” and “2” to denote the upstream and downstream of the boundary respectively, the leading edge relative total pressure conservation boundary condition is:

\[
\delta P'_{t_1} = \delta P'_{t_2}
\]  

(3.34)
Equation 3.34 is a modification of equation A.8 in reference [38]. The expressions for $\delta P_{t1}'$ and $\delta P_{t2}'$ are the same as in references [13, 38]. Substituting these expressions into equation 3.34 will give:

$$\frac{1}{1 + \frac{\gamma - 1}{2} M_1'^2} \left[ (1 - 0.5 M_1'^2) \frac{\delta P_1}{P_1} + 0.5 \gamma M_1'^2 \frac{\delta \rho_1}{\rho_1} + \gamma \frac{\delta V_{x_1}}{a_1} + \gamma M_1' \frac{\delta V_{\theta_1}}{a_1} \right] =$$

$$\frac{1}{1 + \frac{\gamma - 1}{2} M_2'^2} \left[ (1 - 0.5 M_2'^2) \frac{\delta P_2}{P_2} + 0.5 \gamma M_2'^2 \frac{\delta \rho_2}{\rho_2} + \gamma M_2' \frac{\delta W_2}{a_2} \right]$$

Combining the mass conservation and rothalpy conservation boundary conditions to the modified relative total pressure conservation boundary condition, and using equations 2.25 and 2.33 in reference [38] to simplify will give the following $n^{th}$ harmonic leading edge boundary condition:

$$V_{Lk} V_k(x_{LEk}, s) = B_{Lk} B_k(x_{LEk}, s)$$

Equation 3.36 is a modification of equation A.19 (or 2.34) in reference [38]. There is no first order lag term in equation 3.36 because the total pressure loss has been moved from the leading edge to the trailing edge. See Appendix D for details of the matrices in equation 3.36.

3.2.3 Trailing Edge Boundary Conditions

The boundary conditions used at the blade row trailing edge are: mass conservation, rothalpy conservation, relative total pressure loss, and flow turning according to the trailing edge deviation angle. The equations for the trailing edge mass conservation and rothalpy conservation boundary conditions are the same as in references [13, 38]. The relative total pressure and deviation boundary conditions are modified as follows:

- In references [13, 38], all the total pressure loss is modeled at the leading edge and the relative total pressure at the trailing edge is conserved. Here, the relative total pressure loss is modeled at the trailing edge. The effect of air injection on the blade row loss performance characteristic is incorporated by including an actuation dependence.
in the relative total pressure loss coefficient correlation defined by equation 3.18. The modified relative total pressure loss coefficient is: \( \omega_{\text{loss}} = \omega_{\text{loss}}(\beta_{LE}, M_{LE}', U_j) \). Unsteady changes in the relative total pressure loss is modeled using a first order lag with time constant \( \tau_p \). The corresponding expression for the unsteady response of the relative total pressure perturbation at the blade row trailing edge is:

\[
\delta P'_{TE} = \delta P'_{TE\text{Eisen}} - \frac{1}{1 + s\tau_p} \left\{ (\delta P'_{tLE} - \delta P_{LE})\omega_{\text{loss}}(\tilde{\beta}_{LE}, M_{LE}', \tilde{U}_j) \right\} + (P'_{tLE} - P_{LE}) \left[ \frac{\partial \omega_{\text{loss}}}{\partial \tan \beta_{LE}} \delta \tan \beta_{LE} + \frac{\partial \omega_{\text{loss}}}{\partial M_{LE}'} \delta M_{LE}' + \frac{\partial \omega_{\text{loss}}}{\partial U_j} \right] \]

(3.37)

where \( P'_{TE\text{Eisen}} \) is the isentropic relative total pressure at the blade row trailing edge.

- The effect of air injection on the blade row deviation performance characteristic is incorporated by including an actuation dependence in the relative exit flow angle. The modified relative exit flow angle is: \( \beta_{TE} = \beta_{TE}(\beta_{LE}, M_{LE}', U_j) \). Unsteady changes in the relative exit flow angle is modeled using a first order lag with time constant \( \tau_d \). The corresponding expression for the unsteady response of the relative exit flow angle perturbation is:

\[
\delta \beta_{TE} = \frac{1}{1 + s\tau_d} \left[ \frac{\partial \beta_{TE}}{\partial \tan \beta_{LE}} \delta \tan \beta_{LE} + \frac{\partial \beta_{TE}}{\partial M_{LE}'} \delta M_{LE}' + \frac{\partial \beta_{TE}}{\partial U_j} \right] \]

(3.38)

Equations 3.37 and 3.38 are modifications of equations A.25 and A.26 respectively in reference [38]. The relative total pressure loss time constant, \( \tau_p \), and the deviation angle time constant, \( \tau_d \), are assumed to be equal i.e., \( \tau = \tau_p = \tau_d \). Combining the mass conservation and rothalpy conservation boundary conditions to the modified relative total pressure conservation boundary condition, and using equations 2.25 and 2.33 in reference [38] to simplify will give the following \( n^{th} \) harmonic trailing edge boundary condition:

\[
V_{T,k+1} V_k(x_{TE,k}, s) = B_k B_k(x_{TE,k}, s) \begin{bmatrix} \hat{B} \\ \hat{C} \\ \hat{D} \end{bmatrix}_k + (P_{Lk} + \frac{1}{1 + s\tau} P_{T_k}) V_k(x_{LE,k}, s) + \frac{1}{1 + s\tau} b_k \tilde{u}(s)
\]

(3.39)
Equation 3.39 is a modification of equation A.30 (or 2.35) in reference [38]. The main difference is that the modified equation has two additional terms: the $P_{Lk}$ term multiplying the upstream duct disturbance coefficient vector due to the fact that the total pressure loss is moved from the leading edge to the trailing edge, and the actuation term due to the introduction of actuation dependence in the blade row performance characteristics. The details of the matrices in equation 3.39 are included in Appendix D.

### 3.2.4 Blade Row Transmission Matrix

The duct dynamics, leading edge and trailing edge boundary conditions, and blade row dynamics can be used to relate the disturbance coefficients in the ducts upstream and downstream of the blade row. The transmission matrix across the blade row is obtained by substituting equation 3.36 into equation 3.39 to eliminate the blade row disturbance coefficients:

$$
\begin{bmatrix}
B \\
C \\
D \\
E
\end{bmatrix}_{k+1} = 
\begin{bmatrix}
B \\
C \\
D \\
E
\end{bmatrix}_{k} 
A_k(s)
+ b_R(s)\tilde{u}(s)
$$

Equation 3.40 is a modification of equation 2.36 in reference [38]. In addition to the difference in entries of $A_k$ in equation 3.40 and $A_k$ in equation 2.36 of reference [38], the main difference between these two equations is the actuation component. The details of $A_k$ and $b_k$ in equation 3.40 are given in Appendix D.

### 3.2.5 Sudden Expansion or Contraction Boundary Conditions

The flowpath of high speed compressors such as the ADLARF 2-stage military fan in Figure 3-11b have variable cross-sectional areas. The way this change in cross-sectional area is currently accounted for is to treat the duct as a uniform area duct with a representative cross-sectional area. In order to accurately model the change in disturbances arising from expansions or contractions in the compressor flowpath, the annuli can be divided into discrete ducts with different constant areas. This boundary condition does not alter the results in this research because the transonic compressor used and shown in Figure 3-11a has fairly uniform cross-sectional areas. The four duct flow variables in two adjacent ducts of
different areas can be related through the following boundary conditions: mass continuity, constant stagnation pressure, constant stagnation enthalpy (or stagnation temperature), and angular momentum conservation. For a sudden increase in the cross sectional area, the constant total pressure assumption implies that the mean flow speed is not large enough for flow separation (i.e., no losses) to occur at the sharp corners in the transition between the two duct areas. The sudden expansion or contraction boundary condition relating the perturbation coefficients in two adjacent ducts of different areas is given by:

\[
\begin{bmatrix}
B \\
C \\
D \\
E
\end{bmatrix}
_{\text{kd,downstream}} = A_D(s)
\begin{bmatrix}
B \\
C \\
D \\
E
\end{bmatrix}
_{\text{kd,upstream}}
\] (3.41)

where the matrix \(A_D(s)\) is a constant matrix containing mean flow parameters of the two adjacent ducts of different cross-sectional areas. The details of the derivation are given in Appendix D.

3.2.6 End Conditions

The end conditions consist of three inlet boundary conditions and the exit boundary condition. These end conditions define the interaction of the compressor inlet and exit ducts with the rest of the compression system.

Inlet Boundary Conditions: In references [13, 38, 130] the three inlet conditions are: zero total pressure, zero entropy, and zero vorticity perturbations. These conditions model an open, clean, and smooth flow. The inlet end conditions of zero entropic and vortical perturbations are not changed. The total pressure condition acts as an impedance condition for the inlet duct potential waves, and is modified as follows:

- The inlet impedance condition is replaced with a generalized complex acoustic impedance relating the static pressure and velocity perturbations, \(Z_{\text{in}}(\omega) = \frac{\delta P}{\delta V} = R_{\text{in}}(\omega) +\)
The modified inlet impedance condition is:

\[
\begin{bmatrix}
1 & 0 & -Z_{in} \frac{a}{P} \frac{V_x}{V} & -Z_{in} \frac{a}{P} \frac{V_\theta}{V}
\end{bmatrix}
\begin{bmatrix}
\frac{\delta P}{\delta P} \\
\frac{\delta \rho}{\delta P} \\
\frac{\delta V_x}{\delta \theta} \\
\frac{\delta V_\theta}{\delta \theta}
\end{bmatrix}
\bigg|_{in} = 0
\] (3.42)

Applying equation 2.25 from [38] to the inlet impedance condition in equation 3.42 and the other inlet boundary conditions will give:

\[
N(s)
\begin{bmatrix}
B \\
C \\
D \\
E
\end{bmatrix} = 0
\] (3.43)

where \(N(s)\) is a 3x4 matrix. Equation 3.43 is a modification of equation A.33 in reference [38]. The only difference between \(N(s)\) in equation 3.43 and \(N(s)\) in equation A.33 of reference [38] is that they have different first rows.

**Exit Boundary Condition:** The exit condition model depends on what is downstream of the compressor. Feulner [38] assumed that the flow dumps into a plenum. The corresponding exit conditions consists of an open end condition, \(\delta P = 0\), for the non-zeroth harmonic perturbations (non-axisymmetric rotating stall harmonics) applied at the interface of the exit duct and plenum, and the open end condition plus plenum dynamics for the zeroth harmonic perturbations. Weigl [130] found that the most lightly damped rotating stall modes were unaffected by the exit boundary condition. However, for the zeroth harmonic, Weigl showed that an open end condition, \(\delta P = 0\), does not capture the oscillatory low frequency mode, whereas a closed end condition, \(\delta V_x = 0\), does show some low frequency oscillatory behavior. Instead of using the open end or closed end conditions, Paduano [97] suggested the use of a more general complex acoustic impedance end condition relating the static pressure and axial velocity perturbations, \(Z_{ex}(\omega) = \frac{\delta P}{\delta V_x} = R_{ex}(\omega) + jX_{ex}(\omega)\).

Such an impedance boundary condition provides a means for adequately modeling realistic exit conditions. The real component of the complex impedance (resistance) models the
downstream throttle and the imaginary component of the impedance (reactance) models the capacitive (or compressibility) and inertial effects. The modified exit impedance condition is:

\[
\begin{bmatrix}
1 & 0 & -Z_{\text{ex}} \frac{a V_x}{P V} & -Z_{\text{ex}} \frac{a V_{\theta}}{P V}
\end{bmatrix}
\begin{bmatrix}
\frac{\delta P}{P} \\
\frac{\delta \rho}{\bar{\rho}} \\
\frac{\delta V_x}{\bar{a}} \\
\frac{\delta V_{\theta}}{\bar{a}}
\end{bmatrix}_{\text{ex}} = 0
\] (3.44)

The exit boundary condition can be written in the following form:

\[
X(s)
\begin{bmatrix}
B \\
C \\
D \\
E
\end{bmatrix}_{k+1} = 0
\] (3.45)

where \(X(s)\) is a 1x4 matrix whose entries depend on the harmonic number, \(n\). Equation 3.45 is a modification of equation 2.44 in reference [38]. See Appendix D for the details of the complex impedance boundary condition.

**Summary of Compressible Stall Inception Model**

The compressible rotating stall inception model is a two-dimensional linearized stability model for high speed axial-flow compressors. An overview of the procedure for analyzing the compressor system dynamics is shown in Figure 3-13. Compressor geometry, blade row performance correlations, and stagnation properties of the air supplied to the jet actuators are the required information for the compression system dynamic analysis. The first stage for analyzing the compression system dynamics is to determine the steady flow quantities throughout the compressor flowpath using the steady air injection model described in Section 3.1. The steady flow solution from the steady air injection model, the inlet and exit boundary conditions, the unsteady loss and deviation lag parameters, and the compressor sensitivity parameters are the inputs to the compressible rotating stall model. The outputs from the stability model are the eigenvalues of the pre-stall dynamics, the eigenvectors of the pre-stall flow field, and dynamic response of the compression system to external
excitation. Since our objective for this research is to improve the input-output dynamics predictability, we are only interested in the transcendental transfer functions (referred to as the truth model). The procedure for approximating the transcendental functions in the input-output dynamics to generate the eigenvalues, eigenvectors, and state space matrices is well documented in reference [38].

Experimentally measured pre-stall dynamics for NASA Stage 35 are used to validate the modified compressible rotating stall inception model in Section 4.3. The compressible rotating stall inception model is important for studying the effects of compressor design parameters on stability during compressor redesign, and for designing and evaluating feedback control laws.

![Diagram of procedure for analyzing compressor system dynamics](image-url)

**Figure 3-13:** Overview of procedure for analyzing compressor system dynamics (modified from [41]).
CHAPTER 4

EXPERIMENTAL VALIDATION OF AIR INJECTION MODELS

In the previous chapter, methods for modeling the steady and unsteady effects of air injection in high speed compressors were presented. These models are validated in this chapter by comparing the performance predicted by these models with experimental measurements. The experimental setup for the steady and unsteady injection experiments is described in Section 4.1. The experimental validation of the steady air injection model (Section 3.1) is presented in Section 4.2. Finally, the modified compressible stall inception model (Section 3.2) is validated in Section 4.3.

4.1 Description of Experimental Validation Experiments

The steady and unsteady air injection models were experimentally validated using NASA Stage 35 (described in Chapter 2), and also using NASA Rotor 35, which had the same experimental setup as NASA Stage 35 except that the stator was taken out. Steady injection validation experiments consist of compressor speedlines and spanwise profiles of total pressure, static pressure, and total temperature at various axial locations in the compressor flowpath. Unsteady validation experiments consist of forced response input-output dynamics of the pre-stall flow field perturbations.

Aero performance for NASA Stage 35 and Rotor 35 was measured using two 7-element total pressure and two 7-element total temperature rakes at Station K in the instrumentation layout shown in Figure 2-3. The speedlines were then calculated from these aero rake measurements at Station K assuming that the total pressure and total temperature at the
compressor face are uniform radially and equal to the upstream plenum values. The spanwise profiles for NASA Stage 35 used to validate the streamline curvature code in the steady air injection model are measurements reported in a NASA Technical Report [104]. These profiles were generated from survey measurements and a streamline curvature code operating in "design" mode. Similar experimental results generated from survey measurements for NASA Stages 36, 37, and 38 are reported in references [103, 105, 106]. The spanwise profiles for NASA Rotor 35 were determined from survey measurements that were taken at Station J with Kiel-headed total pressure and total temperature area traverse (or survey) probes shown in Figure 2-2.

Forced response measurements of the input-output dynamics of the pre-stall flow field perturbations were used for validating the modified compressible stall inception model in NASA Stage 35. The Fourier series expansion for the actuation commands from a circular array of jet actuators at Station E or the pressure perturbations from an annular array of high bandwidth Kulite sensors at Station F is:

\[
x(\theta) = x_0 + \sum_{n=1}^{\infty} [x_{rn} \cos(n\theta) + x_{tn} \sin(n\theta)]
\]

\[
= \sum_{n=-\infty}^{\infty} x_n e^{jn\theta}
\]

where the real and complex spatial Fourier coefficients are related by:

\[
x_n = x_{rn} + j x_{tn}
\]

Combining the discrete Fourier series expansion in equation 4.1 for a set of measurements at M circumferential locations \((\theta_1, \theta_2, \ldots, \theta_M)\), will give the following matrix relation:

\[
\begin{bmatrix}
  x(\theta_1) \\
x(\theta_2) \\
\vdots \\
x(\theta_M)
\end{bmatrix}
= \begin{bmatrix}
  1 & \cos \theta_1 & \sin \theta_1 & \cos 2\theta_1 & \sin 2\theta_1 \\
  1 & \cos \theta_2 & \sin \theta_2 & \cos 2\theta_2 & \sin 2\theta_2 \\
  \vdots & \vdots & \vdots & \vdots & \vdots \\
  1 & \cos \theta_M & \sin \theta_M & \cos 2\theta_M & \sin 2\theta_M
\end{bmatrix}
\begin{bmatrix}
x_0 \\
x_{r1} \\
x_{t1} \\
x_{r2} \\
x_{t2}
\end{bmatrix}
\]
which can be written in a more compact form as:

$$f(\theta) = \mathbf{F}\hat{\mathbf{f}}$$  \hspace{1cm} (4.4)

where $f(\theta)$ is the vector containing the measurements at the M circumferential locations and $\hat{\mathbf{f}}$ is the vector containing the spatial Fourier coefficients. As shown in references [46, 130], the spatial Fourier coefficients can be computed from the M values of $f(\theta)$ using the pseudo-inverse of $\mathbf{F}$ given by:

$$\hat{\mathbf{f}} = (\mathbf{F}^T\mathbf{F})^{-1}\mathbf{F}^T f(\theta)$$  \hspace{1cm} (4.5)

The output of the identified system transfer function is the spatial Fourier coefficient of the wall static pressure perturbation at Station F, which was obtained by passing the measurements from an array of eight high bandwidth wall static pressure Kulite sensors at Station F through a high pass filter. The input of the identified system transfer function is the spatial Fourier coefficient of the input command to the actuators at Station E from the control PC described in Chapter 2.

4.2 Experimental Validation of Steady Air Injection Model

In Section 4.2.1, the streamline curvature code used in the steady air injection model described in Section 3.1 is first validated using the total pressure and total temperature profiles for NASA Stage 35 with a solid casing (i.e., no injectors) at different mass flow rates. In Section 4.2.2, the steady air injection model is used to predict the total pressure and total temperature profiles at Station J for NASA Rotor 35 with different injection configurations. The results from the model are compared with experimental measurements. In Section 4.2.3, the steady air injection model is then used to predict the speedlines for NASA Stage 35 at various levels of steady air injection and the results from the model are compared with experimental measurements.

4.2.1 Validation of Streamline Curvature Code

The first step in validating the steady air injection model was to validate the streamline curvature code using the experimental data reported in [104] at 100% of design speed. The
streamline curvature code in the steady air injection model is in analysis mode, and the experimental data reported in [104] was generated from survey measurements for NASA Stage 35 and a streamline curvature code in design mode. A streamline curvature code in design mode is used to determine the details of the flow within a compressor (usually at locations where experimental measurements cannot be made) from test measurements combined with a description of the compressor geometry and, where necessary, correlations of blade row performance. On the other hand, a streamline curvature code in analysis mode is used to determine the details of the flow throughout a compressor from inlet flow conditions, a description of the compressor geometry, and correlations of blade row performance.

For each blade row, the performance correlations required as inputs to the streamline curvature code in analysis mode are the spanwise profiles of the following: the relative total pressure loss coefficient, the relative exit flow angle (or equivalently deviation angle), and tangential blockage parameter. These blade row performance correlations were modeled as quadratic functions of the blade row inlet relative Mach number and inlet relative flow angle, and are represented by the polynomial functions given in equations 3.19 and 3.20. The coefficients for the polynomial functions representing the relative total pressure loss coefficient and relative exit flow angle at different spanwise locations of the rotor and stator of NASA Stage 35 were determined by fitting the polynomial functions to the correlation data in [104] using a least squares approach. Figure 4-1a compares the spanwise profiles of the relative total pressure loss coefficient from [104] with predictions from the polynomial fits for both the rotor and stator of NASA Stage 35 at a mass flow rate of 18.26 kg/s. The solid lines represent the predicted profiles from the polynomial fits and the circles represent experimental measurements. The measurements for the rotor are plotted with an error bar of ± 0.02 and the measurements for the stator are plotted with an error bar of ± 0.01. The size of these error bars, which represent about 1% of the mean values, were selected for the purpose of comparison. Similar plots comparing the measured and predicted spanwise profiles of the relative total pressure loss coefficient at mass flow rates of 19.64 kg/s, 20.27 kg/s, 20.64 kg/s, 21.00 kg/s, and 21.10 kg/s are shown in Figures 4-3a, 4-5a, 4-7a, 4-9a, and 4-11a respectively. The small discrepancies between the measured and predicted relative total pressure loss coefficient at the tip region can be explained by the fact that the tip region
is the high Mach number region and will thus require higher order polynomials to capture the loss coefficient. As expected, such discrepancies do not exist for the stator because the Mach number at the stator tip region is subsonic. Figure 4-1b compares the spanwise profiles of the relative exit flow angle from [104] with predictions from the polynomial fits for both the rotor and stator of NASA Stage 35 at a mass flow rate of 18.26 kg/s. The solid lines represent the predicted results from the polynomial fits and the circles represent experimental measurements. The measurements for the rotor are plotted with an error bar of $\pm 2^\circ$ and the measurements for the stator are plotted with an error bar of $\pm 0.5^\circ$. The size of these error bars, which represent about 2% of the mean values, were selected for the purpose of comparison. Similar plots comparing the measured and predicted spanwise profiles of the relative exit flow angle at mass flow rates of 19.64 kg/s, 20.27 kg/s, 20.64 kg/s, 21.00 kg/s, and 21.10 kg/s are shown in Figures 4-3b, 4-5b, 4-7b, 4-9b, and 4-11b respectively. These plots show that the quadratic polynomial fits are sufficient for predicting the relative exit flow angles within 2% accuracy. This 2% uncertainty can be reduced even further by using higher order polynomials.

Figures 4-2a and 4-2b show the spanwise profiles of total pressure and total temperature respectively at the leading and trailing edges of the rotor and stator of NASA Stage 35 at a mass flow rate of 18.26 kg/s. The solid lines represent the predicted spanwise profiles from the streamline curvature code and the circles represent experimental measurements from [104]. The total pressure measurements are plotted with an error bar of $\pm 0.25$ psia and the total temperature measurements are plotted with an error bar of $\pm 2$ K. The size of these error bars are based on the uncertainties reported in Table 2.4 of reference [130] and they represent the estimated errors in the data based on the inherent accuracies of the instrumentation and the recording system. Similar total pressure and total temperature validation profiles at mass flow rates of 19.64 kg/s, 20.27 kg/s, 20.64 kg/s, 21.00 kg/s, and 21.10 kg/s are shown in Figures 4-4, 4-6, 4-8, 4-10, and 4-12 respectively. With the exception of Figure 4-12 representing the spanwise profiles at the choking mass flow rate of 21.10 kg/s, the total pressure and total temperature spanwise profiles predicted by the streamline curvature code agrees very well with the experimental measurements. The discrepancy at the choking point is due to the occurrence of supersonic and subsonic solutions for the same mass flow rate and the extreme sensitivity of velocities to flow rates around the
choking point. The prediction around the choking point can be improved by modifying the streamline curvature program to select the supersonic solution at the choking point instead of the subsonic solution. However, this was not pursued further because we were not interested in predicting the effect of steady air injection at the choking point of the compressor characteristic.
Measured and predicted correlations for the rotor and stator of NASA Stage 35 at 18.26 kg/s.

Figure 4-1:
Figure 4-2: Measured and predicted spanwise profiles for NASA Stage 35 at 18.26 kg/s.
Figure 4-3: Measured and predicted correlations for the rotor and stator of NASA Stage 35 at 19.64 kg/s.
Figure 4-4: Measured and predicted spanwise profiles for NASA Stage 35 at 19.64 kg/s.
Figure 4-5: Measured and predicted correlations for the rotor and stator of NASA Stage 35 at 20.27 kg/s.
Figure 4-6: Measured and predicted spanwise profiles for NASA Stage 35 at 20.27 kg/s.
Figure 4-7: Measured and predicted correlations for the rotor and stator of NASA Stage 35 at 20.64 kg/s.
Figure 4-8: Measured and predicted spanwise profiles for NASA Stage 35 at 20.64 kg/s.
Figure 4-9: Measured and predicted correlations for the rotor and stator of NASA Stage 35 at 21.00 kg/s.
Figure 4-10: Measured and predicted spanwise profiles for NASA Stage 35 at 21.00 kg/s.
Figure 4-11: Measured and predicted correlations for the rotor and stator of NASA Stage 35 at 21.10 kg/s.
Figure 4-12: Measured and predicted spanwise profiles for NASA Stage 35 at 21.10 kg/s.
4.2.2 Application of Steady Air Injection Model to NASA Rotor 35

To characterize the effect of steady air injection on a single rotor, the stator of NASA Stage 35 was taken out. The speedlines that were measured for the single rotor with and without air injection using the sheet injectors at different levels of blowing and orientation are shown in Figure 4-14a. For the sake of comparison, the corresponding speedlines that were measured for the complete stage with the rotor and stator in place are shown in Figure 4-14b. Figure 4-14a (single rotor, air injected at 0° yaw) shows a consistent reduction in the stalling mass flow rate and a consistent increase in the pressure rise as the amount of air injected is increased from 0% to 100% valve opening at design operating conditions. However, when air is injected at -15° yaw, a consistent reduction in the stalling mass flow rate was observed as the amount of air injected is increased, but no noticeable change in pressure rise is observed when the injection level was increased from 50% to 100% valve opening at design operating conditions. This behavior is quite different from that for the complete stage shown in Figure 4-14b where there is a consistent reduction in the stalling mass flow rate and a consistent increase in pressure rise as the level of injection was increased from 0% to 100% valve opening at 0° and -15° yaw of the injector orientation. This difference implies that the interaction between the rotor and stator plays an important role in how air injection affects the blade row characteristics and subsequent compressor performance.

At the operating points marked A, B, C, D, and E on the compressor map in Figure 4-14a, the total pressure, total temperature, and flow angle profiles in the radial and circumferential directions were measured at Station J (10.75 cm or 1.93 chord lengths) downstream of the rotor as described in Section 4.1. These radial and circumferential profiles were used to generate the two-dimensional surveys shown in Appendix B. The radial profiles used to validate the model were obtained by circumferentially mass averaging the total pressure and total temperature survey measurements. The steady injection model described in Section 3.1 was then used to predict these circumferentially mass averaged radial profiles of total pressure and total temperature. The radial total pressure and total temperature profiles for steady air injection were predicted using two sets of blade row performance correlations. The sets of blade row performance correlations are based on different concepts for modeling the effect of air injection. These modeling concepts are illustrated by Figure 4-13. The first modeling concept assumes that the blade row performance correlations are not affected by air injection i.e., do not depend on the injection configuration \( \frac{\partial \omega}{\partial w_i} = 0 \). For this modeling
approach, the polynomial functions determined in Section 4.2.1 for the rotor were used. The second modeling concept assumes that the blade row performance correlations do change with the injection configuration i.e., $\frac{\partial \omega}{\partial u_j} \neq 0$. For this modeling approach, the polynomial functions determined in Section 4.2.1 were modified based on the blade row performance correlations determined from CFD solutions by Hathaway [60].

![Diagram](image)

**Figure 4-13:** Schematic illustrating different concepts for modeling the blade row performance correlation functions with air injection.

Figure 4-15a shows two sets of spanwise blade row performance correlations for NASA Rotor 35 operating at 100% speed with solid casing and a mass flow rate of 18.42 kg/s (i.e., operating point A in Figure 4-14a). Figure 4-15b shows the corresponding spanwise profiles of total pressure and total temperature at Station J predicted by the steady air injection model (with the injectors turned off i.e., $u_j = 0$) using the blade row performance correlations in Figure 4-15a. The circles represent the circumferentially mass averaged measurements at Station J. The dashed lines represent the rotor performance correlations predicted by the polynomial functions from the solid casing data for the complete stage and the corresponding spanwise profiles predicted. The solid lines represent the rotor performance correlations from CFD solutions for the single rotor with solid casing and the corresponding spanwise profiles predicted. Figure 4-16a shows two sets of spanwise blade row performance correlations for NASA Rotor 35 operating at 100% speed, a total mass flow rate of 16.92 kg/s with the sheet injectors at 0° yaw (i.e., operating point C in Figure
while air was supplied at 100 psig and the valves 100% opened. Figure 4-16b shows the corresponding spanwise profiles of total pressure and total temperature at Station J predicted by the steady air injection model using the blade row performance correlations in Figure 4-16a. The dashed lines represent the blade row performance correlations predicted by the polynomial functions from the first modeling concept (which assumes $\frac{\partial \gamma}{\partial u_j} = 0$) and the corresponding spanwise profiles predicted. The solid lines represent the blade row performance correlations predicted by the polynomial functions from the second modeling concept (which assumes $\frac{\partial \gamma}{\partial u_j} \neq 0$) and the corresponding spanwise profiles predicted. Figure 4-17 shows similar plots of the blade row performance correlations, and the total pressure and total temperature spanwise profiles at Station J with the sheet injectors at $-15^\circ$ yaw (i.e., operating point E in Figure 4-14a) while air was supplied at 100 psig and the valves 100% opened. The disparity in rotor tip loss coefficient from the two modeling concepts in Figures 4-16a and 4-17a can be explained by the fact that the blade row losses are much more sensitive to the relative inlet Mach number and incidence with solid casing. With such steeply varying losses, small changes in the inlet relative Mach number and inlet relative flow angle will lead to large changes in loss coefficient. Injecting air thus makes the blade row losses more robust to the inlet relative Mach number and flow angle perturbations. The effect of this can be seen in the predicted profiles in Figures 4-16b and 4-17b which show that using the blade row performance characteristics from solid casing data does not capture the flow redistribution in the blade rows associated with air injection. The total pressure measurements are plotted with an error bar of $\pm 0.25$ psia and the total temperature measurements are plotted with an error bar of $\pm 2$ K. The size of these error bars are based on the uncertainties reported in Table 2.4 of reference [130] and they represent the estimated errors in the data based on the inherent accuracies of the instrumentation and the recording system. Since the total pressure and total temperature spanwise profiles were accurately predicted only when the blade row performance correlations from the second modeling concept were used, it can be concluded that the blade row performance characteristics are changed when high pressure air is being injected into a compressor at a location upstream of the rotor. These results validate the modeling approach employed in Chapter 3 where air injection in high speed axial-flow compressors was modeled as having two main effects: changing the mass, momentum and swirl of the flow coming into the compressor, and changing the compressor blade row performance characteristics.
Figure 4-14: Measured total-to-static pressure characteristics at 100% speed for NASA Stage 35 with a single rotor and the complete stage for different injection configurations.
Figure 4-15: Spanwise profiles of rotor performance correlations, and the corresponding measured and predicted total pressure and total temperature profiles at Station J for NASA Stage 35 operating at 100% speed with a single rotor and solid casing at a mass flow rate of 18.42 kg/s (point A on Figure 4-14a).
Figure 4-16: Spanwise profiles of rotor performance correlations, and the corresponding measured and predicted total pressure and total temperature profiles at Station J for NASA Stage 35 operating at 100% speed with a single rotor, high pressure air supplied at 100 psig, sheet injectors oriented at 0° yaw, valves fully opened, and a total mass flow rate of 16.92 kg/s (point C on Figure 4-14a).
Figure 4-17: Spanwise profiles of rotor performance correlations, and the corresponding measured and predicted total pressure and total temperature profiles at Station J for NASA Stage 35 operating at 100% speed with a single rotor, high pressure air supplied at 100 psig, sheet injectors oriented at $-15^\circ$ yaw, valves fully opened, and a total mass flow rate of 16.80 kg/s (point E on Figure 4-14a).
4.2.3 Application of Steady Air Injection Model to NASA Stage 35

The steady air injection model was used to predict the total pressure characteristics of the complete NASA Stage 35 operating at 100% speed with the sheet injectors used to inject high pressure air at 0° and -15° yaw. The two sets of performance correlations for the rotor, based on the different modeling concepts discussed in Section 4.2.2, and the performance correlations for the stator determined in Section 4.2.1 (i.e., the stator performance correlations were not modified) were used to predict the speedlines for the complete stage with air injection. Figure 4-18a shows the total pressure characteristics predicted by the steady air injection model using the rotor performance correlations from the first modeling concept (i.e., \( \frac{\partial \omega}{\partial u_j} = 0 \)), and the stator performance correlations from Section 4.2.1. Figure 4-18b shows the total pressure characteristics predicted by the steady air injection model using the rotor performance correlations from the second modeling concept (i.e., \( \frac{\partial \omega}{\partial u_j} \neq 0 \)), and the stator performance correlations from Section 4.2.1. In Figure 4-19, the total pressure characteristics from Figure 4-18 are isolated for each configuration and the measurements plotted with ± 2% error bars. The plots on the left are the isolated speedlines from Figure 4-18a and the plots on the right are the isolated speedlines from Figure 4-18b.

The results summarized in Figures 4-18 and 4-19 show that the correct speedline trends with air injection (with the speedlines predicted to within a ± 2% error) were obtained when the rotor performance correlations from the second modeling concept (i.e., \( \frac{\partial \omega}{\partial u_j} \neq 0 \)) were used as inputs to the steady air injection model. However, the saturation in the speedline for -15° yaw was not captured. This is due to the fact that the rotor performance correlations were modeled from a limited sample of CFD solutions. Also, since the stator behaves as a diffuser, it could be possible that the saturation in the speedline for -15° yaw was not captured because the performance correlations for the stator were not modified. From the validation results in this chapter, it can be concluded that air injection affects the blade row performance characteristics and this dependence must be accounted for in order to accurately capture the effects of air injection on high speed compressor performance.
(a) Predictions obtained with polynomial functions from solid casing experiments

(b) Predictions obtained with CFD polynomial functions

**Figure 4-18:** Measured and predicted total pressure characteristics for NASA Stage 35 with no injectors installed, and jet actuators injecting 100 psig high pressure air with the valves 100% opened and sheet injectors at different orientations.
Figure 4-19: Total pressure characteristics from Figure 4-18 with 2% error bars.
4.3 Experimental Validation of Compressible Stall Inception Model

In this section, the modified compressible stall inception model described in Section 3.2 is used to predict the pre-stall dynamics of NASA Stage 35, and the results compared with experimental measurements. Figure 4-20 shows the meridional view of the flowpath for NASA Stage 35 illustrating the components used for applying the compressible model. The flowpath is divided into six constant area ducts (three upstream ducts, an inter-blade row duct, and two downstream ducts) so that the flow through the converging flowpath is accurately modeled. The jet actuators are located at Station E, which was chosen to coincide with the junction separating the first and second ducts, and the wall static Kulites sensors are located at Station F which is in the third duct.

![Figure 4-20: Meridional view of NASA Stage 35 flowpath showing the components for the compressible stall inception model.](image)

The actuator model in equation 3.33, the expansion and contraction model in equation 3.41, the blade row transmission matrix in equation 3.40, the inlet boundary condition in equation 3.43, the exit boundary condition in 3.45, and the static pressure sensor model...
in equation 2.58 of reference [38] were applied to NASA Stage 35 shown in Figure 4-20 to obtain the transfer function relationship from the control input at Station E to the sensed static pressure perturbation output at Station F for each spatial harmonic. Figure 4-21a is a reproduction of Figure 4-32 in reference [130]. It compares the measured first harmonic transfer function to the theoretical input-output dynamics using the original compressible stall inception model. Figure 4-21b is a similar comparison using the modified compressible model. The solid lines represent the polynomial fit from forced response measurements, and the dashed lines represent the predictions from the unmodified and modified compressible model. Even though the modified compressible stall inception model predicted some acoustic resonances that were not identified in the experimental measurements, it was able to capture the incompressible rotating stall mode.
(a) First harmonic transfer function from reference [130]

(b) First harmonic transfer function using the modified compressible model

Figure 4-21: Comparison of theoretical (dashed line) and measured (solid line) first harmonic transfer function at 100% speed.
4.4 Summary of Experimental Validation of Models

Circumferentially mass averaged radial profiles of total pressure and total temperature, and speedlines for NASA Stage 35 were used to validate the steady air injection model. The results from the experimental validation of the steady air injection model suggests that the effect of air injection can be broken down into two main components. The first component is the change in flow properties in the upstream duct due to the jet actuator; this was predicted using a control volume analysis and wind tunnel measurements. The second component is the effect of the actuation on the blade row performance characteristics; this was predicted using a streamline curvature code in analysis mode. The circumferentially mass averaged radial profiles used for validating the steady air injection model and the corresponding two-dimensional profiles in Appendix B show considerable spanwise mixing as the flow goes through the rotor blades. These changes in the fluid redistribution pattern taking place within the blade rows as a result of air injection were captured by modifying the spanwise distribution of blade row performance characteristics using solutions from computational fluid dynamics (CFD).

Comparisons between the theoretical dynamics predicted by the modified compressible stall inception model and transfer functions from forced response measurements indicate that the modal stability and input-output dynamics for the zeroth spatial harmonic are still not yet accurately captured, but there is an improvement in the prediction of modal stability and input-output dynamics for the first harmonic.
In the previous chapter, the steady air injection model was validated using experimental measurements from a transonic compressor. To implement active control using air injectors in an aircraft engine, a feasible way to supply high pressure air to the actuators is to recirculate air from a downstream stage of a multistage compressor. Such a source implies that air will be supplied to the jet actuators at a high temperature. Thus an important application of the verified steady air injection model is to investigate the effectiveness of injecting hot air into the single-stage transonic compressor used in this research. In this chapter, the verified model is used to predict the behavior of the transonic compressor if air is injected at a higher temperature.

5.1 Effect of Hot Air Injection

Presently, the air being injected into the transonic compressor for stabilization has a supply pressure of 100 psig and a supply temperature of 305.6 K. For this application, we are assuming that the air supply pressure is 100 psig as with the present set of control experiments, and the air supply temperature is that from a compressor which can raise the pressure of air from atmospheric pressure to 100 psig, i.e., $\pi_c = 7.8$, and has a polytropic efficiency of 90% which is typical of modern turbomachines. This corresponds to an adiabatic efficiency
of 86.8% and results in an air supply temperature of 553.1 K.\(^1\)

For this research, the mass flow rate from the actuator is assumed to be the same as the current design mass flow rate, stated in Table 2.3. The reason for maintaining the same mass flow rate is that air injection experiments have demonstrated that the blade row performance characteristics change with the amount of air being injected. This constant mass flow rate condition can be obtained by changing the area of the slot orifice. When the valve is operating under choked conditions, the mass flow rate is:

\[ \dot{m}_v = C_d A \frac{P_{s0}}{\sqrt{T_{s0}}} \sqrt{\frac{2\gamma}{R(\gamma - 1)}} \left( P_r^{2/\gamma} - P_r^{(\gamma+1)/\gamma} \right) \]

(5.1)

where \( C_d \) is the discharge coefficient, \( A \) is the orifice area, \( \gamma \) is the ratio of specific heats, \( P_{s0} \) is the air supply pressure, \( T_{s0} \) is the air supply temperature, and \( P_r = \left[ \frac{2}{\gamma+1} \right]^{\gamma/(\gamma-1)} \).

Since the air supply total pressure remains the same and the air supply total temperature is increased by a factor of 1.81, the mass flow rate expression in equation 5.1 shows that the slot orifice area will have to be increased by a factor of 1.35 to maintain the same mass flow rate.

5.1.1 Flow Properties of Exit Jet from Actuator

Using the injector model by Berndt [10], the effect of increasing the temperature of the air supplied to the actuator on the flow properties of the jet coming out from the sheet injector was investigated. It should be noted that the model matches the static pressure of the exit jet to the static pressure in the compressor inlet duct since a static pressure gradient cannot be supported in the upstream duct. The effect of temperature on the Mach number and total pressure of the jet coming out of the sheet injector, for an injector mass flow rate of 0.735 kg/s and a supply pressure of 100 psig, is shown in Figure 5-1. Similar plots, for 1.178 kg/s injector mass flow rate, are shown in Figure 5-2. These plots show that the effect of increasing the temperature of the air supplied to the actuator is to increase the velocity (Mach number) and total pressure of the jet coming out of the actuator.

\(^1\)The expression relating the adiabatic efficiency, \( \eta_c \), and the polytropic efficiency, \( \eta_{pol} \), is: \( \eta_c = \frac{\gamma_c}{\gamma_c - 1} \frac{\pi_c - 1}{\gamma_c \eta_{pol} - 1} \). The compressor adiabatic efficiency is: \( \eta_c = \frac{1}{\gamma_c - 1} \frac{\pi_c - 1}{\gamma_c - 1} \), where \( \pi_c \) is the compressor pressure ratio, \( r_c \) is the compressor temperature ratio, and \( \gamma_c \) is the ratio of specific heats.
Figure 5-1: Effect of air supply temperature on the injector exit Mach number and total pressure for 0.735 kg/s actuator mass flow rate.
Figure 5-2: Effect of air supply temperature on the injector exit Mach number and total pressure for 1.178 kg/s actuator mass flow rate.
5.1.2 Flow Properties at Compressor Inlet

The effect of these high velocity, high pressure, and high temperature jets from the actuator on the upstream duct was investigated using the control volume analysis described as part of the steady air injection model in Chapter 3. In addition to the thermodynamic properties of the freestream in the compressor inlet duct and the injected jet from the actuators, the shape of the partially mixed out flow from wind tunnel tests is required. However, such wind tunnel tests were not conducted and the shape for the case with cold air were used. The rationale for using this is that the static pressure at the injector plane in both cases are the same, so we expect the streamlines to be the same. The justification for this is based on the Munk and Prim substitution principle [56] which states that "for a steady, isentropic flow of a specified geometry and stagnation pressure distribution, any change in the stagnation temperature distribution will leave the streamline shapes and Mach number distribution unaltered". The effects of increasing the actuator air supply temperature on the velocity, absolute Mach number, total pressure, and total temperature of the flow entering the compressor are shown in Figures 5-3a, 5-3b, 5-4a, and 5-4b respectively. These plots show that when the air supply temperature is increased while maintaining the same air supply pressure and mass flow rate, the flow entering the compressor has a higher velocity, higher absolute Mach number, higher total pressure, and higher total temperature.
Figure 5-3: Effect of air supply temperature on the compressor inlet velocity and absolute Mach number for 1.178 kg/s of 100 psig air injected at 0° yaw with the sheet injectors.
Figure 5-4: Effect of air supply temperature on the compressor inlet total pressure and total temperature for 1.178 kg/s of 100 psig air injected at 0° yaw with the sheet injectors.
5.1.3 Compressor Performance

The steady air injection model can be used to predict the spanwise profiles throughout the flow field for any given operating point, but it cannot be used to predict the mass flow rate at which the transonic compressor is going to stall for any given injection configuration. However, based on our knowledge of the behavior of NASA Stage 35 to steady air injection it is safe to say that the higher the velocity being injected at the tip, the more range extension (or the lower the stalling mass flow rate). This is because NASA Stage 35 was found to be a tip critical machine and the higher the velocity being injected at the tip the more the tip region will be unloaded (i.e., the lower the incidence).

The steady air injection model was used to compute the spanwise distribution of total pressure and total temperature for NASA Stage 35 operating at 100% speed with a single rotor at 15.80 kg/s with 1.178 kg/s of 100 psig air at 305.6 K and 553.1 K injected at 0° yaw. Figure 5-5a shows the spanwise distribution of the rotor inlet relative Mach number and relative inlet flow angle (or incidence). Even though the relative velocity is higher at the tip for higher supply temperature, the relative Mach number is smaller because of the high local speed of sound at the tip. The relative inlet flow angle (or incidence) is lower at the tip for higher supply temperature. Thus the rotor blades are unloaded at the tip for higher supply temperature. This unloading at the tip should have a healthy effect for NASA Stage 35 vis-a-vis the reduction in the stalling mass flow rate. Figure 5-5b shows the rotor performance correlations of loss coefficient and relative exit flow angle (or deviation). The changes in rotor loss and deviation correlations are very small. Figure 5-6a shows the spanwise distribution of the total pressure at the rotor inlet and Station J (10.75 cm or 1.93 chord lengths downstream of the rotor). The total pressure at the rotor inlet is higher at the tip for high air supply temperature but the overall pressure rise across the blade row is less for the high air supply temperature. Again, the uniformity in the spanwise distribution of total pressure at Station J reflects the redistribution of the flow in the rotor blades or spanwise mixing that was incorporated in the modified rotor performance characteristics in the previous chapter. Figure 5-6b shows the spanwise distribution of the total temperature at the rotor inlet and Station J. The high temperature region at the tip spreads from 15% span at the rotor inlet to about 20% span at Station J. A similar set of spanwise profiles at -15° yaw are shown in Figures 5-7 and 5-8.
The steady air injection model was then used to predict the 100% speedlines with air at $305.6 \, K$ and $553.1 \, K$, injected at $0^\circ$ yaw and $-15^\circ$ yaw. The computed speedlines are shown in Figure 5-9. These speedlines show that the overall effect of increasing the temperature of the air supplied to the actuators while maintaining the air supply pressure and mass flow rate constant is to reduce the total pressure rise across the compressor. The drop in pressure is more severe when the jet actuators are oriented at $0^\circ$ yaw with the pressure dropping to levels below that with no injection. The predicted speedlines which show that the total pressure with the injectors oriented at $0^\circ$ yaw is lower that the total pressure with the injectors at $-15^\circ$ yaw is surprising because it is contrary to the observed trend in Figure 4-18 for low temperature air. One explanation could be that the blade row loss characteristic for $0^\circ$ injection is steeper than the blade row loss characteristic for $-15^\circ$ injection. With such a steep blade row loss characteristic, the changes in tip relative inlet Mach number and tip relative inlet flow angle for $0^\circ$ injection (which are larger than those for $-15^\circ$ injection as can be seen by comparing Figures 5-5a and 5-7a) will result in higher losses.
(a) Rotor inlet relative Mach number and relative flow angle

(b) Rotor loss coefficient and exit relative flow angle

Figure 5-5: Correlations for NASA Rotor 35 operating at 15.74 kg/s and 100% speed with 1.178 kg/s of 100 psig air at different temperatures being injected at 0° yaw using sheet injectors.
Figure 5-6: Predicted total pressure and total temperature profiles at Station J for NASA Rotor 35 operating at 15.74 kg/s and 100% speed with 1.178 kg/s of 100 psig air at different temperatures being injected at 0° yaw using the sheet injectors.
Figure 5-7: Correlations for NASA Rotor 35 operating at 15.62 kg/s and 100% speed with 1.178 kg/s of 100 psig air at different temperatures being injected at \(-15^\circ\) yaw using sheet injectors.
Figure 5-8: Predicted total pressure and total temperature profiles at Station J for NASA Rotor 35 operating at 15.62 kg/s and 100% speed with 1.178 kg/s of 100 psig air at different temperatures being injected at -15° yaw using the sheet injectors.
Figure 5-9: Computed total pressure characteristics for NASA Stage 35 operating at 100% speed with 1.178 kg/s of 100 psig air at different temperatures injected at 0° yaw and -15° yaw.
5.1.4 Summarizing the Effect of Hot Air Injection

On a mass average level, injecting hot air into NASA Stage 35 is predicted to have two main effects: a potential increase in the operating range (i.e., decrease in the stalling mass flow rate), and a decrease in the total pressure rise across the transonic compressor. The potential for increase in operating range extension can be explained by the fact that the high velocity jet from the actuator unloads the rotor blades at the tip.

The decrease in overall total pressure rise across the transonic compressor can be explained if the overall compressor total pressure rise is broken into its components: total pressure rise in the compressor inlet duct, and the total pressure rise across the blade rows. When the air supply temperature is increased while maintaining the same air supply pressure and mass flow rate, the results in Section 5.1.2 show that the total pressure in the compressor inlet duct increases. However, the total pressure rise across the blade rows decreases. The decrease in total pressure rise across the blade rows can be explained using the generic speedlines for axial flow compressors shown in Figure 5-10.

![Figure 5-10: Generic speedlines for axial flow compressors.](image-url)
When the actuator air supply temperature is increased while maintaining the same air supply pressure and mass flow rate, the total pressure and total temperature at the compressor inlet increases. This implies that the corrected mass flow rate \( \dot{m}_c = \dot{m}\sqrt{\frac{T}{T_{\text{ref}}}} \) remains almost the same whereas the corrected speed \( N_c = N/\sqrt{\theta} \) decreases. \( \theta = \frac{T}{T_{\text{ref}}} \) and \( \delta = \frac{P}{P_{\text{ref}}} \). The decrease in corrected speed at constant or slightly higher corrected mass flow rate will lead to a decrease in total pressure rise. Therefore, the increase in total pressure rise in the compressor inlet duct is not large enough to offset the decrease in total pressure rise across the blade rows due to reduced corrected speed. If the air supplied to the jet actuators in an active stabilization application such as in aeroengines is the recirculated air from a downstream stage of a multistage compressor, then the limitation in pressure rise as a result of hot air injection will make this a less desirable option. One way to get around such a problem would be to install a cooling system that will reduce the temperature (while keeping the pressure constant) of the air being supplied to the jet actuators. It should also be noted that the case investigated in this research is an optimistic one because the slot orifice of the jet actuator was increased in order to maintain the same mass flow rate. However, if no design modifications are made on the jet actuators, equation 5.1 shows that the mass injected will be even less. This implies that the pressure rise across the transonic compressor will be reduced even further.

The changes in vorticity for a fluid particle is given by:

\[
\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \nabla) \vec{u} - \vec{\omega}(\nabla \cdot \vec{u}) - \nabla \times \left( \frac{1}{\rho} \nabla p \right) + \nabla \times \vec{F}_{\text{visc}} + \nabla \times \vec{F}_{\text{body}}
\]  

(5.2)

where \( \vec{\omega} \) is the vorticity, \( \vec{u} \) is the velocity, \( \rho \) is the density, \( p \) is the pressure, \( \vec{F}_{\text{visc}} \) is the sum of viscous forces, and \( \vec{F}_{\text{body}} \) is the sum of body forces. For an incompressible inviscid flow with non-uniform density and conservative body forces, the change in vorticity for a fluid particle is shown in reference [56] to reduce to:

\[
\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \nabla) \vec{u} + \frac{1}{\rho^2} (\nabla \rho \times \nabla p)
\]  

(5.3)

The second term on the right hand side of equation 5.3 shows that vorticity is generated by the interaction of density and pressure gradients i.e., whenever the surfaces of constant density and constant pressure are not aligned or \( \nabla \rho \times \nabla p \neq 0 \). Thus, an aspect of unstedi-
ness not explained by this mass averaged quasi-steady prediction is that a hot streak going through a pressure gradient can lead to the generation of counterrotating vortices in the blade passage. Consult reference [132] for a detailed computational study of the unsteady effects of a density wake convecting through compressor blade rows.

The results from the feasibility study showed that high temperature air will increase the velocity and momentum of the jet from the actuator, reduce the relative Mach number at the compressor inlet, increase the temperature rise across the compressor, and reduce the pressure rise\(^2\) across the compressor. The implications of these for implementing active control is that direct recirculation of high temperature air from a downstream stage is not a good way of implementing active control using air injection. Since the results show that we need to inject high pressure and low temperature air to increase the pressure rise across the compressor, a plausible approach will be to pass the high temperature air from a downstream stage through a heat exchanger so that the temperature is reduced while maintaining the high pressure. However, the weight penalty of such a heat exchanger may outweigh the benefits of active control.

\(^2\)The reduction in pressure rise implies that the the closed loop speedlines with hot air injection will be shallow.
New control concepts, to address various implementation challenges in active control of rotating stall and surge are presented in this chapter. In industrial applications such as aeroengines, implementation of air injection control schemes is constrained by factors such as the “cost” (in terms of efficiency lost) and availability of high pressure air, and the size and weight of the actuation scheme. Air will typically be recirculated from a downstream compressor stage. The challenge, then, is to create control schemes that achieve acceptable performance with less air and fewer actuators. In this chapter, we will show that the net mass injected can be reduced by using a single-sided injection scheme; this concept is presented in Section 6.1. The control schemes for reducing the net mass injected are validated analytically in Section 6.2 using nonlinear simulations, and the control concept for implementing these control schemes in high speed compressors is validated experimentally in Section 7.3. We will also propose that the number of actuators can be reduced by using control laws that incorporate the discrete nature of the jet actuators; these control concepts are presented in Section 6.3, and the feasibility of the actuation configurations selected are demonstrated experimentally in Section 7.4.

6.1 Reducing the Mass Flow for Effective Control

Previous work on high speed compressors by Koch and Smith [76, 75] and Weigl et al. [130, 131] have demonstrated that steady injection of high pressure air can increase the pressure rise and extend the operating range of axial flow compressors. However, steady
injection is the most inefficient way of using the control power provided by high pressure air. Forced response measurements have demonstrated that unsteady air injection provides an effective lever on the pre-stall dynamics of high speed compressors. This led to the use of active feedback control to obtain even further range extension. The current state of the art requires air to be injected about a mean level so that the time average mean injection remains constant. This unsteady injection scheme is referred to as double-sided injection because air is injected at levels above and below the mean. A more effective actuation scheme that combines the advantages of steady and unsteady injection is single-sided injection.

6.1.1 Single-Sided Injection Scheme

With the single-sided injection scheme, air is only injected above a certain baseline level (usually zero). In the double-sided case, in contrast, air is injected above and below the mean available blowing. The main difference between single-sided injection and double-sided injection is that for single-sided injection, the control input can move only in one direction, whereas for double-sided injection, the control input can move in both directions about a constant mean. Thus the mean level of injection is constant with double-sided injection while the mean level of injection with single-sided injection depends on the amplitude of the circumferentially non-uniform component.

Using the single-sided injection scheme for control makes use of the effects of steady injection and unsteady injection to control rotating stall. Since single-sided injection results in an unsteadiness-dependent mean blowing, the time-averaged blowing will increase with unsteadiness. Thus two effects, which have both been demonstrated to extend the stable operating range, are introduced: increasing the stable operating range by increasing the mean level of injection, and stabilization of unstable modes by unsteady injection. Control laws which use single-sided blowing should take advantage of both of these effects, and a cumulative improvement will be possible with less overall blowing. This is demonstrated analytically in Section 6.2.

The single-sided injection scheme is illustrated in Figure 6-1 using a first harmonic constant gain feedback control law similar to the ones implemented by Paduano [96], Haynes [61],
The constant gain feedback control law is given by:

\[
\ddot{u}_n = -Z_n \tilde{\psi}_n
\]  

(6.1)

where \(\tilde{\psi}_n\) is the \(n^{th}\) spatial Fourier coefficient (SFC) of the static pressure perturbations measured at the sensor location, \(\ddot{u}_n\) is the \(n^{th}\) harmonic SFC of the actuation (injection) wave, which is introduced as an \(n^{th}\) harmonic sinusoidal wave, and \(Z_n\) is a constant complex gain. The amplitude of the control gain, \(|Z_n|\), is the scaling between the measured pressure perturbation SFC and commanded harmonic injection SFC, and is fixed for all frequencies. The phase angle of \(Z_n\), \(\angle Z_n\), is the spatial offset between the input and output waves and is also fixed for all frequencies. In Figure 6-1, the first plot shows the first harmonic of the measured perturbation wave. The second plot shows the twelve actuator commands, which have been scaled and offset in the circumferential direction by the control gain, for double-sided injection. The third plot shows the corresponding momentum flux profile injected at the tip by the sheet injector\(^1\). Since the injected mass flow varies linearly with the valve position, the injected mass flow is also sinusoidal about a mean value of 3.6% injection level. The magnitude of the momentum flux with the valves half open is indicated by a dashed line. The momentum is normalized by the upstream duct freestream value\(^2\). The fourth plot shows the twelve actuator commands for single-sided injection with the valves nominally-off, and the bottom plot shows the corresponding momentum flux profile injected at the tip by the sheet injector. The potential for reducing the mass flow for effective control with a single-sided actuation scheme can be easily seen from the analytical expression for the mean injection level of a first harmonic constant gain control law with single-sided actuation which is given by: \(\bar{\gamma}_{cg1s} = \gamma_o + \frac{A}{\pi}\), where \(\gamma_o\) is the baseline injection level\(^3\), and \(A\) is the first harmonic amplitude. At the nominally-off position, \(\gamma_o = 0\) and the mean injection level reduces to \(\bar{\gamma}_{cg1s} = \frac{A}{\pi}\). Thus, control laws which use single-sided unsteady blowing can take advantage of the steady and unsteady effects of air injection vis-a-vis stable operating range extension, and a cumulative improvement will be possible with less overall blowing.

\(^1\)The injected momentum flux profile was measured in a wind tunnel by Berndt [10].
\(^2\)The momentum flux is less than the baseline value of 1.0 for the injectors with low mass flows because the sheet injectors stick out into the compressor inlet duct.
\(^3\)The corresponding mean injection level for double-sided injection is \(\bar{\gamma}_{cg2s} = \gamma_o\).
Figure 6-1: Schematic of constant gain rotating stall control using double-sided and single-sided injection schemes.
6.1.2 Application of Nonlinear Controllers to High Speed Compressors

As discussed in the previous subsection, single-sided injection provides us with an actuation scheme for reducing the amount of mass required to extend the stable operating range for axial compressors. But single-sided injection is inherently a nonlinear actuation scheme because of the changes in time-average blowing. This implies that nonlinear models will be needed to design control laws that maximize the effects of single-sided injection. However, nonlinear control theoretic models suitable for designing control laws exist only for low speed machines.

The existence of multiple modes in high speed compressors has been demonstrated analytically by Bonnaure [13] and Feulner [38], and experimentally by Weigl et al. [130, 131]. These modes are classified as incompressible and compressible. Tryfonidis et al. [121, 122] studied the stall inception data for various compressors and found that stall-inception could be initiated by either the incompressible modes or compressible modes. The incompressible modes are the Moore-Greitzer rotating stall modes that have been identified in low speed machines. The rotation rates of these incompressible rotating stall modes depend on the compressor rotation frequency. The compressible modes are the acoustic modes due to compressibility. The rotation rates of these compressible modes are governed by the speed of sound. The acoustic modes do not play any role in the stall inception of low speed machines because of the wide separation in rotation rates between the incompressible and compressible modes. In high speed machines, the rotation rates of the incompressible modes and compressible (or acoustic) modes are of the same order. Thus, there is the potential for spillover in high speed machines due to the closeness in rotation frequencies between the incompressible and compressible modes. For example, control laws designed based on nonlinear models for low speed machines will be able to stabilize the incompressible rotating stall modes. However, if these nonlinear control laws are implemented in high speed compressors, they could excite and destabilize the acoustic modes while stabilizing the incompressible rotating stall modes. Such a destabilizing effect was observed by Weigl et al. [130, 131] on NASA Stage 35 using a first harmonic constant gain feedback control. Weigl [130] found that even though the stall inception in NASA Stage 35 was initiated by the incompressible rotating stall modes, a constant gain control law was unable to stabilize the transonic compressor at 100% speed. The first harmonic constant gain controller was shown to damp the rotating stall [1, 0] mode while destabilizing some acoustic modes.
The proposed control concept for alleviating this spillover effect is to append an incompressible state estimator to the nonlinear control laws designed based on the low speed models. The rationale for this control concept is to do no harm by making sure that the acoustic modes which do not initiate the stall inception in NASA Stage 35 are not excited. By estimating the incompressible states from the high speed compressor measurements, only the incompressible dynamics are fed back as illustrated in Figure 6-2. Refer to Appendix E for details of how the incompressible states are estimated from the compressible dynamics.

Figure 6-2: Block diagram of control concept for implementing nonlinear control laws.

6.1.3 Nonlinear Control Laws

The linearized theory of Moore and Greitzer was developed for small amplitude perturbations and is valid only in the pre-stall region. However, during stall inception, the waves grow as they evolve into full scale stall and the linearized theory will not be valid under these conditions. During large amplitude transients, the surge and rotating stall modes are coupled. The nonlinear interaction between the modes during stall inception often causes energy to be transferred between the modes. The compression system can thus be driven to instability by small amplitude perturbations through this nonlinear coupling between surge and rotating stall. As will be demonstrated in Section 6.2, control laws which account for the nonlinear coupling between surge and rotating stall will be better equipped to suppress both small and large amplitude perturbations.

Nonlinear coupling between surge and rotating stall is captured by the full state nonlinear distributed model derived in Appendix C. This model can be transformed into a second
order perturbation equation of the form:

\[ H \ddot{x} + C \dot{x} + L x + d = \tau \]  

(6.2)

where \( x \), given by \( x = \frac{1}{s + \lambda_B} \hat{\phi} \) with \( \lambda_B = \frac{n_B}{4B_L c} \), is a filtered version of the vector containing the flow coefficient perturbation, \( \hat{\phi} \), at evenly spaced locations around the compressor annulus, \( H, C, \) and \( L \) are constant matrices, \( d \) is the vector containing the nonlinear components of the compressor dynamics such as the total-to-static pressure rise characteristics and inlet distortion effects, and \( \tau \) is the vector containing the control terms at evenly spaced locations around the compressor annulus. Details on how this second-order perturbation equation was derived from the full state nonlinear distributed model can be found in Section F.1 of Appendix F. Two nonlinear control laws based on this model were investigated in this reasearch: sliding mode control and robust adaptive control. These nonlinear control laws are robust to inaccuracies in the terms included to the model (parametric uncertainties) and inaccuracies in the system order (unmodeled dynamics), and provide alternative approaches to dealing with model uncertainty.

**Sliding Mode Control Law**

Sliding mode control law is a robust controller which consists of a nominal component aimed at inverting the plant dynamics and an additional term aimed at dealing with the model uncertainty. Sliding mode control has been successfully applied to nonlinear control of underwater vehicles, automotive transmission and engines, high performance electric motors, power systems, and robot manipulators. Details of the derivation of the sliding mode control law are presented in Section F.2. The derivation closely follows that in [111], thus the details of the proofs are omitted. Rotating stall stabilization requires that the flow coefficient perturbations, \( \hat{\phi} \), are reduced to zero (as is the case with linear controllers) or prevented from growing above a certain acceptable or manageable level (as is the case with bifurcation controllers [7]). If the goal is to completely damp out the flow coefficient perturbations, stabilization of compressor instabilities can be interpreted as a tracking problem where the desired trajectory is \( \hat{\phi} = 0 \) (\( \iff \) \( x = 0 \)), but if the goal is to prevent the flow coefficient perturbations from growing above certain acceptable levels, then \( \| \hat{\phi} \| \leq \epsilon \) is desired.
We first define a generalized sliding surface as

\[ s = \dot{x} + \Lambda \ddot{x} \]  

(6.3)

where \( \ddot{x} = x - x_d \), \( x_d \) is the desired trajectory, and \( \Lambda \) is a symmetric positive definite matrix (or, more generally, a matrix such that \( \Lambda \) is Hurwitz). \( \Lambda \) is a diagonal matrix whose diagonal elements are equal to the control bandwidth, \( \lambda \). The control bandwidth, \( \lambda \) is typically limited by the frequency of the slowest unmodelled resonances, the neglected time delays such as in actuators, and the processing delays. Although the tuning of the control bandwidth, \( \lambda \), may be done experimentally, it is typically selected to meet the constraints of its limiting factors: \( \lambda \leq \lambda_R \approx \frac{2\pi}{3} \nu_R, \lambda \leq \lambda_A \approx \frac{1}{3\lambda_A}, \) and \( \lambda \leq \lambda_S \approx \frac{1}{3} \nu_{\text{sample}} \) where \( \nu_R \) is the frequency of the slowest unmodelled resonance, \( T_A \) is the actuator time constant, and \( \nu_{\text{sample}} \) is the sampling rate. Ideally, the most effective design corresponds to matching the three limitations of the control bandwidth i.e., \( \lambda = \lambda_R \approx \lambda_A \approx \lambda_S \). If the ideal criteria cannot be met, the desired control bandwidth is the minimum of the three bounds i.e., \( \lambda = \min(\lambda_R, \lambda_A, \lambda_S) \). Since \( s = 0 \) represents a linear differential equation whose unique solution is \( \ddot{x} = 0 \) given initial conditions \( \ddot{x}(0) = 0 \), the tracking problem \( \ddot{x} = 0 \) \( (\Leftrightarrow \ddot{\phi} = 0) \) can equivalently be reduced to keeping the vector \( s = 0 \). Also, the tracking problem \( \| \ddot{x} \| \leq \frac{\Phi}{\lambda} \) \( (\Leftrightarrow \| \ddot{\phi} \| \leq \frac{\Phi}{\lambda} \) can equivalently be reduced to keeping the sliding vector \( \| s \| \leq \Phi \). See Chapter 7 of [111] for the mathematical derivation showing the equivalence of these limits. Thus large values of \( \lambda \) will improve the tracking of the controller by reducing the position errors.

By forcing the closed loop system to follow the sliding surface, the second order tracking problem is replaced by a first order stabilization problem. The first order stabilization problem of keeping the vector \( s \) at zero is achieved by choosing the control law, \( T \), such that outside of the sliding surface, the following sliding condition is satisfied:

\[ \frac{1}{2} \frac{d}{dt} (s^T H s) \leq -\eta (s^T s)^{\frac{1}{2}} \]  

(6.4)

where \( \eta \) is a strictly positive constant. The sliding condition constraints system trajectories to point towards the surface. A geometric interpretation of the sliding condition is that the generalized norm (or 'distance') to the surface decreases along all system trajectories. The
sliding condition ensures that \( s \to 0 \) in finite time \( t_f \leq \frac{s(0)}{\eta} \). The switching controller for this system which satisfies the sliding condition in equation 6.4 is given by:

\[
\tau = \hat{\tau} - k \text{sgn}(s) \tag{6.5}
\]

where \( \hat{\tau} \) is the nominal component of the control term aimed at inverting the plant dynamics (this component will be the stabilizing control law if the model was exact), and the switching term \( k \text{sgn}(s) \) which makes the sliding surface attractive, accounts for the presence of modeling imprecision and of disturbances. The nominal control component, \( \hat{\tau} \), is:

\[
\hat{\tau} = -\hat{H}\hat{x} + (\hat{L} - \hat{C}\hat{A})\hat{x} + \hat{d} \tag{6.6}
\]

and the \( i \text{th} \) element of the column vector \( k \), \( k_i \), is:

\[
k_i = \| \left[ \hat{H}\hat{x} + (\hat{L} - \hat{C}\hat{A})\hat{x} + \hat{d} \right]_i \| + \eta_i \tag{6.7}
\]

where \( \hat{H}, \hat{C}, \hat{L}, \) and \( \hat{d} \) are estimates of \( H, C, L, \) and \( d \) respectively, and the corresponding errors in the estimates are defined as \( \hat{H} = \hat{H} - H, \hat{C} = \hat{C} - C, \hat{L} = \hat{L} - L, \) and \( \hat{d} = \hat{d} - d \).

See Section F.2 in Appendix F for the details on how to obtain the matrices in the sliding mode control law.

Even though the switching control law given in equation 6.5 will lead to perfect tracking i.e., \( \hat{\tau} \to 0 \), the switching controller is not acceptable because the discontinuities in the control law leads to chattering which is undesirable in practice. Chattering is undesirable in practice because it involves high control activity and may excite high-frequency dynamics neglected in the model (such as the unmodeled acoustic modes and neglected time-delays). The chattering problem encountered with the switching control law can be eliminated by smoothing out the control discontinuity in a thin boundary layer neighboring the switching surface

\[
B(t) = \{ x, |s(x; t)| \leq \Phi \} \quad \Phi > 0 \tag{6.8}
\]

where \( \Phi \) is the boundary layer thickness, and \( \epsilon = \frac{\Phi}{\lambda} \) is the boundary layer width. The continuous approximation of the switching control or the sliding mode control law of the
The system is given by:

$$\tau = \dot{\tau} - k \text{sat}(s/\Phi)$$  \hspace{1cm} (6.9)$$

where the discontinuity in the switching control law $\text{sgn}(s)$ is approximated by the saturation function $\text{sat}(s/\Phi)$. Outside of $B(t)$, the sliding mode control law is the same as the switching control law in equation 6.5, which satisfies the sliding condition of equation 6.4. This guarantees the boundary layer to be attractive, hence invariant. Thus the sliding mode control law leads to tracking to within a guaranteed precision, $\epsilon$, rather than 'perfect' tracking as is the case with the switching control law. In the context of rotating stall and surge stabilization, the sliding mode control will reduce the flow coefficient perturbations to within guaranteed levels.

The proof presented for the sliding mode control law places no restrictions on the control term $\tau$. The implication of this is that the closed loop system will be globally asymptotically stable i.e., all trajectories will converge to the equilibrium point. However, the control authority is limited in an actual compression system, thus the control term, $\tau$, is restricted. The bounds of the control term can be estimated from the relation in equation F.30 of Appendix F between the actuator input, $\gamma$, and control term, $\tau$:

$$\tau = [K \dot{\gamma} + \gamma \Psi_{c1}(\phi) + \gamma^2 \Psi_{c2}(\phi)] - [K T \dot{\gamma}^* + \dot{\gamma}^* \Psi_{c1}(T \dot{\phi}^*) + \gamma^* \Psi_{c2}(T \dot{\phi}^*)]$$  \hspace{1cm} (6.10)$$

where $\Psi_{c1}$ and $\Psi_{c2}$ are components of the compressor characteristics with air injection given in equation C.70 and shown in Figure C-6b of Appendix C. By using the appropriate restrictions on the actuator input, $\gamma$, for single-sided injection, the feasibility of single-sided sliding mode control with the valves in the nominally half-open position is demonstrated numerically in Section 6.2.

**Robust Adaptive Control Law**

The adaptive control law described here is similar to the one used for trajectory control of robot manipulators. The robust adaptive control law is composed of a nominal part aimed at inverting out the system dynamics and an additional term aimed at dealing with model uncertainty. In addition, the model is actually updated during operation based on
the measured performance. The main advantage of using the robust adaptive controller for stabilizing rotating stall is that unlike the sliding mode control law, no a priori knowledge of the system parameters of $H$, $C$, $L$ in equation 6.2 are required. Details of the derivation of the robust adaptive control law are presented in Appendix F.

For the robust adaptive control design, the column vector term, $d$, containing the nonlinear components of the compression system dynamics in equation 6.2, is assumed to be an unknown but bounded disturbance,

$$\left| d(x, t) \right| \leq D$$  \hspace{1cm} (6.11)

Thus only the bound of the disturbance (nonlinear components) are required. Let a new tracking error due to the disturbance be defined as:

$$s_\Delta = s - \Phi \text{sat}(s/\Phi)$$  \hspace{1cm} (6.12)

where $s$ is the generalized sliding surface defined in equation 6.3. The robust adaptive control law is given by:

$$\tau = Y \hat{a} - K_D s$$  \hspace{1cm} (6.13)

and the adaptation law is:

$$\dot{a} = -\Gamma Y^T \hat{s}_\Delta$$  \hspace{1cm} (6.14)

where $Y \hat{a} = (\hat{C} - \hat{H} \Lambda) \hat{x} + \hat{L} \hat{x}$. The relation between the controller gain matrix, $K_D$, the bound of the disturbance, $D$, and the boundary layer thickness of the sliding surface, $\Phi$, is: $K_D \Phi - D = 0$. $Y$ is a row vector containing the flow coefficient perturbations and their rates at various evenly spaced locations around the compressor annulus, $a$ is a matrix whose entries are elements from the matrices $(C - H \Lambda)$ and $L$, and $\Gamma$ is the adaptation gain matrix. See [111] for details on the procedures and significance of various choices of the control gain matrix, $K_D$, and the adaptation gain matrix, $\Gamma$. It should be noted that this approach does not necessarily estimate the unknown parameters exactly, but simply generates values that allow the desired task to be achieved.
6.2 Analytical Comparison of Control Schemes

The analytical study presented in this section has several objectives. First, we demonstrate that the modified full state nonlinear distributed model for low speed machines can predict the open loop stall inception of NASA Stage 35 at mean injection. Second, we demonstrate that more operating range extension can be obtained with robustness to compressor disturbances by using a single-sided sliding mode control law. Finally, we demonstrate stable operating range extension with reduced levels of blowing.

The ordinary differential equations characterizing the modal stall inception for low speed machines are presented in Section 6.2.1, the open loop simulation results used to validate the nonlinear distributed model are presented in Section 6.2.2, and the simulation results used to evaluate the effectiveness of different control schemes are presented in Section 6.2.3.

6.2.1 Modal Stall Inception Model for Low Speed Machines

The derivation of the full state nonlinear distributed model representing modal stall inception in low speed machines is given in Appendix C. Lumping the momentum component of the total pressure rise in the upstream duct due to air injection and the total-to-static pressure rise across the compressor blade rows, the full state nonlinear distributed model representing modal stall inception in low speed machines (axial flow compressors) including the actuator dynamics reduces to:

\[
\begin{align*}
E \dot{\phi} &= -A \dot{\phi} + \psi_1(\phi, \gamma_j) - L_r - L_s - T\psi + K \dot{\gamma}_j \\
\dot{\psi} &= \frac{1}{4B^2} \left[ -mS^T \dot{\phi} - \sqrt{2\psi} \right] \\
\tau_{loss} \dot{L}_r &= -J L_r + L_l^s(\phi, \gamma_j) \\
\tau_{loss} \dot{L}_s &= -L_s + L_s^s(\phi, \gamma_j) \\
\dot{x} &= A_{act} x + B_{act} \tilde{\gamma}_c \\
\dot{\gamma} &= C_{act} x + D_{act} \tilde{\gamma}_c \\
\bar{\gamma}_j &= Z_{cfg} \left[ \tilde{\gamma} + \gamma_{c0} \right]
\end{align*}
\]
where $\gamma_c$ is the normalized input command\(^4\), $\gamma$ is the actual valve position\(^5\), $\gamma_{c0}$ is the valve position when there is no input command voltage i.e., $\gamma_{c0} = 0.5$. $A_{act}$, $B_{act}$, $C_{act}$, and $D_{act}$ are the state space matrices for the actuator dynamics. The matrices $E$, $A$, $K$, and $J$, as well as the unsteady loss time constant, $\tau_{loss}$, were computed for NASA Stage 35 using the Moore-Greitzer parameters determined in Appendix C and summarized in Table C.3.5. The nonlinear functions $\Psi_\gamma$, $L_\gamma^s$, and $L_\delta^s$ for NASA Stage 35 were determined from the polynomial fits of the actual characteristics and the computed ideal characteristics given in Appendix C. The matrix $Z_{efg}$ contains the shape function information of the jet actuators and is determine from the momentum profiles measured in wind tunnel [10].

Badmus et al. [3] characterized compression system disturbances into impulsive and persistent disturbances. Impulsive disturbances correspond to system disturbances that perturb the system state from a given equilibrium point without throttling that system equilibrium. Persistent disturbances correspond to system disturbances that throttle the system equilibrium, thereby creating a new system equilibrium (or set of equilibria). These compression system disturbances were incorporated into the simulation by using the generalized initial condition representation in reference [107]:

\[
\begin{align*}
\dot{\phi}_o &= T (\bar{\phi}^* + \delta \phi_l) + \kappa \sin(\theta + \beta_l) \\
\psi_o &= \psi + \delta \psi_l \\
L_{\delta \phi_o} &= L_{\gamma}^{ss}(\phi_o) \\
L_{\delta \phi_o} &= L_{\delta}^{ss}(\phi_o) \\
K_t &= K_t(t)
\end{align*}
\]

where $\bar{\phi}^*$ is the equilibrium mean flow, $\delta \phi_l$ is the mean flow perturbation\(^6\) (or surge disturbance), $\kappa$ is the first mode perturbation amplitude (or rotating stall disturbance), $\beta_l$ is the relative orientation of the first mode flow perturbation with respect to the square-wave-

---

\(^4\)The input command to the actuation system is the voltage with the range ±10 V. The actuation command here is normalized such that: -10V corresponds to $\gamma_c = 0$, 0V corresponds to $\gamma_c = 0.5$, and +10V corresponds to $\gamma_c = 1$.
\(^5\)The valve opening is normalized such that $\gamma = 0$ corresponds to fully closed and $\gamma = 1$ corresponds to fully opened.
\(^6\)A positive mean flow perturbation will drive the compression system towards the stable portion of the compressor characteristic and a negative mean flow perturbation will drive the compression system towards the unstable portion of the compressor characteristic.
shaped inlet distortion ($\beta_l$ does not affect the stability boundaries for a uniform inlet flow i.e., case with no distortion), $\delta \psi_l$ is the plenum pressure perturbation, and $K_t$ is the throttle forcing function representing persistent disturbances. The simulation was implemented on MATLAB using SIMULINK [110]. A schematic showing the simulation setup for the closed loop system is given in Figure 6-3.

![Figure 6-3: Block diagram of closed loop simulation setup.](image)

The state is the flow coefficient at the compressor inlet, $\phi$. The sensors for this experiment are wall static Kulites at Station F, which are located at a nondimensional axial location of $\eta_m = -0.0971$ from the compressor inlet. The sensor readings are used by the control algorithm to determine the input command voltage. Since an input range of $\pm 10V$ is set for all A/D channels, the commanded voltage by the control law is passed through a saturation function to make sure the physical limitations of the actuator are not exceeded. The commanded input voltage is then passed through a servo dynamics to account for the bandwidth constraints of the actuation system.

6.2.2 Validation of Full State Nonlinear Distributed Model on NASA Stage 35
The open loop stall inception data for NASA Stage 35 at mean injection was obtained by turning on all twelve valves to the half opened position and closing the throttle quasi-steadily
until the compressor went into stall. To model this process using the simulation, the initial throttle coefficient was set at $K_t = 2.8730$, which is slightly higher than the throttle coefficient at the open loop stall inception point, $K_{t,\text{inception}} = 2.7398$. This corresponds to operating the compressor in the unstable portion of the compressor characteristic at $\Phi = 0.4000$ which is slightly less than the open loop stall inception flow coefficient of $\Phi_{\text{stall}} = 0.4103$. To initiate rotating stall in the compression system, a small impulsive disturbance on the first mode of the flow coefficient was introduced to the system. This corresponds to internal system noise driving the system into rotating stall. Using the generalized initial condition representation in equation 6.16, the following disturbance parameters were used: $\delta \phi_l = -0.01$, $\kappa = 0.02$, $\beta_l = 0$, $\psi_l = 0$, and $K_t = 2.8730$. The simulated open loop stall inception results are compared with experimental measurements in Figure 6-4. Figure 6-4a shows the static pressure perturbations that were measured at mean injection with the eight pressure kulite sensors around the annulus at Station F (see instrumentation in Figure 2-3), and Figure 6-4b shows the corresponding static pressure perturbations from simulation. To match the rotation frequency of the stall cells, the effective inertia parameters were used instead of the inertia parameters determined from the compressor blade geometry. By studying the stall inception of three different compressors using the nonlinear distributed model, Mansoux et al. [85] showed that the character of the stall inception (to a large extent) is determined by the shape of the compressor characteristic. Since we have measurements only for the stable portion of NASA Stage 35 compressor characteristic, the complete compressor characteristic reported in Appendix C and used for this analysis was obtained by adjusting the unstable portion until the transient from stall inception to fully developed stall matched the experiment. Consistent with the experimental measurements, the simulated open loop stall inception and fully developed stall cell is dominated by the first harmonic incompressible rotating stall $[1, 0]$ mode. These results show that the nonlinear distributed model captures the stall inception at mean blowing but does not capture the associated compressor surge. Having validated the enhanced full state nonlinear distributed model qualitatively using the open loop stall inception data from NASA Stage 35 operating at 100% speed, the model was then used for open loop and closed loop simulations.
Figure 6-4: Measured and predicted open-loop pre/post stall static pressure perturbations of kulite pressure sensors at Station F for 50% injection.
6.2.3 Stable Operating Range Extension

The goal of the closed loop simulations presented in this section is to determine which control scheme will provide more stable operating range extension given the same limitations of mass flow rate. To achieve this goal, three control schemes are investigated: double-sided first harmonic constant gain control, single-sided first harmonic constant gain control with the valves nominally half-open, and single-sided sliding mode control with the valves nominally half-open. These control schemes were selected because they have the same baseline which is the steady mean injection level, and the full state nonlinear distributed model was validated at the steady mean injection level.

To determine the stable operating range extension for each control scheme, the simulation was started at an operating point in the stable portion of the compressor characteristic with no disturbances. The throttle setting, $K_t$, was then increased (closing the throttle valve) systematically (staircase ramp) with each throttle setting kept constant for a time long enough to allow the system to reach steady state (100 rotor revolutions in this case). The throttle setting or persistent disturbance, $K_t(t)$, is shown in Figure 6-5. The corresponding impulsive disturbance parameters used are: $\delta \phi_l = 0$, $\kappa = 0$, $\beta_l = 0$, $\delta \psi_l = 0$.

![Staircase Ramp](image)

**Figure 6-5:** Throttle setting used to determine the stable operating range extension for various control schemes with a baseline at the steady mean injection level.
Steady Mean Blowing: To validate the open loop stall point for steady mean injection, the simulation was run for the throttle forcing shown in Figure 6-5 with all the control schemes turned off. Figure 6-7a shows the open loop characteristics for NASA Stage 35 with steady mean blowing. It shows that the sudden drop in pressure associated with rotating stall occurs when the throttle was closed beyond a flow coefficient of 0.4092, which corresponds to the open loop stall inception point at mean injection. The corresponding amplitudes of the first, second, and third modes are shown in Figure 6-7b. This shows that the open loop stall inception at mean injection is initiated by the first mode i.e., [1, 0] mode.

Double-Sided First Harmonic Constant Gain Control: The double-sided first harmonic constant gain control is a linear control scheme and is the baseline against which other control schemes are compared. The tuned complex gain used for the double-sided first harmonic constant gain control law is $Z_1 = 10 e^{j \frac{\pi}{8}}$. Figure 6-8a shows the closed loop characteristics for NASA Stage 35 with double-sided first harmonic constant gain control. It shows that the sudden drop in pressure associated with rotating stall occurs when the throttle was closed beyond a flow coefficient of 0.3709, which corresponds to a stable operating range extension of 9.36% beyond the mean injection stall point. The corresponding amplitudes of the first, second, and third modes are shown in Figure 6-8b. This shows that the first mode is completely damped out by the controller and the stall inception of the closed loop system is then initiated by the second mode, i.e., [2, 0] mode. Further stable operating range extension will therefore be obtained by controlling this second mode.

Single-Sided First Harmonic Control: The single-sided first harmonic constant gain control scheme with the valves nominally half-open is chosen as an alternative actuation scheme with the same control law as the baseline (double-sided first harmonic constant gain control). The schematic for this control scheme illustrating the first harmonic of the measured pressure perturbation wave, the actuator command, and the corresponding momentum flux profile injected at the tip by the sheet injector is given in Figure 6-6.

The tuned complex gain used for the single-sided first harmonic constant gain control law is $Z_1 = 10e^{j\pi}$. Figure 6-9a shows the closed loop characteristics for NASA Stage 35 with single-sided first harmonic constant gain control with the valves nominally half-open. It shows
that the sudden drop in pressure associated with rotating stall occurs when the throttle was closed beyond a flow coefficient of 0.3709, which corresponds to a stable operating range extension of 9.36% beyond the baseline stalling point. The corresponding amplitudes of the first, second, and third modes are shown in Figure 6-9b. This shows that the single-sided first harmonic constant gain control was able to completely damp out the first mode and thus achieve the maximum operating range extension possible when the first mode is damped. Beyond this point, the stall inception of the closed loop system is initiated by the second mode which must be controlled to achieve further stable operating range extension.

**Single-Sided Sliding Mode Control:** The details of the sliding mode control law are presented in Section 6.1.3. The jet actuators used for implementing the control laws on NASA Stage 35 have a bandwidth of 400 Hz which corresponds to a non-dimensional value of $\lambda_A = 1.4$. Thus the sliding mode controller bandwidth selected was $\lambda = \frac{1}{3} \lambda_A = 0.4667$. The inertia parameters and duct length were assumed to have a 5% uncertainty. The
time taken for the controller to force the trajectory into the boundary layer is given by 
\[ t_f \leq \frac{s(0)}{\eta}. \] Thus, the trajectory of the controlled system will enter the boundary layer faster for large values of \( \eta \). But from the expression for the gain vector, \( k \), in equation 6.7, large values of \( \eta \) will saturate the actuator. A numerically tuned value of \( \eta = 0.033 \) was selected. From Figure D-2 in reference [130], the amplitude of the first harmonic static pressure perturbation is \( |\tilde{p}_1| = 0.04 \) at the stall point of NASA Stage 35 for the linear control scheme. This corresponds to \( |\tilde{v}_1| = 0.02 \) or \( |\tilde{\phi}_1| \leq \frac{\tilde{v}_1}{\Phi} = 0.05 \). Since \( ||\tilde{\phi}|| \leq \frac{A\mu}{A} \Phi \), \( \lambda_B = 0.0891 \), and \( \lambda = 0.4667 \), the boundary layer size is \( |s| \leq \Phi = 0.26 \).

Figure 6-10a shows the closed loop characteristics for NASA Stage 35 with single-sided sliding mode control with the valves nominally half-open. It shows that the sudden drop in pressure associated with rotating stall occurs when the throttle was closed beyond a flow coefficient of 0.3265, which corresponds to a stable operating range extension of 20.42% beyond the mean injection stall point. The corresponding amplitudes of the first, second, and third modes are shown in Figure 6-10b. This plot shows a difference in the closed loop stall behavior for the single-sided sliding mode control scheme and the first harmonic constant gain control schemes. With the first harmonic constant gain control schemes, the first mode is completely damped out and the closed loop stall (Figures 6-8b, 6-9b) is then initiated by the second mode. However, with the single-sided sliding mode control scheme, the closed loop stall inception (Figures 6-10b) is still initiated by the first mode.

Conclusion: The closed loop simulations show that both the double-sided first harmonic constant gain control and the single-sided first harmonic constant gain control with the valves nominally half-open were able to completely damp out the first mode that initiated the stall inception in NASA Stage 35. Thus, both the double-sided first harmonic control scheme and the single-sided first harmonic control scheme with the valves nominally half-open were able to achieve the maximum operating range extension of 9.36% possible when the first mode only is completely damped. However, a stable operating range extension of 20.42% was obtained with the single-sided sliding mode control when the valves were nominally half-open. It can be concluded from these results that for the same mass limitation, more range extension can be obtained from control laws that incorporate the nonlinear interaction between the modes.
Open Loop: Steady Mean Injection

(a) Open loop characteristics

(b) Amplitudes of 1st, 2nd, and 3rd modes

Figure 6-7: Open loop characteristics and mode amplitudes for NASA Stage 35 with steady mean injection, $\Phi_{\text{stat}} = 0.4092$. 
Figure 6-8: Closed loop characteristics and mode amplitudes for NASA Stage 35 with double-sided first harmonic constant gain control, $\Phi_{stall} = 0.3709$. 
Closed Loop: 1-Sided 1ˢᵗ Harmonic Constant Gain Control

(a) Closed loop characteristics

(b) Amplitudes of 1ˢᵗ, 2ⁿᵈ, and 3ʳᵈ modes

Figure 6-9: Closed loop characteristics and mode amplitudes for NASA Stage 35 with single-sided first harmonic constant gain control with the valves nominally half-open, $\Phi_{\text{stall}} = 0.3709$. 

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Figure 6-10: Closed loop characteristics and mode amplitudes for NASA Stage 35 with single-sided sliding mode control with the valves nominally half-open, $\Phi_{s,t,all} = 0.3265$. 
6.2.4 Robustness to Compressor Noise

The goal of the closed loop simulations presented in this section is to determine which control scheme will be more robust to disturbances that can drive the compressor into rotating stall while operating at the open loop stall inception point with mean injection given the same limitations of mass flow rate. This criteria was selected because it gives a measure of the operability properties of a control scheme. Two control schemes are investigated: double-sided first harmonic constant gain control, and single-sided sliding mode control with the valves nominally half-open. The objective is to quantify the domain of attraction about the open loop stall inception point at mean injection for each control scheme. Badmus et al. [3, 4] showed that “enlarging the domain of attraction of linearly stable axisymmetric equilibria of the uncontrolled compressor has the effect of eliminating the abrupt jump in rotating stall amplitude during stall inception, and eliminating the hysteresis loop usually associated with rotating stall”. Since $\beta_1$ does not affect the stability boundaries for a uniform inlet flow, it is set to zero. Based on the generalized initial condition representation in equation 6.16, the compressor noise is characterized by the surge disturbance, $\delta \phi_1$, the rotating stall disturbance, $\kappa$, and the plenum disturbance, $\delta \psi_1$, for a given throttle setting, $K_t$. The throttle coefficient was set at $K_t = 2.8730$ which corresponds to operating the compressor at a flow coefficient $\Phi = 0.4000$ which is slightly less than the open loop stall inception flow coefficient of $\Phi_{\text{stall}} = 0.4103$ at mean injection.

In reference [84], Mansoux derived the following expression for the time rate of change of a Lyapunov function for rotating stall, $\dot{V}$, for a compressor operating in open loop:

$$\dot{V}(\tilde{\phi}, \tilde{\psi}) = \frac{1}{M} \tilde{\phi}^T \cdot \tilde{\Psi}(\tilde{\phi}) - \tilde{\psi} \cdot \tilde{\phi}_T(\tilde{\psi})$$  \hspace{1cm} (6.17)

where $M$ is the number of discrete points around the annulus. This is equation 2.31 in reference [84]. By mapping out the contour of $\dot{V} = 0$, the domain of attraction for the open loop system at mean injection can be determined. But the domain of attraction obtained from equation 6.17 is more optimistic since the first term is an average value. For example, the overall average $\dot{V}$ can be less than zero (signifying stability) at a given operating point where certain localized regions will have $\dot{V} > 0$ while other localized regions have $\dot{V} < 0$. A more conservative domain of attraction can be obtained by using the following localized...
expression for $\tilde{V}$:

$$\tilde{V}(\hat{\phi}, \hat{\psi}) = \max\left[\hat{\phi}_i \cdot \hat{\Psi}_i(\hat{\phi}_i)\right] - \hat{\psi} \cdot \hat{\phi}_P(\hat{\psi}) \quad (6.18)$$

where $i = 1, 2, \ldots, M$. The contours of $\tilde{V}$ given by equations 6.17 and 6.18 for NASA Stage 35 at a flow coefficient of $\Phi = 0.4000$ and mean injection are shown in Figure 6-11. These contours show that the smallest domain of attraction corresponds to $\delta \psi_1 = 0$. To reduce the number of states for the domain of attraction from three-dimensions to two-dimensions, we keep the plenum pressure perturbation constant at zero i.e., $\delta \psi_1 = 0$ as in reference [107]. The resulting domain of attraction will be in the $\delta \phi_1 - \kappa$ space.

For this analysis, the area enclosed by the boundary separating the stable and unstable regions, the negative $\delta \phi_1$-axis, and the positive $\kappa$-axis is used as a numerical estimate for the domain of attraction. Using this criterion, the area of the domain of attraction around the equilibrium operating point of $\Phi = 0.4000$ at mean injection for the uncontrolled compressor is zero i.e., $A_{ol} = 0$. The enlargement of the domain of attraction of the equilibrium operating point $\Phi = 0.4000$ by the double-sided first harmonic constant gain control scheme and single-sided sliding mode control scheme with the valves nominally half-open are shown in Figure 6-12. From these plots, the area of the domains of attraction are estimated as $A_{cgc2s} = 0.00395$ for the double-sided first harmonic constant gain controlled compressor, and $A_{smc1s} = 0.007975$ for the single-sided sliding mode controlled compressor. These results show that when the compressor is operating near the open loop stall inception point, the single-sided sliding mode controller is more effective at rejecting disturbances that can drive the compression system into rotating stall. Therefore, it can be concluded that for the same mass limitations, the single-sided sliding mode controller increases the robustness to rotating stall disturbances and thus has better operability properties than the double-sided first harmonic constant gain controller.
Figure 6-11: Open loop contours of $\hat{V}$ computed using equations 6.17 and 6.18 for NASA Stage 35 with mean injection and $\Phi = 0.4000$. 

(a) Contours of $\hat{V}$ for different values of $\kappa$ from equation 6.17

(b) Contours of $\hat{V}$ for different values of $\kappa$ from equation 6.18
(a) Domain of attraction for double-sided first harmonic constant gain control

(b) Domain of attraction for single-sided sliding mode control with the valves nominally half-open

Figure 6-12: Closed loop domains of attraction for NASA Stage 35 operating at $\Phi = 0.4000$ for double-sided first harmonic constant gain control and single-sided sliding mode control with the valves nominally half-open.
6.2.5 Stable Operating Range Extension with Less Mass Flow

The goal of the closed loop simulations presented in this section is to demonstrate that the stable operating range of NASA Stage 35 can be extended with less mass flow. Figure 6-13a shows the open loop characteristics for NASA Stage 35 with solid casing i.e., no blowing. It shows that the sudden drop in pressure associated with rotating stall occurs when the throttle was closed beyond a flow coefficient of 0.4313, which corresponds to the open loop stall inception point with no injection. The corresponding amplitudes of the first, second, and third modes are shown in Figure 6-13b. The amplitudes of these modes show that the open loop stall inception with solid casing (i.e., no injection) is initiated by the first mode. Thus, the single-sided first harmonic constant gain feedback control with the valves nominally-off is used to demonstrate that the stable operating range can be extended with less mass flow. The closed loop characteristics for NASA Stage 35 with single-sided first harmonic constant gain control when the valves are in the nominally-off position is shown in Figure 6-14a. It shows that the sudden drop in pressure associated with rotating stall occurs when the throttle was closed beyond a flow coefficient of 0.3851, which corresponds to a stable operating range extension of 10.71% beyond the baseline stalling point. The corresponding amplitudes of the first, second, and third modes are shown in Figure 6-14b. Figure 6-15 compares the stable operating range extensions for the double-sided first harmonic constant gain control scheme from Figure 6-8a, and the single-sided first harmonic constant gain control scheme with the valves nominally-off from Figure 6-14a.

The amount of actuator mass required under similar initial conditions was investigated for all four control schemes: double-sided first harmonic constant gain control, single-sided first harmonic constant gain control with the valves nominally half-opened, single-sided sliding mode control with the valves nominally half-opened, and first harmonic constant gain control with the valves nominally-off. Since the baselines for the nominally-off and nominally half-opened valve positions are different, the same size of surge and rotating stall disturbances ($\delta \psi_i = -0.02$, $\kappa = 0.04$, $\beta_i = 0$, $\delta \psi_i = 0$) were introduced to the compression system at equilibrium points ($\Phi_0 = 0.4200$ for the control scheme with the valve nominally-off and $\Phi_0 = 0.4000$ for the control schemes with the valve nominally half-opened) which are slightly below the corresponding open loop stall inception points ($\Phi_{stall} = 0.4313$ for  

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\[7\] The schematic for single-sided first harmonic constant gain control with the valves nominally-off is shown in the fourth and fifth plots of Figure 6-1.
the baseline with the valve in the nominally-off position and $\Phi_e = 0.4000$ for the baseline with the valve in the nominally half-opened position). The total amount of actuator mass required by each control scheme to bring the initial conditions to the equilibrium point (i.e., reject the compressor disturbance) was computed using the following expression:

$$m_{\text{total}} = \sum_{n=1}^{N_{\text{acts}}} \int_0^{t_f} \dot{m}_n(t) \, dt$$  \hspace{1cm} (6.19)$$

where $\dot{m}_n$ is the mass flow rate from the $n^{th}$ actuator, $N_{\text{acts}}$ is the number of actuators (twelve actuators were used), and $t_f$ is a large enough time to allow the system to reach steady state (for the size of the compressor disturbance selected, 20 rotor revolutions was large enough). Figure 6-16 shows the total actuator mass required by the different control schemes to reject the same surge and rotating stall disturbances. The results show that considerably less mass is required by the single-sided first harmonic constant gain control with the valves nominally-off.
Figure 6-13: Open loop characteristics and mode amplitudes for NASA Stage 35 with solid casing, $\Phi_{stall} = 0.4313$. 
Figure 6-14: Closed loop characteristics and mode amplitudes for NASA Stage 35 with single-sided first harmonic constant gain control with the valves nominally-off, \( \Phi_{\text{stall}} = 0.3851 \).
Figure 6-15: Closed loop characteristics for double-sided first harmonic constant gain control and single-sided first harmonic constant gain control with the valves nominally-off.

Figure 6-16: Bar graph comparing the total actuator mass required by the different control schemes to reject the same surge and rotating stall disturbances.
6.3 Reducing the Number of Actuators for Effective Control

With the spatial harmonic control approach which is the current state of the art, the implicit assumption is that the actuators can generate a continuous wave profile. However, the discrete nature of the actuators is demonstrated in Figure 6-1 which plots the momentum flux injected at the tip by the sheet injectors for a first harmonic constant gain control. A continuous actuation wave profile can be generated with an infinite number of actuators, thus the effectiveness of harmonic controllers will increase with the number of actuators. The range extension obtained with the spatial harmonic controllers using twelve actuators demonstrate that twelve actuators is sufficient since it can generate up to the fifth harmonic. In an industrial setting such as an aircraft engine application, a more plausible active control scheme to implement will require about three or four big actuators instead of twelve. For such cases, spatial harmonic control laws will break down. A more effective control scheme will be one that accounts for the discrete nature of the four actuators. To account for the discrete nature of the actuators, the effect of each actuator on each harmonic of the pre-stall perturbations will have to be identified.

6.3.1 Selection of Actuation Configurations

Figure 6-17 shows four actuation configurations that were experimentally investigated in this research. These configurations were selected based on the parametric actuation studies presented in Appendix G involving various configurations for twelve, eight, and six actuators. Actuation configuration I is the baseline configuration of twelve actuators equally spaced around the compressor annulus; actuation configuration II which consists of four ganged actuators (i.e., four groups of two actuators), represents a plausible configuration for four big actuators; actuation configuration III which consists of three ganged actuators (i.e., three groups of two actuators), represents a plausible configuration for three big actuators; and actuation configuration IV is the configuration for implementing the spatial harmonic control with six actuators equally spaced around the compressor annulus. Two criteria are used to quantify the performance of each actuation configuration: the RMS of actuator activity when the closed loop system is subjected to a unit intensity white noise, and the RMS of total control effort required to stabilize the compressor. The performance comparison

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8The RMS of total control effort is defined as the RMS of control activity in excess of the baseline multiplied by the number of actuators.
from the parametric actuation studies in Appendix G for the actuation configurations in Figure 6-17 are shown in Figure 6-18. The RMS of actuator activity shown in Figure 6-18a can be interpreted as the maximum effort required from each actuator to reject compressor noise at a given operating point on the compressor map, and the RMS of total control effort shown in Figure 6-18b can be interpreted as a measure of the total control power needed from each actuation configuration to stabilize the compressor at a given operating point on the compressor map. Over the flow coefficient range of $0.37 \leq \Phi \leq 0.41$, which were demonstrated analytically in the previous section to be accessible with a first harmonic constant gain controller, the parametric simulation results show that for the configurations with fewer than twelve actuators, actuation configuration II requires the least control effort. This actuation configuration is used in the next section to describe the control concept for reducing the number of actuators. The feasibility of the actuation configurations shown in Figure 6-17 are demonstrated experimentally in Section 7.4.

\[\text{Figure 6-17: Configurations for twelve, eight, and six actuators.}\]
Actuation configurations with twelve, eight, and six actuators

(a) RMS of actuator activity for the best configurations with twelve, eight, and six actuators

(b) RMS of total control effort for the best configurations with twelve, eight, and six actuators

Figure 6-18: Performance comparison for actuation configurations with twelve, eight, and six actuators.
6.3.2 Control Concept for Reducing the Number of Actuators

The control concept for reducing the number of actuators required for effective control of compressor instabilities consists of implementing control laws that incorporate the discrete nature of the actuators. This control concept is presented in this section for actuation configuration II (four ganged actuators i.e., four groups of two actuators) in Figure 6-17. The general multi-input multi-output (MIMO) system for this configuration of four big actuators is represented by:

$$\tilde{y}(s) = G(s) \tilde{u}(s)$$  \hspace{1cm} (6.20)

where the output vector $\tilde{y}$ represents the spatial Fourier coefficients (SFCs) of the static pressure perturbations at Station F and the input vector $\tilde{u}$ represents the command from each of the actuators. The transfer function matrix, $G(s)$, is made up of elements $G_{ij}(s)$ which represent the transfer function between the $j^{th}$ actuator input command and the $i^{th}$ harmonic of the pre-stall perturbations. The corresponding multi-input multi-output (MIMO) system for actuation configuration II in Figure 6-17 which consists of four ganged actuators (inputs) and senses the first four harmonics of the static pressure perturbations is given by:

$$\begin{bmatrix}
\tilde{y}_0(s) \\
\tilde{y}_1(s) \\
\tilde{y}_2(s) \\
\tilde{y}_3(s)
\end{bmatrix} =
\begin{bmatrix}
G_{01}(s) & G_{02}(s) & G_{03}(s) & G_{04}(s) \\
G_{11}(s) & G_{12}(s) & G_{13}(s) & G_{14}(s) \\
G_{21}(s) & G_{22}(s) & G_{23}(s) & G_{24}(s) \\
G_{31}(s) & G_{32}(s) & G_{33}(s) & G_{34}(s)
\end{bmatrix}
\begin{bmatrix}
\tilde{u}_1(s) \\
\tilde{u}_2(s) \\
\tilde{u}_3(s) \\
\tilde{u}_4(s)
\end{bmatrix}$$  \hspace{1cm} (6.21)

A state-space representation of the MIMO system given by equation 6.21 can be identified from the transfer functions $G_{ij}(s)$ using a system identification program called FORSE (Frequency dependent Observability Range Space Extension) designed by Jacques [71]. By identifying the elements $G_{ij}(s)$ i.e., the effect of each actuator on the different spatial harmonics, the discrete nature of the actuators are incorporated into the system model. An $H_\infty$ robust control law can then be designed for the MIMO system. The procedure for designing such an $H_\infty$ robust controller is well documented in Chapter 6 of reference [130], and Chapter 4 of reference [93]. Consult references [27, 51, 77] for more information on the $H_\infty$ control design.
6.4 Summary of New Control Concepts

Control concepts for reducing the actuation requirements for the suppression of rotating stall and surge were presented. Single-sided injection was proposed as the most efficient actuation scheme for reducing the amount of mass required for effective control of rotating stall and surge. However, since single-sided injection is inherently nonlinear and nonlinear models are available for incompressible machines only, a control concept was proposed whereby robust estimators are used to extract the incompressible states from measurements of the compressible system outputs. This control concept is validated experimentally in the next chapter. Two nonlinear control laws: sliding mode control and robust adaptive control, were designed based on a nonlinear distributed model for incompressible machines. An analytical study was then conducted using the nonlinear distributed model to demonstrate operating range extension with less mass, and to characterize the effectiveness of the sliding mode control law vis-a-vis operating range extension and robustness to compressor noise.

A parametric study was used to select two actuation configurations with fewer actuators. The feasibility of implementing these actuation configurations are demonstrated experimentally in the next chapter. A control concept that incorporates the discrete nature of the jet actuators was proposed as a means of reducing the number of actuators required for effective control.
CHAPTER 7

RESULTS FROM CONTROL EXPERIMENTS

In this chapter, the results from control experiments that validate some of the control concepts introduced in the previous chapter are presented. The experimental setup for control experiments is described in Section 7.1; the procedures for a typical control run are described in Section 7.2; the results from control experiments validating the control concept of appending incompressible state estimators to control laws that stabilize only the Moore-Greitzer rotating stall mode are presented in Section 7.3; and the results from control experiments validating the actuation configurations selected for this research are presented in Section 7.4.

7.1 Experimental Setup for Control Experiments

The control experiments for this research were conducted on NASA Stage 35, which is described in Chapter 2. Since some of the control experiments in this research were conducted with non-rotating radial and circumferential distortions in the inlet flow, the distortion generators on the test section for NASA Stage 35 are described in this section. The control experiments implementing the control concept requiring estimates of the incompressible dynamics were conducted with a non-rotating radial distortion in the inlet flow, and the control experiments for validating the actuation configurations selected in Figure 6-17 were conducted with a non-rotating circumferential distortion in the inlet flow.

Distortions are large non-uniformities in flow profiles that can be caused by flow separation or non-axisymmetric intake duct geometry. For example, the compressor inlet in aircraft
engines may experience large circumferential flow variations during aircraft maneuvers that involve high angles of attack, yaw angles or cross winds. Inlet distortion adversely affects the performance and stability of compressors and has been identified by Kerrebrock [74] as "one of the most troublesome problems in modern aircraft propulsion systems". Forms of inlet distortion include total and static pressure, flow velocity, flow angle, and temperature distortions. The distortions can be radial or circumferential and rotating or non-rotating. The measured speedlines for NASA Stage 35 operating at 85% speed with different distortions in the inlet flow are summarized in Figure 7-1. These speedlines show the loss in stability associated with the introduction of distortion in the inlet flow. A radial distortion screen in the inlet duct increased the stalling mass flow rate to 16.28 kg/s, an increase of 4.2% from the solid casing stall point of 15.63 kg/s, and the circumferential distortion screen in the inlet duct increased the stalling mass flow rate by 0.5% of the solid casing stall point. With steady mean injection, the stall point with inlet radial distortion was reduced by 9.2%, and the stall point with inlet circumferential distortion was reduced by 6.2%.

![Figure 7-1: Total-to-static pressure rise across NASA Stage 35 at 85% speed for different inlet flow distortions (measurements from [114, 125]).](image-url)
7.1.1 Radial Distortion Generator

A radial distortion in the inlet flow was generated by placing a radial distortion screen at approximately ten chord lengths upstream of the rotor. The radial distortion screen is a circumferentially uniform fine mesh covering about 38% of the radial span in the tip region as shown in the annular cross-section of Figure 7-2. The total pressure drop generated by the distortion screen was approximately one dynamic head, i.e., \( \frac{\Delta P_t}{\frac{1}{2} \rho u_c^2} \approx 1 \).

![Diagram of Tip-radial distortion screen in r - \theta plane.](image)

Figure 7-2: Tip-radial distortion screen in r - \theta plane.

Figure 7-3 shows the measured radial distribution of the flow coefficient, \( \phi = \frac{u}{\bar{u}} \), for the distorted and undistorted inlet flow, and the measured radial distribution of the total pressure drop across the distortion screen in the inlet duct for the distorted and undistorted cases. Due to the circumferential uniformity of the radial distortion screen, the flow profiles with the radial distortion screen are circumferentially uniform. The total pressure drop, \( \Delta P_t = P_t - \bar{P}_t \), is normalized by the mean dynamic head. The measurements show that the radial distortion screen introduces a total pressure loss in the tip region, increases the total pressure in the hub region, reduces the axial velocity in the tip region (loads the tip), and increases the axial velocity in the hub region (unloads the hub).
7.1.2 Circumferential Distortion Generator

A circumferential distortion in the inlet flow was introduced by mounting a circumferential distortion screen on a carrier at approximately ten chord lengths upstream of the rotor (the same axial location as the radial distortion screen). The circumferential distortion screen is a fine mesh with an extent, \( \theta_d \), of 120° covering the full radial span as shown in the annular cross-section of Figure 7-4. The screen can be rotated through 350° around the annulus, and the total pressure drop generated by the distortion screen was about one dynamic head, i.e., \( \frac{\Delta P_t}{\frac{1}{2} \rho c^2} \approx 1 \).
The DC(60)$^1$ descriptor is one of the parameters used to characterize the magnitude of the inlet circumferential distortion. See references [133, 124] for details on this parameter. The distortion screen used for this research corresponds to DC(60)=0.61037 [116]. Figure 7-5 shows the velocity, static pressure, and total pressure profiles measured at mid span for the circumferentially distorted inlet flow with the distortion screen blocking the flow in the range $120^\circ \leq \theta \leq 240^\circ$.

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$^1$As stated in reference [124], $\text{DC}(60) = \frac{\bar{P}_{\text{t}_{\text{360}^\circ}} - \bar{P}_{\text{t}_{\text{worst} 60^\circ}}}{\frac{1}{2} \rho C^2}$, where $\bar{P}_{\text{t}_{\text{360}^\circ}}$ is the mean total pressure over the entire $360^\circ$ sector of the annulus and $\bar{P}_{\text{t}_{\text{worst} 60^\circ}}$ is the mean total pressure over the $60^\circ$ sector of the annulus with the severest distortion effect. In an idealized case where the static pressure is uniform, DC(60)=1 corresponds to zero-velocity flow in a $60^\circ$ sector of the annulus.
Figure 7-5: Measured circumferential profiles of the flow coefficient, total and static pressure at the compressor inlet, and static pressure at the compressor exit for 1 dynamic head distortion and $\dot{m}_{corr} = 15.3 \text{ kg/s}$ (measurements from [114, 125]).

7.2 A Typical Control Run

The general control architecture for harmonic feedback control is given in Figure 7-6. Measurements from an array of eight wall static Kulites at Station F (see instrumentation layout in Figure 2-3) are passed through a high pass filter to obtain the wall static pressure perturbations. For each harmonic, the spatial Fourier coefficient (SFC) of the actuation wave is computed from the corresponding SFC of the static pressure perturbations and the control law. The actuator command is obtained by inverting the SFC of the actuation wave.
Typically, control law implementation requires three types of runs: sensor calibration, system identification, and feedback control. The data acquisition and control software described in Section 2.3 contains programs for each of these runs. All of the control experiments were conducted with the sheet injectors at $-15^\circ$ yaw (see Section 2.2.2).

The first step was to calibrate the 8 wall static Kulite sensors at Station F, which were used to determine the stall precursors (circumferential static pressure perturbations) for active feedback control experiments. The Kulite sensors were calibrated at the beginning of a test session and when the rotor speed was changed. The calibration was done at two mass flows along the speedline (one at a high mass flow near the choked point and one at a lower mass flow near the stall point). At each of these operating points, one second of unsteady data was acquired for all 8 Kulite sensors at Station F. The data point was accepted when the measured value and variance was within $0.1 \ psi$ with a variance of $0.2 \ psi^2$. The calibration constants for each of the 8 Kulite sensors were then determined by using the 4 steady wall static pressure measurements at Station F from the ESCORT system\textsuperscript{2} as a reference. For the control experiments that were conducted with a uniform inlet flow and a radially distorted inlet flow, all of the 8 Kulite sensors at Station F were calibrated at once, using as a reference the average of the absolute static pressure measurements from 4 steady wall static pressure taps at Station F. However, for the experiments with circumferentially distorted inlet flow, the 8 Kulite sensors at Station F were calibrated individually, and the reference for each of the Kulites was interpolated from the 4 steady pressure measurements.

The next step was to identify the compression system pre-stall dynamics. System identification experiments were conducted to obtain models for designing estimators, for determining

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\textsuperscript{2}The ESCORT system is the steady state instrumentation system. It is described in Section 2.1.2.
parameters for the nonlinear control laws, and for designing dynamic compensators required by the model based controllers for the reduced actuation system. All of these tests were conducted at a fixed compressor mass flow rate. Typically, five 9 second sinusoidal excitation sweeps were recorded (the computer RAM limited the circular buffer to 45 seconds with the 3.0 kHz sampling frequency). The magnitude of the forcing was adjusted in real time to achieve the highest possible coherence (measured on an analog system analyzer) without driving the system unstable. A state-space representation of the compression system pre-stall dynamics was identified from the measured transfer functions using a system identification program called FORSE (Frequency dependent Observability Range Space Extension) designed by Jacques [71] and system identification scripts written in MATLAB. Estimators were then designed based on the identified system dynamics using the procedures outlined in Appendix E. Steady state measurements and the identified system dynamics were used to determine the parameters for the nonlinear controllers using the parametric identification procedures outlined in Appendix C. Before the control laws could be implemented with the control computer in the third and final step, the estimators were converted to discrete-time. The order of each estimator was reduced so that all of the necessary calculations could be completed at the 3.0 kHz sampling frequency. Model reduction of the continuous-time estimators was based on the Hankel Singular value decomposition [18]. The reduced order model was then converted to discrete-time using Tustin’s approximation [95] with frequency prewarping. The model reduction and discrete-time approximation were implemented iteratively through a trial and error process until the zero order hold (ZOH) equivalent of the discrete-time controller closely matched the original continuous-time compensator. The state-space model for the accepted discrete-time approximation of the continuous-time compensator was then balanced and the state-space matrices rounded to nine significant digits to reflect the accuracy of the control computer calculations.

The third and final step was to use the control PC described in Section 2.3 to test the control laws. Both steady and unsteady measurements were taken as part of the control experiments. A typical control run for a constant gain controller appended with an estimator proceeded as follows: While the compressor was operating in the stable portion of the speedline (open loop stable), the state-space matrices for the estimator and feedback gains for the constant gain controller were set so that the system began to operate closed loop.
Feedback gains were then changed, to try to allow the system to operate at lower mass flows. The feedback gains were changed iteratively until the lowest possible stalling mass flow was achieved. The closed loop compressor speedline was determined by measuring the steady state compressor performance at several mass flows, from the choked mass flow to the new stall mass flow, using the steady state instrumentation system. To measure the development of the pre-stall perturbations into full scale rotating stall, stall transients were taken using the data acquisition system. Ten seconds of unsteady data were recorded as the throttle was closed at an extremely slow, constant rate from a stable operating point near the stall point into stall. This facility was not allowed to operate continuously with fully developed surge and rotating stall. Hence, the throttle was opened at the fastest possible rate once the compressor went unstable. Therefore, only limited data of the fully developed instability was measured for this research. Stall transients were also taken for open loop stall inception.

7.3 Results from New Control Concepts Experiments

The control concept of appending incompressible state estimators to control laws that stabilize only the Moore-Greitzer rotating stall mode was implemented on NASA Stage 35 with a radially distorted inlet flow. The radial distortion generator is described in Section 7.1.1. The effect of this distortion on stall inception is discussed in reference [115]. Since the radial distortion screen is circumferentially uniform, the flow profiles are also circumferentially uniform and the harmonics of circumferential pressure perturbations are decoupled as is the case with a uniform inlet flow. The first harmonic empirical transfer function estimate (ETFE) at 85% speed and a mass flow rate of 14.79 kg/s (near stall) is shown in Figure 7-7a. At this speed, the tip relative Mach number, $M'_{tip}$, of this compressor is 1.25. The measured transfer function shows the existence of a lightly damped low-frequency rotating stall (or Moore-Greitzer) mode as well as compressible (or acoustic) modes. The top plot in Figure 7-7b shows the corresponding identified first harmonic transfer function obtained by fitting polynomials to the measurements in Figure 7-7a. The middle and bottom plots in Figure 7-7b are, respectively, the corresponding Moore-Greitzer and acoustic disturbance components obtained using the formulation given in equation E.1.
(a) First harmonic ETFE at 85% speed and 14.79 kg/s

(b) Identified first harmonic transfer function at 85% speed and 14.79 kg/s

**Figure 7-7:** Measured and identified first harmonic transfer function of NASA Stage 35 at 85% speed and 14.79 kg/s with inlet radial distortion.
Figure 7-8 shows the open loop stall inception data at 85% speed with mean blowing. The static pressure traces and the accompanying spectrograms show that the mode driving the compressor into stall is the first harmonic incompressible Moore-Greitzer mode traveling around the annulus at about 40% of the rotor speed. The first harmonic compressible \([1, 1]\) mode traveling around the annulus at about 80% of the rotor speed is also present but does not grow. To demonstrate the effectiveness of the control concept of appending incompressible state estimators to controllers that stabilize only the incompressible Moore-Greitzer modes, a root locus analysis was performed using estimated dynamics of the unstable transonic compressor. Since the compressor was driven to instability by the \([1, 0]\) rotating stall mode, the unstable dynamics of the plant was estimated from the identified system dynamics in Figure 7-7a by making the \([1, 0]\) mode unstable. The root locus analysis for the unstable plant using a first harmonic constant gain control law is shown in Figure 7-9a. The results show that the first harmonic constant gain controller stabilizes the incompressible Moore-Greitzer rotating stall \([1, 0]\) mode that was shown to be driving the transonic compressor into stall. However, the first harmonic constant gain controller destabilizes the acoustic modes \([1, 2], [1, -2], [1, -3],\) and \([1, -4]\). As shown by the root locus analysis in Figure 7-9b, the incompressible Moore-Greitzer mode is stabilized and the acoustic modes are not destabilized when an optimal minimax filter is appended to the first harmonic constant gain controller. The transfer function of the optimal minimax filter used in the root locus analysis is shown in Figure 7-10a and the corresponding open loop transfer function of the combined plant and optimal minimax filter is shown in Figure 7-10b. The solid line is the first harmonic Moore-Greitzer mode from Figure 7-7b and the dashed line is the open loop system with the optimal minimax filter. These transfer functions help explain why this control concept works. The optimal minimax filter inverts the acoustic disturbance such that the resulting open loop system is the incompressible Moore-Greitzer mode that can be stabilized with the first harmonic constant gain control. To test this control concept on NASA Stage 35, the first harmonic constant gain control law without any estimators appended to it was first used to stabilize the compressor at 85% speed. No range extension was obtained. This is because some acoustic modes were excited and destabilized by the first harmonic constant gain controller. However, when the first harmonic constant gain controller was appended with the optimal minimax filter, the stalling mass flow rate was decreased to 12.98 kg/s, a reduction of 20.3% from the stall point of 16.28 kg/s for tran-
sonic compressor operating at 85% speed with inlet radial distortion. This corresponds to an additional range extension of 9%\(^3\) above the reduction obtained with steady mean injection.

In an industrial application such as aircraft engines where the engines must operate at different speeds, the robustness of any active control scheme to changes in speed is very important. To evaluate the robustness of this control concept to changes in compressor speed, the optimal minimax filter designed using the identified system dynamics at 85% speed was used in conjunction with the constant gain controller to stabilize the transonic compressor at 75%, 80%, 85%, and 90% speeds. The closed loop speedlines are shown in Figure 7-11a. These speedlines show that the operating range extension increases as the compressor speed is decreased from 90% to 75%, indicating that this control concept is robust to changes in compressor speed. A similar robustness test was performed by Weigl [130] with an \(H_\infty\) controller for a uniform inlet flow at 100% corrected speed. The \(H_\infty\) controller was designed based on the identified system dynamics at 100% speed and the controller was used to stabilize the compressor at 100% and 95% speeds. The speedlines from [130] and Figure 7-11a are reproduced in Figure 7-11b. These results show that the operating range extension at 95% speed is less than the operating range extension at 100% speed, contrary to the observed trend in this research.

The difference in performance can be explained using robustness arguments. The acoustic modes in high speed compressors are modes with axial structure which are introduced due to the compressibility of the flow and as such are relatively independent of the compressor rotation speed. On the other hand, the incompressible rotating stall mode is the component of the high speed compressor stall inception dynamics that is sensitive to changes in compressor speed, since its rotation rate scales with rotor rotation rate. With the eigenvalue perturbation \(H_\infty\) design in [130], uncertainties in the pre-stall eigenmode locations

\(^3\)This is comparable to the additional 9.3% reduction that was obtained with a first harmonic robust \(H_\infty\) controller by Spakovszky et al. [115]. The stalling mass flow rate reduction obtained with steady mean injection was 11.3% and is different from the 9.2% reduction indicated by the open loop speedlines in Figure 7-1a. The difference in stalling mass flow rates is due to the fact that the two sets of measurements were taken under different conditions. When the open loop speedlines in Figure 7-1a were taken, the spanwise profiles in the inlet duct were also measured and used to determine the calibration constants for correcting the mass flow rates through the compressor orifice. However, no spanwise profiles were measured when the closed loop speedlines in Figure 7-11a were taken several weeks later. Therefore, the calibration constants from the open loop speedlines in Figure 7-1a were used to correct the mass flow rates for the closed loop speedlines in Figure 7-11a.
were addressed by incorporating circular regions around the plant eigenvalues in the $H_\infty$ control design process. The resulting control law was able to stabilize the plant for all possible eigenvalue locations within these circular disks. Thus the $H_\infty$ control law is robust to changes in compressor speed that will not drive any of the eigenvalues out of its circular disk. Since the $H_\infty$ control law was designed based on the compressor stall inception dynamics at 100% speed, it is the optimal control law for stabilizing the incompressible rotating stall mode without destabilizing the acoustic modes at that speed. However, at 95% speed, the $H_\infty$ control law designed based on the identified system dynamics at 100% speed is a suboptimal control law with respect to stabilizing the incompressible rotating stall mode, whose eigenvalue location scales with the compressor speed. The notch-based control concept (of appending incompressible state estimators to a controller that can stabilize the incompressible rotating stall mode at all speeds) is a pseudo model-based controller because it consists of a constant gain controller component whose gains are experimentally tuned to stabilize the incompressible rotating stall mode at all compressor speeds, and a model-based filter that notches out the acoustic component of the compressor system stall inception dynamics. Since the acoustic modes are not sensitive to compressor speed, a notch filter designed based on the identified system dynamics at a high enough speed\textsuperscript{4} can be used to notch out the acoustic modes at different speeds. Therefore, the notch-based control concept is more robust\textsuperscript{5} to changes in compressor speed because it stabilizes the component of the compressor system dynamics which is sensitive to rotation speed.

\textsuperscript{4}Since the compressor stall inception dynamics are usually identified over a finite range of frequencies normalized by the rotor speed, $\pm 1.5\Omega_{\text{design}}$ in our case, the notch filter should be designed based on the compressor stall inception dynamics at the highest possible speed so that the notch filter will account for as many acoustic modes as possible.

\textsuperscript{5}These experimental results are encouraging but not conclusive, since they were conducted at different operating conditions.
Open Loop with 50% Injection

Figure 7-8: 85% speed throttle ramp for NASA Stage 35 with inlet radial distortion and steady mean injection.
Figure 7-9: 85% speed root locus analysis with first harmonic constant gain control.
Figure 7-10: Transfer functions of optimal minimax incompressible state estimator and the resulting open loop first harmonic system dynamics.
Figure 7-11: 85% closed loop speedlines from first harmonic notch-based control of NASA Stage 35 with inlet radial distortion, and 100% closed loop speedlines from first harmonic $H_\infty$ control of NASA Stage 35 with uniform inlet flow reproduced from [130].
7.4 Results from Reduced Actuation Control Experiments

Control experiments for different actuation configurations with fewer actuators were implemented on NASA Stage 35 with a circumferentially distorted inlet flow. The circumferential distortion generator is described in Section 7.1.2. The effect of this distortion on stall inception is discussed in [116]. Unlike the cases with uniform inlet flow and radially distorted inlet flow, where the harmonics of the pressure perturbations are decoupled, forced response transfer functions with circumferential distortion showed strong coupling between the harmonics. Coupling between harmonics due to circumferential distortion was observed to have a huge positive effect from a control standpoint, in that a constant gain controller was able to stabilize the compressor. This was in contrast to the cases with uniform flow and radial distortion, where the harmonics are uncoupled and an incompressible state estimator had to be appended with the constant gain controller before the compressor could be stabilized. Using the experimentally tuned optimal gain, the first harmonic constant gain controller was used to evaluate the effectiveness of the different actuator configurations in Figure 6-17.

Open loop stall inception data at 85% speed with circumferential distortion are shown in Figure 7-12. The pressure traces and first harmonic power spectrum show that open loop stall inception is initiated by the incompressible rotating stall or Moore-Greitzer [1, 0] mode, at 0.4Ω, since it is the first mode to go unstable. The first harmonic compressible [1, 1] mode is present but does not grow. A constant gain controller was experimentally tuned to damp the [1, 0] mode. The optimal gain for the constant gain controller was found to be \( Z_1 = 2e^{j\pi} \).

Figure 7-13 shows the unsteady stall inception data for the first harmonic constant gain controller with actuation configuration I. The first harmonic power spectrum shows that the [1, 0] mode is completely damped out. The pressure traces and second harmonic power spectrum show that the second harmonic incompressible rotating stall [2, 0] mode is the first to go unstable. The unsteady stall inception data in Figure 7-14 for the first harmonic constant gain controller with actuation configuration II shows a similar behavior to the closed loop stall inception with actuation configuration I. The only difference is that the closed loop stall inception with actuation configuration II is soft whereas the closed loop stall inception with actuation configuration I is hard (large amplitude perturbations). The unsteady stall inception data for the first harmonic constant gain controller with actuation configurations III and IV are shown in Figures 7-15 and 7-16 respectively. The pressure
traces and first harmonic spectra show that the incompressible rotating stall \([1, 0]\) mode is not completely damped out for both configurations which consist of six actuators and the closed loop stall inception for these configurations are still initiated by the \([1, 0]\) mode at 0.4Ω. Since the actuators were not saturated, higher gains may have been able to completely damp out this mode. These results imply that the gains will have to be increased as the number of actuators are reduced to completely damp out the unstable modes. This observation is consistent with the trend from the parametric simulation results in Figure 6-18 which show that more gain is required to stabilize the compressor for the actuators in actuation configurations III and IV than from the actuators in actuation configurations I and II.
Figure 7-12: 85% throttle ramp for NASA Stage 35 with inlet circumferential distortion and steady mean injection.
Figure 7-13: 85% throttle ramp for NASA Stage 35 with inlet circumferential distortion and first harmonic constant gain control using actuation configuration I.
1st harmonic control with 4x2 actuators

Figure 7-14: 85% throttle ramp for NASA Stage 35 with inlet circumferential distortion and first harmonic constant gain control using actuation configuration II.
Figure 7-15: 85% throttle ramp for NASA Stage 35 with inlet circumferential distortion and first harmonic constant gain control using actuation configuration III.
Figure 7-16: 85% throttle ramp for NASA Stage 35 with inlet circumferential distortion and first harmonic constant gain control using actuation configuration IV.
The reductions in stalling mass flow rate obtained with first harmonic constant gain feedback control using the experimentally tuned optimal gain of $Z_1 = 2e^{j\pi}$ for the four actuation configurations in Figure 6-17 are shown in Figure 7-17. The actuator valves were nominally half-opened, air was being supplied at 100 psig, and the sheet injectors were oriented at $-15^\circ$ yaw. Without feedback, the stalling mass flow rate was reduced by 8.7%$^6$ compared to the case with no injection, which stalled at 15.70 kg/s with circumferential distortion. First harmonic constant gain control reduced the stalling mass flow rate by 18.4%, 14.4%, 13.2%, and 13.5% for actuation configurations I, II, III, and IV in Figure 6-17 respectively.

**Figure 7-17:** 85% closed loop speedlines for NASA Stage 35 with inlet circumferential distortion for actuation configurations in Figure 6-17 using an optimally tuned first harmonic constant gain controller.

---

$^6$This reduction in stall point at mean blowing is different from the 6.2% reduction indicated by the open loop speedlines in Figure 7-1b. The difference is due to the fact that the two sets of measurements were taken several weeks apart. When the open loop speedlines in Figure 7-1b were taken, the circumferential profiles in the inlet duct were also measured and used to determine the calibration constants for correcting the mass flow rates through the compressor orifice. However, no spanwise profiles were measured when the closed loop speedlines in Figure 7-17 were taken several weeks later. Thus the disparity is due to the fact that the calibration constants from the first set of measurements were used to correct all the mass flow rates.
To evaluate the effectiveness of each actuation configuration, the mechanical or kinetic energy from the injected jet for 100 pre-stall revolutions and 100 post-stall revolutions was used as a measure of the control effort. The control effort is thus given by:

\[ E_{\text{control}} = \int_{t-100}^{t_0} \frac{1}{2} \dot{m}_{\text{jet}} V_{\text{jet}}^2 \, dt \]  

(7.1)

The stalling mass flow rate reduction and the corresponding control effort are summarized in Table 7.1. A measure of the effort required from each actuator to stabilize the compressor is the \(kJ\) of control effort per actuator per percentage of stalling mass flow rate reduction. This is summarized in the fourth column of Table 7.1, and can be compared with the parametric simulation results in Figure 6-18a which shows the RMS of actuator activity for the four actuation configurations at different flow coefficients. As expected with harmonic feedback control laws which are based on generating continuous wave profiles, the most effective configuration with the least actuator activity was obtained with actuation configuration I which consists of twelve actuators equally spaced around the compressor annulus. There is agreement between the results from simulation and control experiment for this configuration. The simulation results also show that actuation configuration II which consists of four ganged actuators (i.e., four groups of two actuators) has the second least actuator activity for flow coefficients above 0.37. This trend is also in agreement with the results from control experiments. The simulation results in Figure 6-18a also show that actuation configurations IV and III have the third and fourth least actuator activity respectively for flow coefficients above 0.37. This trend is also in agreement with the results from control experiments. Similar trends are observed when the parametric simulation results in Figure 6-18b which shows the RMS of total control effort are compared with the total control effort per percentage of stalling mass flow rate reduction summarized in the fifth column of Table 7.1. It should be noted that, a comparison between the simulation results in Figure 6-18 and the experimental results in Table 7.1 below a flow coefficient of 0.37 is not valid. This is because the results in Table 7.1 are based on a controller which was not able to extend the stable operating range to such low flow coefficients. Due to the fact that the stable operating range of the transonic compressor was extended with actuation configuration III which consists of three ganged actuators (i.e., three groups of two actuators), it can be concluded that it is feasible to stabilize the compressor with three big actuators.
Table 7.1: 85% speed first harmonic constant gain control experiments on NASA Stage 35 with circumferential distortion for different actuation configurations.

<table>
<thead>
<tr>
<th></th>
<th>% Decrease in Stalling Mass Flow</th>
<th>Jet Mechanical Energy for 100 Revolutions (kJ)</th>
<th>Effort per Actuator per % Decrease in Stalling Mass Flow (kJ / %)</th>
<th>Total Effort per % Decrease in Stalling Mass (kJ / %)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean Injection</td>
<td>8.7</td>
<td>1.0416</td>
<td>0.010</td>
<td>0.120</td>
</tr>
<tr>
<td>CONFIG I</td>
<td>18.4</td>
<td>1.3073</td>
<td>0.006</td>
<td>0.072</td>
</tr>
<tr>
<td>CONFIG II</td>
<td>14.4</td>
<td>1.1779</td>
<td>0.010</td>
<td>0.080</td>
</tr>
<tr>
<td>CONFIG III</td>
<td>13.2</td>
<td>1.1511</td>
<td>0.015</td>
<td>0.090</td>
</tr>
<tr>
<td>CONFIG IV</td>
<td>13.5</td>
<td>1.1479</td>
<td>0.014</td>
<td>0.084</td>
</tr>
</tbody>
</table>

7.5 Summary of Control Experiments

The control concept of appending incompressible state estimators to controllers that consider only the incompressible Moore-Greitzer rotating stall modes was validated experimentally. By appending an optimal minimax filter to a first harmonic constant gain controller, the operating range for NASA Stage 35 was extended by margins comparable to those obtained with a first harmonic robust $H_\infty$ controller. This control concept was also shown to be robust to changes in compressor speed because it stabilizes the component of the compressor system dynamics which is sensitive to rotation speed. Since the control concept was validated for a constant gain control law, the implementation of this control concept with nonlinear control laws still has to be justified.

Control experiments on NASA Stage 35 showed that the coupling between harmonics due to a circumferentially inlet distorted flow had a positive effect from a control standpoint, in that a constant gain controller was able to stabilize the compressor (unlike the cases with uniform inlet flow and radial inlet distortion). The actuation configurations selected for this research were experimentally demonstrated to be feasible using first harmonic constant gain control laws on NASA Stage 35 with an inlet circumferential distortion. It can be concluded from the results that it is feasible to reduce the number of actuators required for active control to three big ones.
In this chapter, a summary of the research work is presented in Section 8.1, the conclusions deduced from the thesis results are presented in Section 8.2, and the recommendations for future research work are presented in Section 8.3.

8.1 Summary

In the first part of this thesis, a steady air injection model was developed, verified experimentally, and used in two engineering applications. The steady air injection model consists of a control volume analysis and a streamline curvature analysis. The control volume analysis uses shape functions from wind tunnel measurements to model the change in mass, momentum, and swirl of the flow in the inlet duct containing the jet actuators. It requires as inputs the stagnation properties of the flow in the compressor inlet duct and the stagnation properties of the jet. It computes the two-dimensional \((r-\theta)\) profiles of the partially mixed out flow at the compressor face. Using this information, the streamline curvature portion of the model determines the response of the compressor blade rows. It requires geometric inputs which include the meridional flowpath coordinates and geometry of each blade row, and aerodynamic inputs which include spanwise performance correlations of the following: relative total pressure loss coefficient, relative exit flow angle, and tangential blockage factor. Using this information, the code computes the spanwise profiles of the steady flow quantities throughout the compressor. This steady air injection model was validated using circumferentially mass averaged total pressure radial profiles, circumferentially mass averaged total temperature radial profiles, and speedlines from a single-stage transonic
compressor.

The steady air injection model was used to investigate the effectiveness of injecting high temperature air in a single-stage transonic compressor. Results from the feasibility study showed that injecting high temperature air will increase the velocity and momentum of the jet from the actuator, reduce the relative Mach number at the compressor inlet, increase the temperature rise across the compressor, and reduce the pressure rise across the compressor. Three effects of hot air injection that are potentially beneficial can be seen. First, the higher velocity and momentum of the jet has the potential to increase the operating range of the single-stage transonic compressor. Second, the high temperatures at the compressor exit resulting from injecting hot air will lead to more power being available for the turbine. Third, the low relative Mach numbers at the compressor inlet resulting from hot air injection will reduce the shocks in the blade passage. However, there are some undesirable effects of injecting high temperature air such as high unsteady loading on the compressor rotor blades resulting from the high velocity jet, reduction in pressure rise across the compressor, and generation of discrete vortices on the blade surfaces as a result of the hot streaks going through a pressure gradient. The implications of these for implementing active control is that direct recirculation of high temperature air from a downstream stage is not a good way of implementing active control using air injection. Since the results show that we need to inject high pressure and low temperature air to increase the pressure rise across the compressor, a plausible approach might be to pass the high temperature air from a downstream stage through a heat exchanger so that the temperature is reduced while maintaining the high pressure. However, the weight penalty of such a heat exchanger may outweigh the benefits of active control.

The steady air injection model was used to improve the predictions of an existing two-dimensional compressible model of compressor rotating stall and surge dynamics [13, 38]. This linearized model was modified as follows:

1. The actuator boundary condition was modified to incorporate the changes in mean flow across the jet actuator.

2. Blade row total pressure losses were modeled to occur at the blade row trailing edges instead of the leading edges.
3. Blade row performance characteristics were made actuation dependent. This resulted in the introduction of new sensitivity parameters, \( \frac{\partial \omega_{iass}}{\partial u_j} \) and \( \frac{\partial \beta_{TE}}{\partial u_j} \) into the compressor stability model, where \( \omega_r \) is the blade row relative total pressure loss coefficient, \( \beta_r \) is the blade row relative exit flow angle or deviation, and \( u_j \) is the actuation command. These new design sensitivity parameters play an important role in modifying the system zeros.

4. A boundary condition was added to model the propagation of disturbances in a flow-path with variable cross-sectional areas.

5. A generalized complex acoustic impedance was added to the end conditions. This impedance relates static pressure and velocity perturbations.

In the second part of this thesis, new control concepts for reducing the air injection requirements (net mass flow and number of actuators) for effective stabilization of compressor instabilities are developed and validated analytically and experimentally. Single-sided actuation was proposed as the actuation scheme for reducing the net mass flow requirements. This was motivated by the fact that single-sided unsteady blowing has two avenues for extending the stable operating range - stabilization of unstable modes, and increased levels of time-averaged blowing as unsteadiness increases. A nonlinear distributed rotating stall and surge model for low speed compressors was modified to incorporate the effects of air injection and verified using open loop stall inception data from a single-stage transonic compressor. Three control schemes were evaluated: double-sided first harmonic constant gain control, single-sided first harmonic constant gain control, and single-sided sliding mode control law. Robustness to compressor disturbances and operating range extension were tested using Matlab simulations. Practical constraints such as servo dynamics, actuator saturation, and sensor limitations were included in the simulation. The simulation results showed that more range extension can be obtained from single-sided (unsteady) blowing than from double-sided blowing if control laws which incorporate the nonlinear coupling between surge and rotating stall are used. Also, the results show that the sliding mode control law is more robust to compressor disturbances than the heuristic control law. The extension of stable operating range was also demonstrated at a reduced level of blowing with the actuator valves nominally-off.
Even though the stall inception of the single-stage transonic compressor used for this research was initiated by the incompressible rotating stall modes, a constant gain feedback control law (which has been demonstrated in low speed compressors to be able to stabilize these incompressible modes) was unable to stabilize the transonic compressor. This observed behavior is due to the spillover effects of the acoustic modes. To alleviate the spillover effects of the acoustic modes, a control concept was proposed whereby incompressible state estimators were appended to control laws that only consider the incompressible rotating stall modes. In this way, only the incompressible dynamics were measured and fed back. This control concept was validated experimentally on a transonic compressor operating at 85% speed with an inlet radial distortion. By implementing a simple first harmonic constant gain controller appended with a robust $H_\infty$ estimator, the stable operating range of the transonic compressor was extended by 9% beyond the stall point at mean injection. The results from the control experiments showed that this control concept maintains the robustness of $H_\infty$ and is robust to changes in the compressor speed. This control approach provides a framework for the conceptualization and implementation of nonlinear control laws such as the sliding mode control law discussed above.

To reduce the number of actuators required for effective stabilization of compressor instabilities, a parametric study was conducted to select the best configurations for eight and six actuators. The feasibility of the actuation configurations selected was demonstrated experimentally on a single-stage transonic compressor operating at 85% speed with an inlet circumferential distortion. A control concept was proposed for actuation systems with reduced number of actuators. The main feature of this control approach is that it incorporates the discrete nature of the jet actuators. By identifying the transfer function of each jet actuator on the spatial Fourier coefficients (SFCs) of the pre-stall static pressure perturbations, a multi-input multi-output (MIMO) system can be generated. An $H_\infty$ robust controller can then be designed for the identified MIMO system.

8.2 Conclusions

The following conclusions were reached based on the results from this research work:
• Air injection changes the mass, momentum, and swirl of the flow in the duct containing the jet actuator, and the blade row performance characteristics. On a mass averaged basis, the pressure rise in the duct due to the jet actuator is small and as a result, most of the pressure rise come from changes in the compressor blade row performance induced by the actuator.

• Injecting air at the tip of the transonic compressor changes the flow redistribution in the blade passage and thus changes the blade row performance characteristics. These changes in blade row performance characteristics were accounted for by introducing an actuation dependence. The actuation dependence of the performance characteristics resulted in the introduction of new design parameters for actuation. These design parameters were shown to play an important role in modifying the zeros of the compression systems input-output characteristics.

• Based on the potential for stable operating range extension and the reduction in total pressure, it can be concluded on a first cut basis that hot air injection is not an effective active control scheme.

• For high speed compressors whose stall inception is initiated by the incompressible rotating stall modes but cannot be stabilized by control laws that consider only the incompressible rotating stall modes due to spillover effects from the acoustic modes, an effective concept for alleviating the spillover effects from the acoustic modes is to append incompressible state estimators to the control laws.

• A major step in reducing the levels of blowing required to stabilize rotating stall and surge in axial flow compressors is to implement control laws using single-sided injection with the valves nominally-off.

• More operating range extension can be obtained from single-sided unsteady blowing than double-sided blowing if control laws which incorporate the nonlinear coupling between surge and rotating stall are used. One of such promising control law is the sliding mode control law which was shown to provide more operating range extension and is robust to compressor disturbances.

• It is feasible to extend the stable operating range of the transonic compressor with three big actuators which have the same control authority as six of the actuators used...
in this research. It is assumed that no additional actuator delays would be incurred by implementing these actuation configurations.

8.3 Recommendations for Future Research

The steady air injection model presented in this research was validated using circumferentially mass averaged radial profiles of total pressure, circumferentially mass averaged radial profiles of total temperature, and speedlines from NASA Stage 35. To validate the model for application to a full-scale engine, the steady air injection model must be tested using similar experimental measurements from high speed multistage compressors.

The steady air injection model should be used to investigate the effectiveness of air injection in multistage compressors. Such a feasibility study can be used to determine the number of compressor stages whose behaviors will be altered by a jet actuator placed at the upstream duct, and can then be used to determine the optimal location for placing the jet actuator in a multistage compressor.

It was demonstrated in Chapter 5 that injecting high temperature air will reduce the total pressure rise across the compressor, and that injecting high pressure air will increase the total pressure rise across the compressor. In a multistage application where air will be recirculated from a downstream stage, a tradeoff study can be performed using the verified steady air injection model to determine the optimal stage from which high pressure and high temperature air should be recirculated.

Even though on a first cut basis the injection of hot air does not look promising, a more detailed feasibility study like the ones in references [108, 120] where the advantages and disadvantages identified in this thesis are quantified in a full scale engine application will be necessary.

More work needs to be done to improve the zeroth harmonic input-output dynamics of the compressible stall and surge inception model. Since the zeroth harmonic has been shown to be more sensitive to the boundary conditions, more attention needs to be paid to the end conditions. Also, the compressible stall inception model should be extended to include both radial and circumferential distortion.
The results from the control experiments showed that the first harmonic constant gain control law did not destabilize the lightly damped compressible modes of the single-stage transonic compressor with an inlet circumferential distortion. To obtain more stable operating range extension, a nonlinear control law such as the sliding mode control law designed in this thesis based on an enhanced incompressible model with the necessary modifications to account for inlet distortion should be implemented without any incompressible state estimators appended to it.
APPENDIX A

CALIBRATION AND CHARACTERIZATION OF ACTUATORS

This appendix outlines the operational procedures for the control unit, the assembly, disassembly and calibration procedures for the jet actuators, the steady and dynamic response of the actuators, and the injector dynamic model. Details of the actuator design, wind tunnel tests, frequency response, and performance of the prototype can be found in reference [10]. The assembly, disassembly, and calibration procedures for the jet actuation system are presented in Section A.1; the steady state characteristics and servo motor dynamics for the twelve jet actuators used in this research are presented in Section A.2; and the Helmholtz resonator model used by Berndt [10] to simulate the injector dynamics is summarized in Section A.3.

A.1 Assembly and Disassembly of Electronic Connections
If the actuator is to be shipped around, the following assembly and disassembly procedures should be followed:

A.1.1 ASSEMBLY
- From Back of Electronics Cabinet

1. Disconnect 208VAC 30AMP Yellow Power Supply Cables (2).

2. Disconnect 115VAC 10AMP Black Power Supply Cable (for fans).
- **Kaman Sensor Electronics Box**

  1. Remove back panel and strain reliefs.
  2. Screw in White Kaman Sensor cables from Channel A (12 labelled connectors).

    | 1 | 2 | 3 | 4 | 5 | 1 | 2 | 3 |
    | NC | NC | NC | - Output | + Output | GND | -15V | +15V |
    |    |    |    | (Black) & Shield | (Colored) | (Black) | (White) | (Red) |

  4. Reattach strain reliefs and back panel.

- **At MOOG Actuators**

  1. Connect Motor Coil Connectors (12)

- **Inside Back of Electronics Cabinet**

  1. Remove back panel with cable through holes and strain relief for power supply cables
  2. Connect Kaman Power cable (3 wire cable from power bus screw terminals)

    +15 V - Red
    GND - Black
    -15 V - White

  3. Connect Banana Plugs for Kaman sensor signals (12 labeled connectors)
  4. Reattach strain relief and back panel

- **From Back of Electronics Cabinet**

  1. Connect 208VAC 30AMP Yellow Power Supply Cables (2)
  2. Connect 115VAC 10AMP Black Power Supply Cable (for fans)

**A.1.2 DISASSEMBLY**

- **From Back of Electronics Cabinet**

  1. Disconnect 208VAC 30AMP Yellow Power Supply Cables (2)
2. Disconnect 115VAC 10AMP Black Power Supply Cable (for fans)

- **Inside Back of Electronics Cabinet**

  1. Remove back panel with cable through holes and strain relief for power supply cables
  2. Remove Banana Plugs for Kaman sensor signals (12 labelled connectors)
  3. Disconnect Kaman Power cable (3 wire cable from power bus screw terminals)
     
     +15 V - Red
     GND - Black
     -15 V - White
  
  4. Reattach strain relief and back panel

- **From MOOG Actuators**

  1. Disconnect Motor Coil Connectors (12)

- **Kaman Sensor Electronics Box**

  1. Remove back panel and strain reliefs
  2. Unscrew White Kaman Sensor cables from Channel A (12 labeled connectors)
  3. Check kaman Sensor Signals (plug in screw connectors for 12 sensors)

     | 1 | 2 | 3 | 4 | 5 | 1 | 2 | 3 |
     |---|---|---|---|---|---|---|---|
     | NC | NC | NC | - Output | + Output | GND | -15V | +15V |
     |   |   |   | (Black) & Shield | (Colored) | (Black) | (White) | (Red) |

  4. Reattach strain reliefs and back panel

**A.1.3 Start-Up and Shut-Down Procedures**

For proper operation of the control unit, the following start-up and shut-down procedures are recommended by the manufacturers at Moog Inc. [102]. These instructions are posted on the control unit.
Start-Up Procedure

1. Switch OFF the Main Power switch on the Console and both Sorensen power supplies.
2. Place the Enable/Disable switch in the DISABLE position.
3. Place the Dynamic/Static switch of each channel in the DYNAMIC position.
4. Connect motor and sensor cables. **DO NOT** connect or disconnect cables if the Enable/Disable switch is in the ENABLE position.
5. Make sure 115VAC/10A and 208VAC/30A power is correctly supplied to the Console.
6. Switch ON the Main Power switch on the Console.
7. Switch ON both Sorensen power supplies.
8. Place the Enable/Disable switch in the ENABLE position.
9. Start-Up complete. System is operational.

Shut-Down Procedure

1. Place the Enable/Disable switch in the DISABLE position.
2. WAIT for the pointers of the Sorensen power supplies to go down all the way to zero.
3. Switch off both Sorensen power supplies.
4. Turn off the Main Power switch on the Console.
5. Shut-down procedure is complete. You are home free now.

Important Points for Operating the Control Unit

- The static knob goes from 0 - 10. The static input voltage corresponding to various knob readings are as follows:

<table>
<thead>
<tr>
<th>Knob Indicator</th>
<th>Static Input Voltage</th>
<th>Valve</th>
<th>Sensor Position</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>+10V</td>
<td>FULLY OPENED</td>
<td>-0.01 inch</td>
</tr>
<tr>
<td>5</td>
<td>0V</td>
<td>HALF OPENED</td>
<td>0 inch</td>
</tr>
<tr>
<td>0</td>
<td>-10V</td>
<td>FULLY CLOSED</td>
<td>+0.01 inch</td>
</tr>
</tbody>
</table>
• Under nominal operating conditions i.e., zero input voltage, the sensor position should be at zero. If it is not, it can be adjusted to zero by either adjusting the proximeter probe on top of the motor or using the Dynamic Trim Knob. \( \frac{5}{16} \) and \( \frac{1}{8} \) wrenches are required for adjusting the proximeter probe.

• Nominal position for the Dynamic Trim Knob is straight up. The purpose of the dynamic trim is to correct the sensor location using a bias current. This will come in handy when correcting the sensor location movement caused by slanting or tilting the motor.

• To convert voltage readings from the output monitors to the actual values in a dynamic test, use the following conversions:

<table>
<thead>
<tr>
<th>Dynamic Input</th>
<th>1 Volt/0.001 inch</th>
</tr>
</thead>
<tbody>
<tr>
<td>Position Monitor</td>
<td>100 Volts/1 inch</td>
</tr>
<tr>
<td>Current Monitor</td>
<td>1 Amp/1 Volt</td>
</tr>
</tbody>
</table>

• When changing back and forth between Dynamic and Static modes, the Static Input Knob must be at 5.0.

• Make sure you are in Dynamic mode before enabling or disabling the system.

• DO NOT disconnect motor coil connectors when system is enabled.

• DO NOT use a SQUARE WAVE input command when running the control unit in Dynamic mode i.e., with the Dynamic/Static switch of each channel in the DYNAMIC position. But if you must, the frequency of the square wave must be at least 13 Hz.

• If the control unit is running in dynamic test mode with the Enable switch on and there is an emergency where you don’t have time to shut-down the system using the procedure outlined above, then you can just turn off the Main Power switch on the console (whose main function is to power the logic). It should be noted again that this is a bad practice and the preferred way to turn off the control unit is to follow the steps outlined above.

• When assembling the sensor leads before turning on the control unit you should keep in mind that the sensor leads are labeled as either “active” or “inactive”. On the
connector end panel "Channel A" is the "active" sensor channel and "Channel B" is the "inactive" sensor channel. Sensors having model numbers ending with '1' will use only the "A" connector while sensors having a model number ending in '2' will use both connectors. See KAMAN KDM-7200 Instruction manual [73] for more details.

A.1.4 Valve Sleeve-Cylinder Centering Procedure

To prevent the valve sleeve from rubbing against the valve cylinder, the centering procedure outlined below should be followed after checking and making sure that valve sleeve surfaces are level. The valve sleeve-cylinder centering procedure makes use of the linearity of the motor. For small amplitudes of the input command signal, the output sleeve displacement is a scaled and delayed version of the input. However, for large amplitude input command signal the behavior of the motor will be nonlinear (due to the open loop characteristics being influenced by the closed loop characteristics). Therefore, for an input command sinusoidal signal of small amplitude a distorted sinusoidal output sleeve displacement will signify rubbing. The centering procedure requires that air be blown through the valve in the reverse direction so that the motor can be adjusted on the valve top plate.

1. Mount the motor onto the valve top plate. Fasten the 4 mounting bolts by hand until there is snug fit.

2. Attach the valve sleeve to the motor shaft and fasten with a socket driver. Make sure the valve sleeve is firmly attached to the motor. Use loctite with care.

3. Unscrew the 4 bolts mounting the motor onto the valve body to loosen the motor (so that it can be moved around).

4. Input a sine wave to the command input and monitor the position of the valve sleeve on an oscilloscope. Since the sleeve is moving freely, the input command and the sleeve position signals should be pure sinusoids. Use a small amplitude (±2.5V), low frequency (10 - 40 Hz) sinusoid to ensure that you are in the linear range of the motor.

5. Turn off the input command to the motor.

6. Clamp the valve top plate (with the loose motor still attached) on a workbench in an upside down position.

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7. Place the spacer, dowell pins, and use the dowell pins to align the valve bottom plate onto the top plate.

8. With the valve body still in the upside down position, and no bolts attaching the bottom and top plates, turn on the input command to the motor.

9. Monitor the valve position on the oscilloscope. If there are any kinks or distortions in the sinusoidal position signal, tap/play with the bottom plate a little.

10. Fasten the 2 center bolts to get a tight fit.

11. Turn off the input command.

12. Attach the external bottom plate with the hose attachment to the valve bottom plate and fasten the 4 corner bolts.

13. Clamp the valve on the workbench in the upright position.

14. Connect the hose to the compressed air supply.

15. Turn on the input command and monitor the position signal.

16. Turn on the feed air supply, and continue to monitor the position signal. Set the feed air supply pressure to low levels (30 psig recommended) in order not to overwork the motor spring.

17. Move/rotate the motor around while monitoring the position signal until there are no kinks/distortions in the position sinusoid.

18. When the position signal on the oscilloscope has no kinks/distortions, fasten the bolts mounting the motor in the valve top plate. When fastening the mounting bolts, adjust adjacent bolts making one or two turns at a time to ensure that the motor is not tilted at anytime.

**NOTE:** If you are unable to get a sinusoid without kinks/distortions, turn off the air supply, turn off the input command to the motor, unscrew the 4 bolts mounting motor to the valve body, rotate the motor by 180°, and repeat steps 4-18.
A.1.5 Meeting Design Flow Requirements

After assembling the actuator components, the next thing to check is whether it meets the flow requirements at minimum (0%), mean (50%), and maximum (100%) injection levels. First make sure the dimensions of the valve components check with the design specifications. Next check for maximum mass flowrate, then mean and minimum (leakage) flowrates.

Maximum Flow Rate

First and foremost, the maximum mass flow rate that is achievable should be determined by measuring the mass flowrate when the valve is fully opened at 100 psig supply pressure. When we were putting the valve together, we found that the maximum mass flow rate was less than 0.10 kg/s at 100% valve opening with an air supply pressure of 100 psig. What we first did was check the orifice flow area to make sure that it was 60 mm² (which should correspond to six 0.125 inch wide by 0.04 inch high slots on the valve cylinder spaced 60° apart).

Secondly, check to see if the valve sleeve clears the slots on the valve cylinder when it is fully closed. One way to determine this is to use a depth micrometer.

- Turn off the air supply.
- Turn the Static Input Knob to Fully Closed position i.e. 0.0.
- Measure the distance from the valve bottom plate surface to the top of the valve sleeve using a depth micrometer. Let us call this reading “a”.
- Measure the distance from the valve bottom plate surface to the bottom of the slots on the valve cylinder. Let us call this reading “b”. For proper operation, the readings “a” and “b” should be identical. If they are not, some dimensional modifications will have to be made. For example, when we were putting the valves together the readings we got were “a = 171”, and “b = 231”. Thus the valve sleeve was 0.060 inch below the slots on the valve cylinder at fully closed position and 0.040 inch below the slots at fully opened position (since the stroke of the motor is 0.02 inch). What we did was to add a 0.060 inch shim on the bottom plate so that the sleeve will clear the slots on the valve cylinder at fully closed position.
If the actuator does not give the required mass flow rate at the desired air supply pressure after exploring the two options mentioned above, then the only other option is to increase the controller gains. It should be noted that increasing the controller gain is not an attractive option because it might alter the frequency response (reduced bandwidth and phase margin) of the control unit. The front panel of the control unit has four cards on the Right Hand Side. Each board has three channels for a total of 12 channels with each channel corresponding to one of the actuators. The channel (pot) arrangements corresponding to actuators 1 through 12 are as follows:

<table>
<thead>
<tr>
<th>Actuator</th>
<th>Channel</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10 7 4 1</td>
</tr>
<tr>
<td>2</td>
<td>11 8 5 2</td>
</tr>
<tr>
<td>3</td>
<td>12 9 6 3</td>
</tr>
</tbody>
</table>

These knobs (called pots) are referred to as GIN for Input Gain to the amplifier command input. GIN takes the input voltage or static input and conditions it such that 10 Volts input corresponds to a sensor position of 0.01 inch. Increasing the gain requires turning the pot COUNTERCLOCKWISE, and decreasing the gain requires turning the pot CLOCKWISE.

To increase the controller gain, the following steps were recommended by Moog [102]:

1. Set the position sensor at zero when the air supply is at 100 psig. This is achieved by adjusting the proximeter probe on top of the motor while monitoring the sensor position on a voltmeter (or an oscilloscope) until the reading is 0.0 or as close to 0.0 as you can.

2. Command the desired channel with a ±10 Volts signal at 50 Hz. Keep monitoring the sensor position on an oscilloscope. This implies that the control unit is in the DYNAMIC setting. The signal you should see is a sinusoidal signal at the same frequency of 50 Hz since this is a linear motor with a bandwidth of 400 Hz.

3. The next step in adjusting the gain is to turn the desired pot. But before you do that, you should MARK the current settings so that if the system response you obtain after adjusting the controller gain is not desirable, you can go back to the original configuration.
4. Gradually TURN the desired pot (for example if I want to increase the controller gain for channel 4, I will be turning the 2nd pot from top on the 2nd board from the right) COUNTERCLOCKWISE to about % of a turn. Adjust the GAIN until you obtain the ±1V i.e. 2V range. It should be noted that each pot has about 20 turns from one end to the other. For actuator #4, we rotated one of the corresponding pot to approximately turns counterclockwise to obtain the desired results. You should keep your eye on the coil current when the valve is at fully opened (FO) and fully closed (FC) positions. The design coil currents are 0.8A and -0.8A at FO and FC valve positions respectively. After adjusting the gains for actuator #4, the coil currents were reading 0.910A and -0.884A at FO and FC valve positions respectively. Whatever you do, these coil currents should NEVER EVER get above ±1 Amp. The current limiters kick in at 1.3 Amps.

5. After adjusting the gain, obtain the following data as a minimum to make sure that the actuator behavior is acceptable:

(a) Current Hysteresis Plot - CURRENT vs. COMMAND
   - AT 0 psig
   - AT 100 psig

(b) Position Hysteresis Plot - POSITION vs. COMMAND
   - AT 0 psig
   - AT 100 psig

(c) Sine Wave History Plot - COMMAND, POSITION, and CURRENT vs. TIME
   - COMMAND = ± 10 Volts at 50 Hz

(d) Frequency Response Plot
   - COMMAND = ± 10 Volts from 1-500 Hz

Mean and Minimum Mass Flow Rates
Each of the valve was designed to have a mass flow rate of 0.02 kg/s at 0% blowing, 0.06 kg/s at 50% blowing, and 0.10 kg/s at 100% blowing when the air supply pressure is 100 psig. After achieving the maximum mass flow rate required at the fully opened position (i.e. position sensor is at -1 Volt), the next step is to make sure that the flow requirements
are met when the valve sleeve is in the mean position (i.e., position sensor is at 0.00 Volts). If the mean flow is not met, a number of 0.001 inch shims are added between the motor and sleeve. The process of adding or removing shims is an iterative one. If the mean mass flowrate (i.e., the mass flowrate when the Static Knob is at 5.0 and the supply pressure is at 100 psig) is above the design specification (0.06 kg/s), you can start with 4 or 5 shims and keep increasing (or decreasing) if the mean mass flow rate is still above (or below) the design specification.

A.2 Valve Characteristics and Servo Motor Dynamics
The actuator calibration consists of measuring the steady state characteristics of the valves and the transfer function of the electro-mechanical servo motor dynamics. In Section 2.2, the steady state valve characteristics and transfer function of the electro-mechanical servo motor dynamics were presented for only one actuator to demonstrate the behavior of a typical actuator. The corresponding calibration plots for all twelve actuators used in this research are presented here. The steady calibration plots of mass flow rate versus cavity pressure (or air supply pressure) for 0%, 50%, and 100% valve opening, and mass flow rate versus input command1 at the design supply pressure of 100 psig for all twelve actuators used in this research are shown in Figures A-1 to A-12. The transfer functions of the electro-mechanical servo motor dynamics at 0 psig and 100 psig for all twelve actuators used in this research are shown in Figures A-13 to A-18.

---

1The input command has been normalized such that a static input voltage of -10 V corresponds to a normalized input of 0 (valve fully closed), a static input voltage of 0 V corresponds to a normalized input of 0.5 (valve half opened), and a static input voltage of +10 V corresponds to a normalized input of 1 (valve fully opened).
Figure A-1: Steady valve characteristics for Actuator Number 01.

Figure A-2: Steady valve characteristics for Actuator Number 02.

Figure A-3: Steady valve characteristics for Actuator Number 03.
(a) Steady valve characteristics

**Figure A-4:** Steady valve characteristics for Actuator Number 04.

(b) Steady characteristics at 100 psig

(а) Steady valve characteristics

**Figure A-5:** Steady valve characteristics for Actuator Number 05.

(b) Steady characteristics at 100 psig

(а) Steady valve characteristics

**Figure A-6:** Steady valve characteristics for Actuator Number 06.
(a) Steady valve characteristics

(b) Steady characteristics at 100 psig

Figure A-7: Steady valve characteristics for Actuator Number 07.

(a) Steady valve characteristics

(b) Steady characteristics at 100 psig

Figure A-8: Steady valve characteristics for Actuator Number 08.

(a) Steady valve characteristics

(b) Steady characteristics at 100 psig

Figure A-9: Steady valve characteristics for Actuator Number 09.

250
(a) Steady valve characteristics

Figure A-10: Steady valve characteristics for Actuator Number 10.

(b) Steady characteristics at 100 psig

Figure A-11: Steady valve characteristics for Actuator Number 11.

(a) Steady valve characteristics

(b) Steady characteristics at 100 psig

Figure A-12: Steady valve characteristics for Actuator Number 12.
Figure A-13: Transfer function of electro-mechanical servo motor dynamics at 0 $psig$ and 100 $psig$ cavity pressures for Actuators 01 and 02.
Figure A-14: Transfer function of electro-mechanical servo motor dynamics at 0 psig and 100 psig cavity pressures for Actuators 03 and 04.
Figure A-15: Transfer function of electro-mechanical servo motor dynamics at 0 psig and 100 psig cavity pressures for Actuators 05 and 06.
Figure A-16: Transfer function of electro-mechanical servo motor dynamics at 0 psig and 100 psig cavity pressures for Actuators 07 and 08.
Figure A-17: Transfer function of electro-mechanical servo motor dynamics at 0 psig and 100 psig cavity pressures for Actuators 09 and 10.
Figure A-18: Transfer function of electro-mechanical servo motor dynamics at 0 psig and 100 psig cavity pressures for Actuators 11 and 12.
A.3 Injector Dynamics

A Helmholtz resonator model shown in Figure A-19 was used by Berndt [10] to simulate the injector dynamics. This model assumes total temperature conservation (i.e. no work is done on the flow going through the actuator), choked valve orifice, zero plenum fluid velocity, incompressible quasi-steady flow at the injector exit, the jet exit static pressure is the same as the static pressure in the compressor inlet duct\(^2\), and negligible static pressure perturbation at the exit of the injector.

The steady state density of the jet from the injector, \( \rho_3 \), can be computed from the following quadratic equation:

\[
0 = (T_{t3}) \rho_3^2 - \left( \frac{P_3}{R} \right) \rho_3 - \left( \frac{\dot{m}_3^2}{2C_pC_d^2A_3^2} \right)
\]

where \( T_{t3} \) is the total temperature of the jet from the injector, \( P_3 \) is the static pressure of the jet from the injector (this is equal to the static pressure in the compressor inlet duct), \( \dot{m}_3 \) is the mass flow rate from the injector, \( A_3 \) is the injector exit area, \( C_d \) is the discharge coefficient, and \( C_p \) is the specific heat at constant pressure. The other thermodynamic properties of the jet from the injector can then be computed from the density and static pressure. The Mach number and total pressure of the jet from the injector computed using this model are compared with measurements in Figures A-20 and A-21. The first order

---

\(^2\)One of the sheet injectors on NASA Stage 35 was instrumented with static and total pressure probes and measurements taken to verify this assumption.
Helmholtz model representing the transfer function of the injector dynamics between the jet exit velocity perturbation, $\ddot{u}_3$, and the injected mass perturbation, $\dot{m}_1$, is:

$$\frac{\ddot{u}_3(s)}{\dot{m}_1(s)} = \frac{1}{\rho_3 A_3} \frac{1}{Ts + 1}$$  \hspace{1cm} (A.2)

where $T$ is the time constant given by $T = \frac{V_2 u_3}{A_3 a_2^2}$; $V_2$ is the volume of the injector plenum, $A_3$ is the injector exit area, $\rho_3$ is the density of the jet from the injector, $u_3$ is the mean velocity of the jet from the injector, and $a_2$ is the speed of sound in the injector plenum. See reference [10] for the details of the derivations of equations A.1 and A.2.
Figure A-20: Injector exit Mach number versus supply pressure for sheet injector at 50% and 100% valve opening (measurements from [14,117]).
Figure A-21: Injector exit total pressure and Mach number versus actuator mass flow for sheet injector (measurements from [14, 117]).
Appendix B

Effects of Steady Injection

Wind tunnel measurements by Berndt [10] show that circumferentially symmetric velocity and total pressure wakes are generated by the jet actuators during steady air injection. Rotor blades passing through these spatially non-uniform velocity wakes will experience unsteady forces. These aerodynamic loading are important because they can lead to high cycle fatigue failure of the compressor blades. The unsteady response of high speed compressors to these stationary wakes is modeled using a diffusing passage subjected to a time varying inlet flow condition in the rotor relative reference frame. Destructive forced vibrations can occur in compressor blades when a periodic aerodynamic force with frequency close to a system natural frequency acts on the blades in a given row. The main purpose for modeling the unsteady response to steady air injection is to estimate the aerodynamic loading on the compressor blades so as to prevent blade failures. In Section B.1, a simple model for the unsteady behavior of high speed axial-flow compressors to steady air injection is presented. In order to quantify the jet wake induced blade loading, a two-dimensional computational fluid dynamic code was used to estimate the force and moment fluctuations on NASA Rotor 35 near the stall point. The computational procedure and results are presented in Section B.2. To characterize the effect of the jet actuators on a transonic rotor, radial and circumferential profiles of total pressure, total temperature, and flow angle were measured at a station downstream of NASA Rotor 35 for different configurations of the jet actuators. The survey measurements on NASA Rotor 35 for different actuator configurations are presented in Section B.3.
B.1 Unsteady Response to Steady Air Injection
When a rotor blade passes through the spatially non-uniform velocity wakes generated by the jet actuators, it experiences an unsteady aerodynamic excitation. This displacement-independent excitation is also known as “gust-load” in literature. In this section, a simple model for blade row response to an upstream wake is presented.

B.1.1 Jet Wake-Rotor Blade Interaction
A simple concept for modeling the interaction between the jet wake and the rotor blade is illustrated in Figure B-1. This is an elementary model of a compressor rotor blade passage moving through a spatial non-uniformity.

![Diagram of jet wake-rotor blade interaction](image)

(a) Absolute reference frame
(b) Rotor relative reference frame

Figure B-1: Interaction of jet wake and rotor in absolute and relative reference frames.

The rotor blade moving through a stationary flow field non-uniformity is equivalent to a one-dimensional flow in a diffusing passage subjected to a time varying inlet flow condition.
in the rotor relative reference frame. Figure B-1a shows the rotor blade going through a spatial non-uniformity generated by the jet wake and Figure B-1b shows the equivalent system in coordinate system relative to the rotor blade. Figure B-2 shows the model of unsteady flow in a diffuser passage with the inlet denoted as station 1 and the exit denoted as station 2. The flow enters the diffuser at a constant inlet flow angle $\beta_1$ and leaves at a constant exit flow angle $\beta_2$. Assuming that the fluid is inviscid, the three equations governing the flow through diffuser passage are the unsteady one-dimensional momentum equation, the conservation of mass, and the conservation of rothalpy (or equivalently the Euler Turbine Equation).

\[ \frac{\partial W}{\partial t} + W \frac{\partial W}{\partial \ell} = -\frac{1}{\rho} \frac{\partial P}{\partial \ell} \quad (B.1) \]

where $W$ is the relative velocity, $\rho$ is the density, and $P$ is the static pressure. The unsteady one-dimensional momentum equation can be integrated from station 1 to 2 to yield:

\[ \int_1^2 \rho \frac{\partial W}{\partial t} d\ell = \left( P_1 + \frac{1}{2} \rho_1 W_1^2 \right) - \left( P_2 + \frac{1}{2} \rho_2 W_2^2 \right) \quad (B.2) \]

\[ \]
This shows that differences in \( P + \frac{1}{2} \rho W^2 \) along the diffuser are due only to unsteadiness. For incompressible flows where the Mach numbers are low, the expression in brackets is the stagnation pressure but that is not applicable for compressible flows where the Mach numbers are high. Applying the mass continuity equation between station 1 and an arbitrary location inside the blade passage gives \( \rho_1 W_1 \cos \beta_1 A_1 = \rho W A \). This mass continuity relation can be used to reduce equation B.2 as:

\[
\ell_e \cos \beta_1 \rho_1 \frac{\partial W_1}{\partial t} = \left( P_1 + \frac{1}{2} \rho_1 W_1^2 \right) - \left( P_2 + \frac{1}{2} \rho_2 W_2^2 \right) \tag{B.3}
\]

where the equivalent length, \( \ell_e \), is defined by \( \ell_e = \int_{1}^{2} \frac{A_1}{A_0} \, dl \). Linearizing equation B.3, and assuming the perturbation is of the form \( e^{i \omega t} \) results in the following momentum perturbation equation:

\[
\delta P_2 + \frac{1}{2} W_2^2 \delta \rho_2 + \bar{\rho}_2 W_2 \delta W_2 = \delta P_1 + \left( \frac{1}{2} W_1^2 - \ell_e \cos \beta_1 \frac{\partial \bar{W}_1}{\partial t} \right) \delta \rho_1 + \left( \bar{\rho}_1 W_1 - j \omega \ell_e \cos \beta_1 \bar{\rho}_1 \right) \delta W_1 + \left( \ell_e \sin \beta_1 \bar{\rho}_1 \frac{\partial \bar{W}_1}{\partial t} \right) \delta \beta_1 \tag{B.4}
\]

**Mass Continuity Equation:** The mass continuity equation between the diffuser inlet and exit is:

\[
\rho_1 W_1 \cos \beta_1 A_1 = \rho_2 W_2 \cos \beta_2 A_2 \tag{B.5}
\]

Linearizing equation B.5 results in the following mass continuity perturbation equation:

\[
(W_1 \cos \beta_1 A_1) \delta \rho_1 + (\bar{\rho}_1 \cos \beta_1 A_1) \delta W_1 + (\bar{\rho}_1 W_1 \sin \beta_1 A_1) \delta \beta_1 = 0 \tag{B.6}
\]

\[
(W_2 \cos \beta_2 A_2) \delta \rho_2 + (\bar{\rho}_2 \cos \beta_2 A_2) \delta W_2 + (\bar{\rho}_2 W_2 \sin \beta_2 A_2) \delta \beta_2 = 0
\]

**Conservation of Rothalpy:** The conservation of rothalpy across the blade row is given by the Euler Turbine Equation which can be reformulated to give:

\[
h_1 + \frac{1}{2} W_1^2 - \frac{1}{2} \Omega^2 r_1^2 = h_2 + \frac{1}{2} W_2^2 - \frac{1}{2} \Omega^2 r_2^2 \tag{B.7}
\]
Using the thermodynamic relation for the enthalpy \( h = C_p T \) and the ideal gas relation \( P = \rho RT \), the rothalpy conservation equation B.7 can be written in terms of \( P, \rho, \) and \( W \). Linearizing the resulting equation gives the following rothalpy conservation perturbation equation:

\[
\frac{C_p \delta P_1}{P_1} \delta P_1 + \frac{C_p \delta T_1}{\rho_1} \delta \rho_1 + \delta W_1 \delta W_1 = \frac{C_p \delta T_2}{P_2} \delta P_2 + \frac{C_p \delta T_2}{\rho_2} \delta \rho_2 + \delta W_2 \delta W_2
\]

(Kutta Condition: It is assumed here that the exit flow angle remains constant at the mean value even with perturbations in pressure, temperature, density, and velocity. This condition is similar to the Kutta condition used in the actuator disk analysis by Cumpsty et al. in reference [17] for subsonic flows leaving the blade passage in the direction of the trailing edges. The Kutta condition gives the following exit flow angle perturbation equation:

\[
\delta \beta_2 = 0
\]

For supersonic outlet flow, the direction of the exit flow will depend on the back pressure and the Kutta condition must then be relaxed. The modification uses the fact that once the flow is choked, the corrected mass flow, \( \dot{m}_c = \frac{\dot{m}_s \sqrt{C_p T_{i0}}}{AP_{o}} \), is constant upstream of the blade row. Therefore, for cases with supersonic exit flows, the Kutta condition is replaced by the linearized equation of the corrected mass flow equation.

Blade Row Response

The linearized momentum equation in B.4, the linearized mass conservation equation in B.6, the linearized rothalpy conservation equation in B.8, and the Kutta condition in equation B.9 provide a description of the blade row behavior from which the response to an upstream wake can be calculated. If we substitute the Kutta condition from equation B.9 into the linearized mass continuity equation in B.6, the linearized equations form the terms of a three-by-four matrix \( \bar{B}_1 \) containing terms at the inlet of the diffuser passage, and a three-by-three matrix \( \bar{B}_2 \) containing terms at the exit of the diffuser passage. The matrix form
of the equations is:

\[
\begin{bmatrix}
\delta P_2 \\
\delta \rho_2 \\
\delta W_2
\end{bmatrix} = \tilde{B}_2^{-1} \tilde{B}_1 \begin{bmatrix}
\delta P_1 \\
\delta \rho_1 \\
\delta W_1 \\
\delta \beta_1
\end{bmatrix} \tag{B.10}
\]

where the three-by-four matrix \( B_1 \) containing the steady mean values at the inlet of the diffuser passage is given by:

\[
B_1 = \begin{bmatrix}
1 & \left( \frac{1}{2} \tilde{W}_1^2 - \ell_e \cos \tilde{\beta}_1 \frac{\partial \tilde{W}_1}{\partial t} \right) & (\tilde{\rho}_1 \tilde{W}_1 - j \omega \ell_e \cos \tilde{\beta}_1 \tilde{\rho}_1) & (\ell_e \sin \tilde{\beta}_1 \tilde{\rho}_1 \frac{\partial \tilde{W}_1}{\partial t})) \\
0 & \tilde{W}_1 \cos \tilde{\beta}_1 A_1 & \tilde{\rho}_1 \cos \tilde{\beta}_1 A_1 & \tilde{\rho}_1 \tilde{W}_1 \sin \tilde{\beta}_1 A_1 \\
& -\frac{C_p T_1}{\tilde{\rho}_1} & \tilde{W}_1 \\
\end{bmatrix}
\tag{B.11}
\]

and the three-by-three matrix \( B_2 \) containing the steady mean values at the exit of the diffuser passage is given by:

\[
B_2 = \begin{bmatrix}
1 & \frac{1}{2} \tilde{W}_2^2 & \tilde{\rho}_2 \tilde{W}_2 \\
0 & \tilde{W}_2 A_2 \cos \tilde{\beta}_2 & \tilde{\rho}_2 A_2 \cos \tilde{\beta}_2 \\
& -\frac{C_p T_2}{\tilde{\rho}_2} & \tilde{W}_2 \\
\end{bmatrix} \tag{B.12}
\]

The steady mean values, and amplitudes of the inlet perturbations in static pressure, density, inlet velocity, and inlet flow angle are obtained from the steady air injection model. The corresponding amplitudes of the perturbations in static pressure, density, and exit velocity at the exit are then used to compute the perturbations in total pressure in the downstream duct.

### B.1.2 Summary of Unsteady Response to Steady Air Injection

The unsteady response of the compressor blade rows to steady air injection was modeled by one-dimensional flow in a diffusing passage subjected to a time varying inlet flow condition in the rotor relative reference frame. The inputs to the model are the steady mean values of the flow quantities and the amplitude of the tangential harmonic forcing resulting from the jet wake at the compressor inlet. These inputs can be obtained from the steady air injection model described in the Section 3.1. The outputs are the tangential flow field.
profiles downstream of the blade rows. It should be noted that the diffuser model does not account for any pressure gradients in the tangential direction. More accurate calculations of the unsteady interaction between the jet wake and blade rows can be obtained from computational methods such as UNSFLO [47, 48, 49].

B.2 Aerodynamic Loading on a Transonic Rotor Blade due to Steady Air Injection

The computations in this section were conducted on NASA Rotor 35 operating at a flow coefficient near the stall point. This corresponds to Reading 3976 in reference [104]. A comparison of the performance parameters at the design point and stalling mass flow point is summarized in Table B.1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Design Values</th>
<th>Reading 3976</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotor total pressure ratio</td>
<td>1.865</td>
<td>2.036</td>
</tr>
<tr>
<td>Stage total pressure ratio</td>
<td>1.820</td>
<td>1.923</td>
</tr>
<tr>
<td>Rotor head rise coefficient</td>
<td>0.273</td>
<td>0.402</td>
</tr>
<tr>
<td>Flow coefficient</td>
<td>0.451</td>
<td>0.340</td>
</tr>
<tr>
<td>Airflow kg/sec</td>
<td>20.188</td>
<td>18.20</td>
</tr>
<tr>
<td>Rotative speed (RPM)</td>
<td>17188.7</td>
<td>17218.5</td>
</tr>
</tbody>
</table>

B.2.1 Computation

The computational fluid dynamic (CFD) program used to obtain the simulation results presented in this section, called UNSFLO, was developed by Giles [48, 49, 50]. It is a two-dimensional procedure, with extensions to include quasi-three-dimensional effects, for calculating unsteady flow in turbomachinery. For this application, the available information include radial flow profiles at the rotor inlet, radial flow profiles at the rotor exit, and wind tunnel measurements of the steady momentum flux distributions due to the sheet and 3-hole injectors. The wake-rotor interaction option is used to compute the aerodynamic loading at three different sections of the blade. In this approach, an unsteady viscous calculation was performed for the rotor blade row with the velocity profiles being specified as unsteady inflow boundary conditions and the blades rigid. The computation is conducted in two
stages. In the first stage, the steady flow measurements at the rotor inlet and rotor exit are used to compute the steady flow field within the blade passages. In the second stage, the velocity wake generated by the jet actuators is then introduced as a boundary condition at the injector plane. Two input files are required to run UNSFLO: one containing the blade geometry information, and the other containing flow data information and unsteady boundary conditions.

**Unsteady Boundary Conditions:** At the inlet boundary, which corresponds to the actuator plane, the unsteady effect due to the incoming jet are specified in the form of velocity profiles. The velocity profiles were obtained from wind tunnel measurements by Berndt [10] for the sheet and 3-hole injectors. To establish the nature of the injection process, Berndt [10] had constructed a rectangular wind tunnel with an area equivalent to \( \frac{1}{12} \)th of the annulus of NASA Stage 35 with flow conditions that simulate those of NASA Stage 35 at stall \((M_f = 0.45)\). The steady momentum flux distributions of the sheet and 3-hole injectors were measured at 63 mm downstream of the injector center-line, which corresponds to the compressor face location. The steady momentum flux distribution ratio obtained for the sheet and 3-hole injectors when the jet actuator was supplied with air at 100 psig with the valve fully opened are shown in Figure B-3. The spatial axes correspond to the radial (0 to 80 mm) and circumferential (-60 to 60 mm) extent of the rectangular wind tunnel.

The jets from both the sheet injector and 3-hole injector occupy about 70% of the circumferential annulus. The radial extent of the sheet injector and 3-hole injector is the outer 15% and 40% of the annulus respectively. The peak momentum ratio for the sheet injector is 4 (corresponding to a velocity ratio of 2) whereas the peak momentum ratio for the 3-hole injector is 2.2 (corresponding to a velocity ratio of 1.5). The velocity profiles generated by the sheet injector were used in calculating the aerodynamic loading on the rotor blades because the sheet injector produced a stronger dynamic effect than the 3-hole injector. The inflow boundary velocity profiles at various radial locations for the sheet injector from Figure B-3a are shown in Figure B-4. Four models are available in UNSFLO for specifying the prescribed velocity profile at the inflow boundary: Sinusoidal, Gaussian, Hodson and Square functions. The wind tunnel measurements were fitted with a Gaussian distribution,
and the corresponding Gaussian wake parameters required by UNSFLO are summarized in Table B.2.

Table B.2: Parameters for the Gaussian wake used to fit the data for the velocity profile of the sheet injector. VWAKE is the fractional velocity defect, and WIDTH is the wake width non-dimensionalized by the wake pitch.

<table>
<thead>
<tr>
<th>% Span</th>
<th>VWAKE</th>
<th>WIDTH</th>
</tr>
</thead>
<tbody>
<tr>
<td>97</td>
<td>-1.620</td>
<td>0.300</td>
</tr>
<tr>
<td>94</td>
<td>-1.090</td>
<td>0.175</td>
</tr>
<tr>
<td>91</td>
<td>-0.800</td>
<td>0.130</td>
</tr>
<tr>
<td>88</td>
<td>-0.415</td>
<td>0.100</td>
</tr>
<tr>
<td>84</td>
<td>-0.105</td>
<td>0.060</td>
</tr>
<tr>
<td>81</td>
<td>-0.009</td>
<td>0.030</td>
</tr>
</tbody>
</table>
(a) Steady momentum flux distribution of sheet injector at maximum injection

(b) Steady momentum flux distribution of 3-hole injector at maximum injection

Figure B-3: Steady momentum flux distribution of the sheet and 3-hole injectors at maximum injection with air supplied at design conditions (from [10]).
Figure B-4: Inflow boundary velocity profiles at various radial locations for the sheet injector at maximum injection. The solid lines correspond to the wind tunnel measurements and the dashed lines correspond to the Gaussian fit.

**Procedure:** An overview of the computational procedure for evaluating the aerodynamic loading on the rotor blades of NASA Stage 35 using UNSFLO is given in Figure B-5. The first step was to perform a steady inviscid calculation. An inviscid grid was generated using the command "GRDINI SEC INV" which used the blade file BLADE.SEC to generate the grid file GRID.INV. The steady inviscid flow was then computed using the command "UNSFLO INV INV" which used the grid file GRID.INV and the input file INPUT.INV (containing boundary conditions such as inflow angle, fluid properties such as the ratio of specific heat capacities, and various numerical parameters such as the number of iterations performed for the steady calculation) to generate the flow file FLOW.INV and the corresponding iteration history file HIST.INV. The purpose of the inviscid calculation was to use
its resulting flow file FLOW.INV to initialize the flow field for the viscous calculation.

The second step was to perform a steady viscous calculation. A viscous grid was generated using the command “GRDINI SEC P1” which used the blade file BLADE.SEC to produce the grid file GRID.P1. The initial flow file was created from the steady inviscid flow calculated in the first step using the command “FLOINT INV INV P1 P1” which used the grid files GRID.INV and GRID.P1, and the flow file FLOW.INV to produce the flow file FLOW.P1. The steady viscous calculation was run using the command “UNSFLO P1 P1” which used the grid file GRID.P1, the flow file FLOW.P1, and the input file INPUT.P1 (containing boundary conditions such as inflow angle, fluid properties such as the ratio of specific heat capacities, and various numerical parameters such as the number of iterations performed for the steady calculation) to produce a new (updated version) flow file FLOW.P1 and the corresponding iteration history file HIST.P1. The iteration history was plotted using the command “HSTPLT P1” whereas the flow results were plotted using the command “FLOPLT P1 P1” which used the grid file GRID.P1 and the flow file FLOW.P1. The flow field generated was for a single passage grid. However, the experiment to be performed will have 12 jet actuators mounted around the circumference of the compressor which has 36 rotor blades. Therefore, each actuator will span 3 rotor blades. To accurately simulate the experimental conditions, a multi-passage grid and its corresponding flow field was then created using the command “FGMULT P1 P1 P2 P2” which used the grid file GRID.P1 and flow file FLOW.P1 for the single passage to produce the grid file GRID.P2 and flow file FLOW.P2 for the multi-passage system.

Third, the unsteady viscous calculation for the multi-passage system was performed using the command “UNSFLO P2 P2” which used the grid file GRID.P2, the flow file FLOW.P2, and the input file INPUT.P2 (containing the unsteady wake boundary conditions and various numerical parameters) to produce a new (updated version) flow file FLOW.P2, the unsteady flow file FLOU.P2, and the corresponding history file HIST.P2. The iteration history was plotted using the command “HSTPLT P2” whereas the flow results were plotted using the command “FLOPLT P2 P2” which used the grid file GRID.P2 and the flow file FLOW.P2. Fourier post-processing was then performed using the command “UNSFPT P2 P2” which used the grid file GRID.P2 and the unsteady flow file FLOU.P2 to produce the Fourier flow file FOUR.P2. Finally, the results were plotted using the command “UNSPLT
P2 P2” which used the grid file GRID.P2 and the Fourier flow file FOUR.P2.

Figure B-5: Calculation procedure using UNSFLO.

B.2.2 Results
The results from UNSFLO include surface and contour plots of point quantities such as density, pressure, Mach number, stagnation enthalpy, stagnation pressure, entropy, vorticity, temperature, turbulent viscosity, turbulent kinetic energy, skin friction, and Nusselt number, and integral quantities such as Lift, Drag, Moment, and Circulation over a period. The components of the aerodynamic loading presented here include the axial forces on the rotor blades, the tangential forces on the rotor blades, and the moment on the leading edge of the rotor blades.

Figures B-6, B-7, and B-8 show the axial force per unit span on rotor sections at the tip, 89% span, and 77% span respectively for the three rotor blades under the influence of the jet actuator. Both the steady loading obtained when the jet is turned off and the unsteady loading obtained at maximum jet injection are shown in the same plot for comparison purposes. The results show that the axial loading on the three rotor blades spanned by a jet actuator fluctuates from 730 $N/m$ to 1963 $N/m$ for the rotor tip section, 1275 $N/m$ to 2300 $N/m$ for the rotor section at 89% span, and 1354 $N/m$ to 1465 $N/m$ for the rotor section at 77% span. These correspond to fluctuation amplitudes of 1233 $N/m$, 1025 $N/m$, and 111 $N/m$ for the rotor sections at the tip, 89% span, and 77% span respectively. Figures
B-9, B-10, and B-11 show the tangential force per unit span on rotor sections at the tip, 89% span, and 77% span respectively for the three rotor blades spanned by a jet actuator.

The results show that the tangential loading on the three rotor blades spanned by the jet actuator fluctuates from $433 \, \text{N/m}$ to $1045 \, \text{N/m}$ for the rotor tip section, $767 \, \text{N/m}$ to $1373 \, \text{N/m}$ for the rotor section at 89% span, and $870 \, \text{N/m}$ to $944 \, \text{N/m}$ for the rotor section at 77% span. These correspond to fluctuation amplitudes of $612 \, \text{N/m}$, $606 \, \text{N/m}$, and $74 \, \text{N/m}$ for the rotor sections at the tip, 89% span, and 77% span respectively. Figures B-12, B-12, and B-12 show the moment per unit span at the leading edge on rotor sections at the tip, 89% span, and 77% span for the three rotor blades spanned by the jet actuator.

The moments for the different blades are different by constant values because the moments were calculated about the origin. The same values will be obtained for the moment if they were taken about the leading edge of each rotor blade. Consult reference [94] for more computations of the unsteady forces and moments on NASA Rotor 35 due to the jet actuation for various orientations of the actuators. The maximum and minimum values of the periodic nonsinusoidal fluctuating loads are summarized in Table B.3.

**Table B.3:** Maximum and minimum values of the periodic fluctuating loads due to jet actuation on three sections of the rotor blades of NASA Stage 35.

<table>
<thead>
<tr>
<th></th>
<th>Rotor Section at 100% span</th>
<th>Rotor Section at 89% span</th>
<th>Rotor Section at 77% span</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axial Force per unit span (N/m)</td>
<td>WITH JET</td>
<td>max 1963</td>
<td>2300</td>
</tr>
<tr>
<td></td>
<td>JET</td>
<td>min 730</td>
<td>1275</td>
</tr>
<tr>
<td></td>
<td>NO JET</td>
<td>1321</td>
<td>1225</td>
</tr>
<tr>
<td>Tangential Force per unit span (N/m)</td>
<td>WITH JET</td>
<td>max 1045</td>
<td>1373</td>
</tr>
<tr>
<td></td>
<td>JET</td>
<td>min 433</td>
<td>767</td>
</tr>
<tr>
<td></td>
<td>NO JET</td>
<td>680</td>
<td>734</td>
</tr>
<tr>
<td>Moment per unit span about origin Blade 1 (Nm/m)</td>
<td>WITH JET</td>
<td>max 64</td>
<td>149</td>
</tr>
<tr>
<td></td>
<td>JET</td>
<td>min 28</td>
<td>60</td>
</tr>
<tr>
<td></td>
<td>NO JET</td>
<td>31</td>
<td>88</td>
</tr>
<tr>
<td>Moment per unit span about origin Blade 2 (Nm/m)</td>
<td>WITH JET</td>
<td>max 97</td>
<td>193</td>
</tr>
<tr>
<td></td>
<td>JET</td>
<td>min 61</td>
<td>116</td>
</tr>
<tr>
<td></td>
<td>NO JET</td>
<td>30</td>
<td>82</td>
</tr>
<tr>
<td>Moment per unit span about origin Blade 3 (Nm/m)</td>
<td>WITH JET</td>
<td>max 235</td>
<td>289</td>
</tr>
<tr>
<td></td>
<td>JET</td>
<td>min 91</td>
<td>170</td>
</tr>
<tr>
<td></td>
<td>NO JET</td>
<td>146</td>
<td>133</td>
</tr>
</tbody>
</table>
Figure B-6: Axial force per unit span on rotor section at the tip (100% span).

Figure B-7: Axial force per unit span on rotor section at 89% span.
Figure B-8: Axial force per unit span on rotor section at 77% span.

Figure B-9: Tangential force per unit span on rotor section at the tip (100% span).
Figure B-10: Tangential force per unit span on rotor section at 89% span.

Figure B-11: Tangential force per unit span on rotor section at 77% span.
Figure B-12: Moment per unit span on the leading edge of rotor section at the tip (100% span).

Figure B-13: Moment per unit span on the leading edge of rotor section at 89% span.
B.2.3 Conclusion

The simulation results show that

- Considerable aerodynamic forces are exerted on the rotor blades as a result of the jet actuation. The fluctuating loading due to steady jet injection are nonsinusoidal and periodic.

- The period of the nonsinusoidal fluctuating load due to steady air injection is $\frac{1}{12}$th the period of revolution of the rotor blades.

- The portion of the rotor blades where there is maximum loading due to the jet injectors are the sections around the tip region.
B.3 Steady Surveys on NASA Rotor 35

To map out the effect of the jet actuators on a transonic compressor rotor, radial and circumferential surveys were measured downstream of NASA Rotor 35 for different levels of steady injection and different orientations of the sheet injectors. The instrumentation for NASA Rotor 35 is shown in Figure B-15. This instrumentation layout for NASA Rotor 35 is identical to the instrumentation layout for the complete NASA Stage 35 discussed in Chapter 2 with the only exception being that the stator was taken out.

![Figure B-15: Meridional cross-sectional view of test section showing the axial stations identified by the standard NASA nomenclature for Rotor 35 survey experiments.](image)

For each actuator configuration, the radial and circumferential profiles were measured at Station J with Kiel-headed total pressure and total temperature area traverse probes. The measured radial and circumferential profiles were then used to generate the survey measurements presented in this section. Figure B-16 shows the total pressure and total temperature surveys at Station J for NASA Rotor 35 operating at 100% speed with solid casing (i.e. when no injectors had been installed), and a mass flow rate of 18.42 kg/s (the mass flow rate at the stall point is 18.18 kg/s). Figure B-17 shows the total pressure and total temperature
surveys at Station J for NASA Rotor 35 operating at 100% speed with the injectors turned off (this configuration is different from the solid casing because the injectors protrude about 7% span into the compressor flowpath), and a mass flow rate of 18.57 kg/s (the mass flow rate at the stall point is 18.18 kg/s). For the experimental setup in this research, there are 12 jet actuators and 36 rotor blades. Thus each jet actuator spans three rotor blades or 30°. The survey measurements presented are repeated to cover the circumferential span of two jet actuators or 60°. Figures B-18, B-19, and B-20 show the total pressure, total temperature, and flow angle surveys respectively at the rotor inlet and Station J for NASA Rotor 35 operating at 100% speed with the sheet injectors at 0° yaw, valves 50% open, air supplied at 100 psig, and freestream mass flow rate of 16.92 kg/s (the mass flow rate from the jet actuators is 0.735 kg/s, and the total mass flow rate at the stall point is 17.20 kg/s). The surveys at the rotor inlet are determined from the control volume analysis discussed as part of the steady air injection model in Section 3.1. Figures B-21, B-22, and B-23 show the total pressure, total temperature, and flow angle surveys respectively at the rotor inlet and Station J for NASA Rotor 35 operating at 100% speed with the sheet injectors at −15° yaw, valves 50% open, air supplied at 100 psig, and freestream mass flow rate of 16.75 kg/s (the mass flow rate from the jet actuators is 0.735 kg/s, and the total mass flow rate at the stall point is 17.10 kg/s). Figures B-24, B-25, and B-26 show the total pressure, total temperature, and flow angle surveys respectively at the rotor inlet and Station J for NASA Rotor 35 operating at 100% speed with the sheet injectors at 0° yaw, valves 100% open, air supplied at 100 psig, and freestream mass flow rate of 15.80 kg/s (the mass flow rate from the jet actuators is 1.178 kg/s, and the total mass flow rate at the stall point is 16.88 kg/s). Figures B-27, B-28, and B-29 show the total pressure, total temperature, and flow angle surveys respectively at the rotor inlet and Station J for NASA Rotor 35 operating at 100% speed with the sheet injectors at −15° yaw, valves 100% open, air supplied at 100 psig, and freestream mass flow rate of 15.62 kg/s (the mass flow rate from the jet actuators is 1.178 kg/s, and the total mass flow rate at the stall point is 16.51 kg/s).

\footnote{The total mass flow rate through the compressor is the sum of the freestream mass flow rate and the mass flow rate from the jet actuators i.e., $m_{\text{total}} = m_{\text{freestream}} + m_{\text{actuator}}$.}
Figure B-16: Total pressure and total temperature surveys downstream of NASA Rotor 35 at 100% speed with solid casing and freestream mass flow rate of 18.42 kg/s.
(a) Total pressure survey at Station J with injectors sticking into flowpath

(b) Total temperature survey at Station J with injectors sticking into flowpath

Figure B-17: Total pressure and total temperature surveys downstream of NASA Rotor 35 at 100% speed with sheet injectors sticking into the compressor flowpath and freestream mass flow rate of 18.57 kg/s (point A on Figure 4-14a).
Figure B-18: Total pressure surveys upstream and downstream of NASA Rotor 35 at 100% speed with sheet injectors at 0° yaw, valves 50% open, air supplied at 100 psig, and freestream mass flow rate of 16.92 kg/s (point B on Figure 4-14a).
(a) Total temperature survey at rotor inlet with mean blowing at 0° yaw

(b) Total temperature survey at Station J with mean blowing at 0° yaw

Figure B-19: Total temperature surveys upstream and downstream of NASA Rotor 35 at 100% speed with sheet injectors at 0° yaw, valves 50% open, air supplied at 100 psig, and freestream mass flow rate of 16.92 kg/s (point B on Figure 4-14a).
(a) Relative flow angle survey at rotor inlet with mean blowing at $0^\circ$ yaw

(b) Absolute flow angle survey at Station J with mean blowing at $0^\circ$ yaw

**Figure B-20:** Flow angle surveys upstream and downstream of NASA Rotor 35 at 100% speed with sheet injectors at $0^\circ$ yaw, valves 50% open, air supplied at 100 psig, and freestream mass flow rate of 16.92 kg/s (point B on Figure 4-14a).
Figure B-21: Total pressure surveys upstream and downstream of NASA Rotor 35 at 100% speed with sheet injectors at $-15^\circ$ yaw, valves 50% open, air supplied at 100 psig, and freestream mass flow rate of 16.75 kg/s (point D on Figure 4-14a).
Figure B-22: Total temperature surveys upstream and downstream of NASA Rotor 35 at 100% speed with sheet injectors at $-15^\circ$ yaw, valves 50% open, air supplied at 100 psig, and freestream mass flow rate of 16.75 kg/s (point D on Figure 4-14a).
Figure B-23: Flow angle surveys upstream and downstream of NASA Rotor 35 at 100% speed with sheet injectors at $-15^\circ$ yaw, valves 50% open, air supplied at 100 psig, and freestream mass flow rate of 16.75 kg/s (point D on Figure 4-14a).
(a) Total pressure survey at rotor inlet with maximum blowing at 0° yaw

(b) Total pressure survey at Station J with maximum blowing at 0° yaw

**Figure B-24:** Total pressure surveys upstream and downstream of NASA Rotor 35 at 100% speed with sheet injectors at 0° yaw, valves 100% open, air supplied at 100 psig, and freestream mass flow rate of 15.80 kg/s (point C on Figure 4-14a).
(a) Total temperature survey at rotor inlet with maximum blowing at 0° yaw

(b) Total temperature survey at Station J with maximum blowing at 0° yaw

Figure B-25: Total temperature surveys upstream and downstream of NASA Rotor 35 at 100% speed with sheet injectors at 0° yaw, valves 100% open, air supplied at 100 psig, and freestream mass flow rate of 15.80 kg/s (point C on Figure 4-14a).
(a) Relative flow angle survey at rotor inlet with maximum blowing at 0° yaw

(b) Absolute flow angle survey at Station J with maximum blowing at 0° yaw

**Figure B-26:** Flow angle surveys upstream and downstream of NASA Rotor 35 at 100% speed with sheet injectors at 0° yaw, valves 100% open, air supplied at 100 psig, and freestream mass flow rate of 15.80 kg/s (point C on Figure 4-14a).
(a) Total pressure survey at rotor inlet with maximum blowing at $-15^\circ$ yaw

(b) Total pressure survey at Station J with maximum blowing at $-15^\circ$ yaw

**Figure B-27:** Total pressure surveys upstream and downstream of NASA Rotor 35 at 100% speed with sheet injectors at $-15^\circ$ yaw, valves 100% open, air supplied at 100 psig, and freestream mass flow rate of 15.62 kg/s (point E on Figure 4-14a).
Figure B-28: Total temperature surveys upstream and downstream of NASA Rotor 35 at 100% speed with sheet injectors at $-15^\circ$ yaw, valves 100% open, air supplied at 100 psig, and freestream mass flow rate of 15.62 kg/s (point E on Figure 4-14a).
(a) Relative flow angle survey at rotor inlet with maximum blowing at $-15^\circ$ yaw

(b) Absolute flow angle survey at Station J with maximum blowing at $-15^\circ$ yaw

**Figure B-29:** Flow angle surveys upstream and downstream of NASA Rotor 35 at 100% speed with sheet injectors at $-15^\circ$ yaw, valves 100% open, air supplied at 100 psig, and freestream mass flow rate of 15.62 kg/s (point E on Figure 4-14a).
APPENDIX C

INCOMPRESSIBLE STALL INCEPTION MODEL

This appendix describes the rotating stall inception model with air injection actuation for low speed axial flow compressors. The model is based on the introduction of small amplitude perturbations on a steady mean flow. In Section C.1, the classical formulation of the 2-D linearized Moore-Greitzer incompressible rotating stall inception model is reformulated to incorporate the effects of air injection. The corresponding full state Nonlinear Distributed Model for rotating stall inception is presented in Section C.2. In Section C.3, the Moore-Greitzer parameters for NASA Stage 35 are identified. Using the identified parameters for NASA Stage 35, the modified incompressible model is then used to predict the stall inception transients for NASA Stage 35 and the predicted results compared with experimental measurements in Figure 6-4.

C.1 Incompressible Stall Inception Model

The two-dimensional Moore-Greitzer stability model has been shown by Paduano [96, 98], Haynes [61, 62], Gysling [57, 59], and Van Schalkwyk [124, 126] to accurately capture the dynamic behavior of small amplitude pre-stall flow field perturbations in low speed compressors. In this section, the incompressible stall inception model is modified to account for the effect of air injection, and the modified (or enhanced) model applied to the transonic compressor for hysteresis analysis and nonlinear control design purposes. The basic incompressible stall inception model for low speed axial compressors was developed by Moore and
Greitzer and is described in references [91, 92]. Consult references [88, 89, 90] for a detailed examination of the theoretical basis for rotating stall, and the development of the quasi-steady compressor model used for modeling the compressor blade rows. The model assumes that fully developed rotating stall is a large amplitude, limit cycle oscillation of an initially linear instability and the flow stability is governed by the linearized compression system dynamics. Each flow quantity consists of a steady mean value and a small perturbation. For example, the axial flow coefficient is given by:

$$\phi(x, \theta, t) = \bar{\phi} + \delta\phi(x, \theta, t)$$  \hspace{1cm} (C.1)

These traveling perturbation waves are decomposed into sinusoidal harmonics:

$$\delta\phi(x, \theta, \tau) = \sum_{n=-\infty}^{n=\infty} \delta\phi_n(x, n, t)e^{i\theta}$$  \hspace{1cm} (C.2)

For a uniform mean flow, the dynamics are solved separately for each harmonic so that each spatial harmonic is a mode of the system with a unique rotation rate and damping ratio. By assessing the stability of small amplitude flow field perturbations which travel around the compressor annulus, the model can be used to determine whether the compressor will stall at a given operating point.

The classical incompressible formulation is useful for analyzing low-speed compressors with constant axial velocity. Here, the formulation has been extended to include variations in axial velocity and density along the compressor. This is modification is to model the flow in compressors with variable area ducts. However, density perturbations are not modeled. An extension of the classical formulation that includes non-uniform axial velocity and density from blade row to blade row and breaks down the overall total-to-static pressure rise characteristic into its components from each blade row is given in [41]. Further extensions to the classical formulation presented here include allowing the overall total-to-static pressure rise characteristic and deviation of the last blade row to be sensitive to the inlet swirl at the compressor face and the actuation. Figure C-1 shows the basic compression system. It is divided into the following six components: upstream duct, actuator, compressor, downstream duct, plenum, and throttle. Each of these components are modeled and the individual components are then coupled to produce the overall compression system model.
C.1.1 Upstream Duct

The density of the upstream duct is assumed to be uniform and constant, and the upstream flow field quantities are assumed to be radially uniform. This reduces the flow field model to two-dimensions with axial and circumferential directions.

Inlet duct dynamics: The inlet duct 2-D flow field is considered irrotational ($\nabla \times \mathbf{V}_{in} = 0$). This implies that the inlet duct velocity is the gradient of a potential ($\mathbf{V}_{in} = \nabla \phi_{in}$), thus the inlet duct is a potential field. Since the flow is also considered incompressible ($\nabla \cdot \mathbf{V}_{in} = 0$), the inlet duct flow field satisfies Laplace’s equation: $\nabla^2 \phi_{in} = 0$ and the corresponding perturbation equation is: $\nabla^2 \delta \phi_{in} = 0$. Since the potential must decay upstream, the harmonic expression for the velocity potential perturbation is:

$$
\delta \phi_{in}(\eta, \theta, \tau) = \sum_{n \neq 0} \delta \phi_{in,n}(\tau)e^{in|\eta|e^{in\theta}}
$$

(C.3)

From the velocity potential definition, the non-dimensional axial velocity perturbation, $\delta \phi_{in}$ is:

$$
\delta \phi_{in} = \frac{\partial}{\partial \eta}(\delta \phi_{in})
$$

$$
= |n|e^{in|\eta|\delta \phi_{in}}
$$

(C.4)
and the non-dimensional tangential velocity perturbation, $\delta \theta_{in}$ is:

$$
\delta \theta_{in} = \frac{\partial}{\partial \theta}(\delta \varphi_{in}) = jn\delta \varphi_{in} = \frac{jn}{|n|} e^{-|n| \eta \delta \phi_{in}}
$$

**Inlet duct momentum equation:** The momentum equation for an inviscid flow with uniform and constant density is:

$$
\frac{\partial \vec{V}_{in}}{\partial t} + \nabla \left( \frac{V_{in}^2}{2} \right) - \vec{V}_{in} \times \vec{\omega} = -\frac{\nabla P_{in}}{\rho_{in}}
$$

(C.6)

Substituting the irrotational flow field relation ($\vec{\omega} = 0$), the upstream duct velocity relation ($\vec{V}_{in} = \nabla \varphi_{in}$), and the total pressure relation for low Mach number flows ($P_{t,in} = P_{in} + \frac{1}{2} \rho_{in} V_{in}^2$) into the momentum equation C.6 and simplifying the resulting expression gives:

$$
\nabla \left[ \frac{\partial \varphi_{in}}{\partial t} + \frac{P_{t,in}}{\rho_{in}} \right] = 0
$$

(C.7)

This implies that the bracketed expression is not a function of space, but remains a function of time. However, since the perturbations decay far upstream, the bracketed expression is actually a constant and the corresponding perturbation equation in non-dimensional form can be shown to be:

$$
\frac{\rho_{in}}{\rho_o} \frac{\partial \varphi_{in}}{\partial \tau} + \frac{\delta P_{t,in}}{\rho_o U_o^2} = 0
$$

(C.8)

Substituting equation C.4 into equation C.8 and rearranging, the inlet total pressure perturbation is:

$$
\frac{\delta P_{t,in}}{\rho_o U_o^2} = -\frac{1}{|n|} e^{-|n| \eta} \frac{\rho_{in} \partial \delta \phi_{in}}{\rho_o \partial \tau}
$$

(C.9)

This equation will be coupled to the compressor dynamics.
C.1.2 Air Injection Model

In an actively controlled compressor, the relationship between pressure and velocity perturbations can be manipulated by the actuator. The effect of injecting high pressure air into a high speed axial-flow compressor is modeled in Chapter 3 and validated experimentally in Chapter 4. It was concluded from the air injection models that the Jet Actuator has two main effects: changing the flow field entering the compressor, and changing the compressor blade row performance characteristics. In the context of a two-dimensional, evenly distributed model that incorporates these two main effects of the Jet Actuator, the injection is modeled as an actuator disk with four properties: mass effect, momentum effect, swirl effect, and blade row characteristic effect. The mass, momentum, and swirl effects model the changes in the flow field entering the compressor, and the blade row characteristic effect models the change in compressor performance characteristics.

**Mass Effect of Actuator:** The mass addition from the actuator alters the total mass flow going through the compressor. This mass addition translates into pressure changes resulting from changing the operating point on the compressor characteristic. From the conservation of mass across the actuator, the mean axial velocity relation across the actuator is:

\[
\frac{\rho_{in}A_{in}}{\rho_oA_o}\phi_{in} + \frac{\rho_jA_j}{\rho_oA_o}\phi_j = \frac{\rho A}{\rho_oA_o}\bar{\phi} = \frac{\rho_{ex}A_{ex}}{\rho_oA_o}\bar{\phi}_{ex}
\]  

(C.10)

where \(\phi_{in}\) is the mean axial velocity at the inlet duct, \(\phi_j = \frac{V_i}{U_o}\) is the mean jet velocity from the injector normalized by the reference wheel speed \(U_o\), \(\bar{\phi}\) is the mean axial velocity at the compressor, and \(\bar{\phi}_{ex}\) is the mean axial velocity at the downstream or exit duct. The corresponding axial velocity perturbation equation across the actuator is:

\[
\frac{\rho_{in}A_{in}}{\rho_oA_o}\delta\phi_{in} + u_j = \frac{\rho A}{\rho_oA_o}\delta\phi = \frac{\rho_{ex}A_{ex}}{\rho_oA_o}\delta\phi_{ex}
\]  

(C.11)

where \(u_j = \frac{\delta m_j}{\rho_oA_oU_o}\) is the control term, with \(m_j\) being the injected mass.

**Momentum Effect of Actuator:** The momentum addition translates into additional pressure rise resulting from the increase in upstream duct dynamic head through velocity amplification due to the presence of the jet actuator. The net effect of the momentum
addition is the shift in the compressor characteristic. From control volume analysis, the
total pressure rise across the actuator is:

\[
\frac{P_{t,x}}{\rho_0 U_0^2} - \frac{P_{t,in}}{\rho_0 U_0^2} = \Psi_{mom}^{tt}(\Phi_{in}, U_j) = \beta_0 U_j^2 + \beta_1 U_j \Phi_{in} + \beta_2 \Phi_{in}^2
\]  

(C.12)

where \( U_j = \frac{m_j}{\rho_0 A_0 U_0} \). Note that \( u_j = \delta U_j \). The corresponding total pressure rise perturbation
equation is:

\[
\frac{\delta P_{t,x}}{\rho_0 U_0^2} - \frac{\delta P_{t,in}}{\rho_0 U_0^2} = \frac{\partial \Psi_{mom}^{tt}}{\partial \Phi_{in}} \delta \Phi_{in} + \frac{\partial \Psi_{mom}^{tt}}{\partial U_j} \delta U_j
\]  

(C.13)

To account for the dynamic effect or non-steady behavior of the actuation system i.e., the
time delay associated with the total pressure build up, a momentum transport time lag is
used for the total pressure rise. The unsteady total pressure rise due to air injection is given
by:

\[
\tau_{mom} \frac{\partial \Psi_{mom}^{tt}}{\partial r} = \Psi_{mom}^{ss} - \Psi_{mom}^{tt}(\tau, \theta)
\]  

(C.14)

where \( \Psi_{mom}^{ss} \) is the steady state total pressure rise in the upstream duct given by equation
C.12, and \( \tau_{mom} \) is the convective time delay which is the delay associated with the time the
injected fluid has to travel the distance from the injector exit to the compressor inlet.

Swirl Effect of Actuator: When the jet actuator is oriented at an angle, the tangential
component of the inlet duct velocity is modified by the high speed jet from the actuator.
This in effect alters the swirl of the fluid (flow incidence) at the compressor inlet thus
changing the operating point on the compressor characteristic. The corresponding pressure
change due to this swirl effect will depend on the swirl sensitivity of the compressor. From
a tangential momentum balance across the jet actuator, the non-dimensional expression
relating the inlet duct and compressor inlet tangential velocities is:

\[
\frac{\rho A}{\rho_0 A_0} \left[ \frac{\phi \delta \theta + \bar{\phi} \delta \phi}{\Phi_{in}} \right] \delta \Phi_{in} + \left[ 2 \bar{\phi}_j \sin(\theta_j) \right] u_j
\]  

(C.15)

Blade Row Characteristic Effect: Blade row characteristics include the total pressure
loss coefficient, and deviation characteristics. How does air injection affect the blade row
characteristics? For this research, the jet actuator is modeled as an actuator disk that introduces a wake (or radial distortion) at the injection plane upstream of the compressor. From the results of steady injection, it was observed that the compressor responds differently to these inlet boundary conditions. Details of the mechanisms involved in the compressor blade row response to a wake like the one being introduced at the tip by the sheet injectors can be found in studies such as the one by Valkov [123] where the impact of upstream rotor wakes and tip leakage vortices on the loss of a stator downstream was investigated. However, it is sufficient for modeling purposes in the context of control to say that the effect of the jet wake introduced by the actuator is to change the blade row performance characteristics (and equivalently the compressor performance characteristics). These changes in blade row performance characteristics can be quantified using the steady injection model and results described in Chapters 3 and 4. Blade rows have a family of characteristic curves and the curve on which the blade row operates will depend on the inlet boundary condition (which determines what fluid redistribution pattern takes place within the blade rows). The effect of moving the blade row characteristics to a new curve in the family of curves is modeled by including an actuation dependence on the blade row performance characteristics. Therefore, the total-to-static pressure compressor characteristic is modeled as a function of the compressor flow coefficient, the inlet flow angle, and the actuation term:

\[ \Psi_{cts}^l = \Psi_{cts}(\Phi, \alpha, U_j) \]  \hspace{1cm} (C.16)

and the exit flow angle characteristic is modeled as a function of the compressor flow coefficient, the inlet flow angle, the wheel speed, and the actuation term:

\[ \alpha_{ex} = \alpha_{ex}(\Phi, \alpha, \Omega, U_j) \]  \hspace{1cm} (C.17)

### C.1.3 Compressor Blade Rows

A "semi-actuator" disk [67] is used to model the behavior of the compressor blade rows. The semi-actuator disk model accounts for the inertia of the fluid in the blade passage by accounting for the acceleration through the unsteady pressure matching condition. Secondly, the actuator disk model assumes that the wavelength of the inlet disturbances are far much greater than the blade pitch such that the inlet disturbance sees the blades as
being smeared to form a continuous disk. Thirdly, the actuator disk model also assumes that the length to circumference or aspect ratio of the blades is small such that there is no flow redistribution.

**Compressor inlet swirl perturbation:** Using the expression for the inlet swirl from the velocity triangle at the compressor inlet \( \tan \alpha = \frac{\dot{V}}{\phi} \), the non-dimensional inlet swirl perturbation equation is:

\[
\delta \alpha = -\frac{\ddot{\varphi}}{\phi^2 + \ddot{\varphi}^2} \delta \phi + \frac{\ddot{\phi}}{\phi^2 + \ddot{\varphi}^2} \delta \theta \quad (C.18)
\]

**Quasi-Steady Compressor Model:** The basic compressor model is the Moore-Greitzer two-dimensional “semi-actuation” description given in [91, 92] with modifications to include swirl sensitivity as in [81] and the injection effect on the loss characteristic as discussed in Section C.1.2. The core flow in the blade rows is assumed to be incompressible and radially uniform with the fluid constrained in one dimension by the blades and the compressor hub and casing. By matching the pressure rise with the fluid acceleration for each blade row and combining the individual blade row pressure rises, the following relation was obtained:

\[
\frac{P_{ex} - P_{t,x}}{\rho_0 U_0^2} = \Psi_c^{ts}(\phi, \alpha, U_j) - \lambda \frac{\partial \phi}{\partial \theta} - \mu \frac{\partial \phi}{\partial \tau} \quad (C.19)
\]

where \( \Psi_c^{ts}(\phi, \alpha, U_j) \) is the steady state compressor characteristic, \( \phi \) is the local unsteady axial velocity flow coefficient at the compressor, \( \mu \) is the total inertia, and \( \lambda \) is the rotor inertia. The corresponding total-to-static pressure perturbation equation is:

\[
\frac{\delta P_{ex}}{\rho_0 U_0^2} - \frac{\delta P_{t,x}}{\rho_0 U_0^2} = \frac{\partial \Psi_c^{ts}}{\partial \phi} \delta \phi + \frac{\partial \Psi_c^{ts}}{\partial \alpha} \delta \alpha + \frac{\partial \Psi_c^{ts}}{\partial U_j} u_j - \lambda \frac{\partial}{\partial \theta} (\delta \phi) - \mu \frac{\partial}{\partial \tau} (\delta \phi) \quad (C.20)
\]

Substituting the expression for \( \delta \alpha \), the compressor inlet flow angle perturbation from equation C.18 into equation C.20 and substituting the expression for \( \delta \theta \), the compressor inlet tangential velocity perturbation from equation C.15 into the resulting expression gives the total-to-static pressure rise perturbation in terms of the compressor inlet flow coefficient perturbation \( \delta \phi \), the inlet duct tangential velocity perturbation \( \delta \theta_{in} \), the inlet duct flow coefficient perturbation \( \phi_{in} \), and the control input \( u_j \). Further simplification of the result-
ing total-to-static pressure rise perturbation by substituting the expressions for $\delta \phi$ from equation C.11 and $\delta \phi_{in}$ from equation C.5 gives:

$$\frac{\delta P_{ex}}{\rho_o U_o^2} - \frac{\delta P_{t,x}}{\rho_o U_o^2} = -\mu \frac{\rho_{in} A_{in}}{\rho A} \frac{\partial \delta \phi_{in}}{\partial \tau} - \mu \frac{\rho_A A_o}{\rho A} \frac{\partial \phi_j}{\partial \tau}$$

$$+ \frac{\rho_{in} A_{in}}{\rho A} \left[ \frac{\partial \psi_{c}^{ts}}{\partial \phi} - \left( \alpha_1 - \frac{j \psi}{n} e^{-jn\alpha_2} \right) \frac{\partial \psi_{c}^{ts}}{\partial \alpha} - jn\lambda \right] \delta \phi_{in}$$

$$+ \frac{\rho_o A_o}{\rho A} \left[ \frac{\partial \psi_{c}^{ts}}{\partial \phi} + \frac{\rho_A}{\rho_o A_o} \frac{\partial \psi_{c}^{ts}}{\partial U_j} - \alpha_3 \frac{\partial \psi_{c}^{ts}}{\partial \alpha} - jn\lambda \right] \phi_{in}$$

where the constants $\alpha_1$, $\alpha_2$, and $\alpha_3$ are defined as:

$$\alpha_1 = \frac{2\bar{\phi}}{\bar{\phi}^2 + \bar{\phi}^2}$$

$$\alpha_2 = \frac{\phi_{in}}{\phi^2 + \bar{\phi}^2}$$

$$\alpha_3 = \frac{2(\bar{\phi} - \phi_j \sin \theta_j)}{\phi^2 + \bar{\phi}^2}$$

**Non-Steady Compressor Model:** The quasi-steady compressor model relies on the flowfield disturbances being of long length scale, and therefore low reduced frequency. Thus for higher harmonics i.e., short wavelength disturbances, the reduced frequency is high and the compressor will respond in a non-steady manner. Haynes [61] and Longley [81] modeled these non-steady effects by using time lags for the loss and deviation. The non-steady version of equation C.19, the total-to-static pressure rise across the compressor blade row is:

$$\frac{P_{ex} - P_{t,x}}{\rho_o U_o^2} = \psi_{ideal}^{ts} - L_r - L_s - \lambda \frac{\partial \phi}{\partial \theta} - \mu \frac{\partial \phi}{\partial \tau}$$

$$\tau_{loss} \left( \frac{\partial L_r}{\partial t} + \frac{\partial L_r}{\partial \theta} \right) = L_r^{ss} - L_r$$

$$\tau_{loss} \frac{\partial L_s}{\partial t} = L_s^{ss} - L_s$$

where $\psi_{c}^{ts} = \psi_{ideal}^{ts} - L_r - L_s$, $\psi_{ideal}^{ts}(\phi, \alpha, U_j)$ is the ideal total-to-static pressure rise across the compressor blade rows, $L_r(\tau, \theta)$ is the unsteady rotor loss, and $L_s(\tau, \theta)$ is the unsteady stator loss. Combining the unsteady rotor loss and unsteady stator loss in equation C.22.
using \( L = L_r + L_s \), and \( L_r = rL \), where \( r \) is the rotor reaction gives:

\[
\frac{P_{ex} - P_{t,x}}{\rho_o U_o^2} = \psi_{\text{ideal}}^{ls} - L - \lambda \frac{\partial \phi}{\partial \theta} - \mu \frac{\partial \phi}{\partial \tau}
\]

\[
\tau_{\text{loss}} \left( \frac{\partial L}{\partial t} + r \frac{\partial L}{\partial \theta} \right) = L^{ss} - L
\]

The unsteady exit flow angle is modeled as:

\[
\tau_{\text{dev}} \frac{\partial \alpha_{ex}}{\partial t} = \alpha_{ex}^{ss} - \alpha_{ex}
\]  

(C.24)

C.1.4 Downstream Duct

The downstream duct density is assumed to be uniform and constant, and the downstream flow field quantities are assumed to be radially uniform, thus the downstream duct flow field is assumed to be two-dimensional.

Exit duct dynamics: The flow in the downstream duct is assumed to be two-dimensional, incompressible, inviscid, but vortical due to the introduction of vorticity by the compressor blade row. Thus the downstream duct flow field is a combination of vortical and potential disturbances. The potential (or pressure) perturbations decay downstream and have the form:

\[
\frac{\delta P_{ex}}{\rho_o U_o^2} = \sum_{n \neq 0} \frac{\delta P_n}{\rho_o U_o^2 (\tau)} e^{-|n| \eta e^{jn\theta}}
\]

(C.25)

Exit duct flow angle perturbation equation: The compressor exit flow angle is taken to be the air flow angle of the last blade row. From the velocity triangle relation at the compressor exit \( (\theta_{ex} = \phi_{ex} \tan \alpha_{ex}) \), the tangential velocity perturbation at the exit duct is:

\[
\delta \theta_{ex} = \bar{\phi}_{ex} \sec^2 \bar{\alpha}_{ex} \delta \alpha_{ex} + \tan \bar{\alpha}_{ex} \delta \phi_{ex}
\]

\[
= \frac{\phi_{ex} \sec^2 \alpha_{ex} \delta \alpha_{ex} + \phi_{ex} \delta \phi_{ex}}{\phi_{ex}}
\]

(C.26)

According to the blade row characteristic effect of the actuator from equation C.17, the compressor exit flow angle is a function of the flow coefficient, the inlet swirl, the wheel
speed, and the injected mass i.e., $\alpha_{ex} = \alpha_{ex}(\phi, \alpha, \Omega, U_j)$. Since the wheel speed is assumed to be constant, i.e., $\delta\Omega = 0$, the perturbation equation for the compressor exit flow angle is:

$$\delta\alpha_{ex} = \frac{\partial\alpha_{ex}}{\partial\phi} \delta\phi + \frac{\partial\alpha_{ex}}{\partial\alpha} \delta\alpha + \frac{\partial\alpha_{ex}}{\partial U_j} u_j$$

Substituting equation C.18 into equation C.27 will give an expression for $\delta\alpha_{ex}$ in terms of $\delta\phi$ and $\delta\theta$. The tangential velocity perturbation between the actuator and the compressor, $\delta\theta$, given in equation C.15 can be expressed as a function of $\delta\phi_{in}$ and $u_j$ by substituting the expressions for $\delta\theta_{in}$ and $\delta\phi$ from equations C.5 and C.11 respectively. Finally, substituting the expression for the flow coefficient perturbation between the actuator and the compressor, $\delta\phi$ from equation C.11 and the derived expression for the tangential velocity perturbation between the actuator and the compressor, $\delta\theta$ into the derived expression for the exit flow angle perturbation equation gives:

$$\delta\alpha_{ex} = \frac{\rho_{in} A_{in}}{\rho A} \left[ \frac{\partial\alpha_{ex}}{\partial\phi} - \left( \frac{2\bar{\theta}}{\bar{n}} \right) e^{-\bar{m}n^2} \frac{\partial\alpha_{ex}}{\partial\alpha} \right] \delta\phi_{in} + \frac{\rho_o A_o}{\rho A} \left[ \frac{\partial\alpha_{ex}}{\partial\phi} + \frac{\partial\alpha_{ex}}{\partial U_j} - 2(\bar{\theta} - \bar{\theta} \sin \theta_j) \frac{\partial\alpha_{ex}}{\partial\phi} \right] u_j$$

**Exit duct continuity equation:** Since the flow in the exit duct downstream of the compressor is incompressible, $\nabla \cdot \vec{V}_{ex} = \frac{1}{r} \frac{\partial}{\partial r} (r V_r) + \frac{1}{r} \frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} = 0$. Neglecting the radial component of the velocity, the non-dimensional continuity perturbation equation is:

$$\frac{\partial}{\partial \eta} (\delta\phi_{ex}) = -\frac{\partial}{\partial \theta} (\delta\theta_{ex})$$

**Exit duct momentum equation:** Applying the unsteady momentum equation for an inviscid flow with uniform and constant density to the exit duct downstream of the compressor:

$$\frac{\partial \vec{V}_{ex}}{\partial t} + \left( \vec{V}_{ex} \cdot \nabla \right) \vec{V}_{ex} = -\frac{\nabla P_{ex}}{\rho_{ex}}$$
Taking the corresponding perturbation equation of the axial component of the momentum equation, neglecting density perturbations, assuming that the exit duct has a uniform steady-state flow, non-dimensionalizing the resulting expression, and using the harmonic relation for the potential (or pressure) perturbation to simplify the spatial derivative of the static pressure perturbation gives:

\[
|n| \frac{\rho_o}{\rho_{ex}} \frac{\delta P_{ex}}{\rho_o U_o^2} = \frac{\partial}{\partial \tau} (\delta \phi_{ex}) + \phi_{ex} \frac{\partial}{\partial \eta} (\delta \phi_{ex}) + \phi_{ex} \frac{\partial}{\partial \theta} (\delta \phi_{ex}) \tag{C.31}
\]

Substituting the expression in equation C.29 for the spatial derivative of the flow coefficient perturbation at the exit duct into the exit duct static pressure perturbation equation C.31, and using the expression from equation C.26 for the exit duct tangential velocity perturbation to simplify the resulting equation gives:

\[
\frac{\delta P_{ex}}{\rho_o U_o^2} = \frac{1}{|n|} \frac{\rho_{ex}}{\rho_o} \frac{\partial \delta \phi_{ex}}{\partial \tau} - \frac{jn}{|n|} \frac{\rho_{ex}}{\rho_o} \phi_{ex} \sec^2 \alpha_{ex} \delta \alpha_{ex} \tag{C.32}
\]

The expression for the exit duct static pressure perturbation in terms of the inlet duct flow coefficient perturbation and the control term is obtained by substituting the exit duct flow angle perturbation equation C.28 into equation C.32:

\[
\frac{\delta P_{ex}}{\rho_o U_o^2} = \frac{1}{|n|} \frac{\rho_{ex}}{\rho_o} \frac{\partial \phi_{ex}}{\partial \tau} - \frac{jn}{|n|} \frac{\rho_{ex}}{\rho_o} A_o \left[ \frac{\partial \alpha_{ex}}{\partial \phi} - \frac{jn}{|n|} \frac{\rho_{ex}}{\rho_o} \alpha_2 \frac{\partial \alpha_{ex}}{\partial \alpha} \right] \delta \phi_{in} \tag{C.33}
\]

where the constant \(\alpha_4\) is defined as:

\[
\alpha_4 = \frac{\rho_{ex}}{\rho_o} \phi_{ex}^2 \sec^2 \alpha_{ex}
\]

C.1.5 Plenum

The plenum is modeled as an isentropic compression process with mass introduced from the compressor duct and removed through the throttle. Thus the time varying pressure in the plenum acts as a spring which couples the mass flow through the compressor and the
throttle. The differential equations that describe the plenum dynamics are:

\[
\ell_c \dot{\phi}_{ex} = \Psi_t^{ts}(\phi, \bar{\alpha}, \bar{U}_j) + \Psi_{mom}^{tt}(\phi_{in}, \bar{U}_j) - \Psi_p
\]

\[
\dot{\psi}_p = \frac{1}{4B^2\ell_c} \left[ \Phi_{ex} - \Phi_t(\Psi_p) \right]
\]

where \( \psi_p \) is the plenum pressure, \( \Phi_t \) is the axial velocity flow coefficient through the throttle, and B is the Greitzer B parameter defined by:

\[
B = \frac{U}{2a} \sqrt{\frac{V_p}{A_c L_c}}
\]

where \( V_p \) is the volume of the compression system plenum, \( L_c \) is the length of the compression system duct, \( A_c \) is the annulus area of the compressor, \( U \) is the compressor mean wheel speed, and \( a \) is the speed of sound, \( \sqrt{\gamma R T} \). The Greitzer B parameter is a measure of the compliance relative to the inertia of the compression system. Compression systems with high B parameters, thus high compliant compression systems will tend to surge. Consult reference [54] for more information on the Greitzer B parameter. Linearing the plenum pressure ODE in equation C.34 about the mean gives:

\[
\delta \psi = \frac{1}{4B^2\ell_c} \left( \delta \phi_{ex} - \delta \phi_t \right)
\]

and the corresponding linearized equation for the axial velocity flow coefficient ODE in equation C.34 is given by:

\[
\ell_c \delta \phi_{ex} = \frac{\partial \Psi_t^{ts}}{\partial \phi} \delta \phi + \frac{\partial \Psi_{mom}^{tt}}{\partial \phi_{in}} \delta \phi_{in} + \left[ \frac{\partial \Psi_t^{ts}}{\partial U_j} + \frac{\partial \Psi_{mom}^{tt}}{\partial U_j} \right] u_j - \delta \psi
\]

C.1.6 Throttle

The throttle is modeled as an actuator disk with the pressure drop determined by a pressure drop characteristic, \( \Psi(\Phi_t) \). In a gas turbine engine, the throttle in the compression model represents the turbine and its characteristic is modeled using the quadratic relation:

\[
\Psi_t(\Phi_{t}) = \frac{1}{2} K_t \Phi_{t}^{2}
\]
where $K_t$ is the throttle constant. Linearizing the throttle characteristics to relate small changes in mass flow through to small changes in pressure drop across the throttle:

$$\delta \phi_t = m_t \delta \psi$$  (C.39)

where $m_t$ is the inverse of the slope of the throttle characteristic i.e., $m_t = \frac{1}{\frac{\mu}{\delta \phi_t}} = \frac{1}{K_t \phi_t}$.

### C.1.7 Moore-Greitzer Rotating Stall Model

The total-to-static pressure rise perturbation from the inlet to exit duct can be broken down into its components consisting of the total-to-total pressure rise from the inlet duct to the rotor inlet across the actuator, and the total-to-static pressure rise across the blade rows:

$$\frac{\delta P_{ex} - \delta P_{t,in}}{\rho_o U_o^2} = \left( \frac{\delta P_{ex}}{\rho_o U_o^2} - \frac{\delta P_{t,in}}{\rho_o U_o^2} \right) + \left( \frac{\delta P_{t,x}}{\rho_o U_o^2} - \frac{\delta P_{t,in}}{\rho_o U_o^2} \right)$$  (C.40)

Substituting equations C.13 and C.21 into equation C.40 gives:

$$\frac{\delta P_{ex} - \delta P_{t,in}}{\rho_o U_o^2} = -\mu \frac{\rho_{in} A_{in}}{\rho A} \frac{\partial \delta \phi_{in}}{\partial \tau} - \mu \frac{\rho_o A_o \partial u_j}{\rho A} \frac{\partial \delta \phi_{in}}{\partial \tau} + \rho_{in} A_{in} \frac{\partial \psi_{ts}}{\partial \psi} + \rho A \frac{\partial \psi_{tmom}}{\partial \psi_{in}} - \left( \alpha_1 - \frac{\mu}{\rho_{in} A_{in}} \frac{\partial \psi_{ts}}{\partial \alpha} \right) - jn \lambda \right] \delta \phi_{in}$$

By combining equations C.9 and C.33, and using the continuity equation C.11 to express the exit duct flow coefficient perturbation $\delta \phi_{ex}$ in terms of the inlet duct flow coefficient perturbation $\delta \phi_{in}$ and the control term $u_j$, the total-to-static pressure rise perturbation from the inlet to the exit duct is given by:

$$\frac{\delta P_{ex} - \delta P_{t,in}}{\rho_o U_o^2} = \frac{1}{n} \frac{\rho_{in}}{\rho} \left[ A_{in} + e^{-|n|j} \right] \frac{\partial \delta \phi_{in}}{\partial \tau} + \frac{1}{n} \frac{A_o}{A_{ex}} \frac{\partial u_j}{\partial \tau} - \frac{\mu}{\rho_{in} A_{in}} \frac{\partial \psi_{ex}}{\partial \psi} - \frac{\partial \psi_{ex}}{\partial \phi} \frac{\partial \alpha_{ex}}{\partial \phi} - \frac{\partial \psi_{ex}}{\partial \phi} \frac{\partial \alpha_{ex}}{\partial \phi} - \frac{\partial \psi_{ex}}{\partial \phi} \frac{\partial \alpha_{ex}}{\partial \phi}$$  (C.42)
Equating C.41 to C.42, and grouping common terms will give the following ODE for each harmonic n of the velocity perturbation at the inlet duct:

\[
\begin{align*}
\left[ \frac{\rho_{in} A_{in}}{\rho A} \mu + \frac{1}{|n|} \frac{\rho_{in}}{\rho} \left( A_{in} + e^{-|n| \eta} \right) \right] \frac{\partial \delta \phi_{in}}{\partial \tau} &= - \left[ \frac{\rho_{o} A_{o}}{\rho A} \mu + \frac{1}{|n|} \frac{A_{o}}{A_{ex}} \right] \frac{\partial u_j}{\partial \tau} \\
&+ \frac{\rho_{in} A_{in}}{\rho A} \left[ \left\{ \frac{\partial \psi_{ts}^c}{\partial \Phi} + \frac{\rho A}{\rho_{in} A_{in}} \frac{\partial \psi_{mom}^{ts}}{\partial \Phi_{in}} - \alpha_1 \frac{\partial \psi_{ts}^c}{\partial \alpha} + \alpha_2 \alpha_4 e^{-|n| \eta} \frac{\partial \alpha_{ex}}{\partial \alpha} \right\} \right] \delta \phi_{in} \\
&- j \left\{ n \lambda - \frac{n}{|n|} \left( \alpha_4 \frac{\partial \alpha_{ex}}{\partial \Phi} - \alpha_1 \alpha_4 \frac{\partial \alpha_{ex}}{\partial \alpha} + \alpha_2 e^{-|n| \eta} \frac{\partial \psi_{ts}^c}{\partial \alpha} \right) \right\} u_j \\
&+ \frac{\rho_{o} A_{o}}{\rho A} \left[ \left\{ \frac{\partial \psi_{ts}^c}{\partial \Phi} + \frac{\rho A}{\rho_{o} A_{o}} \left( \frac{\partial \psi_{ts}^c}{\partial U_j} + \frac{\partial \psi_{mom}^{ts}}{\partial U_j} \right) - \alpha_3 \frac{\partial \psi_{ts}^c}{\partial \alpha} \right\} \right] \delta \phi \\
&- j \left\{ n \lambda - \frac{n}{|n|} \alpha_4 \left( \alpha_4 \frac{\partial \alpha_{ex}}{\partial \Phi} + \frac{\rho A}{\rho_{o} A_{o}} \frac{\partial \alpha_{ex}}{\partial U_j} - \alpha_3 \frac{\partial \alpha_{ex}}{\partial \alpha} \right) \right\} u_j \\
\end{align*}
\]

Substituting the expression for \( \delta \phi_{in} \) from equation C.11 into equation C.43, gives the ODE for each harmonic n of the velocity perturbation at the compressor inlet:

\[
\begin{align*}
\left[ \mu + \frac{1}{|n|} \frac{\rho A}{\rho_{in} A_{in}} \left( A_{in} + e^{-|n| \eta} \right) \right] \frac{\partial \delta \phi}{\partial \tau} &= \frac{1}{|n|} \frac{A_{o}}{A_{ex}} e^{-|n| \eta} \frac{\partial u_j}{\partial \tau} \\
&+ \left\{ \frac{\partial \psi_{ts}^c}{\partial \Phi} + \frac{\rho A}{\rho A} \frac{\partial \psi_{mom}^{ts}}{\partial \Phi_{in}} - \alpha_1 \frac{\partial \psi_{ts}^c}{\partial \alpha} + \alpha_2 \alpha_4 e^{-|n| \eta} \frac{\partial \alpha_{ex}}{\partial \alpha} \right\} \delta \phi \\
&- j \left\{ n \lambda - \frac{n}{|n|} \left( \alpha_4 \frac{\partial \alpha_{ex}}{\partial \Phi} - \alpha_1 \alpha_4 \frac{\partial \alpha_{ex}}{\partial \alpha} + \alpha_2 e^{-|n| \eta} \frac{\partial \psi_{ts}^c}{\partial \alpha} \right) \right\} \delta \phi \\
&+ \frac{\rho_{o} A_{o}}{\rho A} \left[ \left\{ \alpha_1 - \alpha_3 \right\} \frac{\partial \psi_{ts}^c}{\partial \alpha} + \frac{\rho A}{\rho_{o} A_{o}} \left( \frac{\partial \psi_{ts}^c}{\partial U_j} + \frac{\partial \psi_{mom}^{ts}}{\partial U_j} \right) - \alpha_3 \frac{\partial \psi_{ts}^c}{\partial \alpha} \right\} \delta \phi \\
&- j\left\{ \alpha_1 - \alpha_3 \right\} \alpha_4 \frac{\partial \alpha_{ex}}{\partial \alpha} - \alpha_2 e^{-|n| \eta} \frac{\partial \psi_{ts}^c}{\partial \alpha} + \alpha_4 \frac{\rho A}{\rho_{o} A_{o}} \frac{\partial \alpha_{ex}}{\partial U_j} \right\} \delta \phi \\
\end{align*}
\]

These are sets of decoupled ordinary differential equations, with each set relating the \( n^{th} \) SFC of the velocity perturbation, \( \tilde{\phi}_n \), to the \( n^{th} \) SFC of the actuator input, \( \tilde{u}_n \). Parametric representation of equation C.44 gives the following SISO complex coefficient, complex state ODEs:

\[
\dot{\tilde{\phi}}_n = (\sigma_{rs} + j \omega_{rs}) \tilde{\phi}_n + (b_{r} + j b_{i}) \tilde{u}_n + g_{r} \tilde{u}_n
\]
where,

\[
\sigma_{rs} = \frac{\partial \psi_{rs}^t}{\partial \Phi} + \frac{\rho A}{\rho_{in} A_{in}} \frac{\partial \psi_{mem}^t}{\partial \Phi_{in}} - \alpha_1 \frac{\partial \psi_{rs}^t}{\partial \alpha} + \alpha_2 \alpha_4 e^{-|n| \eta} \frac{\partial \psi_{ex}^t}{\partial \alpha} \\
\mu + \frac{1}{|n|} \frac{\rho A}{\rho_{o} A_{in}} \left( \frac{A_{in}}{A_{ex}} + e^{-|n| \eta} \right)
\]

\[
\omega_{rs} = -\frac{n \lambda - |n| \left( \alpha_4 \frac{\partial \psi_{ex}}{\partial \Phi} - \alpha_1 \alpha_4 \frac{\partial \psi_{ex}}{\partial \alpha} + \alpha_2 e^{-|n| \eta} \frac{\partial \psi_{ex}^t}{\partial \alpha} \right)}{\mu + \frac{1}{|n|} \frac{\rho A}{\rho_{o} A_{in}} \left( \frac{A_{in}}{A_{ex}} + e^{-|n| \eta} \right)}
\]

\[
b_r = \frac{\rho_{o} A_o}{\rho A} \left[ (\alpha_1 - \alpha_3) \frac{\partial \psi_{ex}^t}{\partial \alpha} + \frac{\rho A}{\rho_{o} A_{o}} \left( \frac{\partial \psi_{ex}^t}{\partial \Phi_{in}} + \frac{\partial \psi_{mem}^t}{\partial \Phi_{in}} \right) - \frac{\rho A}{\rho_{in} A_{in}} \frac{\partial \psi_{mem}^t}{\partial \Phi_{in}} - \alpha_2 \alpha_4 e^{-|n| \eta} \frac{\partial \psi_{ex}^t}{\partial \alpha} \right] \\
\mu + \frac{1}{|n|} \frac{\rho A}{\rho_{o} A_{in}} \left( \frac{A_{in}}{A_{ex}} + e^{-|n| \eta} \right)
\]

\[
b_i = \frac{|n| \rho_{o} A_o}{|n| \rho A} \left[ (\alpha_1 - \alpha_3) \alpha_4 \frac{\partial \psi_{ex}}{\partial \alpha} - \alpha_2 e^{-|n| \eta} \frac{\partial \psi_{ex}^t}{\partial \alpha} + \alpha_4 \frac{\rho A}{\rho_{o} A_{o}} \frac{\partial \psi_{ex}}{\partial \Phi_{in}} \right] \\
\mu + \frac{1}{|n|} \frac{\rho A}{\rho_{o} A_{in}} \left( \frac{A_{in}}{A_{ex}} + e^{-|n| \eta} \right)
\]

\[
g_r = \frac{\frac{1}{|n|} \frac{A_{o}}{A_{in}} e^{-|n| \eta}}{\frac{1}{|n|} \frac{\rho A}{\rho_{o} A_{in}} \left( \frac{A_{in}}{A_{ex}} + e^{-|n| \eta} \right)}
\]

By taking the Laplace transform of equation C.45, the SISO complex coefficient transfer function from the \( n^{th} \) harmonic actuation input to the \( n^{th} \) harmonic velocity perturbation is given by:

\[
G_n(s) = \frac{\ddot{\phi}_n(s)}{\ddot{\alpha}_n(s)} = \frac{g_r s + (b_r + jb_i)}{s - (\sigma_{rs} + j\omega_{rs})}
\]

where the real part of the eigenvalue, \( \sigma_{rs} \), determines the growth rate of the disturbances and the imaginary part of the eigenvalue, \( \omega_{rs} \), determines the rotation rate of the disturbances. Since the coefficients in \( G_n(s) \) are complex, the system does not obey the typical root-locus, Nyquist, Bode construction and stability criteria, because the poles and zeros do not appear in complex-conjugate pairs. However, a technique where each set of the SISO complex form in equation C.45 can be converted into a real-valued MIMO system was developed by Paduano [96].
The growth rate of the $n^{th}$ harmonic disturbance is:

$$\sigma_{rs} = \frac{\partial \psi_{rs}}{\partial \Phi} + \frac{\rho A}{\rho_{in} A_{in}} \frac{\partial \psi_{m}^{in}}{\partial \Phi_{in}} - \alpha_{1} \frac{\partial \psi_{rs}}{\partial \alpha} + \alpha_{2} e^{-|n| \eta} \frac{\partial \psi_{rs}}{\partial \alpha}$$

$$\mu + \frac{1}{|n|} \frac{\rho A}{\rho_{o} A_{in}} \left( A_{in} e^{-\alpha_{2} e^{-|n| \eta}} \right)$$

The expression for growth rate shows that stability is affected by the compressor swirl sensitivity and momentum added from the actuator. As was shown in [81], increasing the inlet swirl angle decreases the flow turning through the blade rows. From the Euler Turbine Equation (see equation 3.15), a decrease in the flow turning will reduce pressure rise across the compressor. Thus the rate of change of the compressor total-to-static pressure rise with respect to the inlet swirl is negative i.e., $\frac{\partial \psi_{rs}}{\partial \alpha} < 0$. This implies that the compressor swirl sensitivity has a destabilizing effect since it makes the effective slope of the compressor characteristic more positive. The experimental results in Chapter 4 show that the change in turning is small. Therefore, the swirl sensitivity decreases the stability only marginally. Results from the mass sensitivity studies in Figures 3-4a and 3-8a show that increasing the compressor inlet mass flow rate produces a decrease in the total pressure rise in the upstream duct i.e., $\frac{\partial \psi_{m}^{in}}{\partial \psi_{in}} < 0$. This implies that the injection of momentum from the actuator has a stabilizing effect by making the effective slope of the compressor characteristic more negative. Assuming that the neutral stability point of the least stable mode determines the stalling point of the compression system, the modified incompressible model predicts air injection to reduce the stalling mass flow rate. The rotation rate of the $n^{th}$ harmonic disturbance is:

$$\omega_{rs} = \frac{n \lambda - \frac{n}{|n|} \left( \alpha_{4} \frac{\partial \sigma_{rs}}{\partial \Phi} - \alpha_{1} \alpha_{4} \frac{\partial \sigma_{rs}}{\partial \alpha} + \alpha_{2} e^{-|n| \eta} \frac{\partial \psi_{rs}}{\partial \alpha} \right) \right)}{\mu + \frac{1}{|n|} \frac{\rho A}{\rho_{o} A_{in}} \left( A_{in} e^{-\alpha_{2} e^{-|n| \eta}} \right)$$

Arguments similar to the ones discussed above can be used to show that the compressor swirl sensitivity causes an increase in the propagation rate. Jet actuation also affect the growth and propagation rates in that it changes the blade row loss and deviation characteristics which are shown to determine the $n^{th}$ harmonic growth and rotation rates. The unsteady effects modeled by the pressure loss and deviation time lags were shown by Haynes [61] and Longley [81] to have a stabilizing effect by making the effective slope of the compressor characteristic more negative, thus allowing the compressor to be stable at positive slopes.
This stabilizing effect is stronger for the higher circumferential harmonics, and thus makes the lower harmonics to go unstable first as the mass flow is reduced. Haynes [61] and Longley [81] also showed that the pressure loss and deviation time lags cause the propagation rate of the pre-stall disturbances to increase.

**C.1.8 Low-Speed Surge Model**

Substituting the expressions for $\delta \phi_{ex}$ from the continuity equation C.11 and $\delta \phi_t$ from C.39 into equation C.36, and taking the laplace transform of the resulting expression gives:

$$
\delta \psi(s) = \frac{\rho A}{4B^2 \ell_c (s + \frac{1}{4B^2 \ell_m})} \delta \phi(s)
$$

(C.47)

Substituting the expression for the compressor inlet flow angle perturbation $\delta \alpha$ from equation C.18 into the equation C.37 results in an expression in terms of $\delta \phi_{ex}$, $\delta \phi$, $\delta \theta$, $u_j$, and $\delta \psi$. The expressions for $\delta \phi_{ex}$ from C.11, $\delta \theta$ from C.15 (noting that $\delta \theta_{in} = 0$ for $n = 0$), and $\delta \phi_{in}$ from equation C.11, are substituted into the resulting expression to simplify it in terms of $\delta \phi$, $u_j$, and $\delta \psi$. Taking the Laplace transform of the simplified expression gives:

$$
\left( s - \frac{1}{\ell_c} \frac{\rho A}{\rho} \left[ \frac{\partial \psi_{ts}}{\partial \phi} + \frac{\rho A}{\rho_{in} A_{in}} \frac{\partial \psi_{mom}}{\partial \phi_{in}} - \frac{2 \tilde{\sigma}_0}{\sigma_0^2 + \tilde{\sigma}_0^2} \frac{\partial \psi_{ts}}{\partial \alpha} \right] \right) \delta \phi(s) = -\frac{1}{\ell_c} \frac{\rho A A_{ex}}{\rho A} \delta \psi_k(s)
$$

(C.48)

Combining equations C.47 and C.48, the SISO transfer function from the zeroth harmonic actuation input to the zeroth harmonic velocity perturbation at the compressor inlet is given by:

$$
G_o(s) = \frac{\tilde{\phi}_o(s)}{\tilde{u}_o(s)}
$$

(C.49)

$$
G_o(s) = m \gamma \left( s + \frac{1}{4B^2 \ell_m} \right)
$$

$$
= \frac{s^2 - 2\sigma_{surge} s + (\sigma_{surge}^2 + \omega_{surge}^2)}{s^2 - 2\sigma_{surge} s + (\sigma_{surge}^2 + \omega_{surge}^2)}
$$

where the real part of the eigenvalue, $\sigma_{surge}$, determines the growth rate of the surge mode and the imaginary part of the eigenvalue, $\omega_{surge}$, determines the rotation rate of the surge.
mode, and are given by the expressions:

\[
\sigma_{\text{surge}} = -\frac{1}{2} \left[ \frac{1}{4B^2\ell_c m_t} - \frac{m_c}{\ell_c} \right]
\]

\[
\omega_{\text{surge}} = \frac{1}{2} \sqrt{\frac{1}{16B^4\ell_c^2 m_t^2} + \left(1 - \frac{1}{2} \frac{m_c}{m_t}\right) - \frac{m_c^2}{\ell_c^2}}
\]

\[
m_c = \frac{\rho ex A ex}{\rho A} \left[ \frac{\partial \Psi_{ts}^c}{\partial \Phi} + \frac{\rho A}{\rho in A in} \frac{\partial \Psi_{t_mom}^c}{\partial \Phi_{in}} - \frac{2 \Phi_{o}^{2}}{\Phi_{o}^{2} + \Phi_{o}^{2}} \frac{\partial \Psi_{ts}^c}{\partial \alpha} \right]
\]

\[
m_{\gamma} = \frac{1}{\ell_c} \frac{\rho ex A ex}{\rho A} \left[ \frac{\partial \Psi_{ts}^c}{\partial U_j} + \frac{\partial \Psi_{t_mom}^c}{\partial U_j} - \frac{\rho o A o}{\rho in A in} \frac{\partial \Psi_{t_mom}^c}{\partial \Phi_{in}} + \frac{2 \Phi_{o}^{2}}{\Phi_{o}^{2} + \Phi_{o}^{2}} \frac{\rho A}{\rho A} \frac{\partial \Psi_{ts}^c}{\partial \alpha} \right]
\]

### C.2 Full State Nonlinear Distributed Model

The differential equations that make up the full state nonlinear distributed model are obtained using the discretization procedure outlined in references [84, 107]. The discretization procedure consists of combining the surge and rotating stall models described in the previous section using a Fourier transform procedure which relies on the eigenfunction form of the system. For example, the discrete Fourier transform (DFT) of the flow coefficient at a circumferential location, \( \theta_k \), around the compressor annulus is defined as:

\[
\Phi(\theta_k) = \frac{1}{\sqrt{M}} \sum_{n=-N}^{+N} \Phi_n \cdot e^{jn\theta_k} \quad (C.50)
\]

Using this definition of the discrete Fourier transform in equation C.50, a vector \( \Phi \) containing the flow coefficient at \( M \) equally spaced locations around the compressor annulus, and the corresponding vector \( \Phi_n \) containing the spatial Fourier coefficients from \( -N \) to \( +N \) are related by the following matrix equation:

\[
\Phi = F \cdot \Phi \quad (C.51)
\]

where \( F \) is the discrete Fourier transform matrix in complex form given by:

\[
F = \frac{1}{\sqrt{M}} \left[ \begin{array}{ccc}
\vdots & \cdots & W^{nk} \cdots \\
\vdots & \ddots & \vdots \\
\vdots & & \vdots 
\end{array} \right]_{n=-N}^{n=+N} \quad (C.52)
\]

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where \( W = e^{-\frac{2\pi i}{M}} \). The corresponding inverse matrix relationship of equation C.51 is:

\[
\phi = F^{-1} \cdot \tilde{\phi} \quad \text{(C.53)}
\]

where \( F^{-1} = F^T \). The vector containing the flow coefficients at \( M \) equally spaced locations around the compressor annulus, \( \phi \), is given by:

\[
\phi = \bar{\phi} + \delta\phi(\theta_k)_{k=0}^{k=M-1}
\]

where \( \bar{\phi} \) is the mean flow, and \( \theta_k = \frac{2\pi k}{M} \) with \( k = 0, 1, 2, ..., M - 1 \). The vector containing the spatial Fourier coefficients from \(-N\) to \(+N\), \( \tilde{\phi} \), is given by:

\[
\tilde{\phi} = \begin{bmatrix}
\vdots \\
\phi_n \\
\vdots \\
\end{bmatrix}^{n=-N}_{n=+N}
\]

Based on the way the complex matrix \( F \) is normalized in equation C.52, the corresponding relation between the zeroth mode, \( \tilde{\phi}_0 \), and the mean flow, \( \bar{\phi} \), is:

\[
\tilde{\phi}_0(t) = \bar{\phi}(t) \cdot \sqrt{M} \quad \text{(C.54)}
\]

The first step in the discretization technique is to write the surge and rotating stall models in terms of the spatial Fourier coefficients (see Section C.1). Secondly, the surge and rotating stall modes are combined in a vector form and matrix relations similar to the one in equation C.51 are used to replace the vectors containing the spatial Fourier coefficients. Finally, by multiplying the entire system equations by \( F^{-1} \) (i.e., using matrix transformations similar to the one in equation C.53), the original set of variables obtained. By applying the discretization technique to the modified surge and rotating stall models in the previous section, the following differential equations for the full state nonlinear distributed model

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were obtained:

\[
\begin{align*}
E \dot{\phi} &= -A \phi + \Psi_{\text{ideal}}(\phi, \gamma_j) + \Psi_{\text{mom}} - L_r - L_s - T\psi + K \dot{\gamma}_j \\
\dot{\psi} &= \frac{1}{4B^2\ell_c} \left[ mS^T\phi - \sqrt{2}\psi \right] \\
\tau_{\text{mom}} \dot{\psi}_{\text{mom}} &= -\Psi_{\text{mom}} + \Psi_{\text{mom}}^s(\phi, \gamma_j) \\
\tau_{\text{loss}} \dot{L}_r &= -J L_r + L_{r5}^s(\phi, \gamma_j) \\
\tau_{\text{loss}} \dot{L}_s &= -L_s + L_{s5}^s(\phi, \gamma_j) \\
\dot{x} &= A_{\text{act}} x + B_{\text{act}} \gamma_c \\
\ddot{\gamma} &= C_{\text{act}} x + D_{\text{act}} \gamma_c \\
\gamma_j &= Z_{\text{cfg}} \left[ \ddot{x} + \gamma_{c0} \right]
\end{align*}
\]

where \( m = \frac{\rho A}{\rho_{\text{ex}} A_{\text{ex}}} \), \( \gamma_c \) is the normalized input command to the jet actuators, \( \gamma \) is the actual jet actuator valve position, \( \gamma_{c0} \) is the jet actuator valve position when there is no input command voltage i.e., \( \gamma_{c0} = 0.5 \). \( A_{\text{act}}, B_{\text{act}}, C_{\text{act}}, \) and \( D_{\text{act}} \) are the state space matrices for the actuator dynamics. The matrix \( Z_{\text{cfg}} \) contains information about the actuation configuration, and the shape of the injected profile from the injectors. Figure C-2 shows the shape profile for one sheet injector derived from wind tunnel measurements in [10].

![Figure C-2: Shape profile for one sheet injector derived from wind tunnel measurements in [10.](image)](image)
C.3 Moore-Greitzer Parameters for NASA Stage 35

Behnken and Murray [8] have presented systematic procedures for identifying the model parameters for a low speed axial compressor using experimental data from surge cycles and local measurements in the compressor annulus. However, these procedures could not be implemented on NASA Stage 35 because the surge cycles could not be taken for safety precautions. This section summarizes the procedures that were used to determine the parameters for the Moore-Greitzer mode in NASA Stage 35. The semi-actuator disk model which is the basis of the Moore-Greitzer model uses the geometry of the compressor and the axisymmetric compressor characteristic. The required parameters are determined from geometric measurements, speedline measurements, and forced response measurements using the Moore-Greitzer surge and rotating stall models derived in Section C.1.

C.3.1 Geometric Parameters

The reference constants used for non-dimensionalizing flow parameters are:

\begin{align*}
    r_o &= 0.2152 \text{ m} \\
    U_o &= 390.6273 \text{ m/s} \\
    \rho_o &= 1.0982 \text{ kg/m}^3
\end{align*}

where \(r_o\) is the mean radius, \(U_o\) is the mean wheel speed, and \(\rho_o\) is the upstream density at design mass flow rate.

The geometric parameters for the compressor are:

\begin{align*}
    \lambda &= \sum_j \frac{b_{x_j}}{\cos \gamma_{x_j}} = 0.4044 \\
    \mu_s &= \sum_j \frac{b_{y_j}}{\cos \gamma_{y_j}} = 0.1996 \\
    \mu &= \lambda + \mu_s = 0.6040 \\
    \ell_u &= 3.3959 \\
    \ell_d &= 3.2659 \\
    \ell_c &= \ell_u + \ell_d = 6.6619 \\
    \eta_m &= -0.1004
\end{align*}

C.3.2 Compressor Total-to-Static Pressure Characteristics

Both the ideal and actual compressor characteristics are required for simulating the incompressible stall inception model. The ideal characteristics correspond to the performance of
the compressor when the flow is isentropic with no losses in the blade rows, no deviation at
the blade row trailing edges, and the flow is axisymmetric throughout. Since the inlet flow
conditions are known and the compressor blade row performance characteristics are known
for the ideal case, the ideal compressor characteristic can be calculated. On the other hand,
the actual compressor characteristics have to be measured experimentally since the actual
flow in the compressor involves some losses in the blade rows, deviation at the blade row
trailing edges, and the flow is non-axisymmetric.

**Ideal Compressor Characteristics:** An explicit expression for the isentropic compres-
sor characteristic for low speed compressors is derived in [66]. This approach was used
by Yeung [136] to determine the isentropic compressor characteristic for a low speed, single
stage axial flow compressor with air injection. However, these expressions are not valid for
high speed compressors where the flow is compressible. Therefore, the ideal compressor
characteristics for high speed compressors for different injection configurations can be com-
puted using the steady air injection model described in Chapter 3 with the relative total
pressure loss coefficient of each blade row set to zero, the exit flow deviation angle set to
zero, and the tangential blockage parameter set to unity for axisymmetric flow. Instead
of running the steady air injection code for different mass flow rates to map out the ideal
compressor characteristics, a much quicker approach is to perform a meanline calculation
using the mass averaged values of the inlet flow properties as the starting point. Consult
reference [135] or any other thermodynamics textbook for details on the derivation of the
isentropic relations used below. The computational procedure for NASA Stage 35 which
consists of a rotor and a stator is outlined below:

1. **Compressor Inlet Flow Properties:** The flow properties at the compressor inlet
   are computed using the calibration procedure outlined in Section 3.1 for modeling the
effect of the jet actuator on the flow in the upstream duct. The mass averaged values
   of the 2-D profiles are used for the meanline analysis.

2. **Relative Total Temperature at Rotor Trailing Edge:** The relative total tem-
   perature at the rotor trailing edge is computed from the conservation of rothalpy,
\[ I_t = h'_t - \frac{1}{2} \Omega^2 r^2, \text{ across the rotor.} \]

\[ T'_{t,2} = T'_{t,1} + \frac{\Omega^2}{2C_p}(r^2 - r^2_1) \quad (C.56) \]

3. **Relative Total Pressure at Rotor Trailing Edge**: The relative total pressure at the rotor trailing edge is computed from the isentropic relation between the relative quantities at the leading and trailing edges.

\[ P'_{t,2} = P'_{t,1} \left( \frac{T'_{t,2}}{T'_{t,1}} \right)^{\frac{\gamma}{\gamma-1}} \quad (C.57) \]

4. **Relative Mach Number at Rotor Trailing Edge**: The relative Mach number at the rotor trailing edge is computed from mass conservation at the trailing edge.

\[ \frac{M^2_2 \cos(\beta_2)}{\left(1 + \frac{\gamma - 1}{2} M^2_2 \right)^{\frac{\gamma + 1}{\gamma - 1}}} = \frac{\dot{m}}{A_2} \sqrt{T'_{t,2}} \sqrt{\frac{R}{\gamma}} \quad (C.58) \]

where \( A_2 \) is the area at the rotor trailing edge, and \( \beta_2 \) is the relative exit flow angle at the rotor trailing edge (\( \beta_2 \) is equal to the rotor trailing edge metal angle since there is no flow deviation). Since the mass conservation equation in \( C.58 \) is nonlinear, the relative trailing edge Mach number, \( M_2 \), is obtained from an iterative solution.

5. **Absolute Flow Properties at Rotor Trailing Edge**: The absolute flow properties such as stagnation pressure and stagnation temperature at the rotor trailing edge can be computed from the relative total pressure, relative total temperature, and relative Mach number using isentropic relations. The static pressure and static temperature are computed from the following isentropic relations:

\[ \frac{P'_{t,2}}{P_2} = \left[ 1 + \frac{\gamma - 1}{2} M^2_2 \right]^{\frac{\gamma}{\gamma-1}} \quad (C.59) \]

\[ \frac{T'_{t,2}}{T_2} = \left[ 1 + \frac{\gamma - 1}{2} M^2_2 \right]^{\frac{\gamma}{\gamma-1}} \quad (C.60) \]

Using the velocity triangle at the rotor trailing edge shown in Figure C-3, the relative velocity, absolute velocity, and absolute Mach number can be computed from the relative Mach number, local speed of sound, \( a_2 = \sqrt{\gamma R T_2} \), and relative exit flow...
angle, $\beta_2$. The absolute Mach number can then be used in conjunction with the static pressure and static temperature to compute the total pressure and total temperature respectively using the following isentropic relations:

\begin{align}
P_{t,2} & = P_2 \left[ 1 + \frac{\gamma - 1}{2} M_2^2 \right]^{\frac{\gamma}{\gamma - 1}} \\
T_{t,2} & = T_2 \left[ 1 + \frac{\gamma - 1}{2} M_2^2 \right]
\end{align}

\hspace{2cm} (C.61) \hspace{2cm} (C.62)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{velocity_triangles.png}
\caption{Velocity triangles.}
\end{figure}

6. **Absolute Flow Properties at Station K**: The total pressure and total temperature at the rotor trailing edge are respectively equal to the total pressure and total temperature at Station K or any location in the downstream duct i.e., $P_{t,2} = P_{t,3}$ and $T_{t,2} = T_{t,3}$. The absolute Mach number at Station K can be computed from the mass conservation relation:

\[ \frac{M_3 \cos(\beta_3)}{\left(1 + \frac{\gamma - 1}{2} M_3^2 \right)^{\frac{\gamma + 1}{2(\gamma - 1)}}} = \frac{\dot{m} \sqrt{T_{t,3}}}{A_3 P_{t,3}} \sqrt{\frac{R}{\gamma}} \]  

\hspace{2cm} (C.63)

where $A_3$ is the area at Station K, and $\beta_3$ is the exit flow angle at the stator trailing edge ($\beta_3$ is equal to the stator trailing edge metal angle since there is no flow deviation). The static pressure and static temperature can then be computed from isentropic relations. So long as the appropriate area is used, the same analysis will apply to any location in the downstream duct.
The ideal total-to-static compressor characteristics computed for NASA stage 35 at 0%, 50%, and 100% injection with the sheet injector at 15° counterswirl are shown in Figure C-4. To incorporate the actuation dependence, the ideal compressor characteristic is modeled with the following two-dimensional representation:

$$\Psi_{ts}(\Phi, \gamma) = \Psi_{40}(\Phi) + \gamma \Psi_{41}(\Phi) + \gamma^2 \Psi_{42}(\Phi)$$  \hspace{1cm} (C.64)

**Figure C-4:** Ideal compressor characteristics for NASA Stage 35 with different amount of air at 100 psig being injected from the sheet injectors at -15° yaw.

**Actual Compressor Characteristics:** The experimentally measured total-to-static compressor characteristics for NASA stage 35 at 0%, 50%, and 100% injection with the sheet injector at 15° counterswirl are shown in Figure C-5. These compressor characteristics for different levels of injection can be represented by the piecewise polynomial fits given below. With the actuators turned off, the piecewise polynomial fit is given by:

$$\Psi_{ts}(\Phi, \gamma = 0) = \begin{cases} -59.3115\Phi^2 + 51.1633\Phi - 10.7655 & \Phi \geq 0.4313, \\ -19.5986\Phi^3 + 17.0890\Phi^2 - 3.8038\Phi + 0.3022 & 0.15 \leq \Phi \leq 0.4313, \\ 5.4490\Phi^2 - 1.6347\Phi + 0.1726 & \Phi \leq 0.15. \end{cases}$$  \hspace{1cm} (C.65)
The piecewise polynomial fit for 50% injection i.e., when the valves are 50% open and air supplied at 100 psig, is given by:

\[
\Psi_c^{ts}(\Phi, \gamma = 0.5) = \begin{cases} 
-27.3888\Phi^2 + 22.4729\Phi - 4.3079 & \Phi \geq 0.4103, \\
-25.1669\Phi^3 + 21.1515\Phi^2 - 4.6467\Phi + 0.3860 & 0.15 \leq \Phi \leq 0.4103, \\
6.2044\Phi^2 - 1.8613\Phi + 0.2196 & \Phi \leq 0.15.
\end{cases} \tag{C.66}
\]

The piecewise polynomial fit for 100% injection i.e., when the valves are 100% open and air supplied at 100 psig, is given by:

\[
\Psi_c^{ts}(\Phi, \gamma = 1) = \begin{cases} 
-17.5653\Phi^2 + 14.0472\Phi - 2.4897 & \Phi \geq 0.3999, \\
-28.0314\Phi^3 + 23.1217\Phi^2 - 5.0444\Phi + 0.4310 & 0.15 \leq \Phi \leq 0.3999, \\
6.4733\Phi^2 - 1.9420\Phi + 0.2456 & \Phi \leq 0.15. \tag{C.67}
\end{cases}
\]

where \(\Phi\) is the axial velocity flow coefficient through the compressor. The piecewise polynomial fits at 0%, 50%, and 100% injection can be used to represent the compressor characteristic as a function of \(\Phi\), the velocity flow coefficient and \(\gamma\), the level of injection which can be considered as the actuation command or valve sleeve position since it has a linear relationship with the injected mass. To incorporate the actuation dependence, the 2-D compressor characteristic is then represented as:

\[
\Psi_c^{ts}(\Phi, \gamma) = a_3\Phi^3 + a_2\Phi^2 + a_1\Phi + a_0 \tag{C.68}
\]

where the coefficients are parabolic functions of \(\gamma\) given by:

\[
\begin{align*}
    a_0(\gamma) &= a_{00} + a_{01}\gamma + a_{02}\gamma^2 \\
    a_1(\gamma) &= a_{10} + a_{11}\gamma + a_{12}\gamma^2 \\
    a_2(\gamma) &= a_{20} + a_{21}\gamma + a_{22}\gamma^2 \\
    a_3(\gamma) &= a_{30} + a_{31}\gamma + a_{32}\gamma^2
\end{align*}
\]

The compressor characteristics at 0%, 50%, and 100% injection are compared with the piecewise polynomial fits in Figure C-6a.
Substituting the expressions for the coefficients $a_0$, $a_1$, $a_2$, and $a_3$ into the two-dimensional compressor characteristic in equation C.68 gives:

$$
\Psi_{ts}^c(\Phi, \gamma) = \Psi_{c0}(\Phi) + \gamma \Psi_{c1}(\Phi) + \gamma^2 \Psi_{c2}(\Phi)
$$

(C.69)

where the flow dependent components $\Psi_{c0}(\Phi)$, $\Psi_{c1}(\Phi)$, and $\Psi_{c2}(\Phi)$ are given by:

$$
\Psi_{c0}(\Phi) = a_{00} + a_{10} \Phi + a_{20} \Phi^2 + a_{30} \Phi^3
$$

$$
\Psi_{c1}(\Phi) = a_{01} + a_{11} \Phi + a_{21} \Phi^2 + a_{31} \Phi^3
$$

(C.70)

$$
\Psi_{c2}(\Phi) = a_{02} + a_{12} \Phi + a_{22} \Phi^2 + a_{32} \Phi^3
$$

$\Psi_{c0}(\Phi)$ represents the compressor characteristic with solid casing (injectors turned off, no injection), $\Psi_{c1}(\Phi)$ represents the linear component of the shift in compressor characteristic due to air injection, and $\Psi_{c2}(\Phi)$ represents the quadratic component of the shift in compressor characteristic due to air injection. The components of the compressor characteristic $\Psi_{c0}(\Phi)$, $\Psi_{c1}(\Phi)$, and $\Psi_{c2}(\Phi)$ for NASA Stage 35 is shown in Figure C-6b.

Figure C-5: Actual compressor characteristics for NASA Stage 35 with different amount of air at 100 psig being injected from the sheet injectors at $-15^\circ$ yaw.
(a) Polynomial fits of actual compressor characteristics for air injection at $-15^\circ$ yaw

(b) Components of actual compressor characteristics for air injection at $-15^\circ$ yaw

**Figure C-6:** Polynomial fits and components of actual compressor characteristics for NASA Stage 35 operating at 100% speed with air injection at $-15^\circ$ yaw.
C.3.3 Moore-Greitzer Surge Model

From equation C.49, the zeroth harmonic or surge growth rate, $\sigma_{\text{surge}}$, is given by:

$$\sigma_{\text{surge}}(\Phi) = -\frac{1}{2} \left[ \frac{1}{4B^2 \ell_c m_t} - \frac{m_c}{\ell_c} \right]$$  \hspace{1cm} (C.71)

and the zeroth or surge rotation rate, $\omega_{\text{surge}}$, is given by:

$$\omega_{\text{surge}}(\Phi) = \frac{1}{2} \sqrt{-\frac{1}{16B^4 \ell_c^2 m_t^2} + \frac{1 - \frac{1}{2} \frac{m_c}{m_t}}{B^2 \ell_c^2} - \frac{m_c^2}{\ell_c^2}}$$  \hspace{1cm} (C.72)

Reformulating the surge rotation rate equation in C.72 to a quadratic equation in $\frac{1}{B^2}$ gives:

$$\frac{1}{B^4} \left( \frac{1}{16m_t^2} \right) + \frac{1}{B^2} \left( \frac{m_c}{2m_t} - 1 \right) + m_c^2 + 4\omega_{\text{surge}}^2 \ell_c^2 = 0$$  \hspace{1cm} (C.73)

The surge growth and rotation rates from system identification experiments on NASA Stage 35 at three different mass flow rates are given in Table C.3.3. For each flow coefficient, $\Phi$,

<table>
<thead>
<tr>
<th>$\dot{m}$ (kg/s)</th>
<th>$\Phi$</th>
<th>$\sigma_{\text{surge}}$</th>
<th>$\omega_{\text{surge}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>18.8740</td>
<td>0.4357</td>
<td>-0.0107</td>
<td>0.0818</td>
</tr>
<tr>
<td>19.3276</td>
<td>0.4462</td>
<td>-0.0261</td>
<td>0.0826</td>
</tr>
<tr>
<td>20.1803</td>
<td>0.4659</td>
<td>-0.0427</td>
<td>0.0786</td>
</tr>
</tbody>
</table>

*Table C.1*: Moore-Greitzer surge parameters for NASA Stage 35.

The surge rotation rate, $\omega_{\text{surge}}$, the effective compressor characteristic slope, $m_c$, and the throttle coefficient, $m_t$, were used to determine the $B - \ell_c$ relationship from the quadratic equation in C.73. The $B - \ell_c$ curves obtained for the three flow coefficients are shown in Figure C-7a. The values for $B$ and $\ell_c$ were selected at the point where the three $B - \ell_c$ curves intersect or the point where the three curves are closest. The surge parameters obtained using this process are $B = 0.3504$ and $\ell_c = 17.3500$. Figure C-7b shows how the surge growth rates and frequencies predicted using the identified parameters compare with the corresponding experimentally determined values.
(a) $B - l_e$ relationship at three flow coefficients

(b) Predicted and actual surge growth and rotation rates

Figure C-7: Moore-Greitzer surge parameters for NASA Stage 35.
C.3.4 Moore-Greitzer Rotating Stall Model

The rotating stall growth rates and rotation rates from system identification experiments on NASA Stage 35 at different mass flow rates for the first and second harmonics are given below in Table C.3.4.

<table>
<thead>
<tr>
<th>( \dot{m} ) (kg/s)</th>
<th>( \Phi )</th>
<th>Harmonic No.</th>
<th>( \sigma_{stall} )</th>
<th>( \omega_{stall} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>18.8377</td>
<td>0.4349</td>
<td>1</td>
<td>-0.0478</td>
<td>0.3657</td>
</tr>
<tr>
<td>19.3140</td>
<td>0.4459</td>
<td>1</td>
<td>-0.1503</td>
<td>0.3637</td>
</tr>
<tr>
<td>20.1440</td>
<td>0.4650</td>
<td>1</td>
<td>-0.3434</td>
<td>0.3810</td>
</tr>
<tr>
<td>18.4794</td>
<td>0.4266</td>
<td>2</td>
<td>-0.0433</td>
<td>0.8750</td>
</tr>
<tr>
<td>19.3049</td>
<td>0.4457</td>
<td>2</td>
<td>-0.2034</td>
<td>0.8443</td>
</tr>
</tbody>
</table>

Table C.2: Rotating stall growth rates and rotation rates from system identification measurements on NASA Stage 35.

Effective inertia parameters are determined to match the frequencies of the rotating stall cells in Table C.3.4. From the original Moore-Greitzer model, the rotation frequency of the \( n^{th} \) harmonic rotating stall wave is given by:

\[
\omega_{rs}(\Phi) = \frac{n \lambda_{eff}}{\mu_{eff} + \frac{2}{|n|}}
\]

The rotation frequency equation can be casted into the following least squares problem:

\[
\begin{bmatrix}
    n & -\omega_{rs}(\Phi)
\end{bmatrix}
\begin{bmatrix}
    \lambda_{eff} \\
    \mu_{eff}
\end{bmatrix} = \begin{bmatrix}
    \frac{2}{|n|}\omega_{rs}(\Phi)
\end{bmatrix}
\]

(C.74)

Using rotation rates from the estimated eigenvalues at different flow coefficients in Table C.3.4, the effective inertia parameters obtained from the least squares analysis were \( \lambda_{eff} = 2.4292 \) and \( \mu_{eff} = 4.6226 \).

Splitting the total-to-static pressure rise across the compressor blade rows into the ideal component and the unsteady loss, i.e., \( \Psi_{ts}^{c}(\Phi, \gamma) = \Psi_{ts}^{ideal}(\Phi, \gamma) - L(\tau, \theta) \); lumping the total pressure rise in the upstream duct due to the jet actuator with the ideal total-to-static pressure rise across the compressor blade rows; neglecting the sensitivity of the ideal total-to-static characteristic to the inlet flow angle, i.e., \( \frac{\partial \Psi_{ts}^{c}}{\partial \alpha} = 0 \); assuming the exit flow angle is constant i.e., \( \frac{\partial \alpha_{ex}}{\partial \alpha} = 0, \frac{\partial \alpha_{ex}}{\partial \Phi} = 0, \) and \( \frac{\partial \alpha_{ex}}{\partial U_j} = 0 \), the \( n^{th} \) harmonic rotating stall ordinary
differential equation in C.44 reduces to:

\[ \mu_m \ddot{\phi}_n = \left[ \frac{\partial \Phi_{ts}^{\text{ideal}}}{\partial \phi} - jn \lambda \right] \ddot{\phi}_n - \dot{L}_n + \frac{1}{|n|} A_{in} e^{-|n| |\eta_m|} \dot{\gamma}_n + \frac{\partial \Phi_{ts}^{\text{ideal}}}{\partial \gamma} \dot{\gamma}_n \]

(C.75)

where \( \mu_m = \mu + \frac{1}{|n|} \rho A_{in} \left( A_{ex} + e^{-|n| |\eta_m|} \right) \). The corresponding \( n^{th} \) harmonic perturbation equation of the unsteady loss in equation C.23 is:

\[ \tau \ddot{L}_n = \frac{\partial L^{ss}}{\partial \phi} \ddot{\phi}_n - (1 + jnr \tau) \dot{L}_n + \frac{\partial L^{ss}}{\partial \gamma} \dot{\gamma}_n \]

(C.76)

Combining equations C.75 and C.76 in matrix form:

\[
\begin{pmatrix}
\dot{\phi}_n \\
\dot{L}_n
\end{pmatrix} =
\begin{pmatrix}
\frac{1}{\mu_m} \left( \frac{\partial \Phi_{ts}^{\text{ideal}}}{\partial \phi} - jn \lambda \right) & -\frac{1}{\mu_m} \\
\frac{1}{\tau} \frac{\partial L^{ss}}{\partial \phi} & -\left( \frac{1}{\tau} + jnr \right)
\end{pmatrix}
\begin{pmatrix}
\ddot{\phi}_n \\
\ddot{L}_n
\end{pmatrix} +
\begin{pmatrix}
\frac{1}{|n|} A_{in} e^{-|n| |\eta|} & \frac{1}{\mu_m} \frac{\partial \Phi_{ts}^{\text{ideal}}}{\partial \gamma} \\
\frac{1}{\mu_m} & \frac{\partial L^{ss}}{\partial \gamma}
\end{pmatrix}
\begin{pmatrix}
\ddot{\gamma}_n \\
\ddot{\gamma}_n
\end{pmatrix}
\]

(C.77)

The corresponding real-valued system of the complex system in equation C.77 is:

\[ \begin{pmatrix}
\dot{\phi}_{rn} \\
\ddot{L}_{rn}
\end{pmatrix} =
\begin{pmatrix}
\frac{1}{\mu_m} \frac{\partial \Phi_{ts}^{\text{ideal}}}{\partial \phi} & \frac{n \lambda}{\mu_m} & -\frac{1}{\mu_m} & 0 \\
-\frac{n \lambda}{\mu_m} & \frac{\partial \Phi_{ts}^{\text{ideal}}}{\partial \phi} & 0 & -\frac{1}{\mu_m} \\
\frac{1}{\tau} \frac{\partial L^{ss}}{\partial \phi} & 0 & -\frac{1}{\tau} & nr \\
0 & \frac{1}{\tau} \frac{\partial L^{ss}}{\partial \phi} & -nr & -\frac{1}{\tau}
\end{pmatrix}
\begin{pmatrix}
\phi_{rn} \\
\phi_{in} \\
\ddot{L}_{rn} \\
\ddot{L}_{in}
\end{pmatrix}
\]

(C.78)

\[
+ \begin{pmatrix}
\frac{1}{|n|} A_{in} e^{-|n| |\eta|} & 0 & \frac{1}{\mu_m} \frac{\partial \Phi_{ts}^{\text{ideal}}}{\partial \gamma} & 0 \\
0 & \frac{1}{|n|} A_{in} e^{-|n| |\eta|} & 0 & \frac{1}{\mu_m} \frac{\partial \Phi_{ts}^{\text{ideal}}}{\partial \gamma} \\
0 & 0 & \frac{1}{\tau} \frac{\partial L^{ss}}{\partial \gamma} & 0 \\
0 & 0 & 0 & \frac{1}{\tau} \frac{\partial L^{ss}}{\partial \gamma}
\end{pmatrix}
\begin{pmatrix}
\ddot{\gamma}_{rn} \\
\ddot{\gamma}_{in} \\
\ddot{\gamma}_{rn} \\
\ddot{\gamma}_{in}
\end{pmatrix}
\]

The characteristic equation for the eigenvalues of the \( n^{th} \) harmonic rotating stall mode in
The characteristic equation has two unknowns: \( \tau \) and \( r \), and explicit expressions for the eigenvalues in terms of these unknowns cannot be obtained since solutions will exist only for certain values of \( \tau \) and \( r \). Approximate expressions for the eigenvalues can be obtained by using a linearized estimate of the unsteady loss perturbation coefficient, \( \tilde{L}_n \), in equation \( C.75 \). The harmonic loss perturbation coefficient, \( \tilde{L}_n \), in equation \( C.76 \) can be rewritten as:

\[
\tilde{L}_n = \frac{1}{1 + \tau \left( \frac{\partial}{\partial t} + jnr \right)} \left[ \frac{\partial L^{ss}}{\partial \Phi} \tilde{\phi}_n + \frac{\partial L^{ss}}{\partial \gamma} \tilde{\gamma}_n \right] \tag{C.80}
\]

For small values of the lag time, \( \tau \), binomial expansion can be used to give:

\[
\tilde{L}_n \approx \left\{ 1 - \tau \left( \frac{\partial}{\partial t} + jnr \right) \right\} \left[ \frac{\partial L^{ss}}{\partial \Phi} \tilde{\phi}_n + \frac{\partial L^{ss}}{\partial \gamma} \tilde{\gamma}_n \right] \tag{C.81}
\]

\[
\approx -\tau \frac{\partial L^{ss}}{\partial \Phi} \tilde{\phi}_n + (1 - jnr\tau) \frac{\partial L^{ss}}{\partial \Phi} \tilde{\phi}_n - \tau \frac{\partial L^{ss}}{\partial \gamma} \tilde{\gamma}_n + (1 - jnr\tau) \frac{\partial L^{ss}}{\partial \gamma} \tilde{\gamma}_n
\]

Substituting equation \( C.81 \) into equation \( C.75 \) gives:

\[
\left[ \mu_m - \tau \frac{\partial L^{ss}}{\partial \Phi} \right] \tilde{\phi}_n = \left[ \left( \frac{\partial \Psi_{ts}^{ideal}}{\partial \Phi} - \frac{\partial L^{ss}}{\partial \Phi} \right) - jn \left( \lambda - \tau \frac{\partial L^{ss}}{\partial \Phi} \right) \right] \tilde{\phi}_n \tag{C.82}
\]

\[
+ \left[ \frac{1}{|n|} A_0 e^{-|n|\eta_m} + \tau \frac{\partial L^{ss}}{\partial \gamma} \right] \tilde{\gamma}_n + \left[ \left( \frac{\partial \Psi_{ts}^{ideal}}{\partial \gamma} - \frac{\partial L^{ss}}{\partial \gamma} \right) + jnr \tau \frac{\partial L^{ss}}{\partial \gamma} \right] \tilde{\gamma}_n
\]

The \( n^{th} \) harmonic growth rate, \( \sigma_{rs} \), and rotation rate, \( \omega_{rs} \), of the rotating stall disturbance
are given by:

\[ \sigma_{rs}(\Phi) = \frac{\partial \psi_{ideal}^s - \partial L_{ss}^s}{\mu_m - \tau \frac{\partial L_{ss}^s}{\partial \Phi}} \]

\[ \omega_{rs}(\Phi) = \frac{n \left( \lambda - \tau \frac{\partial L_{ss}^s}{\partial \Phi} \right)}{\mu_m - \tau \frac{\partial L_{ss}^s}{\partial \Phi}} \]

These expressions for growth rate and rotation rate can be casted into the following standard least squares problem:

\[
\begin{bmatrix}
0 & -\frac{\partial L_{ss}^s}{\partial \Phi} \sigma_{rs}(\Phi) \\
n \frac{\partial L_{ss}^s}{\partial \Phi} & -\frac{\partial L_{ss}^s}{\partial \Phi} \omega_{rs}(\Phi)
\end{bmatrix} \begin{bmatrix} r \tau \\ \tau \end{bmatrix} = \begin{bmatrix} \frac{\partial \psi_{ideal}^s}{\partial \Phi} - \frac{\partial L_{ss}^s}{\partial \Phi} - \mu_m \sigma_{rs}(\Phi) \\
n \lambda - \mu_m \omega_{rs}(\Phi) \end{bmatrix}
\]

(C.83)

Applying the measured growth rates and rotation rates from Table C.3.4 to the least squares problem in equation C.83, will give estimates of \( r \tau \) and \( \tau \). The rotating stall parameters obtained using this process are \( r = 0.5098 \) and \( \tau = 0.4498 \). The eigenvalues can be computed from the incompressible model by solving the \( n \)th harmonic characteristic equation given in C.79 using these estimates of \( r \) and \( \tau \). Figures C-8a and C-8b show how the stall cell growth rates and rotation rates predicted using the identified parameters compare with the corresponding experimentally determined values.
(a) First harmonic growth rate and rotation rate

(b) Second harmonic growth rate and rotation rate

Figure C-8: Measured and predicted growth rate and rotation rate of first and second harmonics.
C.3.5 Conclusion

The Moore-Greitzer parameters for NASA Stage 35 Axial-Flow Transonic Compressor estimated using the procedures outlined above are summarized in Table C.3.5. The parameters for the MIT three-stage axial compressor from [107] have been included for comparison.

<table>
<thead>
<tr>
<th>PARAMETERS</th>
<th>NASA STAGE 35</th>
<th>MIT 3-STAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>m</td>
<td>2.5464</td>
<td>2</td>
</tr>
<tr>
<td>B</td>
<td>0.3666</td>
<td>0.1000</td>
</tr>
<tr>
<td>$\ell_c$</td>
<td>16.2250</td>
<td>6.6600</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.6040</td>
<td>1.2937</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.4044</td>
<td>0.6787</td>
</tr>
<tr>
<td>$\mu_{\text{eff}}$</td>
<td>4.6226</td>
<td>1.2937</td>
</tr>
<tr>
<td>$\lambda_{\text{eff}}$</td>
<td>2.4292</td>
<td>0.6787</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.4498</td>
<td>0.3000</td>
</tr>
<tr>
<td>$\tau$</td>
<td>0.5098</td>
<td>0.7380</td>
</tr>
</tbody>
</table>

Table C.3: Moore-Greitzer parameters for NASA Stage 35.
The compressible stall inception model consists of models for the ducts, blade rows, and boundary conditions. Modifications are made only to the models for the boundary conditions. In this appendix, the details of the linearized boundary conditions are presented. Modifications are made to the jet actuator model, the end conditions, the blade row leading and trailing edge boundary conditions, and a new boundary condition for modeling changes in flowpath cross-sectional areas is derived. The perturbation relations used to derive the boundary conditions are summarized in Section D.1, the boundary conditions for the jet actuator are presented in Section D.2, the boundary conditions for a variable area duct is presented in Section D.3, the blade row boundary conditions are presented in Section D.4, and the end conditions which consist of the inlet and exit boundary conditions are presented in Section D.5.

D.1 Perturbation Relations

In this section, the perturbation relations for mass flow rate, absolute total pressure, absolute total temperature, relative total pressure, and relative total temperature that will be used to derive the boundary conditions in the sections that follow are summarized. Some of the perturbation relations in reference [38] have been repeated so that a completeness. The perturbation relations for relative quantities have been modified with prime superscripts to differentiate between the absolute and relative frame values. Figure D-1 shows typical
velocity triangles relating velocities in the absolute and relative frames for the flow in axial compressors. The relation between the axial velocities in the absolute and relative frames of reference and their corresponding perturbations is:

\[ W_x = V_x \]
\[ \delta W_x = \delta V_x \] (D.1)

the relative and absolute velocity perturbation equations are:

\[ W_\theta = V_\theta - \Omega r \]
\[ \delta W_\theta = \delta V_\theta \] (D.2)

and the relative flow angle perturbation equation is:

\[ \tan \beta = \frac{W_\theta}{W_x} \]
\[ \delta \tan \beta = \frac{\delta W_\theta}{W_\theta} - \frac{\delta W_x}{W_x} \]
\[ \delta \tan \beta = \tan \beta \left[ M_\theta^{-1} \frac{\delta V_\theta}{a} - M_x^{-1} \frac{\delta V_x}{a} \right] \] (D.3)

where \( V_x \) is the absolute axial velocity, \( W_x \) is the relative axial velocity, \( V_\theta \) is the absolute tangential velocity, \( W_\theta \) is the relative tangential velocity, and \( \beta \) is the relative flow angle.

Figure D-1: Velocity triangles.
Absolute total pressure perturbation (equation A.9 in reference [38]) is:

\[ \delta P_t = \delta P + \frac{\gamma M}{1 + \frac{\gamma - 1}{2} M^2} \delta M \]  

Relative total pressure perturbation is:

\[ \frac{\delta P_t}{\bar{P}_t} = \frac{\delta P}{\bar{P}} + \frac{\gamma M'}{1 + \frac{\gamma - 1}{2} M'^2} \delta M' \]  

Absolute total temperature perturbation (equation A.4 in reference [38]) is:

\[ \delta T_t = \frac{\gamma}{2} \frac{1}{1 + \frac{\gamma - 1}{2} M^2} \delta M \]  

Relative total temperature perturbation is:

\[ \frac{\delta T_t}{\bar{T}_t} = \frac{\delta T}{\bar{T}} + \frac{(\gamma - 1)M}{1 + \frac{\gamma - 1}{2} M^2} \delta M \]  

where the ideal gas relation \( P = \rho R T \) is used to express the static temperature perturbation \( \delta T \) in terms of the static pressure perturbation \( \delta P \) and density perturbation \( \delta \rho \) in equations D.6 and D.7.

### D.1.1 Duct Perturbation Relations

The perturbation relations of mass flow rate, Mach number, total pressure, and total temperature are written in terms of the four fluid quantities of static pressure perturbation \( \delta P \), density perturbation \( \delta \rho \), axial velocity perturbation \( \delta V_x \), and tangential velocity...
perturbation ($\delta V_\theta$) which completely characterize the flow field in the ducts. The mass flow rate perturbation relation in terms of the duct flow quantities (equation A.2 in reference [38]) is:

\[
\dot{m} = \rho A V_x \\
\delta \dot{m} = \delta \rho A V_x + \rho A \delta V_x \\
\frac{\delta \dot{m}}{\dot{m}} = \frac{\delta \rho}{\rho} + M^{-1}_x \frac{\delta V_x}{a} \quad (D.8)
\]

The absolute Mach number perturbation relation in terms of duct flow quantities (equation A.6 in reference [38]) is:

\[
M = \frac{V}{a} = \sqrt{(V_x^2 + V_\theta^2)} \frac{\rho}{\gamma P} \\
\delta M = \frac{1}{2} M \frac{\delta \rho}{\rho} - \frac{1}{2} M \frac{\delta P}{P} + M_x M^{-1}_x \frac{\delta V_x}{a} + M_\theta M^{-1}_\theta \frac{\delta V_\theta}{a} \quad (D.9)
\]

The relative Mach number perturbation relation in terms of duct flow quantities is:

\[
M' = \frac{W}{a} = \sqrt{(W_x^2 + W_\theta^2)} \frac{\rho}{\gamma P} \\
\delta M' = \frac{1}{2} M' \frac{\delta \rho}{\rho} - \frac{1}{2} M' \frac{\delta P}{P} + M_x M'^{-1}_x \frac{\delta V_x}{a} + M_\theta M'^{-1}_\theta \frac{\delta V_\theta}{a} \quad (D.10)
\]

where the velocity perturbation relations in equations D.1 and D.2 have been used to simplify the equation. The absolute total pressure perturbation relation in terms of duct flow quantities, obtained by substituting equation D.9 into equation D.4, (equation A.10 in reference [38]) is:

\[
\frac{\delta P_t}{P_t} = \frac{1}{1 + \frac{\gamma - 1}{2} M^2} \left[ (1 - \frac{1}{2} M^2) \frac{\delta P}{P} + \frac{1}{2} \gamma M^2 \frac{\delta \rho}{\rho} + \gamma M_x \frac{\delta V_x}{a} + \gamma M_\theta \frac{\delta V_\theta}{a} \right] \quad (D.11)
\]

The relative total pressure perturbation relation in terms of duct flow quantities, obtained by substituting equation D.10 into equation D.5, is:

\[
\frac{\delta P'_t}{P'_t} = \frac{1}{1 + \frac{\gamma - 1}{2} M'^2} \left[ (1 - \frac{1}{2} M'^2) \frac{\delta P}{P} + \frac{1}{2} \gamma M'^2 \frac{\delta \rho}{\rho} + \gamma M_x \frac{\delta V_x}{a} + \gamma M_\theta \frac{\delta V_\theta}{a} \right] \quad (D.12)
\]
The absolute total temperature perturbation relation in terms of duct flow quantities, obtained by substituting equation D.9 into equation D.6, (equation A.7 in reference [38]) is:

\[
\frac{\delta T}{T} = \frac{1}{1 + \gamma - 1/2} \left( \frac{\delta P}{P} \frac{\delta \rho}{\rho} + (\gamma - 1) M_x \frac{\delta V_x}{a} + (\gamma - 1) M \frac{\delta V_y}{a} \right) \tag{D.13}
\]

The relative total temperature perturbation relation in terms of duct flow quantities, obtained by substituting equation D.10 into equation D.7, is:

\[
\frac{\delta T_1}{T_1} = \frac{1}{1 + \gamma - 1/2} \left( \frac{\delta P}{P} \frac{\delta \rho}{\rho} + (\gamma - 1) M_x \frac{\delta V_x}{a} + (\gamma - 1) M \frac{\delta V_y}{a} \right) \tag{D.14}
\]

D.1.2 Blade Row Perturbation Relations

The perturbation relations of mass flow rate, Mach number, total pressure, and total temperature are written in terms of the three fluid quantities of static pressure perturbation (\(\delta P\)), density perturbation (\(\delta \rho\)), and relative velocity perturbation (\(\delta W\)) which completely characterize the flow field in the blade rows. The mass flow rate perturbation relation in terms of the blade row flow quantities (equation A.14 in reference [38] with the proper notation used for relative flow properties) is:

\[
\dot{m} = \rho AW \\
\delta \dot{m} = \delta \rho AW + \rho A \delta W \\
\frac{\delta \dot{m}}{\dot{m}} = \frac{\delta \rho}{\rho} + M' \frac{\delta W}{\bar{a}} \tag{D.15}
\]

The relative Mach number perturbation relation in terms of the blade row flow quantities (equation A.15 in reference [38] with the proper notation used for relative flow properties) is:

\[
M' = \frac{W}{a} = W \sqrt{\frac{\rho}{\gamma P}} \\
\delta M' = \frac{1}{2} M' \frac{\delta \rho}{\rho} + \frac{1}{2} M' \frac{\delta P}{P} + \frac{\delta W}{\bar{a}} \tag{D.16}
\]

The relative total pressure perturbation relation in terms of the blade row flow quantities, obtained by substituting equation D.16 into equation D.5, (equation A.17 in reference [38])
with the proper notation used for relative flow properties) is:

\[
\frac{\delta P'}{P'} = \frac{1}{1 + \frac{1}{2} M'^2} \left[ \left( 1 - \frac{1}{2} M'^2 \right) \frac{\delta P}{P} + \frac{1}{2} \gamma M'^2 \frac{\delta \rho}{\bar{\rho}} + \gamma M' \frac{\delta W}{\bar{a}} \right] \tag{D.17}
\]

The relative total temperature perturbation relation in terms of the blade row flow quantities, obtained by substituting equation D.16 into equation D.7, (equation A.16 in reference [38] with the proper notation used for relative flow properties) is:

\[
\frac{\delta T'_t}{T'_t} = \frac{1}{1 + \frac{1}{2} M'^2} \left[ \frac{\delta P}{P} - \frac{\delta \rho}{\bar{\rho}} + (\gamma - 1) M' \frac{\delta W}{\bar{a}} \right] \tag{D.18}
\]

### D.2 Jet Actuator Boundary Condition

Air injection is modeled as having two main effects: changing the flow profiles entering the compressor, and changing the compressor blade row performance characteristics. The flow changes in the upstream duct associated with air injection are modeled by the jet actuator boundary condition, and the effects on the blade row performance characteristics are incorporated in the blade row boundary conditions. We redefine the control term as the perturbation from the mean injected mass flow rate normalized by a reference mass flow rate, \( \bar{m}_{\text{ref}} = \rho_o A_o U_o \):\n
\[
u(s) = \frac{\delta \bar{m}_j}{\rho_o A_o U_o} \tag{D.19}
\]

This new control term is used because it is easily related to measurements of mass flow rates that are readily available for the jet actuators used in this research. The jet actuator boundary condition models the flow changes in the duct across the jet actuator as shown in Figure D-2 and thus provides a relation between the harmonic disturbance coefficients in the ducts upstream and downstream of the actuator.

![Figure D-2: Jet actuator boundary condition.](image)
Changes in the flow coming into the compressor due to air injection are modeled by an actuator disk with mass, momentum, and swirl effects. Thus there will be changes in axial velocity (mass effect), tangential velocity (swirl effect), and total pressure (momentum effect resulting from change in dynamic head) across the jet actuator disk. The changes in compressor inlet velocity and incidence (or swirl) for different configurations of air injection are shown by the velocity triangles in Figure D-3. Since the mean flow properties change across the actuator, the jet actuator separates the original duct into two ducts with different mean flow properties at the location of the actuator. Changes in duct flow variables across the jet actuator disk are quantified by applying the continuity, axial momentum, tangential momentum and energy conservation equations. These are the same boundary conditions across the jet actuator used in references [38, 130] with some modifications.

**Mass Conservation:** The mass conservation or continuity equation across the jet actuator in Figure D-2 is:

\[
\frac{\delta \dot{m}_2}{\dot{m}_{\text{ref}}} = \frac{\delta \dot{m}_1}{\dot{m}_{\text{ref}}} + \dot{u}(s) \tag{D.20}
\]

and the corresponding perturbation equation in terms of the duct flow quantities, obtained by using equation D.8 to simplify equation D.20\(^1\), is:

\[
\frac{\rho_2 a_2}{\rho_0 U_0} \left[ M_{x_2} \frac{\delta \rho_2}{\rho_2} + \frac{\delta V_{x_2}}{a_2} \right] = \frac{\rho_1 a_1}{\rho_0 U_0} \left[ M_{x_1} \frac{\delta \rho_1}{\rho_1} + \frac{\delta V_{x_1}}{a_1} \right] + \dot{u}(s) \tag{D.21}
\]

**Axial Momentum Conservation:** The axial momentum conservation equation across the jet actuator in Figure D-2 is:

\[
P_2 A + \rho_2 V_{x_2}^2 A = P_1 A + \rho_1 V_{x_1}^2 A + \rho_j V_j^2 \cos \theta_j A_j \tag{D.22}
\]

and the corresponding perturbation equation is:

\[
\frac{P_2}{\rho_0 U_0^2} \frac{\delta P_2}{P_2} + \frac{\rho_2 V_{x_2}^2}{\rho_0 U_0^2} \frac{\delta \rho_2}{\rho_2} + 2 \frac{\rho_2 V_{x_2} a_2}{\rho_0 U_0^2} \frac{\delta V_{x_2}}{a_2} = \frac{P_1}{\rho_0 U_0^2} \frac{\delta P_1}{P_1} + \frac{\rho_1 V_{x_1}^2}{\rho_0 U_0^2} \frac{\delta \rho_1}{\rho_1} + 2 \frac{\rho_1 V_{x_1} a_1}{\rho_0 U_0^2} \frac{\delta V_{x_1}}{a_1} + 2 \frac{V_j \cos \theta_j}{U_0} \dot{u}(s) \tag{D.23}
\]

\(^1\)Note that for this application \(A_1 = A_2 = A_o\).
This equation is the modification of equation A.36 in reference [38].

**Tangential Momentum Conservation:** The tangential momentum conservation equation across the jet actuator in Figure D-2 is:

\[
\rho_2 V_\theta V_{xz} A = \rho_1 V_\theta V_{x1} A + \rho_j V_j^2 \sin \theta_j A_j
\]  
(D.24)

and the corresponding perturbation equation is:

\[
\frac{\rho_2 V_\theta V_{xz} \delta \rho_2}{\rho_0 U_0^2} + \frac{\rho_2 V_\theta a_2 \delta V_{\theta 2}}{\bar{a}_2} + \frac{\rho_2 V_{x1} a_2 \delta V_{x2}}{\bar{a}_2} = \\
\frac{\rho_1 V_\theta V_{x1} \delta \rho_1}{\rho_0 U_0^2} + \frac{\rho_1 V_{x1} a_1 \delta V_{\theta 1}}{\bar{a}_1} + \frac{\rho_1 V_{x1} a_1 \delta V_{x1}}{\bar{a}_1} + \frac{2 V_j \sin \theta_j u(s)}{U_0}
\]  
(D.25)

This equation is the modification of equation A.37 in reference [38].

**Energy Conservation:** The energy conservation across the jet actuator in Figure D-2 is:

\[
m_2 C_p T_{t2} = m_1 C_p T_{t1} + m_j C_p T_{tj}
\]  
(D.26)

and the corresponding perturbation equation is:

\[
\frac{\rho_2 V_{x2} T_{x2}}{\rho_0 U_0 T_{t10}} \left[ \frac{\delta P_2}{\bar{P}_2} + \frac{\gamma - 1}{2} M_x^2 \frac{\delta \rho_2}{\bar{\rho}_2} + M_{x1}^2 M_{t2} \frac{\delta V_{x2}}{\bar{a}_2} \right] = \\
\frac{\rho_1 V_{x1} T_{x1}}{\rho_0 U_0 T_{t10}} \left[ \frac{\delta P_1}{\bar{P}_1} + \frac{\gamma - 1}{2} M_x^2 \frac{\delta \rho_1}{\bar{\rho}_1} + M_{x1}^2 M_{t1} \frac{\delta V_{x1}}{\bar{a}_1} \right] + \frac{T_{tj}}{T_{t10}} u(s)
\]  
(D.27)

This equation is the modification of equation A.38 in reference [38]. The corresponding perturbation equation for the energy conservation boundary condition in terms of the duct flow quantities, obtained by substituting the perturbation equation for \( \frac{\delta T_{t1}}{T_{t1}} \) from equation D.13 into equation D.27, is:

\[
\frac{\rho_2 V_{x2} T_{x2}}{1 + \frac{\gamma - 1}{2} M_x^2} \left[ \frac{\delta P_2}{\bar{P}_2} + \frac{\gamma - 1}{2} M_x^2 \frac{\delta \rho_2}{\bar{\rho}_2} + M_{x1}^2 M_{t2} \frac{\delta V_{x2}}{\bar{a}_2} \right] = \\
\frac{\rho_1 V_{x1} T_{x1}}{1 + \frac{\gamma - 1}{2} M_x^2} \left[ \frac{\delta P_1}{\bar{P}_1} + \frac{\gamma - 1}{2} M_x^2 \frac{\delta \rho_1}{\bar{\rho}_1} + M_{x1}^2 M_{t1} \frac{\delta V_{x1}}{\bar{a}_1} \right] + \frac{T_{tj}}{T_{t10}} u(s)
\]  
(D.28)
where $\mathcal{M}_i = 1 + \frac{1}{2}(\gamma - 1)(3M_{x_i}^2 + M_{\theta_i}^2)$ with $i = 1, 2$. Combining the perturbation equations for mass conservation (equation D.21), axial momentum conservation (equation D.23), tangential momentum conservation (equation D.25), and energy conservation (equation D.28) into a matrix will give:

$$
\begin{bmatrix}
\frac{\delta P}{\rho} \\
\frac{\delta \rho}{\rho} \\
\frac{\delta V_x}{a} \\
\frac{\delta V_y}{a}
\end{bmatrix}_2 = \begin{bmatrix}
\frac{\delta P}{\rho} \\
\frac{\delta \rho}{\rho} \\
\frac{\delta V_x}{a} \\
\frac{\delta V_y}{a}
\end{bmatrix}_1 + \mathbf{b} \mathbf{u}(s)
$$

(D.29)

where the indices "ka,upstream" and "ka,downstream" in reference [38] have been replaced by "1" and "2" respectively, and the matrix $\mathbf{J}_i$ with $i = 1, 2$ is defined as:

$$
\begin{bmatrix}
\frac{\rho V_{x_i}}{\rho_0 U_o} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \frac{\rho_i V_{x_i} V_{c_1}}{\rho_0 U_o^2} & 0 \\
0 & 0 & 0 & \frac{\rho_i V_{x_i} T_{i_1}}{\rho_0 U_o T_o}
\end{bmatrix}
\begin{bmatrix}
0 & M_{x_i} & 1 & 0 \\
\frac{P_o}{\rho_0 U_o^2} & \frac{\rho_i V_{x_i}^2}{\rho_0 U_o^2} & 2\frac{\rho_i V_{c_1} G_i}{\rho_0 U_o} & 0 \\
0 & 1 & M_{x_i}^{-1} & M_{\theta_i}^{-1} \\
1 & \frac{x_i - 1}{2} M_i^2 & M_{x_i}^{-1} M_i & (\gamma - 1) M_{\theta_i}
\end{bmatrix}
$$

Equation D.29 is the modification of equation A.40 in reference [38]. The main difference is that in equation D.29, $\mathbf{J}_1 \neq \mathbf{J}_2$ whereas in equation A.40 of reference [38], $\mathbf{J}_1 = \mathbf{J}_2 = \mathbf{J}_{ka}$. $\mathbf{b}$ in equation D.29 is the modification of $\mathbf{b}_{ka}$ in equation A.40 of reference [38]. $\mathbf{J}_1 \neq \mathbf{J}_2$ because the modified actuator model incorporates the changes in mean flow across the jet actuator.

Using equation 2.25 in reference [38] to simplify the modified actuator boundary condition in equation D.29, the relation between the perturbation coefficients in the ducts upstream
and downstream of the jet actuator for the \( n^{th} \) harmonic is:

\[
\begin{bmatrix}
B \\
C \\
D \\
E
\end{bmatrix}_2 = A_j(s) \begin{bmatrix}
B \\
C \\
D \\
E
\end{bmatrix}_1 + b_j(s)\tilde{u}(s)
\]

where the control term, \( \tilde{u}(s) \), is the corresponding spatial harmonic of the unsteady injection (injected mass flow normalized by a reference mass flow, \( \dot{m}_{ref} = \rho_oA_oU_o \), \( A_j(s) = V_2^{-1}(x_a, s)J_2^{-1}J_1V_1(x_a, s) \) and \( b_j(s) = V_2^{-1}(x_a, s)J_2^{-1}b \) are frequency dependent matrices containing mean flow parameters of the ducts upstream and downstream of the actuator.

Figure D-3: Velocity triangles at the compressor inlet for different air injection configurations.
D.3 Boundary Conditions for Variable Area Duct

To model the change in flowpath cross-sectional area, the compressor flowpath can be discretized and the disturbance between adjacent discrete elements with different cross-sectional areas are related using a sudden expansion or contraction boundary shown in Figure D-4. The boundary conditions for a sudden expanding or contracting duct are: mass conservation, total temperature conservation, total pressure conservation, and angular momentum conservation.

Mass Conservation or Continuity: The continuity equation and its corresponding linearized boundary condition relating the mass flow rates in the ducts upstream and downstream of the junction in Figure D-4 is:

\[
\frac{\delta \dot{m}_1}{\dot{m}_1} = \frac{\delta \dot{m}_2}{\dot{m}_2}
\]  
(D.31)

where \( \delta \dot{m}_1 \) and \( \delta \dot{m}_2 \) are the mass flow rate perturbations at the common junction of the upstream and downstream ducts given by equation D.8.

Total Temperature Conservation: The conservation of absolute total temperature in the ducts upstream and downstream of the junction in Figure D-4 and the corresponding
linearized boundary condition is:

\[
\begin{align*}
    T_{t_1} &= T_{t_2} \\
    \frac{\delta T_{t_1}}{T_{t_1}} &= \frac{\delta T_{t_2}}{T_{t_2}}
\end{align*}
\] (D.32)

where \( \delta T_{t_1} \) and \( \delta T_{t_2} \) are the total temperature perturbations at the common junction of the upstream and downstream ducts given by equation D.13.

**Total Pressure Conservation:** The conservation of absolute total pressure in the ducts upstream and downstream of the junction in Figure D-4 and the corresponding linearized boundary condition is:

\[
\begin{align*}
    P_{t_1} &= P_{t_2} \\
    \frac{\delta P_{t_1}}{P_{t_1}} &= \frac{\delta P_{t_2}}{P_{t_2}}
\end{align*}
\] (D.33)

where \( \delta P_{t_1} \) and \( \delta P_{t_2} \) are the total pressure perturbations at the common junction of the upstream and downstream ducts given by equation D.11.

**Angular Momentum Conservation:** The conservation of angular momentum in the ducts upstream and downstream of the junction in Figure D-4 and its corresponding linearized boundary condition is:

\[
\begin{align*}
    r_1 V_{\theta_1} &= r_2 V_{\theta_2} \\
    r_1 \delta V_{\theta_1} &= r_2 \delta V_{\theta_2}
\end{align*}
\] (D.34)

Applying the perturbation relations from equations D.8, D.13, and D.11 to the common junction for the upstream and downstream ducts in Figure D-4:
where the matrix $H_i$ with $i = 1, 2$ is defined as:

$$H_i = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{1}{1 + \frac{1}{2} M_i^2} & 0 & 0 \\
0 & 0 & \frac{1}{1 + \frac{1}{2} M_i^2} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
0 & 1 & M_{x_i}^{-1} & 0 \\
1 & -1 & (\gamma - 1) M_{x_i} & (\gamma - 1) M_{\theta_i} \\
(1 - \frac{1}{2} M_i^2) & \frac{1}{2} \gamma M_i^2 & \gamma M_{x_i} & \gamma M_{\theta_i} \\
0 & 0 & 0 & r_i a_i
\end{bmatrix}$$

Combining the boundary conditions in equations D.31, D.32, D.33, and D.34 into a matrix form and using equation D.35 to simplify the resulting matrix equation:

$$H_2 \begin{bmatrix}
\frac{\delta P}{\bar{\rho}} \\
\frac{\delta P}{\bar{\rho}} \\
\frac{\delta V_x}{\bar{a}} \\
\frac{\delta V_x}{\bar{a}}
\end{bmatrix}_2 = H_1 \begin{bmatrix}
\frac{\delta P}{\bar{\rho}} \\
\frac{\delta P}{\bar{\rho}} \\
\frac{\delta V_x}{\bar{a}} \\
\frac{\delta V_x}{\bar{a}}
\end{bmatrix}_1 \quad (D.36)$$

Using equation 2.25 in reference [38] to simplify equation D.36 will give the following boundary condition for the $n^{th}$ harmonic:

$$\begin{bmatrix}
B \\
C \\
D \\
E
\end{bmatrix}_2 = A_D(s) \begin{bmatrix}
B \\
C \\
D \\
E
\end{bmatrix}_1 \quad (D.37)$$

where $A_D(s) = V_2^{-1}(x_d, s) H_2^{-1} H_1 V_1(x_d, s)$.

### D.4 Blade Row Boundary Conditions

Figure D-5 shows the schematic of a single blade row and its adjacent ducts used for deriving the leading edge and trailing edge boundary conditions. The leading edge boundary conditions are derived in Section D.4.1, the trailing edge boundary conditions are derived in Section D.4.2, and the blade row transmission matrix is derived in Section D.4.3.
D.4.1 Blade Row Leading Edge Boundary Condition
The boundary conditions applied at the blade row leading edge are: mass conservation, rothalpy conservation, and relative total pressure conservation.

Mass Conservation or Continuity: The continuity equation equates the mass flow rates at the locations upstream (duct) and downstream (blade row) of the leading edge. This boundary condition is the same as in reference [38]. Applying the continuity equation to the leading edge of Figure D-5:

\[
\frac{\dot{m}_1}{\dot{m}_2} = \frac{\delta\dot{m}_1}{\delta\dot{m}_2}
\]  

(D.38)

where \(\dot{m}_1\) is the mass flow rate at a point in the duct immediately upstream of the leading edge given by equation D.8 and \(\dot{m}_2\) is the mass flow rate at a point in the blade row immediately downstream of the leading edge given by equation D.15.

Rothalpy Conservation: The conservation of rothalpy, \(I_t = h'_t - \frac{1}{2}\Omega^2 r^2 = C_p T'_t - \frac{1}{2}\Omega^2 r^2\), is equivalent to the conservation of relative total temperature. This boundary condition is the same as in reference [38]. Applying the conservation of relative total temperature to the leading edge of Figure D-5:

\[
\delta T'_{t_1} = \delta T'_{t_2}
\]
where $T'_{t_1}$ is the relative total temperature at a point in the duct immediately upstream of the leading edge given by equation D.14 and $T'_{t_2}$ is the relative total temperature at a point in the blade row immediately downstream of the leading edge given by equation D.18.

**Relative Total Pressure Conservation:** In references [13, 38], all of the blade row total pressure loss is modeled at the leading edge. Here, this assumption is refined to be consistent with the modeling assumptions made in the steady air injection model described in Section 3.1, where all the total pressure loss is modeled at the blade row trailing edge. Since all the total pressure loss is modeled at the trailing edge, the relative total pressure is conserved at the leading edge. The conservation of relative total pressure at the leading edge of Figure D-5 is:

$$\delta P'_{t_1} = \delta P'_{t_2}$$

where $P'_{t_1}$ is the relative total pressure at a point in the duct immediately upstream of the leading edge given by equation D.12 and $P'_{t_2}$ is the relative total pressure at a point in the blade row immediately downstream of the leading edge given by equation D.17.

Applying the perturbation relations from equations D.8, D.14, and D.12 to the duct perturbations in equations D.38, D.39, and D.39 respectively:

$$\begin{bmatrix} \frac{\delta m_k}{m_1} \\ \delta T'_{t_1} \\ \delta P'_{t_1} \end{bmatrix} = \mathbf{V}_{Lk} \begin{bmatrix} \frac{\delta P}{P} \\ \frac{\delta \rho}{\rho} \\ \frac{\delta V_x}{a} \\ \frac{\delta V_a}{a} \end{bmatrix} \quad \text{(D.39)}$$

where the index "1" has been replaced by the index "k" for the $k^{th}$ duct to be consistent with the indices in reference [38], and $\mathbf{V}_{Lk}$ is given by:

$$\mathbf{V}_{Lk} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{T'_{t_1}}{1 + \frac{\gamma - 1}{2} M_1^2} & 0 & 0 \\ 0 & 0 & \frac{P'_{t_1}}{1 + \frac{\gamma - 1}{2} M_1^2} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & M_{z_1}^{-1} & 0 \\ 1 & -1 & (\gamma - 1) M_{z_1} & (\gamma - 1) M_{\theta_1}' \\ (1 - \frac{1}{2} M_1^2) & \frac{1}{2} \gamma M_1^2 & \gamma M_1^2 & \gamma M_{\theta_1}'^2 \end{bmatrix}$$

This expression for $\mathbf{V}_{Lk}$ is the same with that in equation A.13 of reference [38] except that
the appropriate notation for relative flow quantities have been included.

Applying the perturbation relations from equations D.15, D.18, and D.17 to the blade row perturbations in equations D.38, D.39, and D.39 respectively:

\[
\begin{bmatrix}
\frac{\delta h_2}{m_2} \\
\delta T'_{t_2} \\
\delta P'_{t_2}
\end{bmatrix}
= 
\mathbf{B}_{Lk}
\begin{bmatrix}
\frac{\delta P}{P} \\
\frac{\delta \rho}{\rho} \\
\frac{\delta W}{a}
\end{bmatrix}_k
\]  

(D.40)

where the index “2” has been replaced by the index “k” for the \(k^{th}\) blade row to be consistent with the indices in reference [38], and \(\mathbf{B}_{Lk}\) is given by:

\[
\mathbf{B}_{Lk} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & \frac{T'_{t_2}}{1 + \frac{3}{2} \gamma M_2} & 0 \\
0 & 0 & \frac{P'_{t_2}}{1 + \frac{3}{2} \gamma M_2}
\end{bmatrix}
\begin{bmatrix}
0 & 1 & M_2^{-1} \\
1 & -1 & (\gamma - 1)M_2 \\
(1 - \frac{1}{2} M_2^2) & \frac{1}{2} \gamma M_2^2 & \gamma M_2
\end{bmatrix}
\]

This expression for \(\mathbf{B}_{Lk}\) is the same with that in equation A.18 of reference [38] except that the appropriate notation for relative flow quantities have been included.

Combining the boundary conditions from equations D.38, D.39, and D.39 in a matrix form i.e., equating D.39 to D.40, will give the following relation between the \(k^{th}\) duct and the \(k^{th}\) blade row:

\[
\mathbf{V}_{Lk}
\begin{bmatrix}
\frac{\delta P}{P} \\
\frac{\delta \rho}{\rho} \\
\frac{\delta V_L}{a} \\
\frac{\delta V_k}{a}
\end{bmatrix}_k
= 
\mathbf{B}_{Lk}
\begin{bmatrix}
\frac{\delta P}{P} \\
\frac{\delta \rho}{\rho} \\
\frac{\delta V_L}{a} \\
\frac{\delta W}{a}
\end{bmatrix}_k
\]  

(D.41)

Using equations 2.25 and 2.33 in reference [38] to simplify equation D.41 will give the following \(n^{th}\) harmonic leading edge boundary condition:

\[
\mathbf{V}_{Lk} \mathbf{V}_k(x_{L Ek}, s)
= 
\mathbf{B}_{Lk} \mathbf{B}_k(x_{L Ek}, s)
\]  

(D.42)
This is a modification of equation A.19 (or 2.34) in reference [38]. The main difference is that there is no first order lag term in equation D.42 because the total pressure loss has been moved from the leading edge to the trailing edge.

D.4.2 Blade Row Trailing Edge Boundary Condition
The boundary conditions applied at the blade row trailing edge are: mass conservation, rothalpy conservation, relative total pressure loss, and flow turning according to the trailing edge deviation.

Mass Conservation or Continuity: The continuity equation equates the mass flow rates at the locations upstream (blade row) and downstream (duct) of the trailing edge. This boundary condition is the same as in reference [38]. Using the notation in Figure D-5, the mass conservation boundary condition at the trailing edge is:

\[ \frac{\dot{m}_2}{\dot{m}_2} = \frac{\dot{m}_3}{\dot{m}_3} \]  

where \( \dot{m}_2 \) is the mass flow rate at a point in the blade row immediately upstream of the trailing edge given by equation D.15 and \( \dot{m}_3 \) is the mass flow rate at a point in the duct immediately downstream of the trailing edge given by equation D.8.

Rothalpy Conservation: This boundary condition is the same as in reference [38]. The conservation of rothalpy (and equivalently relative total temperature conservation) at the trailing edge of Figure D-5:

\[ \delta T'_t = \delta T'_t \]

where \( T'_t \) is the relative total temperature at a point in the blade row immediately upstream of the trailing edge given by equation D.18 and \( T'_t \) is the relative total temperature at a point in the duct immediately downstream of the leading edge given by equation D.14.
Relative Total Pressure Loss: In references [13, 38], the blade row total pressure loss is modeled at the leading edge. Here, the total pressure loss is modeled at the blade row trailing edge. The effect of air injection on the blade row loss performance characteristic is incorporated by including an actuation dependence in the relative total pressure loss coefficient defined by equation 3.18. The expression for the relative total pressure loss coefficient, $\omega_{loss}$, in equation 3.18 can be rearranged as follows:

$$P'_{tTE} = P'_{tTEisen} - \omega_{loss}(\beta_{LE}, M_{LE}, U_j) (P'_{tLE} - P_{LE})$$  \hspace{1cm} (D.44)$$

where $P'_{tTE}$ is the relative total pressure at the trailing edge, $P'_{tTEisen}$ is the isentropic relative total pressure at the blade row trailing edge, $P'_{tLE}$ is the relative total pressure at the leading edge, and $P_{LE}$ is the static pressure at the leading edge. Unsteady changes in the relative total pressure loss is modeled using a first order lag with time constant $\tau_p$. The corresponding expression for the unsteady response of the relative total pressure perturbation at the blade row trailing edge is:

$$\delta P'_{tTE} = \delta P'_{tTEisen} - \frac{1}{1 + s\tau_p} \left\{ \delta P'_{tLE} - \delta P_{LE} \right\} \omega_{loss}(\beta_{LE}, M_{LE}, U_j)$$  \hspace{1cm} (D.45)$$

$$+ \left( P'_{tLE} - P_{LE} \right) \left\{ \frac{\partial \omega_{loss}}{\partial \tan \beta_{LE}} \delta \tan \beta_{LE} + \frac{\partial \omega_{loss}}{\partial M'_{LE}} \delta M'_{LE} + \frac{\partial \omega_{loss}}{\partial U_j} \delta U_j \right\}$$

where the isentropic relative total pressure perturbation at the blade row trailing edge is:

$$\delta P'_{tTEisen} = \left[ 1 + \frac{\Omega^2}{2C_p} (r'_{TE} - r'_{LE}) \right] \frac{\tau_p}{\gamma - 1} \delta P'_{tLE}$$  \hspace{1cm} (D.46)$$

$$+ \left( \frac{r_{LE}}{\gamma - 1} \right) \left[ 1 + \frac{\Omega^2}{2C_p} (r'_{TE} - r'_{LE}) \right] \left[ 1 + \frac{\Omega^2}{2C_p} (r'_{TE} - r'_{LE}) \right] \frac{1}{\gamma - 1} \delta T'_{tLE}$$

Note that $\delta P'_{tTEisen} = \delta P'_{tLE}$ if $r_{LE} = r_{TE}$. Applying the indices in Figure D-5 to equations D.45 and D.46, using equation D.3 to simplify $\delta \beta_{LE}$, using equation D.10 to simplify $\delta M'_{LE}$.

---

\(^2\)Using equations C.56 and C.57, it can be shown that $P'_{tTEisen} = P'_{tLE} \left[ 1 + \frac{\Omega^2}{2C_p} (r'_{TE} - r'_{LE}) T'_{tLE} \right] \frac{\tau_p}{\gamma - 1}$. 

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using equation D.12 to simplify $\delta P_{\text{LE}}'$, and using equation D.14 to simplify $\delta T_{\text{LE}}'$:

$$
\delta P_3' = \left[ 1 + \frac{\Omega^2}{2C_p} (r_3^2 - r_1^2) \right]^{\gamma-1} \left[ \frac{\gamma P_{t'_1}}{\gamma - 1} \frac{\Omega^2 (r_3^2 - r_1^2)}{2C_p} \left[ 1 + \frac{\Omega^2}{2C_p} (r_3^2 - r_1^2) \right]^{\gamma-1} \delta T_{t'_1} 
- \frac{1}{1 + \sigma_{t'_1}} \left\{ (P_{t'_1} - P_1) \frac{\partial \omega_{\text{loss}}}{\partial \tan \beta_1} + (P_{t'_1} - P_1) \frac{\partial \omega_{\text{loss}}}{\partial M_1'} \frac{\partial \omega_{\text{loss}}}{\partial U_j} \right\} \right]^{\gamma - 1} 
= \left( [r_{3,1} r_{3,2} r_{3,3} r_{3,4}] + \frac{1}{1 + \sigma_{t'_1}} [p_{3,1} p_{3,2} p_{3,3} p_{3,4}] \right) \left[ \begin{bmatrix} \frac{\delta P}{P} \\ \frac{\delta \rho}{\rho} \\ \frac{\delta V_a}{a} \\ \frac{\delta V_0}{a} \end{bmatrix} + \frac{1}{1 + \sigma_{t'_1}} \frac{\partial \omega_{\text{loss}}}{\partial U_j} \right]^{\gamma - 1} (D.47)
$$

where $r_{3,i}$ and $p_{3,i}$ with $i = 1, 2, 3, 4$ are defined as:

$$
r_{3,1} = \left[ 1 + \frac{\Omega^2}{2C_p} (r_3^2 - r_1^2) \right]^{\gamma-1} \left[ \frac{\gamma M_{x_1} P_{t'_1}}{\gamma - 1} \frac{\Omega^2 (r_3^2 - r_1^2)}{2C_p} \left[ 1 + \frac{\Omega^2}{2C_p} (r_3^2 - r_1^2) \right]^{\gamma-1} \right]^{\gamma - 1}
$$

$$
r_{3,2} = \left[ 1 + \frac{\Omega^2}{2C_p} (r_3^2 - r_1^2) \right]^{\gamma-1} \frac{\gamma M_{x_1} P_{t'_1}}{\gamma - 1} \frac{\Omega^2 (r_3^2 - r_1^2)}{2C_p} \left[ 1 + \frac{\Omega^2}{2C_p} (r_3^2 - r_1^2) \right]^{\gamma - 1}
$$

$$
r_{3,3} = \left[ 1 + \frac{\Omega^2}{2C_p} (r_3^2 - r_1^2) \right]^{\gamma-1} \frac{\gamma M_{x_1} P_{t'_1}}{\gamma - 1} \frac{\Omega^2 (r_3^2 - r_1^2)}{2C_p} \left[ 1 + \frac{\Omega^2}{2C_p} (r_3^2 - r_1^2) \right]^{\gamma - 1}
$$

$$
r_{3,4} = \left[ 1 + \frac{\Omega^2}{2C_p} (r_3^2 - r_1^2) \right]^{\gamma-1} \frac{\gamma M_{x_1} P_{t'_1}}{\gamma - 1} \frac{\Omega^2 (r_3^2 - r_1^2)}{2C_p} \left[ 1 + \frac{\Omega^2}{2C_p} (r_3^2 - r_1^2) \right]^{\gamma - 1}
$$

$$
p_{3,1} = \left[ \left( \frac{P_{t'_1}}{1 + \frac{\gamma - 1}{2} M_{M_1'^2}} \right)^2 (1 - \frac{1}{2} M_{M_1'^2}) \right] \omega_{\text{loss}} - \frac{1}{2} M_{M_1'} \left( P_{t'_1} - P_1 \right) \frac{\partial \omega_{\text{loss}}}{\partial M_{M_1'}}
$$

$$
p_{3,2} = -\left[ \frac{1}{2} \gamma M_{M_1'^2} \frac{P_{t'_1}}{1 + \frac{\gamma - 1}{2} M_{M_1'^2}} \omega_{\text{loss}} + \frac{1}{2} M_{M_1'} \left( P_{t'_1} - P_1 \right) \frac{\partial \omega_{\text{loss}}}{\partial M_{M_1'}} \right]
$$

$$
p_{3,3} = -\left[ \frac{P_{t'_1}}{1 + \frac{\gamma - 1}{2} M_{M_1'^2}} \gamma M_{x_1} \omega_{\text{loss}} \right] \left( P_{t'_1} - P_1 \right) \left( -M_{x_1} \tan \beta_1 \frac{\partial \omega_{\text{loss}}}{\partial \tan \beta_1} + M_{x_1} M_{M_1'^2} \frac{\partial \omega_{\text{loss}}}{\partial M_{M_1'}} \right)
$$

$$
p_{3,4} = -\left[ \frac{P_{t'_1}}{1 + \frac{\gamma - 1}{2} M_{M_1'^2}} \gamma M_{x_1} \omega_{\text{loss}} \right] \left( P_{t'_1} - P_1 \right) \left( M_{x_1} \tan \beta_1 \frac{\partial \omega_{\text{loss}}}{\partial \tan \beta_1} + M_{x_1} M_{M_1'^2} \frac{\partial \omega_{\text{loss}}}{\partial M_{M_1'}} \right)
$$

The expressions for $p_{3,i}$ are the same as in reference [38] except that the appropriate notation for relative flow properties have been included.
Exit Flow Deviation: The effect of air injection on the blade row deviation performance characteristic is incorporated by including an actuation dependence in the relative exit flow angle. The modified relative exit flow angle is: \( \beta_{TE} = \beta_{TE}(\beta_{LE}, M'_{LE}, U_j) \). Unsteady changes in the relative exit flow angle is modeled using a first order lag with time constant \( \tau_d \). The corresponding expression for the unsteady response of the relative exit flow angle perturbation is:

\[
\delta \beta_{TE} = \frac{1}{1 + s \tau_d} \left[ \frac{\partial \beta_{TE}}{\partial \tan \beta_{LE}} \delta \tan \beta_{LE} + \frac{\partial \beta_{TE}}{\partial M'_{LE}} \delta M'_{LE} + \frac{\partial \beta_{TE}}{\partial U_j} \delta U_j \right]
\]  

(D.48)

Applying the indices in Figure D-5 to equation D.48, using equation D.3 to simplify \( \delta \beta_{LE} \), and using equation D.10 to simplify \( \delta M'_{LE} \):

\[
\delta \beta_3 = \frac{1}{1 + s \tau_d} \left[ \frac{\partial \beta_3}{\partial \tan \beta_1} \delta \tan \beta_1 + \frac{\partial \beta_3}{\partial M'_1} \delta M'_1 + \frac{\partial \beta_3}{\partial U_j} \delta U_j \right]
\]

\[
= \frac{1}{1 + s \tau_d} \left[ \begin{array}{c}
\frac{\delta p}{\rho} \\
\frac{\delta \rho}{\rho} \\
\frac{\delta V_x}{a} \\
\frac{\delta V_a}{a}
\end{array} \right] + \frac{1}{1 + s \tau_d} \frac{\partial \beta_3}{\partial U_j} \delta U_j
\]

(D.49)

where \( d_{4,1}, d_{4,2}, d_{4,3}, \) and \( d_{4,4} \) are defined as:

\[
d_{4,1} = -\frac{1}{2} M'_1 \frac{\partial \beta_3}{\partial M'_1}
\]

\[
d_{4,2} = \frac{1}{2} M'_1 \frac{\partial \beta_3}{\partial M'_1}
\]

\[
d_{4,3} = -\frac{M'_1}{M^2_{x1}} \frac{\partial \beta_3}{\partial \tan \beta_1} + \frac{M'_{x1}}{M'_1} \frac{\partial \beta_3}{\partial M'_1}
\]

\[
d_{4,4} = \frac{1}{M_{x1}} \frac{\partial \beta_3}{\partial \tan \beta_1} + \frac{M'_1}{M'_1} \frac{\partial \beta_3}{\partial M'_1}
\]

The expressions for \( d_{4,i} \) are the same as in reference [38] except that the appropriate notation for relative flow properties have been included.

Assuming the relative total pressure loss time constant, \( \tau_p \), and the deviation angle time constant, \( \tau_d \), are equal i.e., \( \tau = \tau_p = \tau_d \), applying the perturbation relations from equations D.15 and D.18 to the blade row perturbations in equations D.43 and D.44 respectively, and
combining equations D.47 and D.49:

\[
\begin{bmatrix}
\frac{\delta q_{2}}{m_{2}} \\
\delta T'_{2} \\
\delta P'_{3} \\
\delta \beta_{3}
\end{bmatrix} = B_{Tk} \begin{bmatrix}
\frac{\delta P}{P} \\
\frac{\delta \rho}{\rho} \\
\frac{\delta W}{a} \\
\frac{\delta \beta}{\alpha}
\end{bmatrix}_{k} + (P_{Lk} + \frac{1}{1 + sT} P_{Tk}) \begin{bmatrix}
\frac{\delta P}{P} \\
\frac{\delta \rho}{\rho} \\
\frac{\delta W}{a} \\
\frac{\delta \beta}{\alpha}
\end{bmatrix}_{k} + \frac{1}{1 + sT} b_{ku}(s) \text{ (D.50)}
\]

where the index "k" has been used to represent the \(k^{th}\) duct and \(k^{th}\) blade row in order to be consistent with the indices in reference [38]. \(B_{Tk}, P_{Lk}, P_{Tk}, \text{ and } b_{k}\) are given by:

\[
B_{Tk} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & T_{2} & \frac{1}{1 + \frac{s}{M_{2}^{2}}} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
P_{Lk} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
r_{3,1} & r_{3,2} & r_{3,3} & r_{3,4} \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
P_{Tk} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
p_{3,1} & p_{3,2} & p_{3,3} & p_{3,4} \\
d_{3,1} & d_{3,2} & d_{3,3} & d_{3,4}
\end{bmatrix}
\]

\[
b_{k} = \begin{bmatrix}
0 \\
0 \\
(P'_{t1} - P_{1}) \frac{\delta \rho}{\delta U_{j}} \\
\frac{\delta \beta}{\delta U_{j}}
\end{bmatrix}
\]

The third and fourth rows of \(B_{Tk}\) here are different from those in equation A.29 of reference [38]. The third row elements of \(P_{Tk}\) are the same as the third row of \(P_{k}\) in equation A.19 of reference [38], and the fourth row elements of \(P_{Tk}\) are the same as the fourth row of \(D_{k}\) in equation A.29 of reference [38].

Applying the perturbation relations from equations D.3, D.8, D.12 and D.14 to the duct
perturbations in equations D.43, D.44, D.47, and D.49 respectively:

\[
\begin{bmatrix}
\frac{\delta m_3}{m_3} \\
\delta T_t^3 \\
\delta P_t^3 \\
\delta \beta_3
\end{bmatrix}
= V_{Tk+1}
\begin{bmatrix}
\delta P \\
\delta \beta \\
\delta V_x \\
\delta W
\end{bmatrix}_{k+1}
\]  \hspace{1cm} (D.51)

where the index "3" has been replaced with "k+1" to represent the \(k + 1^{st}\) duct, and \(V_{Tk+1}\) is given by:

\[
V_{Tk+1} = 
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{T_t^3}{1 + \frac{1}{2} M_{t^3}^2} & 0 & 0 \\
0 & 0 & \frac{T_t^3}{1 + \frac{1}{2} M_{t^3}^2} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 1 & M_{x^{-3}} & 0 \\
1 & -1 & (\gamma - 1) M_{x_{3}} & (\gamma - 1) M_{\beta_{3}}' \\
0 & 0 & \frac{1}{2} \gamma M_{3}^2 & \gamma M_{x_{3}} \\
0 & 0 & 0 & -M_{\beta_{3}}' M_{3}^{-2}
\end{bmatrix}
\]  \hspace{1cm} (D.52)

This is a modification of \(V_{Tk+1}\) in equation A.28 of reference [38]. In addition to including the appropriate notation for relative flow quantities, the fourth row has been modified to include the expression for \(\frac{\delta \beta_3}{\delta \tan \beta_3}\) to account for the conversion of \(\delta \tan \beta_3\) to \(\delta \beta_3\).

Combining the boundary conditions from equations D.43, D.44, D.47, and D.49 in a matrix form i.e., equating D.50 to D.51, will give the following relation between the \(k^{th}\) duct, the \(k^{th}\) blade row, and the \(k + 1^{st}\) duct:

\[
V_{Tk+1}
\begin{bmatrix}
\delta P \\
\delta \beta \\
\delta V_x \\
\delta W
\end{bmatrix}_{k+1}
= B_{Tk}
\begin{bmatrix}
\delta P \\
\delta \beta \\
\delta V_x \\
\delta W
\end{bmatrix}_{k} + (P_{Lk} + \frac{1}{1 + s\tau} P_{Tk})
\begin{bmatrix}
\delta P \\
\delta \beta \\
\delta V_x \\
\delta W
\end{bmatrix}_{k}
+ \frac{1}{1 + s\tau} b_k u(s)
\]  \hspace{1cm} (D.52)

Using equations 2.25 and 2.33 in reference [38] to simplify equation D.52 will give the
following $n^{th}$ harmonic trailing edge boundary condition:

$$
\begin{bmatrix}
B \\
C \\
D \\
E
\end{bmatrix}_{k+1} = B_{Tk} B_k (x_{TEk}, s) \begin{bmatrix}
\tilde{B} \\
\tilde{C} \\
\tilde{D}
\end{bmatrix}_k + \left( P_{Lk} + \frac{1}{1 + s\tau} P_{Tk} \right) V_k (x_{LEk}, s) \begin{bmatrix}
B \\
C \\
D \\
E
\end{bmatrix}_k + \frac{1}{1 + s\tau} b_k \tilde{u}(s)
$$

(D.53)

This is a modification of equation A.30 (or 2.35) in reference [38]. The main difference is that the modified equation has two additional terms: the $P_{Lk}$ term multiplying the upstream duct disturbance coefficient vector due to the fact that the total pressure loss is moved from the leading edge to the trailing edge, and the actuation term due to the introduction of actuation dependence in the blade row performance characteristics.

D.4.3 Blade Row Transmission Matrix

The duct dynamics, leading edge boundary condition, trailing edge boundary condition, and blade row dynamics can be used to relate the disturbance coefficients in the $k^{th}$ and $k + 1^{th}$ ducts upstream and downstream of the $k^{th}$ blade row. The transmission matrix across the $k^{th}$ blade row is obtained by substituting equation D.42 into equation D.53 to eliminate the blade row disturbance coefficients:

$$
\begin{bmatrix}
B \\
C \\
D \\
E
\end{bmatrix}_{k+1} = \mathbf{A}_k(s) \begin{bmatrix}
B \\
C \\
D \\
E
\end{bmatrix}_k + \mathbf{b}_R(s) \tilde{u}(s)
$$

(D.54)

where $\mathbf{A}_k(s)$ and $\mathbf{b}_R(s)$ are defined as:

$\mathbf{A}_k(s) = V_{k+1}(x_{TEk}, s) V_{Tk}^{-1} B_{Tk} B_k (x_{TEk}, s) B_k^{-1} (x_{LEk}, s) B_k^{-1} L_k V_k (x_{LEk}, s)$

$\mathbf{b}_R(s) = \frac{1}{1 + s\tau} V_{k+1}(x_{TEk}, s) V_{Tk}^{-1} b_k$

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Equation D.54 is a modification of equation 2.36 in reference [38]. In addition to the difference in entries of $A_k$, the main difference between these two equations is the actuation component.

**D.5 End Conditions Boundary Condition**

The end conditions consist of three inlet boundary conditions and the exit boundary condition. These end conditions define the interaction of the compressor inlet and exit ducts with the rest of the compression system.

**D.5.1 Inlet Boundary Conditions**

In references [13, 38] the three inlet conditions are: zero total pressure, zero entropy, and zero vorticity perturbations. These conditions model an open, clean, and smooth flow. The inlet end conditions of zero entropic and vortical perturbations are not changed. The total pressure condition which acts as an impedance condition for the inlet duct potential waves is modified.

**Impedance Condition:** The inlet impedance condition is replaced with a generalized complex acoustic impedance relating the static pressure and velocity perturbations:

$$Z_{in}(\omega) = \frac{\delta P}{\delta V} = R_{in}(\omega) + jX_{in}(\omega)$$  \hspace{1cm} (D.55)

The corresponding perturbation equation for the modified inlet impedance condition is:

$$\begin{bmatrix} 1 & 0 & -Z_{in} \frac{a}{P} \frac{M_x}{M} & -Z_{in} \frac{a}{P} \frac{M_\theta}{M} \end{bmatrix} \begin{bmatrix} \frac{\delta P}{P} \\ \frac{\delta p}{\delta V} \\ \frac{\delta V_x}{\delta a} \\ \frac{\delta V_\theta}{\delta a} \end{bmatrix}_{in} = 0$$  \hspace{1cm} (D.56)

Applying equation 2.25 from reference [38] to the inlet impedance condition in equation
D.56:

\[
\begin{bmatrix}
N_{1,1} & N_{1,2} & N_{1,3} & N_{1,4}
\end{bmatrix}
\begin{bmatrix}
B \\
C \\
D \\
E
\end{bmatrix}
= 0
\]

(D.57)

where \(N_{1,i}\) with \(i = 1, 2, 3, 4\) are defined as:

\[
\begin{bmatrix}
N_{1,1} & N_{1,2} & N_{1,3} & N_{1,4}
\end{bmatrix} = \begin{bmatrix} 1 & 0 & -z_{in} \frac{a M_x}{P M} & -z_{in} \frac{a M_\theta}{P M} \end{bmatrix} V_1(x_{in}, s)
\]

Zero Vorticity: The irrotational (no vorticity) boundary condition in the inlet duct from reference [38] is unchanged. By linearizing the vorticity equation \(\Omega_{vort} = \frac{\partial v_x}{\partial \theta} - \frac{\partial v_\theta}{\partial x}\) and applying equation 2.25 from reference [38], the inlet vorticity boundary condition for the \(n^{th}\) harmonic (equation A.32 in reference [38]) is:

\[
D_n(s) = 0
\]

(D.58)

Zero Entropy: The no entropy boundary condition in the inlet duct from reference [38] is unchanged. By linearizing the equation for isentropic processes \(P \rho^{-\gamma} = \text{constant}\) and applying equation 2.25 from reference [38], the inlet entropy boundary condition for the \(n^{th}\) harmonic (equation A.31 in reference [38]) is:

\[
E_n(s) = 0
\]

(D.59)

Combining equations D.57, D.58, and D.59 in a matrix form:

\[
N(s)
\begin{bmatrix}
B \\
C \\
D \\
E
\end{bmatrix}
= \begin{bmatrix}
N_{1,1} & N_{1,2} & N_{1,3} & N_{1,4}
\end{bmatrix}
\begin{bmatrix}
B \\
C \\
D \\
E
\end{bmatrix}
= 0
\]

(D.60)

This is a modification of equation A.33 in reference [38]. The main difference between these two equations is that the first rows of \(N(s)\) are different.
D.5.2 Exit Boundary Condition

The exit condition model depends on what is downstream of the compressor. Feulner [38] assumed that the flow dumps into a plenum. The corresponding exit conditions consists of an open end condition, $\delta P = 0$, for the non-zeroth harmonic perturbations (non-axisymmetric rotating stall harmonics) applied at the interface of the exit duct and plenum, and the open end condition plus plenum dynamics for the zeroth harmonic perturbations. The exit condition is replaced with a general complex acoustic impedance end condition relating the static pressure and axial velocity perturbations:

$$Z_{ex}(\omega) = \frac{\delta P}{\delta V} = R_{ex}(\omega) + j X_{ex}(\omega)$$  \hspace{1cm} (D.61)

Such an impedance boundary condition provides a means for adequately modeling realistic exit conditions. The real component of the complex impedance (resistance) models the downstream throttle and the imaginary component of the impedance (reactance) models the capacitive (or compressibility) and inertial effects. The corresponding perturbation equation for the modified exit impedance condition is:

$$\begin{bmatrix} 1 & 0 & -\frac{a}{P} M_x & -\frac{a}{P} M_\theta \end{bmatrix} \begin{bmatrix} \frac{\delta P}{P} \\ \frac{\delta \rho}{\rho} \\ \frac{\delta V_x}{a} \\ \frac{\delta V_\theta}{a} \end{bmatrix}_{ex} = 0$$  \hspace{1cm} (D.62)

Applying equation 2.25 from reference [38] to the exit impedance condition in equation D.62:

$$X(s) = \begin{bmatrix} B \\ C \\ D \\ E \end{bmatrix}_N = \begin{bmatrix} X_{1,1} & X_{1,2} & X_{1,3} & X_{1,4} \end{bmatrix} \begin{bmatrix} B \\ C \\ D \\ E \end{bmatrix}_N = 0$$  \hspace{1cm} (D.63)

where $X_{1,i}$ with $i = 1, 2, 3, 4$ are defined as:

$$X(s) = \begin{bmatrix} X_{1,1} & X_{1,2} & X_{1,3} & X_{1,4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{a}{P} M_x & -\frac{a}{P} M_\theta \end{bmatrix} V_1(x_{ex}, s)$$
APPENDIX E

DESIGN OF ESTIMATORS

In this appendix, an overview of two sets of estimators is presented. These estimators are grouped based on their structures and the assumptions made in deriving them. The problem formulation used in estimating the incompressible dynamics from the rotating stall inception dynamics of high speed compressors is presented in Section E.1. In Section E.2, the first set of estimators are presented. The first set of estimators consist of the Kalman filter, which assumes perfect knowledge of the plant model and noise statistics, and the Minimax filter, which assumes perfect knowledge of the plant model but uncertainties in the noise model. In Section E.3, the second set of estimators are presented. The second set of estimators consist of the $\mu$-estimator described in [2] and the robust filter described in [83], which assume uncertainties in both the plant and noise models. For each group of estimators, the plant model is described, followed by the structure of the filters and the error dynamics. The properties of each of the estimators are then discussed.

E.1 Formulation for Estimating Incompressible Dynamics
The compressible rotating stall model for high speed compressors discussed in Chapter 3 was shown to contain both the lightly damped incompressible rotating stall (or Moore-Greitzer) modes and the compressible (or acoustic) modes. For estimating the incompressible dynamics, the compressible modes are treated as acoustic disturbances to be notched out. Therefore, the single-input single-output (SISO) transfer function representation for each spatial harmonic can be considered to be the sum of these two components i.e.,

$$G(s) = G_{MG}(s) + G_{DIS}(s).$$

The incompressible (Moore-Greitzer) component, $G_{MG}(s)$, is
considered to be the plant and the compressible (acoustic disturbance) component, $G_{\text{DIS}}(s)$, is modeled as the measurement noise disturbance. This implies that the system dynamics for the pre-stall dynamics for high speed compressors can be represented by:

$$y(s) = G_{\text{MG}}(s) u(s) + G_{\text{DIS}}(s) u(s)$$

(E.1)

where $u(t)$ is the input to the compression system consisting of a process disturbance signal, $u_p$, and a measurement disturbance signal, $u_v$. The disturbance signals $u_p$ and $u_v$ are modeled as unit intensity white noise signals. The state-space representation of the plant dynamics (which in this context will correspond to the Moore-Greitzer component or incompressible dynamics) is given by:

$$\dot{x}_p(t) = \left[ A_p + \Delta A_p \right] x_p(t) + B_p u_p(t)$$

$$y(t) = C_p x_p(t) + D_p u_v(t) + v(t)$$

(E.2)

where $x_p$ is the plant state vector, $y$ is the measurement vector, $u_p$ is modeled as a unit intensity process disturbance signal, $u_v$ is modeled as a unit intensity measurement disturbance signal, and $v$ is the acoustic disturbance (or measurement noise corrupting the incompressible dynamics). The state-space representation of the acoustic disturbance, $v$, (which in this context represents the compressible or acoustic component) is given by:

$$\dot{x}_v(t) = \left[ A_v + \Delta A_v \right] x_v(t) + B_v u_p(t)$$

$$v(t) = C_v x_v(t) + D_v u_v(t)$$

(E.3)

### E.2 Estimator Design with Known Plant Model

The implicit assumption of this filter design is that the plant of interest can be described by a linear state-space model, and the linear state-space representation of the plant model is given by:

$$\dot{x}_p(t) = A_p x_p(t) + B_p w(t)$$

$$y(t) = C_p x_p(t) + v(t)$$

$$z(t) = M_p x_p(t)$$

(E.4)
where $x_p$ is the plant state vector, $y$ is the measurement vector, $z$ is the vector containing the states to be estimated, $w$ is the process noise, and $v$ is the measurement noise. The process noise, $w$, is given by the linear system dynamics:

$$
\begin{align*}
\dot{x}_w(t) &= A_w x_w(t) + B_w w_1(t) \\
  w &= C_w x_w(t) + D_w w_1(t)
\end{align*}
$$

(E.5)

where $w_1$ is modeled as a unit intensity white noise signal. The measurement noise is also modeled by the linear system dynamics:

$$
\begin{align*}
\dot{x}_v(t) &= A_v x_v(t) + B_v v_1(t) \\
  v(t) &= C_v x_v(t) + D_v v_2(t)
\end{align*}
$$

(E.6)

where $v_1$ is modeled as a unit intensity process noise signal through the shaping filter dynamics and $v_2$ is modeled as a separate noise in the feedforward term so that the process and measurement noise are uncorrelated.

The system dynamics for the plant model, process noise and measurement noise given by equations E.4, E.5, and E.6 respectively, can be combined to give the following state-space representation of the system dynamics:

$$
\begin{align*}
\dot{x}(t) &= Ax(t) + Bd(t) \\
  y(t) &= Cx(t) + Dd(t) \\
  z(t) &= Mx(t)
\end{align*}
$$

(E.7)

where $x(t)$ is the state vector, $y(t)$ is the measurement vector, $d(t)$ is the white noise disturbance normalized to have unit covariance, and $z(t)$ is the vector containing the states to be estimated. The matrices $B$ and $D$ define how the disturbance enters the state and measurements, thus any covariance characteristic may be obtained by adjusting the values of these matrices. The particular case where the process noise and the measurement noise are uncorrelated is obtained when $BD^T = 0$. This can be obtained by formulating the state-space representation such that $B = \begin{bmatrix} \Gamma_1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & \Gamma_2 \end{bmatrix}$.

The state-space equations of the Kalman and minimax filters for the dynamic system given
by equation E.7 are given by:

\[
\begin{align*}
\dot{x} &= A\hat{x} + K(y - \hat{y}) \\
\dot{y} &= C\hat{x} \\
\dot{z} &= M\hat{x}
\end{align*}
\] (E.8)

where \( \hat{x} \) is the estimate of the state vector, \( \hat{y} \) is the estimate of the measurement vector, \( \hat{z} \) is the estimated states, and \( K \) is the gain matrix for the filter. The structure of the filters is shown in Figure E-1.

By combining the system dynamics given in equation E.7 and the filter dynamics given in equation E.8, the error dynamics for the Kalman and minimax filters is given by the state-space equations:

\[
\begin{align*}
\dot{x} &= A_x\tilde{x} + B_x d \\
e &= M\tilde{x}
\end{align*}
\] (E.9)

where \( \tilde{x} = x - \hat{x} \), \( e = z - \hat{z} \), \( A_x = A - KC \), and \( B_x = B - KD \). Figure E-2 shows the frequency domain representation of the Kalman and minimax filters illustrating the closed-loop transfer function from the noise disturbance, \( d \), to the error, \( e \).

**E.2.1 Kalman Filter Design**

The Kalman filter is an optimal state estimator that minimizes the expected value of the 2-norm of the estimation error squared. The Kalman filter's optimality is based on the as-
The assumption that one has perfect knowledge of the plant model and noise statistics. Therefore, when there are significant modeling errors, the Kalman filter will not be optimal i.e., there is no guarantee that the resulting estimation error from the Kalman filter will be small or bounded.

The cost function for the Kalman filter is represented by:

$$J = E\left[\|e\|^2\right] = E\left[\int_0^\infty e^T(t)e(t)dt\right]$$  \hspace{1cm} (E.10)

Consult references [1, 2, 83] for more details on the derivation and properties of the Kalman filter. The gain matrix for the Kalman filter, $K$, is:

$$K = PC^T\left(\Gamma_2\Gamma_2^T\right)^{-1}$$  \hspace{1cm} (E.11)

where $P$ represents the Kalman filter error covariance, which is found from the Riccati equation:

$$\dot{P} = AP + PA^T + \Gamma_1\Gamma_1^T - PC^T\left(\Gamma_2\Gamma_2\right)^{-1}CP$$  \hspace{1cm} (E.12)

where $P(0) = E[e(0)e^T(0)]$. The steady state Kalman filter has a constant gain matrix which is found from the symmetric, positive definite matrix, $P$, which solves the algebraic Riccati equation:

$$AP + PA^T + \Gamma_1\Gamma_1^T - PC^T\left(\Gamma_2\Gamma_2\right)^{-1}CP = 0$$  \hspace{1cm} (E.13)

By letting $G(s)$ represent the closed-loop transfer function from the noise disturbance, $d$, to the error, $e$, it can be shown using Parseval’s theorem that the 2-norm of the estimation
error is given by:

\[ \|e\|^2 = \|G(s)\|^2 \]  
(E.14)

Therefore, the steady-state Kalman filter is the \( H_2 \) optimal estimator. Since white noise has equal intensity at all frequencies, the Kalman filter minimizes the integral of the singular values of \( G(j\omega) \) over frequency.

### E.2.2 Minimax Filter Design

The minimax filter is an optimal state estimator that minimizes the expected value of the 2-norm of the estimator error squared based on the assumption that one has perfect knowledge of the plant model, as does the Kalman filter, but the noise is assumed to be bounded in energy with an unknown spectrum, unlike the Kalman filter where the noise statistics are assumed to be known. Therefore, the cost function for the minimax filter is the integral of the squared estimation error subject to a norm bound on the process and measurement noises. Restricting the minimax filter to only those that are linear and unbiased, i.e., those where the state estimate is identical to the actual state when no noise is present, the minimax estimator dynamics will have the structure shown in Figure E-1.

A full derivation of the minimax filter design can be found in [2]. The minimax estimator gain is selected to minimize the cost function defined in equation E.10 for the worst-case disturbance, resulting in the constrained minimax problem:

\[
\min_{K} \max_{d} \|e\|^2
\]

given the disturbance norm constraint \( \|d\|^2 \leq D \) and the error dynamics constraint in equation E.9. Using Lagrange multipliers, the disturbance norm constraint and the error dynamics constraint can be added to the cost function to obtain a typical game theory problem where the estimator gain, \( K \), and the noises, \( d \), are selected by two opposing players with conflicting objectives. The extremum and minimizing gain can then be found from the augmented cost function using variational calculus. The gain matrix for the minimax
filter, $K$, is given by:

$$K = P C T \left( \Gamma_2 \Gamma_2^T \right)^{-1}$$  \hspace{1cm} (E.15)

where $P$ represents the minimax filter error covariance, which is found from the modified Riccati equation:

$$\dot{P} = AP + PA^T + \Gamma_1 \Gamma_1^T - P \left[ C T \left( \Gamma_2 \Gamma_2^T \right)^{-1} C - \frac{1}{\gamma^2} M^T M \right] P$$  \hspace{1cm} (E.16)

The steady state minimax filter has a constant gain matrix which is found from the symmetric, positive definite matrix, $P$, which solves the algebraic Riccati equation

$$AP + PA^T + \Gamma_1 \Gamma_1^T - P \left[ C T \left( \Gamma_2 \Gamma_2^T \right)^{-1} C - \frac{1}{\gamma^2} M^T M \right] P = 0$$  \hspace{1cm} (E.17)

The minimax estimator is shown in [2, 83] to achieve an $H_\infty$ norm bound on the closed-loop system

$$\| G(s) \|_\infty < \gamma$$  \hspace{1cm} (E.18)

where $G(s)$ is the closed-loop transfer function from the noise disturbance, $d$, to the error, $e$, and the $H_\infty$ norm is defined as the maximum gain of a transfer function over all frequencies.

Minimax filters are not guaranteed to exist for arbitrary values of the design weight $\gamma$. Since $\gamma$ influences the size of the $\frac{1}{\gamma^2} P M^T M P$ term in the Riccati equation E.17, there will exist values of $\gamma > 0$ for which there is either no solution to the Riccati equation or for which $P$ will not be positive semi-definite. It has been shown that there is a minimum value of $\gamma$, $\gamma_{\text{min}}$, for which the minimax optimization problem still has a solution. Hence useful values of $\gamma$ will lie in the interval $\gamma_{\text{min}} \leq \gamma < \infty$. When $\gamma = \gamma_{\text{min}}$, the minimax filter is equivalent to the $H_\infty$ optimal estimator which is designed to directly minimize the $H_\infty$ norm of the closed-loop system. The $H_\infty$ optimal estimator design based on $H_\infty$ control design methods is presented in [40]. On the other hand, comparing the Riccati equations E.13 and E.17, it can be seen that the minimax filter has the same form as the Kalman filter except that there is an additional $\frac{1}{\gamma^2} P M^T M P$ term. In the limit as $\gamma \to \infty$, the minimax estimator reduces to the Kalman filter. Hence, the suboptimal minimax estimator $\gamma_{\text{min}} < \gamma < \infty$
can be considered as a generalization of the Kalman filter that provides a desired level of robustness to noise modeling error. This explains why the minimax filter is also referred in literature [12] as the $H_{\infty}$ Kalman filter.

### E.3 Estimator Design with Uncertainties in Plant and Noise Models

The optimality of the Kalman filter and the $H_{\infty}$ optimal estimator described above becomes meaningless if the design model is inaccurate. Thus neglecting modeling errors can lead to large degradation in performance. Over the years, several estimators that are robust to both plant and noise modeling uncertainties have been derived by Appleby [2], DeSouza et al. [113], and Mangoubi [83, 82]. The generalized minimax robust filter in [83, 82] has been selected for this research because it can be applied to both time-varying and time-invariant systems, it covers a broader class of model uncertainties, and its robustness to modeling uncertainties and neglected dynamics was demonstrated in reference [70].

Figure E-3 is a general transfer function representation of a nominal plant, $P$, with modeling uncertainties represented by the $A(s)$ term, and an estimator, $F$. $\epsilon$ and $\eta$ are the signals connecting the plant and the perturbation. The dynamics of the $A(s)$ block are considered to be unknown in the robust estimator derivation. It is assumed that the perturbation dynamics are stable and that the error and disturbance signals have been scaled so that the $\infty$-norm of $A(s)$ is less than one.

$$
\|A(s)\|_{\infty} = \max_{\omega} \sigma(\Delta(j\omega)) < 1
$$

The closed-loop transfer function, $G(s)$, is shown in Figure E-4, and can be expressed as a transfer function matrix:

$$
\begin{bmatrix}
\epsilon(s) \\
\phi(s)
\end{bmatrix} =
\begin{bmatrix}
G_{11}(s) & G_{12}(s) \\
G_{21}(s) & G_{22}(s)
\end{bmatrix}
\begin{bmatrix}
\eta(s) \\
\delta(s)
\end{bmatrix}
$$

(E.19)

The nominal closed-loop system is represented by $G_{22}(s)$, and the perturbed closed-loop
Figure E-3: General Representation of Robust Estimation Problem.

The system is given by:

\[ G_{ed}(s) = G_{22}(s) + G_{21}(s)\Delta(s)\left[I - G_{11}(s)\Delta(s)\right]^{-1}G_{12}(s) \]  

(E.20)

The measure of robust performance is the maximum gain from the disturbances to the estimation error, \( \|G_{ed}(s)\|_\infty \), in the presence of the norm-bounded modeling uncertainty.

Figure E-4: Closed-Loop Transfer Function Representation.

**Eigenvalue Perturbation:** For a linear dynamic system whose nominal state-space representation is given by:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

(E.21)
errors in the system model can be accounted for by modeling the perturbed system as:

\[
\dot{x}(t) = \left[ A + \Delta A \right] x(t) + \left[ B + \Delta B \right] u(t) \\
y(t) = \left[ C + \Delta C \right] x(t) + \left[ D + \Delta D \right] u(t) 
\]  
(E.22)

The following simplifying assumptions are made:

- \( \Delta B = 0. \)
- \( \Delta C = 0. \)
- \( \Delta D = 0. \)

These assumptions were made for two reasons. The first reason is to avoid numerical problems. The assumptions reduce the number of states or order of the perturbed plant system dynamics and thus will help avoid the numerical problems usually associated with designing filters for large order systems. The second reason is to take advantage of the 'eigenvalue perturbation' technique developed by Smith [112]. The eigenvalue perturbation technique addresses uncertainty in the eigenmode locations and provides a simple way of writing the state-space equations of the perturbed plant dynamics.

According to the eigenvalue perturbation technique, the error in the \( A \) matrix can be written in a special form so that the modification in the plant state space equation is given by:

\[
\dot{x}(t) = Ax(t) + \underbrace{W_{11} \Delta W_{12} x(t)}_{(A + W_{11} \Delta W_{12}) x(t)} + Bu(t) 
\]  
(E.23)

The eigenvalue perturbations are constructed through a clever choice of the \( A \) matrix and the two constant weighting matrices, \( W_{11} \) and \( W_{12} \). A similarity transform is applied to
cast the $A$ matrix into the following block diagonal modal form:

$$
A = \begin{bmatrix}
\sigma_1 & -\omega_1 \\
\omega_1 & \sigma_1 \\
\vdots & \ddots \\
\sigma_N & -\omega_N \\
\omega_N & \sigma_N \\
\vdots & \ddots \\
\sigma_n & -\omega_n \\
\omega_n & \sigma_n
\end{bmatrix}
$$

(E.24)

where the $i^{th}$ eigenvalue is defined to be $\lambda_i = \sigma_i + j\omega_i$. Equation E.24 represents a plant with $2n$ complex eigenvalues such as the pre-stall dynamics (the eigenvalue perturbation structure is easily extended to real valued eigenvalues). The $A$ matrix is ordered so that the $2N$ modes to be perturbed are in the upper left-hand portion of $A$.

The weighting matrices are structured to form the following diagonal matrix of scalar values:

$$
W = \begin{bmatrix}
w_1 & \ldots & w_N \\
w_1 & \ldots & w_N \\
\vdots & \ddots & \vdots \\
w_N & \ldots & w_N \\
0 & \ldots & 0 \\
\end{bmatrix}
$$

(E.25)

$$
W = \begin{bmatrix}
W_N & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
2N \\
2(n - N)
\end{bmatrix}
$$

(E.26)

$$
W = W_{11}W_{12}
\begin{bmatrix}
W_N \\
0
\end{bmatrix} \begin{bmatrix}
1 & 0
\end{bmatrix}
$$

(E.27)

(E.28)
Smith proves that the scalar value, \( w_i \), in \( W \) is the radius of the circular uncertainty region centered around the \( i^{th} \) eigenvalue, \( \lambda_i \).

**Generalized State-Space Representation of Perturbed Plant:** Applying the 'eigenvalue perturbation' technique to the plant and disturbance state-space dynamics given in equations E.2 and E.3 respectively results in the perturbed plant model shown in Figure E-5. The generalized state-space representation of the perturbed plant model is given by:

\[
\begin{bmatrix}
\dot{x}(t) \\
\epsilon(t) \\
e(t) \\
y(t)
\end{bmatrix} =
\begin{bmatrix}
A & B_1 & B_2 & 0 \\
C_1 & D_{11} & D_{12} & 0 \\
C_2 & 0 & 0 & -C_2 \\
C_3 & D_{31} & D_{32} & 0
\end{bmatrix}
\begin{bmatrix}
x(t) \\
\eta(t) \\
d(t) \\
\hat{x}(t)
\end{bmatrix}
\tag{E.29}
\]

Consult [2] to see how a large class of uncertainties, including parametric uncertainties and neglected dynamics from model reduction can be absorbed into the generalized state-space representation of the perturbed plant given by equation E.29.

**Figure E-5:** Perturbed plant with weighted \( \Delta \) blocks.
Robust Filter Problem Formulation: Combining the output of the perturbation $\Delta$, $\eta$, and the noise disturbance, $d$, into an augmented noise vector $r = \begin{bmatrix} \eta \\ d \end{bmatrix}$, the state-space dynamics of the perturbed plant given by equation E.29 reduces to:

$$
\begin{bmatrix}
\dot{x}(t) \\
\epsilon(t) \\
e(t) \\
y(t)
\end{bmatrix} =
\begin{bmatrix}
A & B & 0 \\
S & T & 0 \\
M & 0 & -M \\
C & D & 0
\end{bmatrix}
\begin{bmatrix}
x(t) \\
r(t) \\
\hat{x}(t)
\end{bmatrix}
$$

(E.30)

where

$$
B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}
$$

$$
T = \begin{bmatrix} D_{11} & D_{12} \end{bmatrix}
$$

$$
D = \begin{bmatrix} D_{31} & D_{32} \end{bmatrix}
$$

$$
S = C_1
$$

$$
M = C_2
$$

$$
C = C_3
$$

The objective of the robust filter is to achieve a bound on the ratio of the estimation error energy to the noise disturbance energy for any perturbation $\Delta$ of bounded induced 2-norm.

The induced norm of a linear operator $T$ is defined by:

$$
\|T\|_i = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{x=1} \|Tx\| = \sup_{x \leq 1} \|Tx\|,
$$

(E.31)

$$
= \inf_{k \in \mathbb{R}^+} \{ \|Tx\| \leq k \|x\| \}
$$

The mathematical representation of the robust filter objective stated above is:

$$
G_{i2} \equiv \sup_{d \neq 0} \frac{\|\epsilon\|}{\|d\|} < \gamma
$$

(E.32)

$$
\forall \Delta \text{ such that } \|\Delta\|_{i2} \equiv \sup_{\epsilon \neq 0} \frac{\|\eta\|_2}{\|\epsilon\|_2} < \frac{1}{\gamma}
$$
By considering the perturbation output, $\eta$, as an additional exogenous disturbance input to the plant, and the perturbation input, $\epsilon$, as an additional error term, a new criterion $\bar{J}_1$ is defined as:

$$\bar{J}_1 < \gamma^2 \quad \forall \Delta \quad \text{such that} \quad \|\Delta\|_{l2} < \frac{1}{\gamma} \quad (E.33)$$

where

$$\bar{J}_1 \equiv \sup_{(\eta,d) \neq 0} \frac{\|e\|^2 + \|\epsilon\|^2}{\|d\|^2 + \|\eta\|^2}$$

It is shown in [82] that the new criterion in equation E.33 is sufficient to meet the bound of equation E.32. This criterion may be expressed as:

$$J_2 = \|e\|^2 + \|\epsilon\|^2 - \gamma^2 (\|r\|^2) < 0 \quad \forall \mathbf{r} \neq 0 \quad (E.34)$$

This enables us to define a minimax or game-theoretic estimation problem that would minimize the modified cost function given by equation E.34 with respect to the state estimate, $\hat{x}$, in the presence of the worst possible input in a manner analogous to the minimax filter for a known plant model. The corresponding mathematical representation of the minimax or game-theoretic estimation problem is:

$$\min_{\hat{x}} \max_{\mathbf{r}} J_2$$

subject to the constraints of the perturbed plant represented by the state-space equations in E.30.

**Robust Filter Equations:** The equations for the robust filter are derived in two stages with each stage requiring a solution to a Riccati equation. The first stage is the transformation stage where the terms in the objective function that were introduced for robustness and were not affected by the estimate, $\hat{x}$, are bounded. In references [6, 79, 82], a 'completing the square' argument is used to prove that

$$\|\epsilon\|^2 - \gamma^2 (\|r\|^2) = -\gamma^2 (\|\bar{r}\|^2) \quad (E.35)$$
if and only if the modified Riccati equation

$$\dot{X} = X(A + BZ^{-1}T^T S) + (A + BZ^{-1}T^T S)^T X + \gamma^{-2}XBZ^{-1}B^T X + S^T S$$

$$X(t_f) = 0$$  \hspace{1cm} (E.36)

has a solution such that $X \geq 0$, and $Z > 0$, $\forall t$, where

$$\bar{r} \equiv Z^{\frac{1}{2}} r - r^*$$

$$r^* \equiv \gamma^{-2}Z^{\frac{1}{2}}(B^T X + S^T T)x$$

$$Z \equiv I - \gamma^{-2}(T^T T)$$

The goal of the second stage is to seek a state estimate, $\hat{x}$, such that $J_2 < 0$, and it consists of solving a transformed problem for the optimal estimate. Now, the estimation problem is defined in terms of the new disturbance $\bar{r}$. By substituting equation E.35 into equation E.34, the objective function $J_2$ is redefined as:

$$J_2 = ||e||^2_2 - \gamma^2 ||\bar{r}||^2_2$$  \hspace{1cm} (E.37)

Expressing the state and observation equations in E.30 in terms of the new disturbance $\bar{r}$ (by substituting $r = Z^{-\frac{1}{2}}(\bar{r} + r^*)$ into these equations), the transformed state-space equations are given by:

$$\dot{x} = \bar{A}x + \bar{B}\bar{r}$$

$$y = \bar{C}x + \bar{D}\bar{r}$$  \hspace{1cm} (E.38)

where

$$\bar{A} = A + \gamma^{-2}B(B^T X + T^T S)$$

$$\bar{B} = BZ^{-\frac{1}{2}}$$

$$\bar{C} = C + \gamma^{-2}D(B^T X + T^T S)$$

$$\bar{D} = DZ^{-\frac{1}{2}}$$
The estimation problem has now been reduced to a formulation similar to the minimax filter with known plant dynamics. As was shown in [82], the optimal filter has the form:

\[
\dot{x} = (\tilde{A} - K\tilde{C})x + Ky
\]  

(E.39)

with the error dynamics given by:

\[
\dot{x} = \tilde{A}x + \tilde{B}r \\
e = M\tilde{x}
\]  

(E.40)

where $\tilde{A} = (\bar{A} - K\tilde{C})$, and $\tilde{B} = (\bar{B} - K\tilde{D})$. The optimal gain matrix for the robust filter, $K$, is given by:

\[
K = (P\tilde{C}^T + \tilde{B}\tilde{D}^T)(\tilde{D}\tilde{D}^T)^{-1}
\]  

(E.41)

where $P$ satisfies a second modified Riccati equation:

\[
\dot{P} = \tilde{A}P + P\tilde{A}^T + \gamma^{-2}PM^TMP + \tilde{B}\tilde{B}^T
\]  

P(0) = P_0

(E.42)
APPENDIX F

NONLINEAR CONTROL LAWS

In this appendix, the full state nonlinear distributed model derived in Appendix C is converted to a form suitable for nonlinear control design, and nonlinear control laws that are robust to uncertainties in the model parameters are derived. The procedure for converting the full state nonlinear distributed model to a second order perturbation form suitable for nonlinear control design is presented in Section F.1; the derivation of a sliding mode control law is presented in Section F.2; and the derivation of a robust adaptive control law is presented in Section F.3.

F.1 Second Order Perturbation Equation

The unsteady full state nonlinear distributed model for rotating stall and surge inception from Appendix C (equations in C.55 without the actuator dynamics) is:

\[
\begin{align*}
    \dot{\phi} &= -A \phi + \Psi_{\text{ideal}}(\phi, \gamma) + \Psi_{\text{mom}} - L_{tr} - L_s - T \psi + K \dot{\gamma} \quad \text{(F.1)} \\
    \dot{\psi} &= \frac{1}{4B^2\ell_c} [mS^T \phi - \phi_t(\psi)] \quad \text{(F.2)} \\
    \tau_{\text{mom}} \dot{\Psi}_{\text{mom}} &= -\Psi_{\text{mom}} + \Psi_{\text{mom}}^{ss}(\phi, \gamma) \quad \text{(F.3)} \\
    \tau_{\text{loss}} \dot{L}_{tr} &= -J L_{tr} + L_{tr}^{ss}(\phi, \gamma) \quad \text{(F.4)} \\
    \tau_{\text{loss}} \dot{L}_s &= -L_s + L_s^{ss}(\phi, \gamma) \quad \text{(F.5)}
\end{align*}
\]

Using the technique outlined in reference [107], the state equations F.1, F.2, F.3, F.4, and F.5 can be transformed into a new local coordinate in which the origin is located at
an equilibrium point \((\tilde{\phi}^*, \psi^*, \tilde{\gamma}^*)\), whose stability is being investigated using the following transformations:
\[
\begin{align*}
\tilde{\phi} &= \phi - T \tilde{\phi}^* \\
\tilde{\gamma} &= \gamma - T \tilde{\gamma}^* \\
\tilde{\psi} &= \psi - \psi^* \\
\tilde{\phi}_t &= \phi_t - \tilde{\phi}^* \approx m_t \tilde{\psi} \\
\tilde{\Psi}_{cu} &= \Psi_{cu}(\tilde{\phi}, \tilde{\gamma}) - \Psi_{cu}(T \tilde{\phi}^*, T \tilde{\gamma}^*)
\end{align*}
\] (F.6)

where the nonlinear components have been lumped into \(\tilde{\Psi}_{cu}\) which is defined as: \(\tilde{\Psi}_{cu} = \Psi_{ideal}(\tilde{\phi}, \tilde{\gamma}) + \Psi_{mom} - L_r - L_s\). The nonlinear component can be written as:
\[
\tilde{\Psi}_{cu} = \Psi_i(\phi, \gamma) - \frac{L_r - L_s}{\tau + J}
\]

\(= \frac{rL_s(\phi, \gamma)}{\tau + J} - \left(1 - r\right)\frac{\tau_{ss}(\phi, \gamma)}{\tau_{ss} + 1}
\)

\(= \Psi_i(\phi, \gamma) - r\frac{\Psi_i(\phi, \gamma) - \Psi_i(\phi, \gamma)}{\tau + J} - \left(1 - r\right)\frac{\Psi_i(\phi, \gamma) - \Psi_i(\phi, \gamma)}{\tau + J}
\)

\(= \Psi_i(\phi) - r\frac{\Psi_i(\phi)}{\tau + J} - \left(1 - r\right)\frac{\Psi_i(\phi)}{\tau + J}
\)

\[\quad + \gamma \cdot \left[\Psi_{i1}(\phi) - r\frac{\Psi_{i1}(\phi) - \Psi_{i1}(\phi)}{\tau + J} - \left(1 - r\right)\frac{\Psi_{i1}(\phi) - \Psi_{i1}(\phi)}{\tau + J}\right]
\]

\[\quad + \gamma^2 \cdot \left[\Psi_{i2}(\phi) - r\frac{\Psi_{i2}(\phi) - \Psi_{i2}(\phi)}{\tau + J} - \left(1 - r\right)\frac{\Psi_{i2}(\phi) - \Psi_{i2}(\phi)}{\tau + J}\right]
\]

where the momentum component of the total pressure rise in the upstream duct and the total-to-static pressure rise across the blade rows have been lumped together \(\Psi_i = \Psi_{ideal}(\phi, \gamma) + \Psi_{mom}\); the computed ideal compressor characteristic representation from equation C.64 i.e., \(\Psi_i = \Psi_i(\phi) + \gamma\Psi_{i1}(\phi) + \gamma^2\Psi_{i2}(\phi)\), and the measured compressor characteristic representation from equation C.69 i.e., \(\Psi_c = \Psi_c(\phi) + \gamma\Psi_{c1}(\phi) + \gamma^2\Psi_{c2}(\phi)\) are used. Transforming equations F.1 and F.2 using the transformation equations in F.6 gives:
\[
\begin{align*}
E \frac{\ddot{\phi}}{\tau} &= -A \dot{\phi} + \tilde{\Psi}_{cu} - T \tilde{\psi} + K \dot{\gamma} \\
\frac{\ddot{\psi}}{\tau} &= \frac{1}{4B^2E_c} \left(mS^T \dot{\phi} - m_t \dot{\psi}\right)
\end{align*}
\] (F.8) (F.9)
The perturbation equation F.9 can be rearranged into:

\[
\ddot{\psi} = \frac{\dot{m}_t}{4B^2t_c} S^T \ddot{\phi}
\]  \hspace{1cm} (F.10)

Substituting the expression for \(\ddot{\psi}\) from equation F.10 into equation F.8, and simplifying the resulting expression gives:

\[
E \frac{\ddot{\phi}}{dt} + \left[ A + \frac{m_t}{4B^2t_c} E \right] \frac{\dot{\phi}}{dt} + \frac{1}{4B^2t_c} \left[ m_t A + m T S^T \right] \frac{\dot{\phi}}{dt} + \frac{\ddot{\psi}}{m_t} = \ddot{\psi}_{ct} + K \dot{\gamma}
\]  \hspace{1cm} (F.11)

The second order nonlinear differential equation for rotating stall and surge inception given in equation F.11 can therefore be written as:

\[
H \ddot{x} + C \dot{x} + L x + d = \tau
\]  \hspace{1cm} (F.12)

where the following substitutions have been made:

\[
x = \frac{\ddot{\phi}}{dt} + \frac{m_t}{4B^2t_c}
\]  \hspace{1cm} (F.13)

\[
\tau = K \dot{\gamma} + \gamma \left( \frac{\Psi_{i1}(\phi)}{\tau s + J} - \frac{\Psi_{i1}(\phi) - \Psi_{c1}(\phi)}{\tau s + I} \right)
\]  \hspace{1cm} (F.14)

\[
d = - \left[ \Psi_{i0}(\phi) - \frac{\Psi_{i0}(\phi) - \Psi_{c0}(\phi)}{\tau s + J} - (1 - r) \frac{\Psi_{i0}(\phi) - \Psi_{c0}(\phi)}{\tau s + I} - \psi_{c0}(T\ddot{\phi}^*) \right]
\]  \hspace{1cm} (F.15)

\[
H = E
\]  \hspace{1cm} (F.16)

\[
C = \left[ A + \frac{m_t}{4B^2t_c} E \right]
\]  \hspace{1cm} (F.17)

\[
L = \frac{1}{4B^2t_c} \left[ m_t A + m T S^T \right]
\]  \hspace{1cm} (F.18)

---

**F.2 Sliding Mode Control Law**

The goal of rotating stall and surge stabilization is to completely damp out or prevent the flow coefficient perturbations from growing beyond a certain acceptable limit. Therefore,
the stabilization problem can be seen as a tracking problem with the desired trajectory \( \mathbf{x}_d = 0 \). The second order tracking problem in F.12 can be transformed to an equivalent first order stabilization problem by defining a sliding surface, \( \mathbf{s} \), as:

\[
\mathbf{s} = \dot{\mathbf{x}} + \Lambda \mathbf{\ddot{x}} = \dot{\mathbf{x}} - \dot{\mathbf{x}}_r
\]  

where \( \Lambda \) is a symmetric positive definite matrix (or more generally a \( \Lambda \) is Hurwitz) representing the bandwidth of the controller, \( \dot{\mathbf{x}}_r = \dot{\mathbf{x}}_d - \Lambda \dot{\mathbf{x}} \), and \( \mathbf{\ddot{x}} = \mathbf{x} - \mathbf{x}_d \). Since \( \mathbf{s} \equiv 0 \) represents a linear differential equation whose unique solution is \( \dot{\mathbf{x}} \equiv 0 \) given initial conditions \( \dot{\mathbf{x}}(0) = 0 \), the tracking problem \( \dot{\mathbf{x}} = 0 \) can be equivalently reduced to keeping the vector \( \mathbf{s} = 0 \) and the tracking problem \( \| \mathbf{s} \| \leq \epsilon \) can be equivalently reduced to keeping the sliding vector \( \| \mathbf{s} \| \leq \Phi \). The larger the controller bandwidth, \( \lambda \), the smaller the tracking error. The first order stabilization problem of keeping the vector \( \mathbf{s} \) at zero is achieved by choosing the control law, \( \tau \), such that the sliding surface is made attractive.

The sliding condition that must be satisfied to make the sliding surface attractive is:

\[
\frac{1}{2} \frac{d}{dt} (\mathbf{s}^T \mathbf{H} \mathbf{s}) \leq -\eta (\mathbf{s}^T \mathbf{s})^{\frac{1}{2}}
\]  

where \( \eta \) is a strictly positive constant. Geometrically, the sliding condition can be interpreted as the generalized norm (or 'distance') to the surface decreases along all system trajectories and it constraints trajectories to point towards the surface. The sliding condition ensures that \( \mathbf{s} \rightarrow 0 \) in finite time \( t_f \leq \frac{s(0)}{\eta} \). Thus the larger the value for \( \eta \), the faster the switching control system trajectory will reach the sliding surface. The switching control law is:

\[
\tau = \dot{\tau} - \mathbf{k} \text{sgn}(\mathbf{s})
\]  

\[
= \mathbf{H} \mathbf{\ddot{x}}_r + \mathbf{C} \mathbf{\ddot{x}}_r + \mathbf{L} \mathbf{x} + \mathbf{d} - \mathbf{k} \text{sgn}(\mathbf{s})
\]

\[1\] For this application, the matrix \( \Lambda \) is a diagonal matrix with the diagonal elements equal to \( \lambda \)

\[2\] For the sliding mode approximation of the switching controller, the larger the value for \( \eta \), the faster the trajectory will enter the boundary layer.
F.2.1 Lyapunov Analysis

Lyapunov analysis is used to proof the stability of the switching control law. The Lyapunov Theorem for Global Stability is stated as follows:

**Theorem F.1 (Global Stability)** Assume that there exists a scalar function $V$ of the state $x$, with continuous first order derivatives such that

- $V(x)$ is positive definite
- $\dot{V}(x)$ is negative definite
- $V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$

then the equilibrium at the origin is globally asymptotically stable.

For control systems for which $\dot{V}$, the derivative of the Lyapunov function candidate, is only negative semi-definite, the Global Invariant Set is used instead of the Global Stability Theorem. A set $G$ is an invariant set for a dynamic system if every system trajectory which starts from a point in $G$ remains in $G$ for all future time. In other words, a set is invariant if, for every initial state in the set, a suitable initial time can be found such that the resulting trajectory stays in the set at all future times. The Global Invariant Set Theorem is stated as follows:

**Theorem F.2 (Global Invariant Set Theorem)** Consider the autonomous system $\dot{x} = f(x)$, with $f$ continuous, and let $V(x)$ be a scalar function with continuous first partial derivatives. Assume that

- $\dot{V}(x) \leq 0$ over the whole state space
- $V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$

Let $R$ be the set of all points where $\dot{V}(x) = 0$, and $M$ be the largest invariant set in $R$. Then all solutions globally asymptotically converge to $M$ as $t \rightarrow \infty$.

For more details on these theorems and their proofs, consult any nonlinear analysis text such as [111, 127].
The Lyapunov function for the stall inception system is:

$$V = \frac{1}{2} \dot{s}^T H s$$ \hspace{1cm} (F.22)

Differentiating the Lyapunov function in equation F.22, and using equations F.12 and F.19 to simplify:

$$\dot{V} = \dot{s}^T H \dot{s}$$ \hspace{1cm} (F.23)

$$= \dot{s}^T (H \ddot{x} - H \ddot{x}_r)$$

$$= \dot{s}^T (H \ddot{x} - H \ddot{x}_r)$$

$$= \dot{s}^T \left[ T - C \ddot{x} - L x - d - H \ddot{x}_r \right]$$

$$= \dot{s}^T \left[ T - C (s + \ddot{x}_r) - L x - d - H \ddot{x}_r \right]$$

$$= -\dot{s}^T C s + \dot{s}^T \left[ \ddot{H} \ddot{x}_r + \ddot{C} \ddot{x}_r + \ddot{L} x + \ddot{d} - k \operatorname{sgn}(s) - (H \ddot{x}_r + C \ddot{x}_r + L x + d) \right]$$

$$= -\dot{s}^T C s + \dot{s}^T \left( \ddot{H} \ddot{x}_r + \ddot{C} \ddot{x}_r + \ddot{L} x + \ddot{d} \right) - \sum_i k_i |s_i|$$

If the gain $k$ is chosen such that:

$$k_i \geq \left| \ddot{H} \ddot{x}_r + \ddot{C} \ddot{x}_r + \ddot{L} x + \ddot{d} \right| + \eta_i$$ \hspace{1cm} (F.24)

then the derivative of the Lyapunov function, $\dot{V}$ is:

$$\dot{V} = -\dot{s}^T C s - \sum_i \eta_i |s_i|$$ \hspace{1cm} (F.25)

$$= -\dot{s}^T C s - \eta (\dot{s}^T s)^{1/2}$$

Note that $C = \left[ A + \frac{m \omega^2}{4 B^2 I_c} E \right]$. Therefore, $C$ is positive definite because $E$ is positive definite and $A$ is a rotation matrix (all eigenvalues of $A$ lie on the $j \omega$ axis i.e., $x^T A x$). Since $C$ is positive definite, $\dot{s}^T C s > 0$, and $\eta_i > 0$, the largest invariant set in the set of all points where $\dot{V}(x) = 0$ is the sliding surface $s = 0$. Also, the Lyapunov function is radially unbounded, i.e., $V(x) \to \infty$ as $\|x\| \to \infty$. Therefore, according to the Invariant Set Theorem, all solutions globally asymptotically converge to $s = 0$ as $t \to \infty$ i.e., the sliding surface $s$ is
F.2.2 Continuous Approximation of Switching Control Laws

Eventhough the switching control law given in equation F.21 will lead to perfect tracking i.e., \( \tilde{\phi} \rightarrow 0 \), the switching controller is not acceptable because the discontinuity in the control law leads to chattering which is undesirable. Chattering is undesirable in practice because it involves high control activity and may excite high-frequency dynamics neglected in the model (such as the unmodeled acoustic modes and neglected time-delays). The chattering problem encountered with the switching control law can be eliminated by smoothing out the control discontinuity in a thin boundary layer neighboring the switching surface

\[
B(t) = \{ \mathbf{x}, |s(\mathbf{x}; t)| \leq \Phi \} \quad \Phi > 0 \tag{F.26}
\]

where \( \Phi \) is the boundary layer thickness, and \( \varepsilon = \frac{\Phi}{\lambda} \) is the boundary layer width, as illustrated in Figure F-1.

![Figure F-1: Thin boundary layer neighboring the switching surface, \( s = 0 \).](image)

The discontinuous \( \text{sgn}(s) \) function in the switching control law in equation F.21 is approximated by the continuous saturation function \( \text{sat}(s/\Phi) \) shown in Figure F-2.
The expression for the continuous approximation of the discontinuity is:

\[
\text{sat}(y) = \begin{cases} 
\text{sgn}(y) & \text{if } |y| \geq 1, \\
y & \text{if } |y| < 1.
\end{cases}
\]  

(F.27)

The continuous approximation of the switching control law in equation F.21 or sliding mode control law is:

\[
\tau = \dot{\xi} - k \text{sat}(s/\Phi)
\]

\[
= \dot{H} \dot{x}_r + \dot{C} \dot{x}_r + \dot{L} x + \dot{d} - k \text{sat}(s/\Phi)
\]

(F.28)

Outside of the boundary layer, \(B(t)\), the sliding mode control law in equation F.28 is the same as the switching control law in equation F.21 which satisfies the sliding condition of equation F.20. This guarantees the boundary layer to be attractive, hence invariant. Thus the sliding mode control law leads to tracking to within a guaranteed precision, \(\epsilon\), rather than 'perfect' tracking as is the case with the switching control law. In the context of rotating stall and surge stabilization, the sliding mode control will reduce the flow coefficient perturbations to within guaranteed levels. From the expression for the gain, \(k\), in equation F.24, it can be seen that the higher the error estimates, the harder the controller will push to force the system trajectory into the boundary layer.

From a practical implementation point of view, the number of states for the controller can be reduced considerably by using the effective parameters which incorporate the unsteady
effects. Thus the estimate \( \hat{d} \) of the nonlinear component \( d \) (defined in equation F.15) used for the nominal component, \( \hat{\tau} \), of the control law is:

\[
\hat{d} = - \left[ \Psi_{c0}(\phi) - \Psi_{c0}(T\phi^*) \right]
\]

Also, the actuator input, \( \gamma \), can be estimated from the approximation of the control term:

\[
\tau = \left[ K \gamma + \gamma^2 \Psi_{c1}(\phi) + \gamma^2 \Psi_{c2}(\phi) \right] - \left[ K T\gamma^* + \gamma^* \Psi_{c1}(T\phi^*) + \gamma^* \Psi_{c2}(T\phi^*) \right]
\]

which is an approximation of the control term given in equation F.14.

F.2.3 Static Pressure Perturbations and Velocity Perturbations

To implement the sliding mode control law, an estimate of the flow coefficient is required. However, NASA Stage 35 is instrumented with wall static Kulites instead of hot wire probes. Therefore, the flow coefficient has to be estimated from the static pressure measurements. The expression for obtaining the flow coefficient from static pressure measurements is given in this section. In NASA Stage 35, the wall static Kulites are placed in the upstream duct at Station F.

Surge Modes, \( n = 0 \): The steady total pressure in the upstream duct is equal to the total pressure in the upstream plenum which is constant, i.e., \( P_{t,in} = P_{in} + \frac{1}{2} \rho_{in} U_{in}^2 = P_{atm} \). The relation between the zeroth harmonic static pressure and velocity perturbations is:

\[
\frac{\delta P_{in}}{\rho_o U_o^2} + \frac{\rho_{in} \delta U_{in}}{\rho_o U_o} = 0
\]

Using nondimensionalized variables, the corresponding relation between the zeroth harmonic static pressure and velocity perturbations is:

\[
\frac{\rho_{in}}{\rho_o} \frac{\delta \phi_{in}}{\phi_{0,in}} = -\psi_{0,in}
\]
Rotating Stall Modes, $n \neq 0$: The upstream duct total pressure perturbation expression from equation C.9 in Subsection C.1.1 is:

$$\frac{\delta P_{t,in}}{\rho_0 U_o^2} = -\frac{1}{|n|} e^{-|n|\eta_s} \frac{\rho_{in}}{\rho_o} \frac{\partial \delta \phi_{in}}{\partial r}$$  \hspace{1cm} (F.33)

From the total pressure relation for incompressible flows which is given by $P_{t,in} = P_{in} + \frac{1}{2} \rho_{in} U_{in}^2$, the total pressure perturbation equation is:

$$\delta P_{t,in} = \delta P_{in} + \rho_{in} U_{in} \delta U_{in}$$  \hspace{1cm} (F.34)

Substituting equation F.34 into equation F.33, simplifying the resulting expression, and using nondimensionalized variables, the relation between the $n^{th}$ harmonic (for $n \neq 0$) velocity and static pressure perturbations is:

$$\rho_{in} \left[ \frac{1}{|n|} e^{-|n|\eta_s} \frac{\partial \phi_{n,in}}{\partial r} + \phi_{in} \phi_{n,in} \right] = -\psi_{n,in}$$  \hspace{1cm} (F.35)

where $\eta_s$ is the nondimensional axial location of the wall static Kulite sensors.

Distributed Ordinary Differential Equation: The distributed differential equation relating static pressure and velocity can be obtained by combining equations F.32 and F.35 using the discretization technique described in references [84, 107]. Using this discretization procedure, the resulting distributed ordinary differential equation is:

$$E_s \Phi + \Phi \dot{\Phi} = -\frac{1}{\rho_{in}} \psi$$  \hspace{1cm} (F.36)

where $E_s = F^{-1} \cdot D_E \cdot F$, $F$ is defined in equation C.52, $F^{-1} = F^T$, and $D_E$ is given by:

$$D_E = \begin{bmatrix} \frac{1}{|n|} e^{-|n|\eta_s} \\ \vdots \\ \frac{1}{|n|} e^{-|n|\eta_s} \end{bmatrix}_{n=-N}^{n=+N}$$  \hspace{1cm} (F.37)
F.3 Robust Adaptive Control Law

The derivation and proof of the robust adaptive control is similar to the sliding mode control law discussed in the previous section. The robust adaptive control law is derived in two parts. For the first part, an adaptive control law is derived for the second order system in equation F.12 with the disturbance term, $d$, set to zero. For the second part, the disturbance term, $d$, is introduced and the procedure in the first part is repeated using a modified definition of the tracking error due to the disturbance.

For the first part in the derivation, the following second order nonlinear differential equation is considered:

$$\mathbf{H} \ddot{\mathbf{x}} + \mathbf{C} \dot{\mathbf{x}} + \mathbf{L} \mathbf{x} = \tau$$  \hspace{1cm} (F.38)

This second order equation is different from equation F.12 in that it does not have the disturbance term, $d$. The control law for the second order system in equation F.38 is:

$$\tau = \mathbf{Y} \hat{\mathbf{a}} - \mathbf{K}_D \mathbf{s}$$  \hspace{1cm} (F.39)

and the corresponding adaptation law is:

$$\dot{\hat{\mathbf{a}}} = -\Gamma \mathbf{Y}^T \mathbf{s}$$  \hspace{1cm} (F.40)

where $\mathbf{a}$ is a vector containing the system parameters which are unknown but assumed to be constant, $\hat{\mathbf{a}}$ is the estimate of the system parameters, $\mathbf{Y}$ is matrix whose components can be derived from the states, $\mathbf{x}$, $\mathbf{Y} \mathbf{a} = \mathbf{H} \ddot{\mathbf{x}} + \mathbf{C} \dot{\mathbf{x}} + \mathbf{L} \mathbf{x}$, $\mathbf{Y} \hat{\mathbf{a}} = \mathbf{H} \ddot{\hat{\mathbf{x}}} + \mathbf{C} \dot{\hat{\mathbf{x}}} + \mathbf{L} \hat{\mathbf{x}}$, $\mathbf{s}$ is the sliding surface defined by equation F.19, $\mathbf{K}_D$ is the controller gain matrix, and $\Gamma$ is the adaptation gain matrix. $\mathbf{K}_D$ is positive semi-definite, and $\Gamma$ is symmetric.

Lyapunov analysis is used to proof that the control law in equation F.39 and the adaptation law in F.40 can stabilize the system in equation F.38. The Lyapunov function candidate, $V$, selected for the proof is:

$$V = \frac{1}{2} \mathbf{s}^T \mathbf{H} \mathbf{s} + \frac{1}{2} \dot{\hat{\mathbf{a}}}^T \Gamma^{-1} \dot{\hat{\mathbf{a}}}$$  \hspace{1cm} (F.41)

This Lyapunov function represents the sum of the square distance from the sliding surface.
(which indirectly represent the size of the flow coefficient perturbations) and the square parameter error. Differentiating the Lyapunov function in equation F.41 and simplifying the resulting expression with the expressions for the control law in equation F.39, the adaptation law in equation F.40, $\dot{a} = \dot{a} - a$, and $\ddot{a} = 0$:

$$
\dot{V} = s^T H \dot{s} + \dot{a}^T \Gamma^{-1} \ddot{a} \quad \text{(F.42)}
$$

$$
= s^T H (\ddot{x} - \ddot{x}_r) + \dot{a}^T \Gamma^{-1} \ddot{a}
$$

$$
= s^T (H\ddot{x} - H\ddot{x}_r) + \dot{a}^T \Gamma^{-1} \ddot{a}
$$

$$
= s^T [\tau - C\dot{x} - Lx - H\ddot{x}_r] + \dot{a}^T \Gamma^{-1} \ddot{a}
$$

$$
= s^T [\tau - (H\ddot{x}_r + C\ddot{x} + L\ddot{x})] + \dot{a}^T \Gamma^{-1} \ddot{a}
$$

$$
= s^T [\tau - Ya] + (-\Gamma Y_T s) s^T \Gamma^{-1} \ddot{a}
$$

$$
= s^T [Y \dot{a} - K_D s - Ya] - s^T Y T \Gamma T \Gamma^{-1} \ddot{a}
$$

$$
= s^T [Y (\dot{a} - a) - K_D s] - s^T Y T \Gamma \Gamma^{-1} \ddot{a}
$$

$$
= s^T Y \dot{a} - s^T K_D s - s^T Y \ddot{a}
$$

$$
= -s^T K_D s
$$

For the second part of the robust adaptive control law derivation, the nonlinear term, $d$, is introduced to obtain the complete second order perturbation equation:

$$
H \ddot{x} + C \dot{x} + Lx + d = \tau \quad \text{(F.43)}
$$

where $d(\phi, t) \leq D$ is an unknown but bounded disturbance, with known bound $D$. The robust adaptive control law for the second order system in equation F.43 is:

$$
\tau = Y \dot{a} - K_D s \quad \text{(F.44)}
$$

and the corresponding adaptation law is:

$$
\dot{a} = -\Gamma Y_T s \Delta \quad \text{(F.45)}
$$
where $s_\Delta$ is a modified sliding surface due to the disturbance, and is defined as: $s_\Delta = s - \Phi \text{sat}(s/\Phi)$ and is depicted in Figure F-3. Note that $s_\Delta = 0$ for $|s| \leq \Phi$, and $\dot{s}_\Delta = \dot{s}$. It should be noted that even though the slope of $s_\Delta$ is discontinuous, the slope of $s_\Delta^2$ is continuous since it does not have the sharp corners.

The modified Lyapunov function candidate, $V$, which incorporates the tracking error due to the disturbance is:

$$
V = \frac{1}{2} s_\Delta^T H s_\Delta + \frac{1}{2} \ddot{a}^T \Gamma^{-1} \ddot{a} 
$$

(F.46)

Differentiating the Lyapunov function in equation F.46 and simplifying the resulting expression with expressions for the control law in equation F.44, the adaptation law in equation
The expression for $\dot{V}$ in equation F.47 can be further simplified using the following:

$$\dot{V} = s_\Delta^T K_D \Phi \text{sat}(s/\Phi) = |s_\Delta^T| K_D \Phi$$  \hspace{1cm} (F.48)

$$|s_\Delta^T d| \leq |s_\Delta^T| D$$  \hspace{1cm} (F.49)

Substituting the relations in equations F.48 and F.49 into the derivative of the Lyapunov function, $\dot{V}$, in equation F.47:

$$\dot{V} = -s_\Delta^T K_D s_\Delta - s_\Delta^T K_D \Phi \text{sat}(s/\Phi) - s_\Delta^T d$$

$$\leq -s_\Delta^T K_D s_\Delta - |s_\Delta^T| K_D \Phi - s_\Delta^T d$$

$$\leq -s_\Delta^T K_D s_\Delta - |s_\Delta^T| K_D \Phi + |s_\Delta^T| D$$

$$\leq -s_\Delta^T K_D s_\Delta - |s_\Delta^T| (K_D \Phi - D)$$

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If the gain matrix $K_D$ is selected such that $K_D \Phi - D = 0$ i.e., $\Phi = K_D^{-1} D$, the derivative of Lyapunov function, $\dot{V}$, can be simplified to:

$$\dot{V} \leq -s_\Delta^T K_D s_\Delta$$  \hspace{1cm} (F.50)

Since $K_D$ is positive semi-definite, $s_\Delta^T K_D s_\Delta > 0$. From equation F.50, $\dot{V} \leq 0$ and the largest invariant set in the set of all points where $\dot{V}(x) = 0$ is the modified sliding surface $s_\Delta = 0$ which is equivalent to $|s| \leq \Phi$. Also, the Lyapunov function is radially unbounded, i.e., $V(x) \to \infty$ as $||x|| \to \infty$. Therefore, according to the Invariant Set Theorem, all solutions globally asymptotically converge to $s_\Delta = 0$ as $t \to \infty$ i.e., $|s| \leq \Phi$ is attractive.
Appendix G

Determination of Actuator Configuration

In this appendix, a parametric study is performed to demonstrate the feasibility of reducing the number of actuators required for effective control of compressor instabilities. Six actuation configurations consisting of twelve, eight, and six actuators are investigated and compared. The state space model used for this parametric study is presented in Section G.1, and the procedure for evaluating the actuation configurations and results are presented in Section G.2.

G.1 State Space Model

The full state nonlinear distributed model in equation C.55 of Section C.2 is used for this parametric study. Two adjustments are made. First, the momentum component of the total pressure rise in the upstream duct due to air injection, $\Psi_{mom}$, is lumped with the total-to-static pressure rise. Second, since the actuator has a bandwidth of 400 Hz (or 1.4 $\Omega$), and the actuator command and valve position have been normalized such that the DC gain is 1.0, the servo dynamics for each actuator is approximated with a first order time-lag system: $\tau_a \dot{\gamma}_j = -\gamma_j + z_j \gamma_c$, where $\gamma_c$ is the normalized input command, $\gamma_j$ is the valve opening which is linearly related to the injected mass, and $z_j$ is the jet gain shape profile for each actuator. The resulting ordinary differential equations for the nonlinear distributed
model including the actuator servo dynamics are as follows:

\[
\begin{align*}
\mathbf{E} \ddot{\phi} &= -\mathbf{A} \ddot{\phi} + \mathbf{Psi}(\phi, \gamma_j) - \mathbf{L}_r - \mathbf{L}_s - T\psi + \mathbf{K} \gamma_j \\
\dot{\psi} &= \frac{1}{4B^2\ell_c} \left[ m_\text{S}T \phi + \sqrt{\frac{2\phi}{K_1}} \right] \\
\tau_{\text{loss}} \ddot{\mathbf{L}}_r &= -\mathbf{J} \mathbf{L}_r + \mathbf{L}_r^{ss}(\phi, \gamma_j) \\
\tau_{\text{loss}} \ddot{\mathbf{L}}_s &= -\mathbf{J} \mathbf{L}_s + \mathbf{L}_s^{ss}(\phi, \gamma_j) \\
\tau_{\text{act}} \dot{\gamma}_j &= -\gamma_j + \mathbf{Z}_{\text{cfg}} \gamma_c \\
\end{align*}
\] (G.1)

\[Z_{\text{cfg}}\] is the matrix containing the shape of the injected velocity profile from the injectors and is based on wind tunnel measurements by Berndt [10]. The corresponding linearization of equation G.1 is:

\[
\begin{align*}
\mathbf{E} \ddot{\phi} - \mathbf{K} \dot{\gamma}_j &= \left[ -\mathbf{A} + \frac{\partial \mathbf{Psi}}{\partial \phi} \right] \ddot{\phi} - T\psi - \mathbf{L}_r - \mathbf{L}_s + \frac{\partial \mathbf{Psi}}{\partial \gamma_j} \dot{\gamma}_j \\
\dot{\psi} &= \frac{1}{4B^2\ell_c} \left[ m_\text{S}T \phi + m_\text{t} \psi \right] \\
\tau_{\text{loss}} \dot{\mathbf{L}}_r &= \frac{\partial \mathbf{L}_r^{ss}}{\partial \phi} \ddot{\phi} - \mathbf{J} \mathbf{L}_r + \frac{\partial \mathbf{L}_r^{ss}}{\partial \gamma_j} \dot{\gamma}_j \\
\tau_{\text{loss}} \dot{\mathbf{L}}_s &= \frac{\partial \mathbf{L}_s^{ss}}{\partial \phi} \ddot{\phi} - \mathbf{J} \mathbf{L}_s + \frac{\partial \mathbf{L}_s^{ss}}{\partial \gamma_j} \dot{\gamma}_j \\
\tau_{\text{act}} \dot{\gamma}_j &= -\gamma_j + \mathbf{Z}_{\text{cfg}} \gamma_c \\
\end{align*}
\] (G.2)

This linearized system of ordinary differential equations can be represented in state space form as follows:

\[
\begin{pmatrix}
\dot{\phi} \\
\dot{\psi} \\
\dot{\mathbf{L}}_r \\
\dot{\mathbf{L}}_s \\
\dot{\gamma}_j
\end{pmatrix} = \begin{pmatrix}
\mathbf{A} \\
\mathbf{B} \gamma_c
\end{pmatrix}
\] (G.3)
where the matrices $A$ and $B$ are defined as:

$$
A = \begin{bmatrix}
E & 0 & 0 & 0 & -K \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & \tau I & 0 & 0 \\
0 & 0 & 0 & \tau I & 0 \\
0 & 0 & 0 & 0 & \tau_a I
\end{bmatrix}^{-1}
\begin{bmatrix}
-A + \frac{\partial \psi_i}{\partial \phi} & -T & -I & -I & \frac{\partial \psi_i}{\partial \tau_i} \\
\frac{mST}{4B^2T_e} & -\frac{mT}{4B^2T_e} & O & O & O \\
\frac{\partial L_e}{\partial \phi} & 0 & -J & O & \frac{\partial L_e}{\partial \tau_j} \\
\frac{\partial L_e}{\partial \phi} & 0 & O & -I & \frac{\partial L_e}{\partial \tau_j} \\
0 & 0 & 0 & 0 & -I
\end{bmatrix}
$$

$$
B = \begin{bmatrix}
E & 0 & 0 & 0 & -K \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & \tau I & 0 & 0 \\
0 & 0 & 0 & \tau I & 0 \\
0 & 0 & 0 & 0 & \tau_a I
\end{bmatrix}^{-1}
\begin{bmatrix}
O \\
0 \\
O \\
O \\
Z_{cfg}
\end{bmatrix}
$$

Six actuation configurations are evaluated for this study: a baseline configuration consisting of all twelve actuators equally spaced around the compressor annulus, two configurations with eight actuators, and three configurations with six actuators. The actuation configurations evaluated are shown in Figure G-1. The first configuration which consists of twelve actuators is the baseline against which all other actuator configurations are measured. The second configuration consists of four groups of two actuators. This configuration is selected because it provides a configuration for four big actuators. The third configuration consists of two groups of four actuators and provides a configuration for two big actuators. The fourth configuration consists of six actuators equally spaced around the compressor annulus, the fifth configuration consists of three groups of two actuators which provides a configuration for three big actuators, and the sixth configuration consists of two groups of three actuators which provides a configuration for two big actuators. All these configurations can be represented by the state space equation in G.3. The only change that has to be made from one configuration to another is the shape function matrix, $Z_{cfg}$, which depends on the actuation configuration. It should be noted that grouping the actuators imply that the grouped actuators will have the same command and will thus act as one input. Therefore, actuation configurations I, II, III, IV, V, and VI will have 12, 4, 2, 6, 3, and 2 inputs respectively.
Figure G-1: Actuation configurations for twelve, eight, and six actuators.
G.2 Parametric Studies and Results

This section presents the evaluation technique and results used for selecting the best actuation configurations. The first step consists of determining the optimal gain matrix required to stabilize the compression system at a given operating point. This is achieved by using a Linear Quadratic Regulator (LQR) design for continuous time systems. The second step consists of determining the rms value of the actuator activity for a white noise excitation of the closed loop system using the LQR controller. This procedure is then repeated at different operating points on the compressor map for all the actuation configurations.

G.2.1 Optimal Control Gain Matrix

The performance index or cost function of interest is the control power and amplitude of the flow coefficient perturbations. Therefore, the Linear Quadratic Regulator design is used to determine the optimal feedback gain matrix, $K_{opt}$, such that the state-feedback law:

$$u(t) = -K_{opt}x(t),$$

minimizes the cost function:

$$J = \int_0^{\infty} [z^T(t)z(t) + u^T(t)Ru(t)]dt$$

subject to the constraint equation: $\dot{x} = Ax + Bu$. $z(t) = Qx(t)$ is a linear combination of the states to be kept small, and these states of interest are weighted relative to the amount of control action in $u(t)$ through the weighting matrix, $R$ in the cost function $J$. Since the goal for this application of rotating stall control is to determine the minimum control activity to prevent the flow coefficient perturbations from growing, $R$ is selected to be ten orders larger than $Q$.

G.2.2 RMS of Control Activity

A white noise disturbance source introduced at the compressor input is reflected as shown in Figure G-2.

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The resulting state space representation of the closed loop system with the noise source is:

\[
\dot{x}(t) = A x(t) + B \left[ u(t) + \xi(t) \right] = [A - BK_{opt}] x(t) + B \xi(t)
\]

\[
y(t) = u(t) = -K_{opt} x(t)
\]

where the system output is the control term, \( u \). The rms of the actuator activity for a unit intensity white noise can then be obtained by solving the following Lyapunov matrix equation:

\[
A X + X A^T = -BB^T
\]  

\( (G.6) \)

### G.2.3 Results from Parametric Studies

Figure G-3a shows the actuator gain\(^1\) required to stabilize the compression system at different points on the compressor map for the actuation configurations with eight actuators. The actuator gain can be interpreted as a measure of the control power needed from each actuator to stabilize the compression system at a given operating point on the compressor.

\(^1\)The actuator gain is defined as the infinity norm of the gain matrix divided by the number of actuators. The infinity norm is selected because it represents the maximum sum of the rows of the control gain matrix.
Figure G-3b shows the RMS (root mean square) of the actuator activity at different points on the compressor map for the actuation configurations with eight actuators. This plot of actuator activity can be interpreted as a measure of the effort required for each actuator to reject compression system disturbances at a given operating point on the compressor map. Thus the actuator activity can be interpreted as a measure of operability. A similar set of measures of performance and operability for actuation configurations with six actuators are given in Figure G-4. The performance and operability measures for the best actuation configurations with twelve, eight, and six actuators are summarized in Figure G-5. From the plots of the actuator gain, it can be observed that for flow coefficients above the stall point at mean injection, \( \Phi = 0.4103 \), no control gain is required since the system is stable. The actuator gain increases as the compressor starts operating in the unstable portion of the compressor map. Both the actuator gain and the RMS of actuator activity in Figure G-5 show the same trends. The preferred actuation configurations are those for which the actuator gain and RMS of actuator activity are small. Thus the preferred actuation configuration order is: I (twelve actuators equally spaced around the compressor annulus), II (four groups of two actuators), IV (six actuators equally spaced around the compressor annulus), and V (three groups of two actuators). The parametric simulation results show that the groups of two actuators are preferred to the groups of three and four actuators, and the actuator gain required for stabilizing the compression system increases as the number of actuators is reduced.
Actuation configurations with eight actuators

(a) $\|K_{opt}\|_\infty$ for configurations with eight actuators

(b) RMS of actuator activity for configurations with eight actuators

Figure G-3: Performance comparison for actuation configurations with eight actuators.
Actuation configurations with six actuators

(a) \( \| K_{opt} \|_\infty \) for configurations with six actuators

(b) RMS of actuator activity for configurations with six actuators

Figure G-4: Performance comparison for actuation configurations with six actuators.
Figure G-5: Performance comparison for actuation configurations with twelve, eight, and six actuators.


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[124] VAN SCHALKWYK, C. M. “Active Control of Rotating Stall with Inlet Distortion”. GTL Report No. 222, Massachusetts Institute of Technology, June 1996.


