Analysis and Design of a Two-Axis Noncontact Position Sensor

by

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Submitted to the Department of Mechanical Engineering in partial fulfillment of the requirements for the degree of

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Abstract

This thesis presents the design, analysis and construction of a two-axis noncontact position sensor. We use this sensor in a magnetic levitation stage where we levitate a 0.64 mm (0.25 inch) diameter, 1 mm (0.04 inch) wall steel tube. The sensor has a circular opening approximately 13 mm (0.5 inch) in diameter through which the tube passes.

A three-pole arrangement with a three-phase input current generates a flux which changes as a function of the tube position. We model the sensor with a magnetic circuit and use this model to predict the relationship between the tube position and the flux behavior. We then use a signal processing board to convert the raw output from the sensor into two voltages, dependent on the x and y position of the tube, respectively.

In this thesis we also describe the design and construction of a three-phase signal generator which drives the three-phase field, the operation of the closed-loop current supply, and the design and construction of the signal processing board.

Thesis Supervisor: David L. Trumper
Title: Rockwell International Associate Professor of Mechanical Engineering
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For Larissa
# Contents

## 1 Introduction
1.1 Overview ........................................ 21
1.2 Background .................................... 22
  1.2.1 Sensor .................................... 23
  1.2.2 Actuator and Controls ...................... 26
  1.2.3 Results .................................. 27
1.3 Thesis Layout ................................... 31

## 2 Background Theory
2.1 Notation ......................................... 34
2.2 Maxwell's Magnetoquasistatic Equations ........... 34
  2.2.1 MQS Assumption ............................ 34
  2.2.2 Example Problem ............................ 35
  2.2.3 Transfer Functions ......................... 38
2.3 Magnetic Circuit Analogy ......................... 39
2.4 Magnetic Diffusion Equation ....................... 42
  2.4.1 Skin Depth ................................ 43
  2.4.2 Attraction or Repulsion ..................... 46
  2.4.3 Shielding ................................ 51

## 3 Sensor Development
3.1 Introduction ..................................... 55
3.2 LVDT ............................................ 56
  3.2.1 Operation .................................. 56
  3.2.2 Terminal Relations ......................... 57
3.3 E-Pickup ......................................... 62
  3.3.1 Experimental E-Pickup Setup ................. 66
3.4 Modified E-pickup ................................ 67
  3.4.1 Experimental Data From Modified E-pickup .... 72
3.5 Final Design .................................... 74

## 4 Field Analysis .................................... 77
4.1 Introduction ..................................... 77
4.2 Magnetic Circuit Analogy ......................... 77
  4.2.1 System Model ............................... 78
# 7 Conclusions and Suggestions for Further Work

- **7.0.3 Experimental Issues**
- **7.1 Closing Thoughts**
- **7.2 Suggestions for Further Work**

## A Matlab Code

- **A.1 Magnetic Vector Potential and Field Lines**
  - **A.1.1 Case 1: Uniform Field**
  - **A.1.2 Case 2: Fourier Series Field**
- **A.2 Predicted Output from Magnetic Circuit Analysis**
List of Figures

1-1 Photograph of one of the quartz ovens which cure the paint on the tube. The tube diameter is approximately 22 mm. A levitation station can be seen behind the oven. .............................................. 23

1-2 Two levitation stations spaced approximately 3.5 meters apart stand on either side of an induction heater. The electrostatic powder paint coating station is just visible at the far right. The tube moves from right to left in this picture at a velocity of 1-2 m/sec. .............................................. 24

1-3 Benchtop scale model showing five sensor and actuator pairs. We use an aluminum rail to position the components. As shown in the photo, the position of the sensor relative to the actuator alternates down the length of the setup....................................................... 25

1-4 Layout of the noncontact position sensor. Three current sources each drive a primary coil, which creates a flux read by the secondary coils. The flux paths, and thus the voltages induced on the secondary coils, are functions of the tube position............................................. 26

1-5 Photograph of the noncontact position sensor. The outer ring and inner shapes are aluminum shielding. We wrap the laminations in Teflon tape to prevent scoring the edges of the coil wire (creating a short circuit from scraping off the insulation); the laminations are therefore white in this photo. We pot each coil in epoxy for increased rigidity and resistance to scoring......................................................... 27

1-6 Photograph of the signal processing board. The BNC output jacks connect to the control computer. We can tune the output voltages using the potentiometers on either side of the jacks..................................... 28

1-7 Experimental hardware, including the sensor, the signal processing board, and the mounting hardware. The metal bracket is aluminum, and the translucent bracket is plexiglass......................................................... 28

1-8 Trace of path followed by the tube relative to the sensor which we use to estimate the linearity of the sensor output. The tube is held fixed and the sensor is moved around it so that the relative motion is as suggested by the lines in black. We then repeat this exercise for the x-direction using a similar but horizontal pattern......................................................... 29
1-9 Trace from the oscilloscope showing experimental data. The trace shows the $x$- and $y$-voltages output from the signal processing board plotted against each other as we move the tube and sensor relative to each other as shown in Figure 1-8. The ideal output is an exact trace of the path in Fig. 1-8.

2-1 Example magnetic circuit. A voltage $V_n$ across the primary coil drives a flux which links the secondary coil. We assume the air gap is small enough that we may ignore the fringing of the field.

2-2 Magnetic circuit element representation of the example problem. The resistive and voltage source elements model the behavior of the sensor, allowing us to solve for the terminal relations using traditional circuit analysis methods.

2-3 Field $H_x^o$ incident upon a conductor. Because of the conductivity, an opposing field is created in the conductor to repel the original field, thus limiting the penetration of the imposed field into the conductor.

2-4 Skin depth of a magnetic field in a conductor. On the $\frac{x}{y} = 0$ edge, the induced field inside the conductor instantaneously equals the applied $H$ field. The imposed field varies sinusoidally with time; the field in the conductor is a traveling wave.

2-5 Permeable, conducting, hollow cylinder in a uniform, time-varying field imposed as the vector potential at surface (e). We assume no variation in the $z$-direction, and finite permeability and conductivity in the tube region.

2-6 Field lines for a steel tube. The imposed field has frequency $\approx 0$ Hz; this is the DC case. We impose a sinusoidal vector potential along the outermost surface and use the transfer relations to calculate the distribution throughout the regions.

2-7 Field lines for a steel tube with an imposed field frequency of 5 kHz. The field inside the tube is now concentrated near the surface, but the field in the air-gap is similar to the DC case above.

2-8 Field lines for a steel tube with an imposed field frequency of 100 kHz. The field in the tube continues to concentrate in a thinner layer of the tube while the field in the air-gap remains essentially the same.

2-9 Field lines for an aluminum tube with an imposed field frequency of 100 kHz. We plot this case as a check on our solution, and we see the conductor repels the field as we expect.

2-10 Various flux paths: 1) through air-gap, 2) entering pole mid-way, 3) through secondary coil. Paths 1) and 2) are acceptable, but path 3) is not.

3-1 Cross section of a cylindrical LVDT. We assume the flux $\Phi$ travels directly from the highly permeable piston to the highly permeable outer core.
3-2 Schematic of a cylindrical LVDT, and associated radial field at the outer surface of the piston as a function of position.

3-3 Close-up of LVDT secondary coil. For the local axes we define the $z = 0$ point at the tip of the piston for computation of the flux linked by the secondary coil.

3-4 E-core schematic. A current supply drives the primary coil to create a flux dependent upon the piston position. The piston moves in the x-direction only.

3-5 Schematic of E-pickup sensor with magnetic circuit overlay. We lump all the leakage flux into a single term, however in the actual sensor there is leakage flux along the entire height of the poles.

3-6 Circuit diagram of the E-pickup sensor. Representing the magnetic elements with MCA equivalents allows us to use Kirchoff’s laws to calculate the flux.

3-7 E-core without coils. We use electrical tape to prevent scoring the coils on the square edges of the ferrite, and we mount the sensor in plastic to minimize the effect of the mount on the field distribution.

3-8 Physical dimensions of E-core used in first experiment. The depth into the page (in the z-direction) is 19 mm.

3-9 Experimental output from E-pickup sensor. Increasing the tube height from 0.5 mm to 1.5 mm decreases the sensitivity by about half.

3-10 Schematic of second E-core sensor. The secondary poles are now above the tube to give a different flux path. The depth of the sensor in the z-direction is 20.4 mm.

3-11 Modified E-pickup with copper shielding. The wires at the bottom of the photo are shielded in aluminum.

3-12 Result of demodulating a shifted sinusoid. The input signal is $\sin(\theta - \theta_0)$, and the reference is $\sin(\theta)$. We show the case for $\theta_0 = 1$ radian. If $\theta_0$ were $\frac{\pi}{2}$ radians, the DC component would be zero. Also note the primary frequency is now twice the original.

3-13 Experimental data from the modified E-pickup sensor without shielding, showing output voltage vs. lateral displacement. The legend shows tube height above primary pole face; we take each series of data at a different height as noted in the legend. The data series tend to spread towards the edges of tube travel.

3-14 Experimental data from the modified E-pickup sensor with shielding in place. Apart from the case at height 1 mm which is close to linear in both cases, the output is more uniform, with less deviation as the tube nears the edges of the sensor range. The difference is much more dramatic if we ignore the data series for the 1 mm case.

3-15 Experimental data from the modified E-pickup sensor comparing shielded case with unshielded case for tube heights of 1 mm and 9 mm. The 1 mm data series follow each other closely, with primarily a DC offset. The 9 mm series tend to differ more towards the edges of measurement.

3-16 Final design of position sensor.
3-17 Photograph of position sensor.

4-1 Equivalent magnetic circuit of the sensor. We assume the shielding will constrain the flux to the paths above; which we denote as reluctances, $\mathcal{R}$.  

4-2 Simplified magnetic circuit. The delta-wye transformations allow us to reduce the ten reluctances of the previous circuit to three.

4-3 Geometry for calculating a lumped-parameter reluctance. The shaded area is the area over which we integrate to find an exact solution. The cross-hatched area is the rectangular approximation.

4-4 Cutout of center of sensor showing the geometry we use to calculate the lumped parameter reluctances. The cross-hatched area represents the aluminum shielding, shaded areas $\mathcal{R}_{LB}$ and $\mathcal{R}_{1t}$ represent the assumed flux paths. The dimension $w_p$ is one-third of the lamination pole thickness.

4-5 Phasor components of fluxes through poles $1$, $2$ and $3$ normalized to $\frac{\Phi_1 + \Phi_2 + \Phi_3}{\sqrt{3}N_p I}$. We see from the symmetry of the phasors that $\Phi_1 + \Phi_2 + \Phi_3 = 0$, as we expect from driving the sensor with a three-phase signal.

4-6 Flux $\Phi_B$ (in black) and corresponding components (in grey), normalized to $\frac{-\sqrt{3}}{N_p I}$, along with output voltage $V_B$ normalized to $\frac{-\sqrt{3}}{\omega N_p N_s I}$. We show the case for the tube in the center of the sensor such that the reluctances $\mathcal{R}_{Aeq}, \mathcal{R}_{Beq}, \mathcal{R}_{Ceq}$ are all equal.

4-7 Geometry for calculating reluctance gap lengths. We define the tube position in Cartesian coordinates with origin at the center of the sensor to facilitate control about the operating point $(0,0)$. The shaded blocks are the ends of the laminate poles.

4-8 Geometry for calculating leakage reluctance path area. Three shaded blocks are the ends of the poles; we omit the rest of the sensor for clarity.

4-9 Contour plot of lines of constant voltage signal magnitude $|V_B|$ across secondary terminals. These are lines of constant voltage from our theoretical analysis. We allow the center of the tube to move inside the radius $r_o - r_t$, such that we plot the voltage corresponding to the location of the center of the tube. We overlay a path of constant radius from pole one.

4-10 Plot of $V_x, V_y$ as a function of position, output from actual sensor. The signal processing board combines the three voltages from the secondary coils into two voltages $V_x$ and $V_y$.

4-11 Plot of $V_x, V_y$ as a function of position using a first order approximation and the exact theoretical values from the Magnetic Circuit Analysis.

4-12 Plot of $V_x, V_y$ as a function of position using the adjusted values of the reluctances $\mathcal{R}_{1t,2t,3t}$. This is also a first order approximation.

4-13 Plot of $V_x, V_y$ as a function of position using all terms. We see that the pattern is slightly closer to a perfect grid output, and that the magnitude is roughly two orders of magnitude greater.
Contour plot of theoretical phase of voltage $V_B$ across secondary terminals. We see that the phase has a different effect than the magnitude: as the tube moves in the $x$-direction, the phase change of $V_B$ is nearly linear in the inner area of the sensor. The units of the legend are radians.

Plot of $V_x$ and $V_y$ using the phase of the voltages instead of the magnitude. The output looks even more linear than when using the magnitude.

Permeable, conducting, hollow cylinder in a uniform, time-varying field imposed as the vector potential at surface (e). We assume no variation in the $z$ direction, and finite permeability and conductivity in the tube region.

Layout of the field problem showing the geometry we use for calculating the Fourier series representation of the field component $H_x$.

Magnetic field (radial-component) at surface (e) as a function of angle $\theta$. This plot shows the complex amplitudes of the step functions. We observe the real part of these amplitudes, so they will never all be the same at any given moment in time because of the three-phase nature of the excitation currents.

Derivative of radial component $H^z$ with respect to $\theta$ at surface (e), plotted as a function of $\theta$.

Magnetic scalar potential at surface (e) at time $t = 0$. Here we use a Fourier series of 51 terms to approximate the field, which we assume has discontinuous steps at the pole faces and is zero otherwise. The overshoot at the step locations is known as the Gibbs phenomenon.

Magnetic field lines showing flux density inside sensor. For computational reasons we only use an 11-term Fourier series approximation of the vector potential at the outer boundary as detailed above. In this case we use the actual geometry from the bench-top scale model with the 6.4 mm tube, and a 5 kHz excitation frequency.

Magnetic field lines showing flux density inside sensor for a tube much smaller than the sensor opening. We impose a 11-term Fourier series vector potential at the outer surface and assume the potential in the middle is zero. The leakage flux clearly dominates.

Complete electronics setup. The signal generator supplies three voltage signals to the current sources, which power the primary coils of the sensor. The signal processing board converts the voltages from the secondary coils into two voltages $V_x$ and $V_y$ which supply position information to the control computer.

Schematic of the three phase signal generator. The same counter drives all three EPROMs, so that the resultant signals retain the proper phase relationships.

Detailed view of the output from the DAC. The two current outputs drive the first op-amp, and the second op-amp removes the DC component of the signal. As configured, the output has a range of $\pm 5$ V.
5-4 Photograph of the top and bottom of the three-phase signal generator circuit board. We use a combination of wire wrap and solder to build the circuit. The tape strips labeled “A”, “B” and “C” cover the UV window on the EPROMs so that the ambient light does not erase the memory over time.

5-5 Circuit diagram of the current amplifier. The inductive load is eight primary coils from eight sensors. Three identical current supplies drive the three phases separately.

5-6 Current amplifier control block diagram.

5-7 Loop transmission of the plant and proportional controller.

5-8 Complete transfer function from the input to the output of the current supply.

5-9 Demodulation circuit layout.

5-10 Demodulation circuit wiring diagram. The top figure is the signal layer, complete with silk screen printing which includes the text and the component outlines; the bottom figure is the power and ground layer, the thinner wires will be absorbed in the copper pour.

5-11 Top view of a populated board and bottom view of a bare board. Except for a few signal wires, the bottom of the board is exclusively for power and ground circuits.

6-1 Photograph of the modified E-pickup without coils or shielding, shown with the shielding pieces. We ground the shielding at a single point to avoid closing a conductive loop around a flux path.

6-2 Photograph of the lamination pieces. Also shown are three sections arranged in a circle, wrapped in Teflon tape to prevent the sharp edges from scoring the coil wire.

6-3 Photograph of the complete sensor, showing the coils, shielding, laminations and the plastic ring which positions the components.

6-4 Five sensor/actuator stations along the aluminum rail. We mount the feet of the sensor brackets flush along one side of the rail (the right side as seen in the photograph).

6-5 Photograph of the sensor mount and printed circuit board. We mill a slot out of the side of the mounting bracket for the wire to pass through. Three set-screws allow for some final positioning of the sensor as necessary.

7-1 Tube levitated with two sensor-actuator stations. We add magnetic shielding to the sensor to reduce the effect of the field from the actuator on the sensor output.
List of Tables

2.1  Selected properties of materials used in the construction of the sensor. 35
2.2  Summary of Maxwell's equations under the magnetooquasistatic assumption. 35
2.3  Skin depth $\delta$ as a function of frequency for selected materials. We list the frequencies in Hertz, but convert to radians/sec for use with equation (2.29). 46
5.1  Values of components used in the demodulation circuit, as seen in Figure 5-9. 136
Chapter 1

Introduction

1.1 Overview

This thesis presents a 2-axis, noncontact position sensor for permeable steel tubes. Manufacturing processes often simultaneously require closed-loop position control and noncontact position sensing. Noncontact sensing is essential where contact might undesirably alter the surface, or where the surface is coated or contaminated in a way which makes contact problematic. Traditional electromagnetic sensors usually sense only in one direction; our design combines three one-dimensional sensors to give a two dimensional position reading with some redundant information which we use for error correction. This sensor provides the feedback device in a magnetic levitation setup using eight such sensors and eight two-axis actuators to levitate a steel tube. Along with the electromechanical design and construction of the sensor, we also present the electronic circuits which drive the sensor input and process the sensor output. These are the three-phase signal generator which commands the current supply, the current supply which drives the sensor, and the signal processing board which calculates the relevant voltages dependent on the tube position.
1.2 Background

Early in 1995, Professor David Trumper began a consulting relationship with the American Metal Handle company (AMH) to help develop a production line for metal broom and mop handles\(^1\). AMH manufactures handles in a continuous process, beginning with a flat strip of steel which is formed and seam-welded into a tube. The tube exits the forming mill at 1-2 meters per second, is cleaned, coated with powdered paint, heated to cure the paint, quenched in a water bath, and finally parted with a flying cutoff mechanism. From the time the paint powder is applied to the time the cured tube is quenched in water, it cannot be touched without marring the surface finish. Ten magnetic levitation stations spaced over an approximately 35 meter span levitate the metal handle during processing. Figures 1-1 and 1-2 show the actual AMH production line with the tube suspended.

Each of the ten levitation stations uses electromagnets for suspension and commercial eddy-current position sensors for position measurement. The suspension system is difficult to tune, but was eventually stabilized after much trial and error. While consulting for AMH, Professor Trumper realized there was a lack of general theory for noncontact sensing and actuating and submitted a proposal to the National Science Foundation; this research is conducted under funding from the resulting grant (DMI-9700973).

The goals of the present phase of this research are three-fold: 1) design a noncontact sensor, 2) design an efficient noncontact actuator, and 3) derive the control theory for stabilizing a flexible structure supported at a number of discrete locations. Doctoral candidate Ming-chih Weng and myself, along with Professor Trumper, are the principal researchers. Mr. Weng has focused on the actuator design and control system design; the sensor is the topic of this thesis. To test our results we have constructed a \(\frac{1}{16}\) scale model using eight sensor-actuator stations to levitate a 6.4 mm steel tube. Figure 1-3 shows the bench-top scale model built in our laboratory.

\(^1\)Conrad Smith, American Metal Handle, 511 Vulcan Dr., Irondale, AL 35210
1.2.1 Sensor

We developed the sensor through a succession of design iterations. The final sensor design, shown schematically in Figure 1-4 and in a photograph in Figure 1-5, uses a differential magnetic flux measurement to determine the position of the tube, which passes through the center of the sensor. Current sources drive the three primary coils with sinusoidal currents $I_1$, $I_2$ and $I_3$ at a frequency of 5 kHz. Because of the geometry of the sensor and the magnetic permeability of the tube, the flux path depends upon the tube position. The voltages across the secondary coil terminals depend on the amount of flux linked by the coils, and thus by reading the AC voltages across the
secondary coils we can determine the tube position.

The primary coils drive a 5 kHz sinusoidal field which circulates through the air-gap in the center of the sensor, then back around through the laminate core. A three-phase current supply drives the sensor such that the current in each primary coil is out of phase by 120° from the neighboring primary coils. Because the poles are arranged geometrically at 120° intervals, the induced magnetic field is a 5 kHz traveling wave. A magnetic field at this frequency will only penetrate aluminum to a depth of about 1.2 mm. We design the aluminum shielding in Figure 1-4 thicker than this depth, therefore the shielding guides the flux by not allowing it to escape the air-gap region without returning through a lamination pole. Similarly, aluminum plates sandwich the sensor to reduce leakage fields in the z-direction, i.e., out of the plane of the sensor.

The voltages induced on the secondary coils also vary sinusoidally at 5 kHz; chang-
Figure 1-3: Benchtop scale model showing five sensor and actuator pairs. We use an aluminum rail to position the components. As shown in the photo, the position of the sensor relative to the actuator alternates down the length of the setup.

ing the tube position affects the amplitude and phase of these signals. The tube position has the greatest effect on the amplitude of the signal and consequently in the present set of electronics we only use the amplitude to predict the tube position. Although this means we discard the phase information, we still use three signals to find two position measurements, which allows for error averaging.

To determine the position of the tube we use the voltages from the three secondary coils. We analyze the magnetic fields in the sensor to relate the output voltages to the tube position; inverting these relations allows us to find the tube position in terms of the output voltages. The analog circuit board shown in Figure 1-6 performs this conversion. The circuit rectifies and combines the voltage signals from the sensor and outputs two voltages proportional to the x- and y-position of the tube, respectively.
Figure 1-4: Layout of the noncontact position sensor. Three current sources each drive a primary coil, which creates a flux read by the secondary coils. The flux paths, and thus the voltages induced on the secondary coils, are functions of the tube position.

1.2.2 Actuator and Controls

We use this position information and the electromagnetic actuators to levitate the tube. A Bernoulli-Euler beam model describes the tube dynamics and predicts the mode shapes and vibration frequencies. Effective placement of the actuators and sensors depends on this dynamic behavior. Placing a sensor too near a node will leave that mode unobservable, while placing an actuator there will leave the mode uncontrollable. Additionally, of necessity the actuator applies a force at a position near to, but not exactly at the position being sensed. This noncollocation means that any mode with a period smaller than twice the noncollocation distance will not be readily controllable.

In addition to the placement of the components, we must decide whether to control each station independently, or to use a state space model of the entire tube system...
Figure 1-5: Photograph of the noncontact position sensor. The outer ring and inner shapes are aluminum shielding. We wrap the laminations in Teflon tape to prevent scoring the edges of the coil wire (creating a short circuit from scraping off the insulation); the laminations are therefore white in this photo. We pot each coil in epoxy for increased rigidity and resistance to scoring.

and control it as a whole. Doctoral candidate Ming-chih Weng is addressing this set of challenges and it is thus not the main focus of my thesis.

1.2.3 Results

Figure 1-7 is a photo of the experimental hardware, including the final design of the sensor, the signal processing circuit board, and the mounting bracket for positioning the sensor on the rail. A total of eight sensors and eight actuators form the complete setup. We fabricated an aluminum rail and sensor-actuator mounts to support and align the components. Figure 1-3 is a photo of five of these sensor-actuator pairs
Figure 1-6: Photograph of the signal processing board. The BNC output jacks connect to the control computer. We can tune the output voltages using the potentiometers on either side of the jacks.

Figure 1-7: Experimental hardware, including the sensor, the signal processing board, and the mounting hardware. The metal bracket is aluminum, and the translucent bracket is plexiglass.
mounted to the alignment rail.

To show the linearity of the sensor output we plot the $x$- and $y$-voltages against each other as we move the tube in a grid pattern inside the sensor. To accomplish this we mount the sensor on a micrometer table with horizontal and vertical travel, (travel in the $x$- and $y$-directions respectively corresponding to the coordinate frame in Fig. 1-4). The tube is held fixed with reference to the base of the micrometer table, passing through the opening of the sensor as it would during operation. Figure 1-8 shows the path followed to give the output seen in Figure 1-9.

Figure 1-8: Trace of path followed by the tube relative to the sensor which we use to estimate the linearity of the sensor output. The tube is held fixed and the sensor is moved around it so that the relative motion is as suggested by the lines in black. We then repeat this exercise for the $x$-direction using a similar but horizontal pattern.

Beginning with the tube in the sensor, centered in the $y$-direction and almost at the rightmost position in the $x$-direction as in Fig. 1-8, we move the sensor vertically in the negative $y$-direction. As the tube reaches the top of the opening we move the sensor one millimeter to the right, then vertically in the positive $y$-direction until reaching the lower extent of travel. We repeat this procedure until the entire inside of the sensor is covered in the $y$-directed lines. Repeating the same procedure for the
x-direction gives a 1 mm grid tracing of the opening.

During this tracing procedure the oscilloscope records $x$- and $y$-voltages from the signal processing board; we call these $V_x$ and $V_y$ respectively. Using two inputs to the scope allows us to plot the data in $x$ vs. $y$ format (as opposed to the default $x$ vs. time format). The “infinite persistence” setting on the scope keeps the entire trace history on the screen. We save this data trace and export it to a file. This is the plot shown in Figure 1-9.

The deviation from linearity is obvious in the output. If the output were perfectly linear the plot would resemble a perfect grid truncated by the circular shape of the sensor opening, i.e., the path traced by the tube. When the tube is near the center of the sensor, the grid shape is clearly visible. However, as the tube nears the edges of the sensor aperture, and especially as it nears the poles, the deviation from linearity increases. The three “points” seen in the experimental output are a result of the increased sensitivity as the tube nears one of the three poles. In these areas, 1 mm of
movement results in a larger change in the output voltage than that corresponding to 1 mm of movement in the center of the sensor. For use with the control system we adjust the output gain on the signal processing board so that 2 mm of displacement on either side of center spans the output voltage range of ± 10 V.

1.3 Thesis Layout

This chapter outlines the goals of the research, the topics covered in this thesis, and gives a brief description of the final sensor design. We organize the rest of the thesis as follows. Chapter 2 presents a background in the necessary theory to familiarize the reader with the concepts used in the rest of the analysis. We present relevant sensor operation principles and topologies in Chapter 3. In Chapter 3 we also describe the design evolution of the sensor. Chapter 4 develops the mathematical analysis of the sensor’s operation and compares this to the experimental results. This comparison also helps to refine the parameters used in the analysis. Chapter 5 presents the design and construction of the electronic circuits necessary for driving the sensor and processing its output waveforms, including the signal generator board, current supply, and the signal processing board. Chapter 6 presents the sensor construction, including material choice and machining. Chapter 7 summarizes the results presented in this thesis and discusses possibilities for future work.
Chapter 2

Background Theory

In this chapter we establish the theory used to analyze the electromagnetic systems developed in this thesis. Maxwell's equations, as simplified by the magnetoquasistatic assumption, form the basis of the analysis from which we explore effects such as skin depth and magnetic diffusion. We develop a lumped parameter revision of these equations via the Magnetic Circuit Analogy (MCA), which is useful where the geometry of the problem is complex and we prefer to work with lumped reluctances [10].

Because many references discuss Maxwell's equations, only a brief introduction follows in this thesis. We encourage the interested reader to investigate [2, 7] and [16] for a more thorough discussion. We first demonstrate the usage of these equations in magnetic circuits by way of a simple example problem, i.e., the air-gap transformer seen in Figure 2-1. When the geometry of the problem does not facilitate the direct solution of Maxwell's equations, we can frequently take a lumped-parameter approach for simplification. We introduce one such lumped parameter method, the MCA, in the context of the example problem in order to compare the two methods.

When the system involves conducting materials, alternating magnetic fields give rise to alternating currents, which in turn affect the field distribution. We present these magnetic diffusion effects as well; specifically with respect to shielding, skin depth and the issue of what we can consider "perfect" conduction.
2.1 Notation

Many of the variables in this analysis are sinusoidal signals which we represent using complex exponential notation as described in [7]. These consist of spatial functions which vary sinusoidally with time. For example we represent a sinusoidal \( \Phi(x, y, z, t) \) as

\[
\Phi(x, y, z, t) = \text{Re}\{\Phi(x, y, z)e^{j\omega t}\},
\]

where \( \Phi(x, y, z) \) is the complex amplitude of the signal. Similarly, when we use a variable in polar coordinates which has a complex amplitude that varies sinusoidally with \( \theta \) and time, we use

\[
\Phi(r, \theta, z, t) = \text{Re}\{\Phi(r, z)e^{j(\omega t - m\theta)}\}.
\]

The variable \( m \) is the angular wave number, and assumes only integer values.

2.2 Maxwell’s Magnetoquasistatic Equations

2.2.1 MQS Assumption

Maxwell’s equations summarize the rules electromagnetic fields have been found to obey. Depending on the system parameters we may make some simplifications; the most helpful in this situation is the magnetoquasistatic (MQS) assumption. We will consider a system MQS if the characteristic time of interest (here, the reciprocal of the excitation frequency) is much larger than the time it takes an electromagnetic wave to propagate over a characteristic length. Dividing the characteristic length by the speed of light and comparing to the characteristic time, the equation

\[
\frac{L}{c} << \tau
\]

holds for MQS systems. Here, \( c \) is the speed of light in a vacuum, \( \tau \) is the reciprocal of the frequency (in our case the frequency is 5 kHz), and \( L \) is the characteristic
length. We use \( L = 0.075 \) m, the largest dimension in the system. Substituting in these numbers gives

\[
\frac{0.075 \text{m}}{3 \times 10^8 \text{m/s}} = 2.5 \times 10^{-10} \ll 2.0 \times 10^{-4} \text{sec.} \quad (2.4)
\]

Therefore we may analyze this system using MQS techniques.

Also, we assume the materials we use to construct the sensor are magnetically linear, such that \( B = \mu H \) and \( \mu \) is constant. Table 2.1 shows typical values for the permeability \( \mu \) and the conductivity \( \sigma \) of the materials we use for the sensor construction.

<table>
<thead>
<tr>
<th>Material</th>
<th>Conductivity ( (\frac{1}{\Omega \text{m}}, 20^\circ \text{C}) )</th>
<th>Permeability ( (\frac{\mu \text{s}}{\text{Am}}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum</td>
<td>( 3.54 \times 10^7 )</td>
<td>( \approx 4\pi \times 10^{-7} = \mu_o )</td>
</tr>
<tr>
<td>Steel Tube</td>
<td>( 0.75 \times 10^7 )</td>
<td>( \approx 5 \times 10^3 \mu_o )</td>
</tr>
<tr>
<td>Silicon Iron Lamination</td>
<td>( 2.1 \times 10^6 )</td>
<td>( \approx 7 \times 10^4 \mu_o )</td>
</tr>
</tbody>
</table>

Table 2.1: Selected properties of materials used in the construction of the sensor.

To summarize the MQS versions of Maxwell’s equations we present Table 2.2.

<table>
<thead>
<tr>
<th></th>
<th>Integral Form</th>
<th>Differential Form</th>
<th>Boundary Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ampere</td>
<td>( \oint_C \vec{H} \cdot d\vec{s} = \int_S \vec{J} \cdot \vec{n} , da )</td>
<td>( \nabla \times \vec{H} = \vec{J} )</td>
<td>( n \times [\vec{H}] = \vec{K} )</td>
</tr>
<tr>
<td>Faraday</td>
<td>( \oint_C \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int_S \vec{B} \cdot \vec{n} , da )</td>
<td>( \nabla \times \vec{E} = -\frac{d}{dt}\vec{B} )</td>
<td></td>
</tr>
<tr>
<td>Gauss</td>
<td>( \oint \vec{B} \cdot \vec{n} , da = 0 )</td>
<td>( \nabla \cdot \vec{B} = 0 )</td>
<td>( n \cdot [\vec{B}] = 0 )</td>
</tr>
</tbody>
</table>

Table 2.2: Summary of Maxwell’s equations under the magnetoquasistatic assumption.

### 2.2.2 Example Problem

We analyze an example problem to compare the direct application of Maxwell’s equations to a solution using the Magnetic Circuit Analysis. Figure 2-1 shows a permeable core transformer with a small air gap. A voltage \( V_m \) drives the primary coil causing
Figure 2-1: Example magnetic circuit. A voltage $V_{in}$ across the primary coil drives a flux which links the secondary coil. We assume the air gap is small enough that we may ignore the fringing of the field.

A current $I_{in}$ to flow through the coil. This current induces a magnetic field $H_c$ in the core and magnetic field $H_g$ in the air gap. We ignore the fringing fields by assuming that the gap $g$ is much smaller than either dimension of the cross-sectional area $A_g$. For simplicity we assume the cross-sectional area of the air gap is the same as the cross-sectional area of the core, i.e., $A_c = A_g$. If this were not so, the change in flux density as a function of position along path $l_L$ would be inversely proportional to the change in area, as $\vec{B}A$ remains constant along the path $l_L + g$ (because of the high core permeability we assume $\vec{B}$ is always perpendicular to the path). By assuming constant area, we impose a constant flux density $\vec{B}$. We will often carry the two areas $A_g$ and $A_c$ through the calculations as distinct parameters to clearly distinguish to which area we refer.

**Ampere’s Law**

Applying Ampere’s law along path $l_L + g$ gives
\[ H_c I_L + H_g g = N_p I_{in}. \] (2.5)

Since we model the secondary coil as an open circuit, no current flows in this wire. If there were a current in the secondary coil, it would show up on the left side of equation (2.5) as an additional term \( N_s I_s \).

**Gauss’ Law (Magnetic)**

Gauss’ law specifies that the net flux passing through a closed surface is zero. We select a surface which encloses only the top half of the core and passes through the air gap, and use the constitutive law for linear magnetic materials to give

\[ \mu_0 H_g A_g = \mu H_c A_c. \] (2.6)

**Faraday’s Law**

Faraday’s law relates the electric field along a closed contour to the flux passing through the contour. In this case the contour is \( C_1 \), which goes from point 2 to point 1 across the voltage \( V_m \), then through the coil back to point 2. We can break up the total integral into the integral across the terminals plus the integral along the wire. Employing Ohm’s law for conductive materials \( (J = \sigma E) \) in the wire gives

\[
\oint_{C_1} E \cdot dl = \int_{\text{terminals}}^1 E \cdot dl + \int_{\text{coil}}^2 \frac{J}{\sigma} \cdot dl. \tag{2.7}
\]

The electric field across the terminals is simply the voltage \( V_{in} \) divided by the distance separating the terminals, so integrating from 2 to 1 returns \(-V_{in}\). Along the wire, the current density \( J \) is the current \( I_{in} \) divided by the cross-sectional area of the wire, \( A_{xc} \). Integrating along the length of the wire returns \( I_{in}R \), where we define \( R \) as

\[ R = \frac{I_{wire}}{\sigma A_{xc}}. \] (2.8)

Thus, the left side of Faraday’s equation becomes
\[ \int_{C_1} E \cdot dl = -V_{in} + I_{in} R. \]  

(2.9)

The right side of Faraday’s equation concerns the magnetic flux density linked by the coils. We assume the flux is constant across the area and parallel to the surface normal \( \vec{n} \). As a result, the dot product in the integral returns the magnitude of the magnetic flux, which we denote simply as \( B \). The result of the integral is thus the product \( BA_cN_p \), where \( N_p \) is the number of turns of the primary coil. Therefore

\[ -\frac{d}{dt} \int_A \vec{B} \cdot \vec{n} \, dA = -\frac{d}{dt}(BA_cN_p), \]  

(2.10)

and the result of applying Faraday’s law is

\[ V_{in} - I_{in} R = A_cN_p \frac{dB}{dt}. \]  

(2.11)

Recall that the cross-sectional areas of the air gap and coil are the same; this means the magnetic flux density \( B \) is constant throughout the magnetic circuit.

To find the voltage \( V_{out} \) across the secondary coil we again use Faraday’s law, but this time the terminals are an open circuit so that the current is zero. Assuming the flux density \( \vec{B} \) is constant over the cross sectional area of the laminations (and perpendicular as above) Faraday’s law reduces to

\[ V_{out} = N_sA_c \frac{dB}{dt}. \]  

(2.12)

### 2.2.3 Transfer Functions

We may now write two important transfer functions using the above relationships. Solving equation (2.6) for \( H_c \) and substituting into (2.5) gives

\[ N_pI_{in} = H_g \left( g + \frac{\mu_oA_l}{\mu_A} \right). \]  

(2.13)

We now combine this with (2.12). Replacing the time derivative with the Laplace variable \( s \) to simplify the equations results in the transfer function
Solving (2.11) for $V_{in}$ and substituting equation (2.13) gives an equation for $V_{in}$ in terms of $H_g$,

\[ V_{in} = H_g \left[ s(\mu_o N_p A_c) + \frac{R}{N_p} \left( g + \frac{\mu_o A_g l_L}{\mu A_c} \right) \right]. \]  

(2.15)

Dividing equation (2.12) by (2.15) and using $B = \mu_o H_g$ results in the transfer function from $V_{in}$ to $V_{out}$,

\[ \frac{V_{out}}{V_{in}} = \frac{s(\mu_o N_p N_s A_g)}{s(\mu_o N_p A_c) + \frac{R}{N_p} \left( g + \frac{\mu_o A_g l_L}{\mu A_c} \right)}. \]  

(2.16)

An important point is immediately evident: using a current source for $I_{in}$ (2.14) instead of a voltage source for $V_{in}$ (2.16) results in a much simpler transfer function. Because the volumes and areas involved here are rather basic, we can easily compute the integrals in Maxwell’s equations. Sometimes the geometry is more complex. In this case we can adopt the Magnet Circuit Analogy formulation to allow us to write the equations in a more direct manner.

### 2.3 Magnetic Circuit Analogy

In the preceding analysis we assumed that the quantities $g$ and $A_g$ were known and constant. Often in magnetic circuits, the exact path of the flux (defined by $g$ and $A_g$) is unknown, and is a function of the position of some part of the circuit (as in an electromagnetic actuator where the flux travels through the moving target). Combining the unknowns $g$ and $A_g$ with the permeability $\mu_g$ into a single unknown quantity can simplify the analysis. We define the reluctance $\mathcal{R}$ as:

\[ \mathcal{R} = \frac{g}{\mu_g A_g}. \]  

(2.17)

Recall from Ampere’s law that the line integral of a magnetic field around a closed
loop is equal to the current density passing through the surface area bounded by the loop. When the flux density $\vec{B}$ is constant over an area $A$, we can simplify the flux $\Phi$ to the product $BA$, where we define the surface area normal vector $\vec{n}$ parallel to the direction of the magnetic flux so that the vector dot product $\vec{B} \cdot \vec{n}$ returns the magnitude $B$. Also, we use the relation $\vec{B} = \mu_0 \vec{H}$ and rewrite Ampere’s law as

$$\oint \frac{\Phi}{\mu_0 A} \cdot ds = \int \vec{J} \cdot \vec{n} da = JA_{xc}. \quad (2.18)$$

Here we assume the current density is perpendicular to the cross-sectional area of the wire, giving the same simplification described above. The product $JA_{xc}$ equals the total current; if the wire carries current $i$ and is arranged as a coil with $N$ turns we can also express this as $Ni$.

For the MCA, we define sections of flux path as reluctances, and assume that the quantities $\mu_0, A$ are constant in each section; this allows us to pull them out of the integral above. When a connected path of $n$ such reluctances encircles a total current density of $Ni$, we can write equation (2.18) as a sum of these reluctances,

$$\sum_{j=1}^{n} \Phi_j R_j = Ni. \quad (2.19)$$

In the magnetic circuit analogy we treat the current density $Ni$ (or magnetomotive force) as a voltage, the reluctance as a resistance, and the flux as a current. The above equation (2.19) is thus analogous to Kirchoff’s Voltage Law. Similarly, Gauss’ law supplies the analogous equation for Kirchoff’s Current Law, by requiring that the net flux (current) entering a closed surface (node) be zero. Equipped with these new relations we model the above magnetic circuit by overlaying the components with their MCA equivalents, as indicated in Figure 2-2.

The proper choice of $g$ and $A_g$ is important for the accuracy of the solution; choosing these assumes detailed knowledge of the flux paths. Across small air gaps, the choice is obvious; but for large gaps and non-uniform geometry we must guess $g$ and $A_g$. We can describe the system in terms of known inputs $Ni$ and unknown reluctances which are a function of geometry; the better our guess for $g$ and $A_g$, the
more accurate the model. The rules for addition of reluctances in series and in parallel directly follow those for resistances in the traditional circuit.

To find the flux, we solve the circuit of Figure 2-2 using standard circuit analysis methods with the reluctances as shown in the figure. As before we assume the fringing of the field across the air gap to be negligible. Applying Kirchoff’s Voltage Law to the loop in Figure 2-2 gives

$$\Phi = \frac{N_p I}{(\frac{I_L}{\mu A_c} + \frac{g}{\mu_0 A_g})}. \quad (2.20)$$

The secondary coil, of inner area $A_c$ and $N_s$ turns, encircles this flux, resulting in $V_{out}$ across the terminals. Applying Faraday’s law results in
\[ V_{\text{out}} = \frac{-d}{dt}(N_s \Phi), \quad (2.21) \]

where the terminals are still an open circuit so the current in the wire is zero. Solving for the relevant transfer function,

\[ \frac{V_{\text{out}}}{I_{\text{cm}}} = s \left( \frac{N_p N_s}{\mu L c} \right) = s \left( \frac{\mu_0 N_p N_s A_g}{g + \frac{\mu_0 A_g L c}{\mu A c}} \right), \quad (2.22) \]

which is the same as we found using the integral forms of Maxwell’s equations. The main simplification is the lumping of the geometric parameters into a single reluctance term. The validity of the answer still depends on our choice for these reluctances, but these are now simple lumped parameters rather than areas of integration. Most importantly, we may use traditional circuit analysis rules to derive the system relations.

### 2.4 Magnetic Diffusion Equation

A time-varying magnetic field can induce an electric field in a conductor; this interaction is called magnetic diffusion. Like Maxwell’s equations in the previous section, we limit this derivation to the extent relevant to our application. For a more complete analysis, we direct the reader to [16, 2] and [7].

For the derivation of the magnetic diffusion equation, the differential form of Maxwell’s equations are most convenient. The first step is to combine Ampere’s law with the constitutive law for Ohmic materials,\(^1\)

\[ \nabla \times \vec{H} = \vec{J} = \sigma \vec{E}. \quad (2.23) \]

We take the curl of both sides, and substitute Faraday’s law in for \( \nabla \times \vec{E} \), giving

\[ \nabla \times \nabla \times \vec{H} = -\sigma \frac{\partial \mu \vec{H}}{\partial t}. \quad (2.24) \]

\(^1\)Here the conductor is stationary; for a moving conductor, Ohm’s law is \( \vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B}) \).
Finally, a vector identity \(^2\nabla \times \nabla \times \vec{A} = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}\) reduces the magnetic diffusion equation to its most familiar form,

\[
\nabla^2 \vec{H} = \sigma \mu \frac{\partial \vec{H}}{\partial t}.
\]  (2.25)

The magnetic field inside a conductor will satisfy this equation.

### 2.4.1 Skin Depth

To predict whether the flux will be repelled from the tube or attracted to it, we examine the rules of magnetic diffusion. Because of the effect described by Ampere's law, an imposed field can induce a volume current in the conductor which will tend to repel the original field. The field decays exponentially with depth into the conductor. The depth to which the field penetrates a conductor, known as the skin depth, depends on the frequency of excitation and the material properties of the conductor. For the ideal "perfect conductor" with infinite conductivity and with permeability \(\mu_o\), the field is completely repelled; while for an insulator with zero conductivity and permeability \(\mu_o\), the field will pass straight through. For this analysis we assume the surrounding medium is air, which is nonconducting and has permeability \(\mu_o\).

This derivation loosely follows those presented in [15] pp. 442-443 and [16] pp. 358-360. The field inside a conductor must satisfy (2.25). Assuming an \(x\)-directed, sinusoidally varying magnetic field is imposed tangentially upon a conductor as shown in Figure 2-3, we propose the resulting field inside the conductor will take the general form

\[
H_x(y, t) = \text{Re}\{\tilde{H}_x(y)e^{j\omega t}\}.
\]  (2.26)

Substituting this into (2.25) gives

\[
\frac{d^2 \tilde{H}_x}{dy^2} = j \sigma \mu \omega \tilde{H}_x.
\]  (2.27)
Figure 2-3: Field $H_0^\circ$ incident upon a conductor. Because of the conductivity, an opposing field is created in the conductor to repel the original field, thus limiting the penetration of the imposed field into the conductor.

Here we use the fact that $H_z$ only varies spatially with $y$ to simplify the Laplacian in (2.25). Substituting a general solution of the form $\tilde{H}_z(y) = A_1 e^{(a+jb)y} + A_2 e^{(a-jb)y}$ into the above equation and cancelling like terms gives

$$(a \pm jb)^2 = j\mu\omega\sigma,$$  \hspace{1cm} (2.28)

which we simplify by defining the skin depth $\delta$ as

$$\delta = \sqrt{\frac{2}{\mu\omega\sigma}}.$$  \hspace{1cm} (2.29)

Using the relation $\sqrt{j} = \frac{1+j}{\sqrt{2}}$ and solving equation (2.28) for $a \pm jb$ in terms of $\delta$ results in a solution of the form

$$\tilde{H}_z(y) = A_1 e^{\frac{(1+j)y}{\delta}} + A_2 e^{-\frac{(1+j)y}{\delta}},$$  \hspace{1cm} (2.30)

where $A_1$ and $A_2$ are constants determined by the boundary conditions. In this case, we may set $A_1$ to zero because we assume the field decays as depth $y$ increases. The total field is now a product of exponentials, and rearranging them makes the behavior of the magnetic diffusion wave more obvious:
\[ H_z = \Re\{A_2 e^{-\frac{y}{\delta}} e^{i(\omega t - \frac{y}{\delta})}\}. \]  

(2.31)

The wave is a product of one exponential which decays at a rate \( \frac{y}{\delta} \) and another which oscillates at a frequency \( \omega \) at a fixed location, and travels with phase velocity \( y = \delta \omega \).

![Figure 2-4: Skin depth of a magnetic field in a conductor. On the \( \frac{y}{\delta} = 0 \) edge, the induced field inside the conductor instantaneously equals the applied \( H \) field. The imposed field varies sinusoidally with time; the field in the conductor is a traveling wave.](image)

In a lumped parameter analysis the skin depth is a fixed quantity; the magnetic field in a conductor is assumed constant to a depth \( \delta \) and zero afterwards. This differs from the exact solution detailed above, which is a diffusion wave. Figure 2-4 shows the lumped parameter estimation for the skin depth, along with the diffusion wave at four different moments in time. We obtain these curves by varying the time \( t \) and keeping the frequency \( \omega \) constant, such that the product \( \omega t \) takes on the values \( 0, \frac{\pi}{4}, \frac{\pi}{3} \) and \( \frac{\pi}{2} \). The horizontal axis in the figure is the distance into the conductor, normalized to \( \delta \); the vertical axis is the \( x \)-component of the magnetic field, normalized to the incident
field magnitude $H_2^*$. These normalizations follow naturally from the derivation above when we recognize that the boundary condition described by Ampere's Law forces the constant $A_2$ in equation (2.31) to equal the incident field magnitude. This is because we model the conductor as having volume currents but not surface currents. As seen in (2.31), even at time $t = 0$ when the product $\omega t$ is zero, (assuming steady-state conditions exist), the field still varies as an exponentially decreasing sinusoid in the conductor.

In Table 2.3 we calculate the lumped parameter skin depths for the three main materials used in the sensor for three different frequencies, using (2.29).

<table>
<thead>
<tr>
<th>Material</th>
<th>$f = 1$ kHz</th>
<th>$f = 5$ kHz</th>
<th>$f = 10$ kHz</th>
<th>$f = 100$ kHz</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum</td>
<td>2.675 mm</td>
<td>1.196 mm</td>
<td>0.846 mm</td>
<td>0.267 mm</td>
</tr>
<tr>
<td>Silicon-Iron</td>
<td>0.0417 mm</td>
<td>0.035 mm</td>
<td>0.0132 mm</td>
<td>0.00417 mm</td>
</tr>
<tr>
<td>Steel</td>
<td>0.0784 mm</td>
<td>0.0186 mm</td>
<td>0.0248 mm</td>
<td>0.00784 mm</td>
</tr>
</tbody>
</table>

Table 2.3: Skin depth $\delta$ as a function of frequency for selected materials. We list the frequencies in Hertz, but convert to radians/sec for use with equation (2.29).

2.4.2 Attraction or Repulsion

As a general rule for a conductor with permeability close to $\mu_0$, whenever the thickness of the material is greater than the skin depth we model it as a perfect conductor, which will repel an imposed field. The field inside a real conductor decays as frequency increases but is never zero throughout, even inside a superconductor [16] pp. 450-451. However, we may safely approximate it as zero if it is much smaller in magnitude than other fields of concern in the system; in this case we say the conductor repels the field. For the steel tube we use in our setup, the skin depth at 5 kHz is $\delta = 0.0186$ mm and the wall thickness is 1 mm, implying that the field in the tube would be negligible if the steel were non-permeable. However the permeability of the steel is much higher than $\mu_0$, and experimental evidence shows the field is still attracted to the tube at 5 kHz. We confirm this experimentally since as we move the tube towards
a pole in the experimental setup, the flux through the nearest two secondary coils increases. Similarly, moving the tube away from a pole causes the flux through the nearest coils to decrease. From this we can see the field is at least partially attracted to the tube.

Another way to characterize the situation is with magnetic time constant $\tau_m$ [2]. This is a measure of the characteristic time with which a magnetic field diffuses into a conductor. We compare it to $\tau_e$, the inverse of the excitation frequency. When $\tau_m$ is much greater than $\tau_e$, the magnetic field does not have time to diffuse into the conductor before the applied field alternates direction. Thus we approximate the field as zero inside the tube for $\tau_m >> \tau_e$. We define the magnetic time constant as

$$\tau_m = \mu \alpha l^2,$$  \hspace{1cm} (2.32)

where we choose the characteristic length $l$ to be 1 mm, the tube wall thickness [2]. Using the material constants for the steel tube we arrive at

$$\tau_m = 0.0518 \text{ sec},$$  \hspace{1cm} (2.33)

which when inverted gives a frequency of about 20 Hz. We might consider a frequency "much larger" than this to be repelled, but this still leaves our excitation frequency of 5 kHz somewhere in the transition range from attraction to repulsion. To get a more precise idea of the field behavior, we examine an exact solution to the problem. We will see that although the field does not penetrate far into the tube, the permeability is high enough that the reluctance on this path is lower than the alternate air path.

**Conducting Tube in a Uniform Field**

Using Maxwell’s equations we solve for the field distribution of the conducting tube depicted in Figure 2-5, using cylindrical coordinates as suggested by the geometry. We denote the boundary surfaces as (a), (b), (c), (d) and (e); field components defined at these surfaces correspond to the values just inside or outside, as the case may be, of the associated boundary. In this chapter we only present the results of the solution
Figure 2-5: Permeable, conducting, hollow cylinder in a uniform, time-varying field imposed as the vector potential at surface (e). We assume no variation in the \( z \)-direction, and finite permeability and conductivity in the tube region.

for a tube in a uniform field, so that we may see the behavior of the field at various frequencies. We derive the entire solution for the general case in Chapter 4 where we then use a Fourier series to approximate the field at the outer boundary (e).

**Field Solution**

We wish to impose a uniform field at boundary (e) so we set the vector potential at radius \( e \) to be \( A^e = \hat{A}^e e^{j(\omega t - m\theta)} = A_0 \cos \theta e^{j(\omega t - m\theta)} \). Equation (4.50) and equations (4.91) through (4.94) give the vector potential at each boundary. We calculate the vector potential throughout the regions using (4.95) through (4.97) and plot the lines of constant vector potential; these are also field lines for the flux density. The Matlab code in Appendix A reproduces these calculations and produces the figures shown below. We plot these cases for a given instant in time, specifically for \( t = 0 \). The
key on the right of each figure shows the correspondence of the line color to the magnitude of the vector potential. The imposed sinusoidal potential at the boundary has an amplitude of 10 weber/meter.

![Field lines for a steel tube](image)

**Figure 2-6:** Field lines for a steel tube. The imposed field has frequency \( \approx 0 \) Hz; this is the DC case. We impose a sinusoidal vector potential along the outermost surface and use the transfer relations to calculate the distribution throughout the regions.

Figures 2-6 through 2-8 show that the field in the air gap is not largely dependent on excitation frequency when the permeability of the tube is high. The field *inside* the tube becomes concentrated at the outer skin as the frequency increases, which we expect from the skin depth analysis in the previous section. However, since the field in the air gap is relatively independent of the excitation frequency, we expect the flux paths to still be a significant function of the tube position at a frequency of 5 kHz. In an actual steel tube, there is a limit to the maximum field density, after which the steel is saturated. When this saturation level is reached the model will not accurately represent the field distribution.

When the material is conducting but has permeability \( \mu_o \), e.g. aluminum, the tube repels the field. We show this case for comparison in Figure 2-9, for an aluminum tube in a 100 kHz field. The tube wall thickness is 1 mm, and we see that the field...
Figure 2-7: Field lines for a steel tube with an imposed field frequency of 5 kHz. The field inside the tube is now concentrated near the surface, but the field in the air-gap is similar to the DC case above.

Figure 2-8: Field lines for a steel tube with an imposed field frequency of 100 kHz. The field in the tube continues to concentrate in a thinner layer of the tube while the field in the air-gap remains essentially the same.
Figure 2-9: Field lines for an *aluminum* tube with an imposed field frequency of 100 kHz. We plot this case as a check on our solution, and we see the conductor repels the field as we expect.

The flux penetrates slightly less than one third of the wall thickness. Our earlier skin depth prediction using the lumped-parameter analysis equation (2.29) was $\delta = 0.267$ mm. Also, we see where the field reappears towards the interior of the tube. Looking back at the behavior of the field in the conductor, we see that it oscillates with depth into the conductor. For example, the curve in Figure 2-4 corresponding to $\omega t = 0$ has a minimum at approximately $\frac{r}{\delta} = 2.5$, where the amplitude of the field reaches a minimum then increases again with depth. This corresponds to the depth into the tube wall at which the field reappears.

### 2.4.3 Shielding

The flux will take the path of least reluctance possible to close in on itself and form a continuous loop. For the sensor to be most effective, the flux should pass through the tube and not simply return to the laminations as soon as possible. To direct the flux we place aluminum shielding between the poles as seen in Figure 1-4.

Our biggest concern is that the flux path must not enter the secondary coil midway,
because then the voltage across the secondary terminals might not accurately reflect
the effect of the tube position on the field. However if the flux enters the pole midway
instead of at the pole face, this is acceptable. Figure 2-10 shows different paths the
flux might take; paths 1 and 2 are acceptable, and path 3 is not.

Figure 2-10: Various flux paths: 1) through air-gap, 2) entering pole mid-way, 3) through secondary coil. Paths 1) and 2) are acceptable, but path 3) is not.

At 5 kHz the skin depth in aluminum is about 1.2 mm, but the aluminum in the
shielding pieces is roughly 15 mm thick between the secondary coils and the inner
sensor opening. This means that the flux will be shielded from entering the secondary
coil midway.

One final concern regarding the shielding is that it must not complete a circuit
enclosing a flux path. Ampere’s Law describes this case; a current will be established
in the conductor to cancel the flux. As this will act to shield out the flux in the sensor
and decrease the sensitivity we must avoid enclosing a flux path with a conducting
circuit. For this reason we insulate the shielding pieces with tape to isolate them from one another to avoid closing any conductive loops around the flux paths.
Chapter 3

Sensor Development

3.1 Introduction

Many instruments use magnetic fields to measure displacements, for example the inductive sensor, the eddy-current sensor, the hall-effect sensor, and the linear and rotary differential transformers, (LVDT) and (RVDT) [12]. Differential flux measurements usually depend on a variable inductance in a magnetic circuit. Recall the example in Chapter 2 where \( L = \frac{\mu_0 N_p N_s A}{g} \); with permeability and number of turns constant, geometry is the only variable. Usually the air gap is used as the specific geometric variable. For an inductive position sensor, the inductance is thereby used to measure the position of an object.

To prepare a foundation for such devices, we begin this chapter with a presentation of the Linear Variable Differential Transformer (LVDT) and the E-pickup. Although less common than the LVDT, the E-pickup, as introduced and described in [11], is closely related to our prototype sensor. Both of these sensors measure position using a piston constrained to allow only 1-dimensional movement. In each case, a single primary coil excites a field which is measured at two places via the voltages induced on two secondary coils. The difference between these two voltages depends on the amount of flux through each coil. The flux, in turn, depends on the position of the piston. In this chapter we analyze these two instruments to clarify some of the relevant concepts associated with differential flux measurements. We explain each of
the two devices below and derive the terminal relations to understand the relationship between output voltage and position.

The E-pickup is very similar to the first sensor we built. The difference is that in our sensor we replace the piston constrained to move in one direction with the tube we wish to measure. In this chapter we also describe the evolution of the sensor design from the initial proof-of-concept version to the final version we use in experiments which levitate the tube. Beginning with an unshielded E-pickup made of ferrite (Figure 3-7), we make successive modifications until we arrive at the final version seen in Figure 3-16. Our first modification is to rearrange the sensor poles to increase the sensitivity of the output voltage to tube movement (Figure 3-10). Next we add shielding to minimize leakage flux (Figure 3-11), thereby improving linearity. The final change is in essence a combination of three sensors into one; each oriented 120° from the others (Figures 3-16, 3-17). We design the final version to fit our scaled-down benchtop model, and construct it out of silicon-iron laminations rather than ferrite.

3.2 LVDT

The LVDT is a commonly used, high-precision, position measurement device. The most prevalent benefits are linearity, long life, and high resolution (the literature even goes so far as to claim infinite resolution!)[11]

3.2.1 Operation

LVDTs function like transformers; the difference in the LVDT is that part of the magnetic circuit is a movable piston. The outer core and the piston are both highly permeable and we assume that the flux, \( \Phi \), travels directly from the piston to the core as seen in Figure 3-1.

An alternating current drives the primary coil. When the piston is centered, an equal amount of flux links both secondary coils, and the differential output voltage is zero; when it is off-center, the output is non-zero. The sign of the voltage depends
on the direction of displacement.

3.2.2 Terminal Relations

To show the dependence of the output voltage on the piston position, we derive the terminal relations below. This derivation can also be found in [3, 1, 5] and [8].
Assumptions

We simplify the problem by making the following assumptions. First, the outer core and the piston are both highly permeable, so we set the magnetic field $\vec{H}$ inside both of them to zero. Second, we neglect the fringing fields by assuming the flux travels directly from the piston to the tube (i.e., only a radial component of the field exists). Third, we assume the field in the secondary coil region does not vary with position along the piston: it is constant with respect to $z$ as shown in the lower section of Figure 3-2. Finally, the secondary coils are an open circuit so we assume zero current therein.

Field Solution

The input to the primary coil is a sinusoidally varying current $I = I_0 e^{i\omega t}$. Applying Ampere's law along path (i) in Figure 3-2 relates the enclosed current to the magnetic field $\vec{H}$ along path (i). Recall that we assume the field travels directly from the piston to the outer core, such that the field has only a radial component (i.e., $\vec{H} = H_r(r, t)\hat{r} = \tilde{H}_r(r) e^{i\omega t} \hat{r}$) which is sinusoidally varying at the same frequency as the input current. When the field is a function of the radius, we use the subscript $r$, e.g., $H_r$. When the field is evaluated at the outer piston radius we use the subscript $r_o$, e.g., $H_{r_o}$. Applying Ampere’s law along path (i) results in

$$
\int_{r_i}^{r_o} (\tilde{H}_r^{L1}(r) - \tilde{H}_r^{L2}(r)) dr = N_p I,
$$

with $N_p$ the number of turns in the primary coil and $\tilde{H}_r^{L1,L2}(r)$ the radial components of the field through the secondary coils 1 and 2, respectively. To find an explicit formula for these fields as a function of radius, we relate a field at radius $r$, ($r_i < r < r_o$) to a field at the surface of the piston. Imagine a cylindrical annulus with inner radius $r_i$ and outer radius $r$ enclosing the piston in the secondary coil region. Gauss’ law imposes conservation of flux, such that the flux entering the inner cylindrical area equals the flux exiting the outer cylindrical area. In other words, the flux density $\tilde{B}_r \hat{r}$ times the area of the inner surface area equals the flux density $\tilde{B}_r \hat{r}$ times the surface
area at radius \( r \) (when \( B \) is constant along the length of the cylinder as it is in our case). This gives

\[
2\pi rl \tilde{B}_{ri} = 2\pi rl \tilde{B}_r,
\]

(3.2)

for a cylindrical annulus of length \( l \). Because the permeability is \( \mu_o \) throughout the region, we may cancel it from both sides. Cancelling like terms leaves

\[
\tilde{H}_r(r) = \frac{r_i}{r} \tilde{H}._{ri}.
\]

(3.3)

We substitute this into (3.1) and compute the integral, giving

\[
r_i (\tilde{H}_{ri}^{L_1} - \tilde{H}_{ri}^{L_2}) \ln(\frac{r_o}{r_i}) = N_p I,
\]

(3.4)

which we reduce to

\[
\tilde{H}_{ri}^{L_1} - \tilde{H}_{ri}^{L_2} = \frac{N_p I}{r_i \ln(\frac{r_o}{r_i})}.
\]

(3.5)

Recall that the field is constant along the sections \( L_1 \) and \( L_2 \) as stated in the assumptions above. A similar analysis along path (ii) gives

\[
\tilde{H}_{ri}^{L_1} - \tilde{H}_{ri}^{L_2} = \frac{L_p N_p I}{b r_i \ln(\frac{r_o}{r_i})} = \frac{L_p}{b} (\tilde{H}_{ri}^{L_1} - \tilde{H}_{ri}^{L_2}),
\]

(3.6)

where we include the term \( \frac{L_p}{b} \) to reflect the portion of the current density enclosed by path (ii).

Next we apply Gauss' law to a cylindrical surface which encloses the entire piston:

\[
2\pi r_i L_1 \tilde{B}_{ri}^{L_1} + \int_0^b \tilde{B}_{ri}^{L_1} dA + 2\pi r_i L_2 \tilde{B}_{ri}^{L_2} = 0,
\]

(3.7)

which we reduce to

\[
L_1 \tilde{H}_{ri}^{L_1} + \int_0^b \tilde{H}_{ri}^{L_1} dL_p + L_2 \tilde{H}_{ri}^{L_2} = 0.
\]

(3.8)
We solve (3.6) for $\tilde{H}_{ri}^{L_1}$ and substitute this into (3.8) to get

$$L_1\tilde{H}_{ri}^{L_1} + \int_0^b \left[ \tilde{H}_{ri}^{L_1} - \frac{L_p}{b} (\tilde{H}_{ri}^{L_1} - \tilde{H}_{ri}^{L_2}) \right] dL_p + L_2\tilde{H}_{ri}^{L_2} = 0,$$

(3.9)

which we reduce to

$$\tilde{H}_{ri}^{L_2} = -\frac{\tilde{H}_{ri}^{L_1}(2L_1 + b)}{(2L_2 + b)}.$$

(3.10)

We can solve for the fields $\tilde{H}_{ri}^{L_1}$ and $\tilde{H}_{ri}^{L_2}$ by combining (3.5) and (3.10):

$$\tilde{H}_{ri}^{L_1} = \frac{N_p I(2L_2 + b)}{L_a r_i \ln(\frac{z_a}{r_i})},$$

(3.11)

$$\tilde{H}_{ri}^{L_2} = -\frac{N_p I(2L_1 + b)}{L_a r_i \ln(\frac{z_a}{r_i})},$$

(3.12)

where we use the fact that $L_1 + L_2 + b = L_a$.

Our task now is to find the voltages induced across the secondary coils. For this we use Faraday’s law, and apply it to secondary coil 1 as seen in Figure 3-2:

$$\oint_C \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int_S \vec{B} \cdot \vec{n} \, da.$$

(3.13)

We break up the left side of the equation into a path along the secondary coil wires plus a path from one terminal to the other, exactly as we did in the example problem of Chapter 2. As in that problem, because the current is assumed zero in the coil, evaluation of this integral returns the negative of the voltage across the terminals, which we define as $-V_1$.

For the right side of Faraday’s law, we must find the total flux linked by the secondary coil. To do this we integrate the flux linked by a sum of elementary coils, each of width $dz$, from zero to the piston penetration depth $z_1$. For a single elementary coil, e.g., as seen in Fig. 3-3, the flux linked is
Figure 3-3: Close-up of LVDT secondary coil. For the local axes we define the $z = 0$ point at the tip of the piston for computation of the flux linked by the secondary coil.

\[ \int_0^{2\pi} \int_0^z B_{r_i}^{L_1} r_i dz d\theta = 2\pi r_i z B_{r_i}^{L_1}. \quad (3.14) \]

where we again assume $\vec{B}$ has only a radial component. The total number of turns in the secondary coil is $N_s$, and the width of the coil in the $z$ direction is $m$. Thus, the number of turns in each elemental coil of width $dz$ is $\frac{N_s}{m} dz$. To find the total amount of linked flux we integrate over the section of secondary coil covering the piston, i.e., from zero to $z_1$ as seen in Figure 3-3. This gives

\[ \int_0^{z_1} 2\pi r_i B_{r_i}^{L_1} N_s z \frac{dz}{m} \frac{\pi r_i N_s B_{r_i}^{L_1}}{m} z_1^2. \quad (3.15) \]

We now use (3.12) to write the voltage across secondary coil 1 in terms of the geometry of the system. Recall that the input current is varying sinusoidally with time, so that the time derivative in Faraday’s law simply brings down a $j\omega$ term. The derivation for secondary coil 2 is nearly identical. Thus we have
\[ \tilde{V}_1 = j\omega\mu_0 N_p N_s I \left( \frac{2L_2 + b}{mL_a \ln \left( \frac{r_a}{r_i} \right)} \right) z_1^2, \]  
(3.16)

\[ \tilde{V}_2 = j\omega\mu_0 N_p N_s I \left( \frac{2L_1 + b}{mL_a \ln \left( \frac{r_a}{r_i} \right)} \right) z_2^2, \]  
(3.17)

The differential voltage output, \( \tilde{V}_{out} = \tilde{V}_1 - \tilde{V}_2 \), is commonly written as

\[ \tilde{V}_{out} = K_1 z \left( 1 - \frac{z^2}{K_2} \right), \]  
(3.18)

where

\[ z = \frac{z_1 - z_2}{2}, \]  
(3.19)

\[ K_1 = \frac{j4\omega\mu_0 N_p N_s I [(b - 2d)z_o + z_0^2]}{mL_a \ln \left( \frac{r_a}{r_i} \right)}, \]  
(3.20)

\[ K_2 = (b + 2d)z_o + z_o^2, \]  
(3.21)

\[ z_o = \frac{z_1 + z_2}{2}. \]  
(3.22)

\( K_1 \) is known as the sensitivity of the transformer, and \((1 - x^2/K_2)\) is the linearity coefficient [1]. We see that the transfer function from the piston displacement \( z \) to the output voltage \( V_{out} \) has both a linear term and a non-linear term. The output will be linear as long as the linear term dominates, i.e., for small displacements \( z \).

### 3.3 E-Pickup

In the LVDT, the air gap between the piston and the outer core, (filled with copper coils of permeability \( \mu_o \)), drives the magnetic field. This gap does not change as the piston moves in the \( z \)-direction. In an E-pickup, shown in Figure 3-4, the total air gap of the magnetic circuit does change with the position of the piston. We analyze an E-pickup made of ferrite, as our prototype sensor is also made of ferrite.
Figure 3-4: E-core schematic. A current supply drives the primary coil to create a flux dependent upon the piston position. The piston moves in the x-direction only.

In this case, the terminal relations depend on a more complicated representation of the flux path across the air gap. Because of the complex shape of the E-pickup magnetic system, we use a magnetic circuit to calculate these relations.

In Figure 3-5 we show the E-pickup with a magnetic circuit overlay. The reluctances $R_{1,2,3}$ represent the flux paths from the E-pickup to the piston; $R_L, R_F$ represent the leakage reluctance and the ferrite core reluctance, respectively. As with the LVDT, the piston and the E-core are highly permeable. We use the term “E-pickup” to mean the complete sensor including the coils; the E-core is the ferrite “E” shape alone. As the piston moves in the x-direction with respect to the primary coil, the reluctances $R_2$ and $R_3$ change, thereby changing the flux through the secondary coils. $R_1$ remains constant because we assume that the distance from the primary pole to the piston, as well as the area of the flux path, is constant. This is true as long as the piston does not move so far laterally as to uncover the primary pole.

We see the layout of the circuit problem in Figure 3-6. To solve the circuit we use standard circuit analysis techniques. A set of four loop equations, as suggested in the figure, yield the circuit equations. We write these in matrix form as
Figure 3-5: Schematic of E-pickup sensor with magnetic circuit overlay. We lump all the leakage flux into a single term, however in the actual sensor there is leakage flux along the entire height of the poles.

Figure 3-6: Circuit diagram of the E-pickup sensor. Representing the magnetic elements with MCA equivalents allows us to use Kirchoff's laws to calculate the flux.
Because the permeability of ferrite is much higher than that of air (in our case, \( \mu = 2700\mu_o \)), we set the reluctances \( \mathcal{R}_F \) to zero. We now invert the above matrix (using the commercially available software Maple) and write the equations for \( \Phi_1 \) and \( \Phi_2 \):

\[
\Phi_1 = N_pI \left( \frac{1}{\mathcal{R}_L} + \frac{\mathcal{R}_3}{\mathcal{R}_L \mathcal{R}_1 + \mathcal{R}_2 \mathcal{R}_3 + \mathcal{R}_1 \mathcal{R}_3} \right),
\]

and

\[
\Phi_2 = N_pI \left( \frac{1}{\mathcal{R}_L} + \frac{\mathcal{R}_2}{\mathcal{R}_L \mathcal{R}_1 + \mathcal{R}_2 \mathcal{R}_3 + \mathcal{R}_1 \mathcal{R}_3} \right).
\]

Subtracting these gives the differential flux

\[
\Phi_1 - \Phi_2 = N_pI \left( \frac{\mathcal{R}_3 - \mathcal{R}_2}{\mathcal{R}_L \mathcal{R}_1 + \mathcal{R}_2 \mathcal{R}_3 + \mathcal{R}_1 \mathcal{R}_3} \right).
\]

Recall that the imposed current varies sinusoidally with time, i.e., we can write it as \( I = i_0 e^{j\omega t} \). Faraday’s law reduces this differential flux to a voltage by bringing down the imaginary frequency from the time derivative. Finally, multiplying by \( N_s \) (the number of linked secondary coil turns) gives

\[
V_1 - V_2 = V_{out} = j\omega N_p N_s I \left( \frac{\mathcal{R}_3 - \mathcal{R}_2}{\mathcal{R}_L \mathcal{R}_1 + \mathcal{R}_2 \mathcal{R}_3 + \mathcal{R}_1 \mathcal{R}_3} \right).
\]

Because \( I \) varies sinusoidally, the imaginary number \( j \) simply changes the phase by 90 degrees; physically we observe the real part of this complex sinusoid. We see
that the leakage fluxes cancel, which we expect when taking a differential voltage from a symmetric circuit. However, the flux through each secondary coil consists of a constant portion from the leakage flux plus a changing portion dependent on the piston position. As the piston moves farther away from the E-pickup in the y-direction, the leakage flux will dominate and the flux will tend to avoid the piston completely, so the differential voltage will approach zero. To decrease the leakage flux relative to the non-leakage flux, we modify the geometry as seen below in section 3.4. But first we present the experimental results of our initial proof-of-concept sensor, an E-pickup which uses a steel tube in place of the permeable piston.

3.3.1 Experimental E-Pickup Setup

We construct our initial experimental version of the sensor from an off-the-shelf E-core. Figure 3-7 shows the ferrite E-core alone, without coils. Ferrite is highly permeable yet nonconducting so that eddy currents do not impede the alternating magnetic field. The magnetic circuit is identical to that presented in Section 3.3 except that we replace the piston with the tube. Because the tube is highly permeable steel, the reluctances through the air gap will dominate those through the tube and the ferrite, just as in the previous E-pickup.

We use a 10 kHz sinusoidal signal from a Kepco power amplifier to drive the primary coil. We had initially set up the system using the Kepco as a current source, but an unstable high frequency ringing in the current signal made the flux characteristics unintelligible. Switching to a voltage source solved the problem, and this being the initial proof-of-concept sensor, a voltage source sufficed to give the required information.

Experimental Data From E-pickup Sensor

We mount the \( \frac{1}{2} \) in diameter, seam-welded, steel tube above the E-pickup, which we secure to an \( x-z \) table. Connecting the secondary coil terminals in series gives the differential voltage output, which we read with an oscilloscope as we move the sensor
under the tube. We show the physical dimensions in Figure 3-8. The input voltage amplitude is 0.1 Volts, the secondary coils have $N_s = 100$ turns and the primary coil has $N_p = 50$ turns.

Figure 3-9 shows the experimental data, where we plot the amplitude of the differential voltage as we move the tube in the $x$-direction. We perform the experiment for two different heights above the primary pole face: 0.5 mm and 1.5 mm. We see that the output voltage does change as a function of tube position, and the sensitivity of the output decreases as we increase the tube height above the pole face. This agrees with the trend predicted by the magnetic circuit analysis of Section 3.3. To increase the sensitivity we explore a different design.

### 3.4 Modified E-pickup

To make the sensor more sensitive to the tube position we move the secondary poles to the far side of the tube. This makes the leakage path longer with respect to the path
Figure 3-8: Physical dimensions of E-core used in first experiment. The depth into the page (in the z-direction) is 19 mm.

Figure 3-9: Experimental output from E-pickup sensor. Increasing the tube height from 0.5 mm to 1.5 mm decreases the sensitivity by about half.
through the tube, which in turn makes the output voltage more sensitive to changes in the tube position. Figure 3-10 shows the new geometry, which we construct out of ferrite E-core pieces. Figure 3-11 is a photograph of this sensor.

![Figure 3-10: Schematic of second E-core sensor. The secondary poles are now above the tube to give a different flux path. The depth of the sensor in the z-direction is 20.4 mm.](image)

The magnetic circuit is essentially the same as before, except that the relative values of the reluctances are different. We present two cases below, one set of data taken from the sensor without any shielding, and another with the shielding in place. For each case the input is a 1 kHz, sinusoidal voltage with an amplitude of 1 Volt. The primary coil has \( N_p = 70 \) turns, and the secondary coils have \( N_s = 185 \) turns. For the second set of data we add shielding to force the flux through the tube and not allow it to simply return directly to the ferrite, as discussed in Chapter 2. We again connect the secondary coils in series to obtain a differential voltage, but this time we demodulate and low-pass filter this voltage to give a DC signal dependent on the tube position.
Figure 3-11: Modified E-pickup with copper shielding. The wires at the bottom of the photo are shielded in aluminum.

Demodulation

Demodulation is a common method for finding the in-phase component of two signals. It involves multiplying the phase-shifted input signal e.g., \( \sin(\theta - \theta_o) \), by the sign (\( \pm \)) of a reference signal, e.g., \( \sin(\theta) \). Figure 3-12 shows a reference signal, a phase-shifted signal, and a corresponding demodulated signal.

We assume that the DC component of the signal is equal to the total area under the curve; thus we can find the DC component of the demodulated signal by computing the integral

\[
2 \int_0^{\pi} \sin(\theta - \theta_o) \, d\theta = 2 \int_0^{\pi} (\sin\theta\cos\theta_o - \cos\theta\sin\theta_o) \, d\theta
\]

\[
= -2\cos\theta_o\cos\theta\bigg|_0^{\pi} - 2\sin\theta_o\sin\theta\bigg|_0^{\pi}
\]

\[
= 4\cos\theta_o.
\]
Figure 3-12: Result of demodulating a shifted sinusoid. The input signal is \( \sin(\theta - \theta_0) \), and the reference is \( \sin(\theta) \). We show the case for \( \theta_0 = 1 \) radian. If \( \theta_0 \) were \( \frac{\pi}{2} \) radians, the DC component would be zero. Also note the primary frequency is now twice the original.

So for a demodulated signal, the area under the resultant curve is four times the cosine of the phase-shift angle. When we simply rectify the signal, the area under the curve is four. In other words,

\[
2 \int_0^\pi |\sin(\theta - \theta_0)| \, d\theta = 4. 
\] (3.31)

So we see that the change in the DC amplitude demodulation reduces the DC component by the cosine of the phase-shift angle.

Multiplying a sinusoidal signal by the sign of a reference signal of the same frequency and phase results in a signal with a positive DC component and a primary frequency twice the original: this is the case where we rectify the signal. If the reference signal is of the same frequency but 90 degrees out of phase, the result of
demodulation is a signal with no DC component. If the reference signal is 180° out of phase, the result is a signal with a negative DC component. Also, demodulating a signal with a reference signal of a different frequency returns a signal with no DC component. In our modified E-pickup, the input signal is the differential voltage from the secondary coils and the reference signal is the voltage from the signal generator which provides the input to the primary coil drive amplifier.

3.4.1 Experimental Data From Modified E-pickup

Figure 3-13: Experimental data from the modified E-pickup sensor without shielding, showing output voltage vs. lateral displacement. The legend shows tube height above primary pole face; we take each series of data at a different height as noted in the legend. The data series tend to spread towards the edges of tube travel.

Figures 3-13, 3-14 and 3-15 show the experimental data. Each plot shows output voltage (after being demodulated and low-pass filtered) against tube position. We obtain these data sets by moving the tube horizontally while keeping the vertical position constant; each set corresponds to a different height as noted in the legends. The data shows that the vertical position has a far lesser effect on the output voltage than the horizontal position. As the tube gets closer to the secondary poles (at the
Figure 3-14: Experimental data from the modified E-pickup sensor with shielding in place. Apart from the case at height 1 mm which is close to linear in both cases, the output is more uniform, with less deviation as the tube nears the edges of the sensor range. The difference is much more dramatic if we ignore the data series for the 1 mm case.

Figure 3-15: Experimental data from the modified E-pickup sensor comparing shielded case with unshielded case for tube heights of 1 mm and 9 mm. The 1 mm data series follow each other closely, with primarily a DC offset. The 9 mm series tend to differ more towards the edges of measurement.
extrema of the data series) the deviation from linearity increases. Also note that the zero voltage point is not at the center of the range \((x = 0)\). This is most likely due to a lack of symmetry in the coil windings, shielding or ferrite pieces. Such difficulty in winding symmetric coils is noted in [13]. Because of the difficulty in making a sensor with perfect symmetry, we will include a variable resistance in the summing circuit to zero the output in future iterations.

Comparing the shielded case to the unshielded, we see the general trend is for the shielding to make the output voltage less susceptible to displacement in the \(y\)-direction, and more linear with displacement in the \(x\)-direction. The effect is slight, but noticeable, especially towards the edges of the sensing range as the tube nears a pole. In Figure 3-14 we see that the data series for heights 3 mm to 9 mm are closer together and more linear than in the unshielded case in Fig. 3-13. In Figure 3-15 we see that the difference between shielded and unshielded is negligible at a height of 1 mm, as only a DC offset distinguishes the two. However for the 9 mm height the difference is more pronounced, as the unshielded case tends to be less linear with movement in the \(x\)-direction.

3.5 Final Design

Because of the near linearity of the output and the decoupling of \(x\)- and \(y\)-position sensing, we choose a differential flux arrangement to sense the tube position. However our task is to sense position in two directions; the devices presented above sense only a single direction. Rather than construct a separate sensor for each direction we design a new configuration which provides all the necessary information. This new design, seen schematically in Figure 3-16 and as a photograph in Figure 3-17, can be thought of as three differential flux sensors (rotated by 0°, 120° and 240°) combined into one.

We note three major differences from earlier designs. First, whereas in the previous designs the primary pole and secondary poles had dedicated functions, in this design each of the three poles acts as both a primary and a secondary. Second, we place the primary coils on the poles to direct the flux into the center of the sensor, and
Figure 3-16: Final design of position sensor.

Figure 3-17: Photograph of position sensor.
locate the secondary coils away from the center to make room for more shielding and to minimize any unwanted cross coupling effects from the primary coils. Also, we use silicon-iron laminations to form the sensor instead of ferrite; we present a more detailed discussion of this design choice in Chapter 6. The last major difference is the scale. We designed the previous sensors to accommodate a full scale tube of about 22 mm diameter, but we design the final version for a tube of 6.4 mm so it can be used in the benchtop-scale model which we are fabricating. The important dimensional constraint is the relation between the pole face area and the tube diameter. The larger the pole face area, the greater the leakage flux in relation to the flux through the tube, so the tube should be larger in diameter than the pole face width in order to increase sensitivity.

We explore two options for driving the three primary coils. The first is to drive each coil with a different frequency. This means that the output from each secondary coil is the sum of three sine waves of three different frequencies. The demodulator then filters out one particular frequency signal from the sum. However, driving the coils at three different frequencies will give each circuit different characteristics, making the analysis and operation unnecessarily complicated. Also, the three frequencies would have to be different enough not to affect each other, and this would mean less freedom to chose the sensor bandwidth. The lowest frequency will determine the bandwidth of the sensor, while the response of the current supply and the speed of the signal generator will limit the highest frequency. Finally, magnetic nonlinearities in the sensor iron laminations will lead to possible cross-coupling of excitation harmonics.

A second option is to drive the three signals at the same frequency, but 120° out of phase from each other. Combined with the geometry of the sensor this results in a rotating traveling wave. The output voltages will then have this same characteristic. Operating the three circuits at the same frequency means the magnitude and phase of the signals will all correspond to tube movement in similar ways, which makes tuning the output and summing the signals much easier. Because of it's symmetry and simplicity, we choose the second option. We present the field analysis of this sensor in Chapter 4, and the electronic circuits which drive the sensor in Chapter 5.
Chapter 4

Field Analysis

4.1 Introduction

In this chapter we derive the terminal relations for the three-phase sensor. We use two different methods to analyze the sensor: the Magnetic Circuit Analogy, and an exact solution using transfer relations. The lumped parameter method using the MCA is the most general of the two so we present this first. This method allows us to plot the predicted output of the sensor as the tube is moved throughout the aperture of the sensor; we compare this output to the experimental output in Section 4.2.6. A more exact solution using the vector potential requires certain boundary conditions and symmetries, we present this analysis for the case where the tube is at the center of the aperture. The advantage of this analysis is that it yields the field solution as a function of space throughout the region of interest. This allows us, for example, to study the field penetration in the tube as a function of the excitation frequency.

4.2 Magnetic Circuit Analogy

The MCA is useful for predicting the amount of flux passing through discrete parts of the sensor, but less helpful when we require an exact solution for the field distribution. In this section we model the sensor with a magnetic circuit as we did with the E-pickup in the previous chapter. We use circuit analysis techniques to find the flux
passing through the outer lamination ring of the sensor in terms of the tube position. We then use this flux to derive equations for the voltages across the secondary coils in terms of the tube position, and invert these relations to find equations for the tube position in terms of the output voltages. We plot the results to compare them with the actual output from the sensor.

4.2.1 System Model

![Figure 4-1: Equivalent magnetic circuit of the sensor.](image)

Figure 4-1: Equivalent magnetic circuit of the sensor. We assume the shielding will constrain the flux to the paths above; which we denote as reluctances, $\mathcal{R}$.

Figure 4-1 depicts the magnetic circuit representation of the sensor. As before, we represent the flux paths with reluctances, and the current-carrying coils with magnetomotive forces. As the flux exits a pole, it can travel through the tube or around it; the reluctances $\mathcal{R}_{1t,2t,3t}$ represent the paths from the poles to the tube, the
reluctances $\mathcal{R}_{LA, LB, LC}$ represent the leakage paths around the tube. $\mathcal{R}_{TA, TB, TC}$ and $\mathcal{R}_{A, B, C}$ represent the flux paths through the tube and lamination sections respectively. We neglect any other leakage paths by assuming the shielding will constrain the flux to the above-mentioned paths.

### 4.2.2 Circuit Analysis

To ease the analysis we simplify the system using the “delta-wye” equivalence relations [6]. The new equivalent reluctances are

\[
\begin{align*}
\mathcal{R}_1 &= \frac{\mathcal{R}_{LA}\mathcal{R}_{LB}\mathcal{R}_{LC}}{\mathcal{R}_{LA}\mathcal{R}_{1eq} + \mathcal{R}_{comb}} + \frac{\mathcal{R}_{LC}}{\mathcal{R}_{LC}\mathcal{R}_{2eq} + \mathcal{R}_{comb}} + \frac{\mathcal{R}_{A}\mathcal{R}_{C}}{\mathcal{R}_{A} + \mathcal{R}_{B} + \mathcal{R}_{C}}, \\
\mathcal{R}_2 &= \frac{\mathcal{R}_{LA}\mathcal{R}_{LB}\mathcal{R}_{LC}}{\mathcal{R}_{LA}\mathcal{R}_{2eq} + \mathcal{R}_{comb}} + \frac{\mathcal{R}_{LC}}{\mathcal{R}_{LC}\mathcal{R}_{1eq} + \mathcal{R}_{comb}} + \frac{\mathcal{R}_{A}\mathcal{R}_{B}}{\mathcal{R}_{A} + \mathcal{R}_{B} + \mathcal{R}_{C}}, \\
\mathcal{R}_3 &= \frac{\mathcal{R}_{LA}\mathcal{R}_{LB}\mathcal{R}_{LC}}{\mathcal{R}_{LA}\mathcal{R}_{3eq} + \mathcal{R}_{comb}} + \frac{\mathcal{R}_{LC}}{\mathcal{R}_{LC}\mathcal{R}_{2eq} + \mathcal{R}_{comb}} + \frac{\mathcal{R}_{A}\mathcal{R}_{C}}{\mathcal{R}_{A} + \mathcal{R}_{B} + \mathcal{R}_{C}},
\end{align*}
\]

where

\[
\begin{align*}
\mathcal{R}_{1eq} &= \mathcal{R}_{1t} + \frac{\mathcal{R}_{TA}\mathcal{R}_{TC}}{\mathcal{R}_{TA} + \mathcal{R}_{TB} + \mathcal{R}_{TC}}, \\
\mathcal{R}_{2eq} &= \mathcal{R}_{2t} + \frac{\mathcal{R}_{TA}\mathcal{R}_{TB}}{\mathcal{R}_{TA} + \mathcal{R}_{TB} + \mathcal{R}_{TC}}, \\
\mathcal{R}_{3eq} &= \mathcal{R}_{3t} + \frac{\mathcal{R}_{TB}\mathcal{R}_{TC}}{\mathcal{R}_{TA} + \mathcal{R}_{TB} + \mathcal{R}_{TC}},
\end{align*}
\]

and

\[
\mathcal{R}_{comb} = \mathcal{R}_1\mathcal{R}_2 + \mathcal{R}_2\mathcal{R}_3 + \mathcal{R}_3\mathcal{R}_1.
\]

This leaves the simplified circuit of Figure 4-2.

This model includes all reluctances, but if some are much smaller than the others we may approximate them as zero. From the geometric symmetry, the reluctances of the paths through the tube and through the lamination sections should be equal (to
within manufacturing consistency), but the others are functions of the tube position. To get a clearer idea of which we may neglect and which we must include in the analysis, we calculate typical reluctance values with the 6.4 mm (¼ in) tube in the center of the sensor.

Calculating Reluctance Values

The general equation for calculating the reluctance of a flux path involves the gap length, the area, and the permeability, which may all may be a function of position [10]. The first simplification we make is the assumption that the gap length is perpendicular to the cross-sectional area. This results in an equation for a differential reluctance of a path of length \( dx \) as

\[
d\mathcal{R} = \frac{dx}{\mu(x)A_{xc}(x)}. \tag{4.4}\n\]

To find the total reluctance of a path from, e.g., point \( a \) to point \( b \), we integrate (4.4) to get

\[
\mathcal{R} = \int_{a}^{b} \frac{dx}{\mu(x)A_{xc}(x)}. \tag{4.5}\n\]

Figure 4-3 shows the geometry we use to calculate an example reluctance. Applying (4.5) to the situation seen in Figure 4-3 results in
Figure 4-3: Geometry for calculating a lumped-parameter reluctance. The shaded area is the area over which we integrate to find an exact solution. The cross-hatched area is the rectangular approximation.

\[
A_{xc}(x) = d(w_1 + \frac{w_2 - w_1}{g} x)
\]

where \( m \) is the slope of the line,

\[
m = \frac{w_2 - w_1}{g}.
\]

We assume no variation in the z-direction, that the cross-sectional area is equal to the depth \( d \) in the z-direction times the length in the \( x, y \) plane. Solving the above integral yields

\[
\mathcal{R} = \frac{1}{\mu_0 d} \int_0^g dx \frac{dx}{w_1 + mx},
\]

\[
= \frac{1}{\mu_0 dm} \int_0^g m dx.
\]

81
\[
\begin{align*}
\frac{1}{\mu_0 dm} \ln(w_1 + mx) \bigg|_{x=g}^{x=0}, \\
\frac{1}{\mu_0 dm} (\ln(w_2) - \ln(w_1)),
\end{align*}
\]  

(4.8)

where we assume the permeability is independent of position. If the cross-sectional area is also independent of position, we may pull it out of the integral as well. This results in a simpler expression for the reluctance, which we write as

\[
\mathcal{R} = \frac{g}{\mu_0 A_{xc}}.
\]  

(4.9)

This is the case for a reluctance path with constant rectangular cross-sectional area \(A_{xc}\). Because we assume the flux paths in our system do not vary in the z-direction, the cross-sectional area will be some function of the geometry in the \(x, y\) plane times the depth of the sensor in the z-direction. For the quadrilateral prism shapes which we use in our analysis, we simplify equation (4.5) by approximating the cross-sectional area as 1/2 the sum of the two base areas. The cross-hatched shape in Figure 4-3 shows this case. Using this geometry, the total reluctance along the gap is

\[
\mathcal{R} = \frac{g}{\mu_0 \frac{w_1 + w_2}{2}}.
\]  

(4.10)

Figure 4-4 shows the geometry of the assumed flux paths which we use to calculate \(\mathcal{R}_{1t}\) and \(\mathcal{R}_{LB}\). The figure shows the tube off-center to make the geometry more obvious, but for the numerical calculations below we assume the tube is in the center. For \(\mathcal{R}_{1t}\), the gap is the vertical distance from pole one to the tube \((gt)\), and the characteristic length is half the sum of the tube diameter \(2r_t\) and the length \(wp\):

\[
\mathcal{R}_{1t} = \frac{gt}{\mu_0 d\frac{2r_t + wp}{2}}.
\]  

(4.11)

For \(\mathcal{R}_{LB}\), the gap is the distance between two poles \((gL)\), and the characteristic length is half the sum of the distance from the tube to the shielding \(L_B\) and the width \(wp\):

\[
\mathcal{R}_{LB} = \frac{gL}{\mu_0 d\frac{L_B + wp}{2}}.
\]  

(4.12)
Figure 4-4: Cutout of center of sensor showing the geometry we use to calculate the lumped parameter reluctances. The cross-hatched area represents the aluminum shielding, shaded areas $R_{LB}$ and $R_{11}$ represent the assumed flux paths. The dimension $w_P$ is one-third of the lamination pole thickness.

We choose $\frac{L_B + w_P}{2}$ as the characteristic length here because it represents the volume between the tube and the shielding which we assume is filled by the leakage flux, and because it tends to zero as the tube approaches either of the two nearest poles (poles 2 and 3 in Fig. 4-4). We expect the leakage flux to behave in the same manner.

For $R_{TA, TB, TC}$, we use skin depth as the characteristic length (for steel with a frequency of 5 kHz), with $\frac{1}{3}$ of the tube circumference as the gap. We calculate $R_{A, B, C}$ using the lamination section thickness for the characteristic length and $\frac{1}{3}$ the circumference of the outer lamination ring plus twice the length of a pole as the gap. Using these values to calculate the reluctances gives
\[ \mathcal{R}_{1t} = \frac{l}{\mu_0 A_{xc}} = \frac{.0048 m}{(4\pi \times 10^{-7} \frac{Vs}{Am})(2.02 \times 10^{-5} m^2)} = 1.90 \times 10^8 \frac{A}{Vs}, \]

\[ \mathcal{R}_A = \frac{l}{\mu A_{xc}} = \frac{.0283 m}{(7.0 \times 10^4 \mu_0)(1.68 \times 10^{-5} m^2)} = 1.91 \times 10^4 \frac{A}{Vs}, \]

\[ \mathcal{R}_{T,A} = \frac{l}{\mu A_{xc}} = \frac{.0066 m}{(5.5 \times 10^3 \mu_0)(1.57 \times 10^{-7} m^2)} = 6.08 \times 10^6 \frac{A}{Vs} \]

\[ \mathcal{R}_{LA} = \frac{l}{\mu_0 A_{xc}} = \frac{.0137 m}{(4\pi \times 10^{-7} \frac{Vs}{Am})(1.52 \times 10^{-5} m^2)} = 7.15 \times 10^8 \frac{A}{Vs}. \quad (4.13) \]

Because the reluctance of the paths through tube and the laminations are two and four orders of magnitude (respectively) smaller than the others, we may ignore them in the following derivation. We thus reduce the equivalent reluctances to

\[ \mathcal{R}_{1t} = \frac{\mathcal{R}_{LA} \mathcal{R}_{LC} \mathcal{R}_{comb}}{\mathcal{R}_{LA} \mathcal{R}_{3t} + \mathcal{R}_{comb}} + \frac{\mathcal{R}_{LB}}{\mathcal{R}_{LA} \mathcal{R}_{1t} + \mathcal{R}_{comb}} + \frac{\mathcal{R}_{LC}}{\mathcal{R}_{LC} \mathcal{R}_{2t} + \mathcal{R}_{comb}}; \]

\[ \mathcal{R}_{2t} = \frac{\mathcal{R}_{LA} \mathcal{R}_{LB} \mathcal{R}_{comb}}{\mathcal{R}_{LA} \mathcal{R}_{3t} + \mathcal{R}_{comb}} + \frac{\mathcal{R}_{LB}}{\mathcal{R}_{LA} \mathcal{R}_{1t} + \mathcal{R}_{comb}} + \frac{\mathcal{R}_{LC}}{\mathcal{R}_{LC} \mathcal{R}_{2t} + \mathcal{R}_{comb}}; \quad (4.14) \]

\[ \mathcal{R}_{3t} = \frac{\mathcal{R}_{LA} \mathcal{R}_{LB} \mathcal{R}_{comb}}{\mathcal{R}_{LA} \mathcal{R}_{3t} + \mathcal{R}_{comb}} + \frac{\mathcal{R}_{LB}}{\mathcal{R}_{LA} \mathcal{R}_{1t} + \mathcal{R}_{comb}} + \frac{\mathcal{R}_{LC}}{\mathcal{R}_{LC} \mathcal{R}_{2t} + \mathcal{R}_{comb}}; \]

with

\[ \mathcal{R}_{comb} = \mathcal{R}_{1t} \mathcal{R}_{2t} + \mathcal{R}_{2t} \mathcal{R}_{3t} + \mathcal{R}_{3t} \mathcal{R}_{1t}. \quad (4.15) \]

### 4.2.3 Solving the Magnetic Circuit

Kirchoff's Current Law at the left node of Figure 4-2 along with Kirchoff's Voltage Law around two closed loops gives us the following linear system of equations:

\[
\begin{bmatrix}
\mathcal{R}_1 & 0 & -\mathcal{R}_3 \\
0 & -\mathcal{R}_2 & \mathcal{R}_3 \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
\Phi_1 \\
\Phi_2 \\
\Phi_3
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & -1 \\
0 & -1 & 1 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
N_p I_1 \\
N_p I_2 \\
N_p I_3
\end{bmatrix}. \quad (4.16)
\]
Recall that the fluxes and currents vary sinusoidally with time. Because the input currents $I_1, I_2, I_3$ are all of the same magnitude and frequency but differ in phase from each other by $\frac{2\pi}{3} = 120^\circ$ (e.g., $I_2 = Re(\tilde{I}_o e^{j(\omega t + \frac{2\pi}{3})}) = I_1 e^{j\frac{2\pi}{3}}$), we may pull the common factor $I_1 = \tilde{I}_o e^{j\omega t}$ out in front, leaving the phase component. Inverting the reluctance matrix to solve for the fluxes gives:

\[
\begin{bmatrix}
\Phi_1 \\
\Phi_2 \\
\Phi_3
\end{bmatrix} = N_p I_1 \begin{bmatrix}
R_1 & 0 & -R_3 \\
0 & -R_2 & R_3 \\
1 & 1 & 1
\end{bmatrix}^{-1} \begin{bmatrix}
1 & 0 & -1 \\
0 & -1 & 1 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
1 \\
e^{j\frac{2\pi}{3}} \\
e^{j\frac{4\pi}{3}}
\end{bmatrix}, \tag{4.17}
\]

Combining factors of $R_1, R_2$ and $R_3$ leaves:

\[
\begin{bmatrix}
\Phi_1 \\
\Phi_2 \\
\Phi_3
\end{bmatrix} = \frac{N_p I_1}{R_1 R_2 + R_2 R_3 + R_3 R_1} \begin{bmatrix}
R_2 + R_3 - R_3 e^{j\frac{2\pi}{3}} - R_2 e^{j\frac{4\pi}{3}} \\
-R_3 + (R_1 + R_3) e^{j\frac{2\pi}{3}} - R_1 e^{j\frac{4\pi}{3}} \\
-R_2 - R_1 e^{j\frac{2\pi}{3}} + (R_1 + R_2) e^{j\frac{4\pi}{3}}
\end{bmatrix}. \tag{4.18}
\]

To get a clearer view of the phasor representation of the values, Figure 4-5 depicts the components of each of the three fluxes.

To find fluxes $\Phi_{A,B,C}$ through the outer sections of the laminations, we note that $\Phi_{1,2,3}$ in Fig. 4-5 are the same as in Fig. 4-1. We apply Kirchoff’s voltage law to the outer circuit loop in Fig. 4-1 and use the fact that $R_A = R_B = R_C$ to give $\Phi_A + \Phi_B + \Phi_C = 0$. Recall that these reluctances are much smaller than others in the circuit, but still non-zero. We apply Kirchoff’s current law at nodes (i),(ii) and (iii) to give

85
Figure 4-5: Phasor components of fluxes through poles 1, 2 and 3 normalized to $\frac{R_1 R_2 + R_3 R_4 + R_5 R_6}{\sqrt{3} N_p I_1}$. We see from the symmetry of the phasors that $\Phi_1 + \Phi_2 + \Phi_3 = 0$, as we expect from driving the sensor with a three-phase signal.

\[
\Phi_1 = \Phi_C - \Phi_A, \\
\Phi_2 = \Phi_A - \Phi_B, \\
\Phi_3 = \Phi_B - \Phi_C. 
\]  

(4.20)

We combine these equations with $\Phi_A + \Phi_B + \Phi_C = 0$ to give

\[
\Phi_2 - \Phi_1 = \Phi_A - \Phi_B - \Phi_C + \Phi_A = 3\Phi_A, \\
\Phi_3 - \Phi_2 = \Phi_B - \Phi_C - \Phi_A + \Phi_B = 3\Phi_B, \\
\Phi_1 - \Phi_3 = \Phi_C - \Phi_A - \Phi_B + \Phi_C = 3\Phi_C. 
\]  

(4.21)
which we rewrite as

\[
\Phi_A = \frac{\Phi_2 - \Phi_1}{3}, \\
\Phi_B = \frac{\Phi_2 - \Phi_3}{3}, \\
\Phi_C = \frac{\Phi_1 - \Phi_3}{3}.
\] (4.22)

Combining this with equation (4.19) results in equations for each of the fluxes in terms of the reluctances \( \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \)

\[
\begin{bmatrix}
\Phi_A \\
\Phi_B \\
\Phi_C
\end{bmatrix} = \frac{-N_p I_1}{\sqrt{3}(\mathcal{R}_1 \mathcal{R}_2 + \mathcal{R}_2 \mathcal{R}_3 + \mathcal{R}_3 \mathcal{R}_1)} \begin{bmatrix}
e^{j\frac{\pi}{2}} & e^{j\frac{\pi}{6}} & 2e^{j\frac{11\pi}{6}} \\
e^{j\frac{5\pi}{6}} & e^{j\frac{\pi}{6}} & e^{j\frac{5\pi}{6}} \\
e^{j\frac{3\pi}{2}} & 2e^{j\frac{5\pi}{6}} & e^{j\frac{5\pi}{6}}
\end{bmatrix} \begin{bmatrix}
\mathcal{R}_1 \\
\mathcal{R}_2 \\
\mathcal{R}_3
\end{bmatrix}. \quad (4.23)
\]

One final delta-wye transformation simplifies the circuit equation even more,

\[
\begin{bmatrix}
\Phi_A \\
\Phi_B \\
\Phi_C
\end{bmatrix} = -\frac{N_p I_1}{\sqrt{3}} \begin{bmatrix}
e^{j\frac{3\pi}{2}} & e^{j\frac{\pi}{6}} & 2e^{j\frac{11\pi}{6}} \\
e^{j\frac{5\pi}{6}} & e^{j\frac{\pi}{6}} & e^{j\frac{5\pi}{6}} \\
e^{j\frac{3\pi}{2}} & 2e^{j\frac{5\pi}{6}} & e^{j\frac{5\pi}{6}}
\end{bmatrix} \begin{bmatrix}
\frac{1}{\mathcal{R}_{Beq}} \\
\frac{1}{\mathcal{R}_{Ceq}} \\
\frac{1}{\mathcal{R}_{Aeq}}
\end{bmatrix}. \quad (4.24)
\]

where

\[
\mathcal{R}_{Aeq} = \frac{\mathcal{R}_1 \mathcal{R}_2 + \mathcal{R}_2 \mathcal{R}_3 + \mathcal{R}_3 \mathcal{R}_1}{\mathcal{R}_3} = \mathcal{R}_{LA} + \mathcal{R}_{1t} + \mathcal{R}_{2t} + \frac{\mathcal{R}_{1t} \mathcal{R}_{2t}}{\mathcal{R}_{3t}},
\]

\[
\mathcal{R}_{Beq} = \frac{\mathcal{R}_1 \mathcal{R}_2 + \mathcal{R}_2 \mathcal{R}_3 + \mathcal{R}_3 \mathcal{R}_1}{\mathcal{R}_1} = \mathcal{R}_{LB} + \mathcal{R}_{2t} + \mathcal{R}_{3t} + \frac{\mathcal{R}_{2t} \mathcal{R}_{3t}}{\mathcal{R}_{1t}}, \quad (4.25)
\]

\[
\mathcal{R}_{Ceq} = \frac{\mathcal{R}_1 \mathcal{R}_2 + \mathcal{R}_2 \mathcal{R}_3 + \mathcal{R}_3 \mathcal{R}_1}{\mathcal{R}_2} = \mathcal{R}_{LC} + \mathcal{R}_{1t} + \mathcal{R}_{3t} + \frac{\mathcal{R}_{1t} \mathcal{R}_{3t}}{\mathcal{R}_{2t}}. \quad (4.26)
\]

The fluxes \( \Phi_A, \Phi_B \) and \( \Phi_C \) link their respective secondary coils, each of \( N_s \) turns.
Faraday’s law describes the voltage induced on the coil terminals as

\[ V = -\frac{\partial \Phi}{\partial t}. \]  

(4.27)

Because only the sinusoidal term \( e^{j\omega t} \) varies with time, the differentiation simply brings down the \( j\omega \) term. Also, since we take the integral across all the turns of the coil we include the term \( N \). The relation is now:

\[ V = -j\omega N \Phi. \]  

(4.28)

Solving for the output voltages in terms of the reluctances we find

\[
\begin{bmatrix}
V_A \\
V_B \\
V_C
\end{bmatrix} = \frac{\omega N_p N_t I_1}{\sqrt{3}} \begin{bmatrix}
1 & e^{j\frac{2\pi}{3}} & 2e^{j\frac{\pi}{3}} \\
-2 & e^{j\frac{2\pi}{3}} & e^{j\frac{4\pi}{3}} \\
1 & 2e^{j\frac{5\pi}{3}} & e^{j\frac{4\pi}{3}}
\end{bmatrix} \begin{bmatrix}
\frac{1}{R_{eq}} \\
\frac{1}{R_{B}} \\
\frac{1}{R_{A}}
\end{bmatrix},
\]

(4.29)

where we distribute the \( j \) term through the matrix to simplify the expression. Figure 4-6 details the components of the flux \( \Phi_B \), along with the corresponding output voltage \( V_B \).

4.2.4 Output Voltage As A Function Of Tube Position

We reduce the reluctances of the previous section to their dependencies on the \( x, y \) position of the tube. To do this we define the gap and the characteristic length as discussed in section 4.2.2. For \( R_{1t,2t,3t} \) representing the paths from the poles to the tube, the relevant variable to solve for is the distance from the pole to the tube, as we assume the cross-sectional area remains constant. For \( R_{LA,LB,LC} \), the relevant variable is the cross-sectional area, because we assume constant path length.

Using the geometry shown in Figure 4-7, we calculate the distances \( d_1, d_2, d_3 \) in terms of the tube position in the \( x, y \) coordinate frame. The parameters \( r_t \) and \( r_o \) are the radii of the tube and sensor aperture, respectively. From this layout,
Figure 4-6: Flux $\Phi_B$ (in black) and corresponding components (in grey), normalized to $\frac{-\sqrt{3}}{N_p I_1}$; along with output voltage $V_B$ normalized to $\frac{-\sqrt{3}}{\omega N_p N_s I_1}$. We show the case for the tube in the center of the sensor such that the reluctances $R_{Aeq}, R_{Beq}, R_{Ceq}$ are all equal.

$$d_1 = \sqrt{x_o^2 + (r_o + y_o)^2} - r_t,$$
$$d_2 = \sqrt{\left(\frac{\sqrt{3}r_o}{2} - x_o\right)^2 + \left(\frac{r_o}{2} - y_o\right)^2} - r_t,$$
$$d_3 = \sqrt{\left(\frac{\sqrt{3}r_o}{2} + x_o\right)^2 + \left(\frac{r_o}{2} - y_o\right)^2} - r_t. \quad (4.30)$$

We present a similar layout in Figure 4-8 for calculating the cross-sectional area of the leakage flux paths. Calculating the lengths $L_{A,B,C}$ in terms of the tube position $(x_o, y_o)$ gives

$$L_A = \sqrt{r_o^2 - A_1^2} + \sqrt{x_o^2 + y_o^2 - A_1^2} - r_t,$$
$$L_B = \sqrt{r_o^2 - x_o^2 - y_o} - r_t,$$
$$L_C = \sqrt{r_o^2 - A_2^2} + \sqrt{x_o^2 + y_o^2 - A_2^2} - r_t. \quad (4.31)$$
Figure 4-7: Geometry for calculating reluctance gap lengths. We define the tube position in Cartesian coordinates with origin at the center of the sensor to facilitate control about the operating point (0,0). The shaded blocks are the ends of the laminate poles.

Where

\[ A_1 = \frac{\sqrt{3} y_o + x_o}{2}, \quad A_2 = \frac{\sqrt{3} y_o - x_o}{2}. \]  

(4.32)

Using the definition for the reluctance and substituting in \( d_{1,2,3} \) and \( L_{A,B,C} \) gives:

\[ R_{it} = \frac{d_i}{\mu_0 d^2 r_i + w_p} \quad \text{for } i = 1, 2, 3, \]  

(4.33)

\[ R_{LB} = \frac{\sqrt{3} r_o}{\mu_0 d^2 r_{1i} + w_p} \quad \text{for } i = A, B, C. \]  

(4.34)

We substitute this into equations (4.25) to arrive at
Figure 4-8: Geometry for calculating leakage reluctance path area. Three shaded blocks are the ends of the poles; we omit the rest of the sensor for clarity.

\[
\begin{align*}
R_{Aeq} &= \frac{2}{\mu_0 d(2r_t + w_p)} \left( \sqrt{3} r_o (2r_t + w_p) + d_1 + d_2 + \frac{d_1 d_2}{d_3} \right), \\
R_{Beq} &= \frac{2}{\mu_0 d(2r_t + w_p)} \left( \frac{\sqrt{3} r_o (2r_t + w_p)}{L_B + w_p} + d_2 + d_3 + \frac{d_2 d_3}{d_1} \right), \\
R_{Ceq} &= \frac{2}{\mu_0 d(2r_t + w_p)} \left( \frac{\sqrt{3} r_o (2r_t + w_p)}{L_C + w_p} + d_1 + d_3 + \frac{d_1 d_3}{d_2} \right),
\end{align*}
\]

(4.35)

where \( d \) is the thickness of the sensor.

4.2.5 Tube Position As A Function Of Output Voltage

Because the equations (4.35) are highly nonlinear in \( x \) and \( y \) we turn to a combination of numerical and experimental methods to solve for an equation of the tube position in terms of the three secondary coil voltages. Because we must assume paths for the
reluctance, but do not know them exactly, we expect some discrepancy between our theoretical results and our experimental results. Therefore we use these two methods together, using the experimental data as a benchmark for the theoretical data.

For our initial estimate we look at the analytical output voltages across the secondary coils. We calculate a matrix which represents these voltages as a function of the tube position, such that the row and column indices are functions of the y- and x-coordinates, respectively. In other words, each entry in the matrix is the predicted value of the voltage when the tube is at the position corresponding to the indices of the entry. Using (4.35) we calculate the reluctances $R_{Aeq, Beq, Ceq}$ as a function of tube position, and combine these to form the output voltages as in equation (4.29). We show the contour plot of the magnitude of the resultant voltage $V_B$ in Figure 4-9. The voltages $V_A$ and $V_C$ are very similar, but rotated 120° clockwise and counter-clockwise, respectively.

![Figure 4-9: Contour plot of lines of constant voltage signal magnitude $|V_B|$ across secondary terminals. These are lines of constant voltage from our theoretical analysis. We allow the center of the tube to move inside the radius $r_0 - r_1$, such that we plot the voltage corresponding to the location of the center of the tube. We overlay a path of constant radius from pole one.](image)

The main trend we notice is that for the upper two-thirds of the opening, the magnitude of the voltage is nearly constant as the tube moves at a constant radius from
pole face one (see path (i) in Fig. 4-9), but decreases as the distance \(d_1\) increases. The distance from pole one is the same \(d_1\) as in Figure 4-7. We see this same trend with voltages \(V_A\) and \(V_C\) and their corresponding pole faces. In the experimental setup we see the voltage is even closer to a direct correspondence to \(d_1\), in fact this trend is valid throughout the entire range of the sensor, i.e., the output voltage decreases monotonically as the distance from the relevant pole face increases for all values of \(d_1\). This differs from the above plot of \(|V_B|\), where the voltage begins to decrease after \(d_1\) is less than about half its maximum value.

If we assume the output voltage is inversely proportional to the distance from the opposite pole face, i.e., the magnitude of voltage \(V_B\) increases as the distance \(d_1\) decreases, we can use this relationship to solve for the tube position in terms of these voltages. First we invert equations (4.30) to give

\[
\begin{align*}
x_o &= \frac{(d_3 + r_t)^2 - (d_2 + r_t)^2}{2\sqrt{3}r_o}, \\
y_o &= \frac{2(d_1 + r_t)^2 - (d_2 + r_t)^2 - (d_3 + r_t)^2}{6r_o}.
\end{align*}
\]

which we expand to

\[
\begin{align*}
x_o &= \frac{d_3^2 - d_2^2 + 2r_t(d_3 - d_2)}{2\sqrt{3}r_o}, \\
y_o &= \frac{2d_1^2 - d_2^2 - d_3^2 + 2r_t(2d_1 - d_2 - d_3)}{6r_o}.
\end{align*}
\]

If we now combine the voltages in this way, such that we replace \(d_1\) with \(|V_B|\), \(d_2\) with \(|V_C|\) and \(d_3\) with \(|V_A|\), we obtain a first estimate for the voltages \(V_x\) and \(V_y\), which depend on the \(x\)- and \(y\)-position of the tube, respectively:

\[
V_x = \frac{|V_A|^2 - |V_C|^2 + 2r_t(|V_A| - |V_C|)}{2\sqrt{3}r_o},
\]

\(93\)
\[ V_y = \frac{2|V_B|^2 - |V_C|^2 - |V_A|^2 + 2r_i(2|V_B| - |V_C| - |V_A|)}{6r_o}. \] (4.41)

4.2.6 Experimental Data From Three-Phase Sensor

Experimentally we implement the first order terms of equations (4.40) and (4.41). We use demodulation chips to rectify the voltage signals and op-amps to sum the signals as desired. To check the accuracy of the output we plot it as described in Chapter 1, where we move the tube in a grid pattern inside the sensor and plot the two output voltages \( V_x \) and \( V_y \) against each other. Figure 4-10 shows our experimental results.

![Plot of \( V_x, V_y \) as a function of position, output from actual sensor. The signal processing board combines the three voltages from the secondary coils into two voltages \( V_x \) and \( V_y \).](image)

As mentioned in Chapter 1, we see that the output is more linear towards the center of the sensor, and that as the tube nears the edges of the opening the data becomes increasingly nonlinear. When we first plot the analytical estimate of the sensor output, it looks more ideal than the experimental output. Figure 4-11 shows a plot of the
first order terms of equations (4.40) and (4.41), where we use the reluctance values as previously calculated. To make the theoretical output correspond to the experimental output more closely, we multiply the reluctances $R_{1t,2t,3t}$ by $\frac{1}{7}$ and plot the predicted output using the first order terms of equations (4.40) and (4.41). Figure 4-12 shows this plot.

![Plot of $V_x, V_y$ as a function of position using a first order approximation and the exact theoretical values from the Magnetic Circuit Analysis.](image)

Figure 4-11: Plot of $V_x, V_y$ as a function of position using a first order approximation and the exact theoretical values from the Magnetic Circuit Analysis.

We see that our adjusted analytical solution is more similar to the experimental results. That we had to adjust the reluctances $R_{1t,2t,3t}$ to make this data match more closely suggests our initial guess of these reluctances was too large. This implies that more flux travels through the tube than we initially estimated. The most likely sources for this discrepancy are our choice of areas for the reluctances, the magnetic circuit itself, or a combination of the two. When modeling the circuit we assume a basic flux path instead of using the exact solution because we do not know the true behavior of the flux; it may well be that the area we chose for our calculations was too small. Also, the magnetic circuit may be missing some reluctances which are in fact relevant. Either way, our adjusted model is close enough for us to observe the major trends and used these to process the voltages into useful signals.

If we include the squared terms in the output plot, we see that the magnitude of
the output is greater, but the shape of the plot is very similar. Because the signal processing board has an adjustable gain stage at the output, the gain of the output voltages matter less than the correspondence to tube position. Figure 4-13 shows the result of plotting $V_x$ vs. $V_y$ against each other using all the terms (squared and linear) of equations (4.40) and (4.41). Appendix A contains the code which we use to simulate the sensor output and produce Figures 4-11 through 4-15.

**Phase of Output Voltage**

In the above analysis we only use the magnitude of the signal. If we look at the phase of the output voltage, as in Figure 4-14, we see another valuable source of information. The phase changes by about $\pm15^\circ$ as we move the tube from one side of the opening to the other, but remains mostly constant as we move the tube from top to bottom. If we assume that the phase of the voltage $V_B$ is dependent only on the $x$-position of the tube (and that this trend holds for the other secondary poles as well), we can derive similar relations to those above to give us an estimate of the tube position in
terms of the phase of the voltages on the secondary coils. These equations are:

\[
V_x = \frac{\angle V_B - \angle V_C - \angle V_A}{2}, \quad (4.42)
\]
\[
V_y = \frac{\angle V_C - \angle V_A}{\sqrt{3}}. \quad (4.43)
\]

We use these equations to plot our theoretical output of voltages \( V_x \) and \( V_y \) as we did in the previous section. Figure 4-15 shows this plot. We see that the output is indeed closer to the ideal grid shape.

We choose not to use the phase to find the tube position because of the difficulty of accurately sensing the phase. The phase only changes about five degrees per millimeter of tube movement. The extra circuitry required to use this method, combined with the fact that the output is still linear only in the middle area of the sensing range leads us to choose to use only the magnitude and not the phase. A possible improvement on future sensors would be to include the phase information in the sig-
Figure 4-14: Contour plot of theoretical phase of voltage $V_B$ across secondary terminals. We see that the phase has a different effect than the magnitude: as the tube moves in the $x$-direction, the phase change of $V_B$ is nearly linear in the inner area of the sensor. The units of the legend are radians.

Figure 4-15: Plot of $V_x$ and $V_y$ using the phase of the voltages instead of the magnitude. The output looks even more linear than when using the magnitude.
nal processing. Theoretically it is possible to get six position measurements using the magnitude and phase of the three voltages, which could lead to a much more accurate sensor.

**Demodulation Against Input Signal**

In our present set of electronics we use the demodulation ICs, but we only use them to rectify the signal. Experimentally we find that the output is less linear when we use these chips to actually *demodulate* the voltage signals. A look at the magnitude and phase makes the reason for this clear.

In Chapter 3 we showed that the result of demodulating a signal shifted by $\theta_o$ degrees from the reference signal will reduce the DC magnitude by $\cos\theta_o$. If we look again at Figure 4-9 we see that as the tube moves away from the sensor center in the positive $x$-direction, the magnitude $|V_B|$ decreases. If we now look at the phase plot in Figure 4-14, we see that the phase is zero at $x = 0$ and increases as the tube moves away from the center in the positive $x$-direction. If we were to demodulate the signal $V_B$ against a reference signal of zero phase (i.e., a signal in phase with $V_B$ when the tube is at the center of the sensor), we would make the DC level decrease *even more* as the tube moves in the $x$-direction. What we would prefer is for the demodulation to increase the DC voltage level as we move the tube away from center. Because the demodulation of the secondary coil voltages does not improve the output of our sensor, we use the chips as synchronous rectifiers instead.

### 4.3 Exact Solution With Tube Centered

We now turn to another approach for solving the field problem. With the tube at the center position, (see Figure 4-16), we calculate an exact solution for the magnetic fields inside the tube and air gap. For this derivation we follow the procedure presented in [7], Specifically, pp. 2.34-2.45 for the nonconducting regions and pp. 6.11-6.14 for the conducting regions. Pages cited outside this range will be noted in the text. The main simplification we make from the derivation in [7] is that our conductor is stationary
with respect to the coordinate frame of the magnetic fields, such that the relative angular velocity \( \Omega \) is zero.

Figure 4-16: Permeable, conducting, hollow cylinder in a uniform, time-varying field imposed as the vector potential at surface (e). We assume no variation in the \( z \) direction, and finite permeability and conductivity in the tube region.

Our derivation proceeds as follows. We begin with the definition of the vector potential and derive the equations which it satisfies in the conducting region and the nonconducting regions. Then we present the boundary conditions and transfer relations for the magnetic field in these regions, and derive the Fourier series representation of the assumed form of the field \( \vec{H}_m \). Finally we solve for the field components at each boundary and use these to plot the field lines throughout the three regions. Appendix A contains the code which we use to plot the field lines.
4.3.1 Vector Potential

Because the tube is conducting we represent the flux density with the magnetic vector potential,

\[ \vec{B} = \nabla \times \vec{A}, \]  

(4.44)

and set the Coulomb gauge such that \( \nabla \cdot \vec{A} = 0 \). In the conducting region where currents may exist, the vector potential satisfies the vector Poisson’s equation

\[ \nabla^2 \vec{A} = -\mu \vec{J}. \]  

(4.45)

In the nonconducting air regions where \( \vec{J} \) is zero, the vector potential satisfies the vector Laplace equation,

\[ \nabla^2 \vec{A} = 0. \]  

(4.46)

We could instead use the scalar potential in the nonconducting region, but because we must use the vector potential inside the conducting tube, our analysis is simpler if we use the vector potential throughout. We assume no variation in the \( z \) direction and choose a solution of the form

\[ \vec{A} = \vec{A}_z = \hat{A}_z(r)e^{i(\omega t - m\theta)}\hat{z}. \]  

(4.47)

We use the “hat” (\( \hat{\ } \)) above a quantity to denote the complex amplitude of a value which varies sinusoidally with both time \( t \) and angle \( \theta \). The variable \( m \) is the angular wave number, and it will come into play when we implement the Fourier series representation. Substituting (4.47) into (4.44) gives

\[ \mu(H_r \hat{r} + H_\theta \hat{\theta}) = \frac{-jm}{r} \hat{A}_z \hat{r} - \frac{\partial \hat{A}_z}{\partial r} \hat{\theta}, \]  

(4.48)

or equivalently
\[ \vec{H} = -\frac{jm}{\mu r} \vec{A}_z \hat{r} - \frac{1}{\mu} \frac{\partial \vec{A}_z}{\partial r} \hat{\theta}. \] (4.49)

Therefore, given the complex amplitude of the radial field at radius \( r_k \) is \( \vec{H}_r^k \) and the permeability in the region is \( \mu_k \), the complex amplitude of the vector potential \( \vec{A}_z^k \) at \( r_k \) is given by

\[ \vec{A}_z^k = \frac{\mu_k r_k}{-jm} \vec{H}_r^k. \] (4.50)

**Vector Potential in Nonconducting Regions**

Laplace’s equation holds in the nonconducting regions (I) and (III). Substituting (4.47) into (4.46) results in the following equation:

\[ \nabla^2 \vec{A} = \left( -\frac{1}{r \partial r} \frac{\partial \vec{A}_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \vec{A}_z}{\partial \theta^2} + \frac{\partial^2 \vec{A}_z}{\partial z^2} \right) \hat{r}_z = 0, \] (4.51)

where we use the fact that the vector potential has only a \( z \)-component to simplify the vector Laplacian. Given that the problem has no \( z \)-variation, and that all terms have the same sinusoidal variation \( e^{i(\omega t - m \theta)} \), we may reduce this to

\[ \frac{1}{r} \left( r \frac{\partial^2 \vec{A}_z}{\partial r^2} + \frac{\partial \vec{A}_z}{\partial r} \right) + \frac{(-jm)^2}{r^2} \vec{A}_z = 0, \] (4.52)

and thus

\[ \frac{\partial^2 \vec{A}_z}{\partial r^2} + \frac{1}{r} \frac{\partial \vec{A}_z}{\partial r} + \frac{-m^2}{r^2} \vec{A}_z = 0. \] (4.53)

The solutions to (4.53) are of the form \( r^{\pm m} \), and for a cylindrical annulus of inner radius \( \beta \), outer radius \( \alpha \), and infinite \( z \) extent, the vector potential satisfies

\[ \vec{A}_z(r) = \tilde{A}^\beta \left[ \left( \frac{\beta}{\alpha} \right)^m - \left( \frac{\alpha}{\beta} \right)^m \right] + \tilde{A}^\alpha \left[ \left( \frac{\alpha}{\beta} \right)^m - \left( \frac{\alpha}{\beta} \right)^m \right]^*, \] (4.54)

where \( \tilde{A}^\beta \) and \( \tilde{A}^\alpha \) are the complex amplitudes of the \( m \)th spatial harmonics of the
vector potential at the inner and outer boundaries, respectively.

**Vector Potential in Conducting Region**

To derive the equation for the vector potential in the conducting region (II), we begin with the magnetic diffusion equation, (2.25), into which we substitute the definition $\vec{H} = \frac{1}{\mu} \vec{B} = \frac{1}{\mu} (\nabla \times \vec{A})$. This results in

$$\nabla^2 (\nabla \times \vec{A}) = \mu \sigma \frac{\partial (\nabla \times \vec{A})}{\partial t},$$

or equivalently,

$$\nabla \times \left[ \nabla \times (\nabla \times \vec{A}) + \mu \sigma \frac{\partial \vec{A}}{\partial t} \right] = 0.$$  \hspace{1cm} (4.56)

We use the same vector identity\(^1\) as before to arrive at this result; recall that we assume magnetically linear media, and again set the Coulomb gauge such that $\nabla \cdot \vec{A} = 0$. Because the curl of the gradient of a scalar is zero, we may write the above term enclosed in brackets as the gradient of a scalar in order to find a solution (see [7], pp. 6.11 - 6.13). We choose this scalar to be zero and again use the vector identity\(^1\) to reduce this to

$$\nabla^2 \vec{A} = \mu \sigma \frac{\partial \vec{A}}{\partial t}.$$  \hspace{1cm} (4.57)

We substitute our assumed form for $\vec{A}$ into (4.57) to arrive at

$$\frac{1}{r} \left( r \frac{\partial^2 \hat{A}_z}{\partial r^2} + \frac{\partial \hat{A}_z}{\partial r} \right) + \frac{(-jm)^2}{r^2} \hat{A}_z = j\mu \sigma \omega \hat{A}_z,$$

or

$$\frac{\partial^2 \hat{A}_z}{\partial r^2} + \frac{1}{r} \frac{\partial \hat{A}_z}{\partial r} - (\gamma^2 + \frac{m^2}{r^2}) \hat{A}_z = 0,$$ \hspace{1cm} (4.59)

with $\gamma = \sqrt{j \sigma \mu \omega}$. When $\gamma$ and $m$ are non-zero, (4.59) is known as Bessel's equation;\(^1\)

\[^1\nabla \times \nabla \times \vec{A} = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}\]
standard solutions to which are Bessel and Hankel functions. We also see that \( \sigma = 0 \) reduces (4.59) to (4.53), the nonconducting case.

We again use the transfer relation method to solve for the field distribution. For a conducting cylindrical annulus with inner radius \( \beta \), outer radius \( \alpha \), and infinite \( z \) extent, the vector potential inside will be of the form ([7], pg. 6.14)

\[
\hat{A}_z(r) = \hat{A}^\alpha \frac{[H_m(j\gamma\beta)J_m(j\gamma\alpha) - J_m(j\gamma\beta)H_m(j\gamma\alpha)]}{[H_m(j\gamma\beta)J_m(j\gamma\alpha) - J_m(j\gamma\beta)H_m(j\gamma\alpha)]} + \hat{A}^\beta \frac{[H_m(j\gamma\alpha)J_m(j\gamma\beta) - H_m(j\gamma\alpha)J_m(j\gamma\beta)]}{[J_m(j\gamma\alpha)H_m(j\gamma\beta) - H_m(j\gamma\alpha)J_m(j\gamma\beta)]}.
\]

(4.60)

The terms \( \hat{A}^\beta \) and \( \hat{A}^\alpha \) represent the values of the vector potential at the inner and outer boundaries, respectively; \( H_m \) and \( J_m \) are Bessel and Hankel functions of the first kind, respectively.

4.3.2 Boundary Conditions

To define the field in the sensor we impose two boundary conditions. The first is that the vector potential at the outer radius (\( \epsilon \)) is nonzero at the pole faces but zero along the shielding sections. We choose this because we assume that the aluminum shielding effectively blocks out any radial component of the field. The second is that the vector potential in the center of the tube is zero. The outer boundary will always be periodic in 2\( \pi \) because of our Fourier series approximation, therefore the fields will be zero at \((x, y) = (0, 0)\) because of symmetry.

**Fourier Series Representation of Magnetic Field**

The principle of superposition allows us to define the boundary condition at surface (\( \epsilon \)) as a sum of sinusoidal terms, or Fourier series. We add these terms to give an estimate of the resultant field distribution. Figure 4-17 shows the layout of the outer boundary, with the pole face locations and relevant parameters. We pick the coordinate frame to make the math more straightforward, such that \( \theta = 0 \) at the
To derive the Fourier series representation we begin by specifying the radial component of the field in terms of step functions spaced along the $\theta$ axis as seen in Figure 4-18. We assume that the field is zero except at the pole faces, where it is constant across the pole face surface. In these areas of constant field, we assume a field of the form

$$H_r^e = \tilde{H}_0 e^{j(\omega t - \delta)} e^{j\delta},$$  

where $\delta$ is the phase lag term: $\delta = 0$ for pole one, $\delta = \frac{2\pi}{3}$ for pole two, and $\delta = \frac{4\pi}{3}$ for pole three. The amplitude of each step function varies sinusoidally with time; the phase of each step corresponds to the phase of the current driving the primary coil at the corresponding pole. When the tube is at the center, symmetry dictates that the amplitude of the field at each pole is the same.

Finding the Fourier series is much easier if we write the theta-derivative of the field,
Figure 4-18: Magnetic field (radial-component) at surface (e) as a function of angle $\theta$. This plot shows the complex amplitudes of the step functions. We observe the real part of these amplitudes, so they will never all be the same at any given moment in time because of the three-phase nature of the excitation currents.

Figure 4-19: Derivative of radial component $\hat{H}_r$ with respect to $\theta$ at surface (e), plotted as a function of $\theta$.

which we show using impulse functions in Figure 4-19. Recall that since the only variation in the $\theta$ direction is due to the $e^{-j\theta}$ term, differentiation with respect to $\theta$ simply brings down the term $-jm$.

In general, we can write a signal $X$ by means of a Fourier series expansion as

$$X = \sum_{m=-\infty}^{\infty} X_m e^{-j2\pi m \theta}.$$  \hspace{1cm} (4.62)

Because our system is periodic in $2\pi$, we set $l = 2\pi$. The subscript $m$ denotes the amplitude of the $m^{th}$ component, where we find $X_m$ using

$$X_m = \frac{1}{2\pi} \int_{0}^{2\pi} X e^{j2\pi m \theta} d\theta;$$  \hspace{1cm} (4.63)
the limits may be over any period of length $2\pi$. We substitute the theta-derivative of $\dot{H}_r^e$ into the above equation to compute the amplitude terms $\dot{H}_{rm}^e$:

\[
\frac{\partial \dot{H}_{rm}^e}{\partial \theta} = -j m \dot{H}_{rm}^e = \frac{1}{2\pi} \int_{-\theta_o}^{2\pi-\theta_o} \frac{\partial \dot{H}_r^e}{\partial \theta} e^{j m \theta} d\theta, \quad (4.64)
\]

where

\[
\frac{\partial \dot{H}_r^e}{\partial \theta} = \dot{H}_o \left\{ \left[ \delta(\theta - (-\theta_o)) - \delta(\theta - \theta_o) \right] + \\
\left[ \delta(\theta - \left( \frac{2\pi}{3} - \theta_o \right)) - (\delta(\theta - \left( \frac{2\pi}{3} + \theta_o \right)) e^{i\frac{2\pi}{3}} + \\
\left[ \delta(\theta - \left( \frac{4\pi}{3} - \theta_o \right)) - (\delta(\theta - \left( \frac{4\pi}{3} + \theta_o \right)) e^{i\frac{4\pi}{3}} \right] \right\}. \quad (4.65)
\]

Substituting (4.65) into (4.64) and evaluating the integral gives

\[
\dot{H}_{rm}^e = \frac{\dot{H}_o}{-2jm\pi} \left\{ (e^{-jm\theta_o} - e^{jm\theta_o}) + (e^{jm(-\frac{2\pi}{3} - \theta_o)} - e^{-jm(\frac{2\pi}{3} + \theta_o)}) e^{i\frac{2\pi}{3}} + \\
(e^{-jm(\frac{4\pi}{3} - \theta_o)} - e^{-jm(\frac{4\pi}{3} + \theta_o)}) e^{i\frac{4\pi}{3}} \right\}, \quad (4.66)
\]

which we reduce to

\[
\dot{H}_{rm}^e = \frac{\dot{H}_o}{m\pi} \sin(m\theta_o) \left\{ 1 + e^{j\frac{2\pi}{3}(m-1)} + e^{j\frac{4\pi}{3}(m-1)} \right\}, \quad (4.67)
\]

thus

\[
\dot{H}_r^e = \sum_{m=-\infty}^{\infty} 3\frac{\dot{H}_o}{m\pi} \sin(m\theta_o) e^{-jm\theta}. \quad (4.68)
\]

Figure 4-20 shows the radial component of the field at surface (e), plotted as a function of angle $\theta$.
Field Components Across Material Boundaries

The boundary condition described by Gauss’ law requires that the normal component of the magnetic flux density $\vec{B}$ be constant across a boundary. Also, because we assume no surface currents exist at any boundary, (we model the conductor as having volume currents only), the tangential component of the magnetic field $\vec{H}$ is constant as required by Ampere’s law. Summarizing these boundary conditions:

1. $\hat{H}^a_{\theta m} = \hat{H}^b_{\theta m}$

2. $\hat{H}^c_{\theta m} = \hat{H}^d_{\theta m}$

3. $\mu_0 \hat{H}^a_{rm} = \mu \hat{H}^b_{rm}$

4. $\mu \hat{H}^c_{rm} = \mu_0 \hat{H}^d_{rm}$

We use the subscript $m$ to denote the amplitude of a Fourier term as in the previous section; these boundary conditions hold for each value of $m$. 

Figure 4-20: Magnetic scalar potential at surface (e) at time $t = 0$. Here we use a Fourier series of 51 terms to approximate the field, which we assume has discontinuous steps at the pole faces and is zero otherwise. The overshoot at the step locations is known as the Gibbs phenomenon.
Transfer Relations

For a pair of concentric cylindrical surfaces with a permeable, conducting medium in between, the transfer relations relating the \( r \) and \( \theta \) directed components of the magnetic field at the boundaries are

\[
\begin{bmatrix}
\hat{H}_0^\alpha \\
\hat{H}_0^\beta
\end{bmatrix} = \frac{j}{m} \begin{bmatrix}
 f_m(\beta, \alpha, \gamma) & g_m(\alpha, \beta, \gamma) \\
g_m(\beta, \alpha, \gamma) & f_m(\alpha, \beta, \gamma)
\end{bmatrix} \begin{bmatrix}
\alpha \hat{H}_r^\alpha \\
\beta \hat{H}_r^\beta
\end{bmatrix}, \tag{4.69}
\]

where \( \beta \) and \( \alpha \) are the inner and outer radii, respectively. We modify the transfer relations presented in [7] slightly to write the theta-components in terms of the radial components for simplicity. The reference [7] uses the vector potential instead, with the conversion between \( \hat{H}_r \) and \( \hat{A}_r \) as seen in equation (4.50). When the medium is not conducting, \( \gamma = 0 \) and the transfer relations take a simpler form,

\[
\begin{bmatrix}
\hat{H}_0^\alpha \\
\hat{H}_0^\beta
\end{bmatrix} = \frac{j}{m} \begin{bmatrix}
 f_m(\beta, \alpha) & g_m(\alpha, \beta) \\
g_m(\beta, \alpha) & f_m(\alpha, \beta)
\end{bmatrix} \begin{bmatrix}
\alpha \hat{H}_r^\alpha \\
\beta \hat{H}_r^\beta
\end{bmatrix}. \tag{4.70}
\]

We define the functions \( f_m, g_m \) as

\[
f_m(x, y) = \frac{m}{y} \left[ \left( \frac{x}{y} \right)^m + \left( \frac{y}{x} \right)^m \right], \tag{4.71}
\]

\[
g_m(x, y) = \frac{2m}{x} \left[ \left( \frac{x}{y} \right)^m - \left( \frac{y}{x} \right)^m \right], \tag{4.72}
\]

\[
f_m(x, y, \gamma) = j\gamma \frac{[H_m(j\gamma x)J_m'(j\gamma y) - J_m(j\gamma x)H_m'(j\gamma y)]}{[J_m(j\gamma x)H_m(j\gamma y) - J_m(j\gamma y)H_m(j\gamma x)]}, \tag{4.73}
\]

\[
g_m(x, y, \gamma) = \frac{-2j}{\pi x} \frac{1}{[J_m(j\gamma x)H_m(j\gamma y) - J_m(j\gamma y)H_m(j\gamma x)]}. \tag{4.74}
\]

The prime above a function denotes the total derivative with respect to the entire argument. We remove the dependence on the primed functions and simplify (4.73) using the relation

109
\[ uR'_m(u) = -mR_m(u) + uR_{m-1}(u) \tag{4.75} \]

where \( R_m \) can be either \( H_m \) or \( J_m \) ([7] pg. 2.37). Applying this to (4.73) gives

\[
\begin{align*}
&f_m(x, y, \gamma) = \\
&\left\{ \frac{H_m(j\gamma x)[J_{m-1}(j\gamma y) - \frac{m}{j\gamma y} J_m(j\gamma y)] - J_m(j\gamma x)[H_{m-1}(j\gamma y) - \frac{m}{j\gamma y} H_m(j\gamma y)]}{[J_m(j\gamma x)H_m(j\gamma y) - J_m(j\gamma y)H_m(j\gamma x)]} \right\} \\
&\left\{ \frac{J_m(j\gamma x)H_m(j\gamma y) - J_m(j\gamma y)H_m(j\gamma x)}{J_m(j\gamma x)H_m(j\gamma y) - J_m(j\gamma y)H_m(j\gamma x)} \right\}.
\end{align*}
\tag{4.76}
\]

which reduces to

\[
\begin{align*}
&f_m(x, y, \gamma) = \\
&\frac{m}{y} + j\gamma \left\{ \frac{H_m(j\gamma x)J_{m-1}(j\gamma y) - J_m(j\gamma x)H_{m-1}(j\gamma y)}{[J_m(j\gamma x)H_m(j\gamma y) - J_m(j\gamma y)H_m(j\gamma x)]} \\
&\right\}.
\end{align*}
\tag{4.77}
\]

### 4.3.3 Field Solution

We use transfer relation (4.70) to write the transfer relation between surfaces (d) and (e) in the nonconducting region (III):

\[
\begin{bmatrix}
\hat{H}_\theta^e_m \\
\hat{H}_\theta^d_m
\end{bmatrix} = \frac{j}{m} \begin{bmatrix}
f_m(c, e) & g_m(e, c) \\
g_m(e, c) & f_m(e, c)
\end{bmatrix} \begin{bmatrix}
e\hat{H}_r^c \\
c\hat{H}_r^d
\end{bmatrix}.
\tag{4.78}
\]

Similarly, we use (4.69) to write the relations between surfaces (b) and (c) in the conducting region (II) as:

\[
\begin{bmatrix}
\hat{H}_\theta^c \\
\hat{H}_\theta^b_m
\end{bmatrix} = \frac{j}{m} \begin{bmatrix}
f_m(a, c, \gamma) & g_m(c, a, \gamma) \\
g_m(a, c, \gamma) & f_m(c, a, \gamma)
\end{bmatrix} \begin{bmatrix}
c\hat{H}_r^c \\
a\hat{H}_r^b
\end{bmatrix}.
\tag{4.79}
\]

We use the boundary conditions to rewrite the above relations. Specifically, we substitute \( \hat{H}_\theta^d_m \) in for \( \hat{H}_\theta^c_m \), \( \hat{H}_\theta^a_m \) in for \( \hat{H}_\theta^b_m \), \( \frac{J_m}{\mu H_m} \) in for \( \hat{H}_r^c_m \), and \( \frac{J_m}{\mu H_m} \) in for \( \hat{H}_r^b_m \). This results in
\[
\begin{bmatrix}
\hat{H}_a^{d_m} \\
\hat{H}_a^{a_m}
\end{bmatrix} = \frac{j}{m} \begin{bmatrix}
f_m(a, c, \gamma) & g_m(c, a, \gamma) \\
g_m(a, c, \gamma) & f_m(c, a, \gamma)
\end{bmatrix} \begin{bmatrix}
\frac{a \mu_o}{\mu} \hat{H}_r^{d_m} \\
\frac{a \mu_o}{\mu} \hat{H}_r^{a_m}
\end{bmatrix} .
\tag{4.80}
\]

We again use (4.70) in the nonconducting region (I) to relate the field at surface (a) to the field in the center:

\[
\begin{bmatrix}
\hat{H}_a^{a_m} \\
\hat{H}_a^{\text{center}}
\end{bmatrix} = \frac{j}{m} \begin{bmatrix}
f_m(0, a) & g_m(a, 0) \\
g_m(0, a) & f_m(a, 0)
\end{bmatrix} \begin{bmatrix}
a \hat{H}_r^{a_m} \\
0
\end{bmatrix} .
\tag{4.81}
\]

We now have two equations for \( \hat{H}_a^{d_m} \) and two for \( \hat{H}_a^{a_m} \). For \( \hat{H}_a^{d_m} \) we have

\[
\hat{H}_a^{d_m} = c \hat{H}_r^{d_m} g_m(c, e) + c \hat{H}_r^{d_m} f_m(e, c),
\tag{4.82}
\]

\[
\hat{H}_a^{d_m} = \frac{c \mu_o}{\mu} \hat{H}_r^{d_m} f_m(a, c, \gamma) + \frac{a \mu_o}{\mu} \hat{H}_r^{a_m} g_m(c, a, \gamma);
\tag{4.83}
\]

for \( \hat{H}_a^{a_m} \) we have

\[
\hat{H}_a^{a_m} = \frac{c \mu_o}{\mu} \hat{H}_r^{d_m} g_m(a, e) + \frac{a \mu_o}{\mu} \hat{H}_r^{a_m} f_m(e, c),
\tag{4.84}
\]

\[
\hat{H}_a^{a_m} = a \hat{H}_r^{a_m} f_m(0, a).
\tag{4.85}
\]

We equate the two above equations for \( \hat{H}_a^{d_m} \), and equate the two above equations for \( \hat{H}_a^{a_m} \). This gives

\[
e \hat{H}_r^{a_m} g_m(c, e) + c \hat{H}_r^{d_m} f_m(e, c) = \frac{c \mu_o}{\mu} \hat{H}_r^{d_m} f_m(a, c, \gamma) + \frac{a \mu_o}{\mu} \hat{H}_r^{a_m} g_m(c, a, \gamma)
\tag{4.86}
\]

for \( \hat{H}_a^{d_m} \) and

\[
\frac{c \mu_o}{\mu} \hat{H}_r^{d_m} g_m(a, c, \gamma) + \frac{a \mu_o}{\mu} \hat{H}_r^{a_m} f_m(c, a, \gamma) = a \hat{H}_r^{a_m} f_m(0, a)
\tag{4.87}
\]
for $\hat{H}^c_{rm}$. Solving the above two equations for $\hat{H}^a_{rm}$ results in the two equations

$$
\hat{H}^a_{rm} = \frac{e\hat{H}^e_{rm} g_m(c, e) + (cf_m(e, c) - \frac{\mu a}{\mu} f_m(a, c, \gamma))\hat{H}^d_{rm}}{\frac{\mu a}{\mu} g_m(c, a, \gamma)},
$$

(4.88)

$$
\hat{H}^a_{rm} = \left(-\frac{\mu a}{\mu} g_m(a, c, \gamma) \left( -m - \frac{\mu a}{\mu} f_m(c, a, \gamma) \right) \right) \hat{H}^d_{rm}.
$$

(4.89)

where we use the fact that $f_m(0, a) = -\frac{m}{a}$. Equating these two equations gives

$$
e\hat{H}^e_{rm} g_m(c, e) + (cf_m(e, c) - \frac{\mu a}{\mu} f_m(a, c, \gamma))\hat{H}^d_{rm} = \frac{\mu a}{\mu} g_m(c, a, \gamma) \left( -m - \frac{\mu a}{\mu} f_m(c, a, \gamma) \right) \hat{H}^d_{rm}.
$$

(4.90)

Solving for $\hat{H}^d_{rm}$ yields

$$
\hat{H}^d_{rm} = \frac{e\hat{H}^e_{rm} g_m(c, e)}{\frac{\mu a}{\mu} g_m(c, a, \gamma) \left( -m - \frac{\mu a}{\mu} f_m(c, a, \gamma) \right) - (cf_m(e, c) - \frac{\mu a}{\mu} f_m(a, c, \gamma))}.
$$

(4.91)

Similarly, solving for $\hat{H}^a_{rm}$ yields

$$
\hat{H}^a_{rm} = \frac{\frac{\mu a}{\mu} g_m(a, c, \gamma) \left( -m - \frac{\mu a}{\mu} f_m(c, a, \gamma) \right) e\hat{H}^e_{rm} g_m(c, e)}{\frac{\mu a}{\mu} g_m(c, a, \gamma) \left( -m - \frac{\mu a}{\mu} f_m(c, a, \gamma) \right) - (cf_m(e, c) - \frac{\mu a}{\mu} f_m(a, c, \gamma))}.
$$

(4.92)

And we can easily find $\hat{H}^c_{rm}$ and $\hat{H}^b_{rm}$ using the boundary conditions $\frac{\mu a}{\mu} \hat{H}^d_{rm} = \hat{H}^c_{rm}$ and $\frac{\mu a}{\mu} \hat{H}^a_{rm} = \hat{H}^b_{rm}$. This gives

$$
\hat{H}^b_{rm} = \frac{\frac{\mu a}{\mu} g_m(c, a, \gamma) \left( -m - \frac{\mu a}{\mu} f_m(c, a, \gamma) \right) e\hat{H}^e_{rm} g_m(c, e)}{\frac{\mu a}{\mu} g_m(c, a, \gamma) \left( -m - \frac{\mu a}{\mu} f_m(c, a, \gamma) \right) - (cf_m(e, c) - \frac{\mu a}{\mu} f_m(a, c, \gamma))},
$$

(4.93)
\[ \hat{H}_{rm}^c = \frac{\mu_0 e \hat{H}_{rm}^c g_m(c, e)}{\frac{\mu_0}{\mu} g_m(c, a, \gamma) \left( \frac{\frac{\mu_0}{\mu} g_m(a, c, \gamma)}{-m - \frac{\mu_0}{\mu} f_m(c, a, \gamma)} \right) - (c f_m(e, c) - \frac{\mu_0}{\mu} f_m(a, c, \gamma))}. \]  

(4.94)

To plot the magnetic flux density field lines, we note that these are the same as the lines of constant vector potential. The vector potential in the three regions is given by the solutions presented earlier: (4.54) for the nonconducting regions, and (4.60) for the conducting region. Substituting in the relevant values gives

\[ \hat{A}_{III}^m = \hat{A}_m^c \left[ \left( \frac{e}{r} \right)^m - \left( \frac{e}{c} \right)^m \right] + \hat{A}_m^d \left[ \left( \frac{r}{c} \right)^m - \left( \frac{e}{c} \right)^m \right] \]  

(4.95)

for the nonconducting region III and

\[ \hat{A}_{II}^m = \hat{A}_m^c \left[ H_m(j \gamma a) J_m(j \gamma c) - J_m(j \gamma a) H_m(j \gamma c) \right] + \hat{A}_m^d \left[ J_m(j \gamma c) H_m(j \gamma r) - H_m(j \gamma c) J_m(j \gamma r) \right] \]  

(4.96)

for the conducting region II [7]. The vector potential in region I is simpler because the vector potential (and hence the radial component of the field) is zero at the center. Because the inner radius of this region is also zero, the vector potential reduces to

\[ \hat{A}_{I}^m = \hat{A}_m^a \left( \frac{r}{a} \right)^m. \]  

(4.97)

Substituting our Fourier series amplitudes $\hat{H}_{rm}^e$ into equations (4.91) - (4.94) results in the corresponding Fourier amplitudes $\hat{H}_{rm}^a$, $\hat{H}_{rm}^b$, $\hat{H}_{rm}^c$, and $\hat{H}_{rm}^d$. Using these with (4.50) gives the vector potential at each surface, and the relations (4.95) - (4.97) are now the solution to the vector potential throughout the sensor.

We use superposition to find the total field solution: we solve (4.95) - (4.97) for each value of $m$ and then sum the terms to arrive at the final solution which we express as

113
We present this solution for two different cases below. In Figure 4-21 we use the geometry of the actual sensor; in Figure 4-22 we increase the outer radius by a factor of ten to show how this changes the flux paths.

\[
\hat{A}_I = \sum_{m=-\infty}^{\infty} \hat{A}_{Im} e^{-jm\theta},
\]

(4.98)

\[
\hat{A}_{II} = \sum_{m=-\infty}^{\infty} \hat{A}_{II} m e^{-jm\theta},
\]

(4.99)

\[
\hat{A}_{III} = \sum_{m=-\infty}^{\infty} \hat{A}_{III} m e^{-jm\theta}.
\]

(4.100)

Figure 4-21: Magnetic field lines showing flux density inside sensor. For computational reasons we only use an 11-term Fourier series approximation of the vector potential at the outer boundary as detailed above. In this case we use the actual geometry from the bench-top scale model with the 6.4 mm tube, and a 5 kHz excitation frequency.

For the geometry which matches the tube, we see the tube attracts a good amount of the flux. In the previous section we increased the reluctances $R_{1t,2t,3t}$ so that our predicted data matched the experimental data more closely. In Figure 4-21 we see that the cross-sectional area of the flux path from the pole to the tube is indeed a
Figure 4-22: Magnetic field lines showing flux density inside sensor for a tube much smaller than the sensor opening. We impose a 11-term Fourier series vector potential at the outer surface and assume the potential in the middle is zero. The leakage flux clearly dominates.

few times larger than we originally assumed when calculating the parametric values in Section 4.2.4. Earlier we assumed the cross-sectional areas of the reluctance paths were triangular in shape, however they bow outwards in the figure. In our field solution we assume that the radial component of the field along surface (e) is zero away from the pole face areas, when in reality there will always be some radial field component for all values of $\theta$. This means that the flux in the ideal case of Fig. 4-21 will be more attracted to the tube than in the actual sensor.

### 4.4 Summary

We see that the magnetic circuit representation of the sensor can predict the experimental output, as long as we choose the parameters well. In this case, our initial guess is not quite correct, but good enough for us to develop a signal processing scheme. We still have to adjust the reluctance values, but this is a small price to pay for the simplicity afforded by the magnetic circuit analysis.

Also, the exact solution confirms the expected behavior of the field in the sensor.
We see that the flux is indeed attracted to the tube, and that the larger the tube, the more flux is attracted to it. As in Chapter 2, we assume the field does not saturate the tube. This assumption is supported by the fact that we have to decrease our lumped parameter estimation of the reluctance from the pole to the tube in order to have a closer correspondence to the experimental output. If the tube were saturated in the actual sensor, we would have to raise the reluctances from the poles to the tube to account for this, as saturation would tend expel the field from the tube.
Chapter 5

Electronics

5.1 Introduction

In this chapter we describe the design and implementation of the electronic systems associated with the sensor operation. We discuss three main topics: 1) the generation of the three-phase signal; 2) the amplification of the signal to a current which drives the primary coils; and 3) the processing of the voltages from the secondary coils into voltages proportional to the tube position. We show a schematic of the entire setup in Figure 5-1.

5.2 Three Phase Signal Generation

To drive the sensor we require a three-phase signal generator capable of frequencies above 5 kHz; we explore the following ideas.

Our first idea is to generate a signal of the proper frequency with an analog oscillator, then use two precision phase-shifters to obtain the desired three waveforms. In this case the frequency is not limited by access times of digital IC’s, and each sine wave output is a single pure frequency, not a stepped representation with high-frequency components. However the analog components will generally have some drift due to temperature changes, which means the output signal will change with temperature as well.
Figure 5-1: Complete electronics setup. The signal generator supplies three voltage signals to the current sources, which power the primary coils of the sensor. The signal processing board converts the voltages from the secondary coils into two voltages $V_x$ and $V_y$ which supply position information to the control computer.
The second idea is to use a digital signal processor (DSP) to generate three sine waves with the proper phase relations and output these signals through three digital to analog converters (DACs). This method is immune to drift due to temperature changes, because the digital signal generator determines the frequency and phase shift of each signal. However this option requires programming the DSP board, which can be time consuming.

We show the last idea in Figure 5-2; this is the method we use in this project. An oscillator clock drives an 8-bit binary counter, which simultaneously indexes three UV-erasable, programmable, read-only memory (EPROM) integrated circuits. We program a sine table into each EPROM so that as the counter indexes consecutive memory locations, the output is a stepped sine wave. The sine tables use 256 steps to approximate the wave. The more steps in the sine table the better the approximation to a continuous signal, but using more steps means that the circuit must run at a higher clock frequency to produce the same 5 kHz output. The discrete steps are a source of high frequency noise, but the first stage of the current source is a low-pass filter. This filter attenuates frequencies higher than the filter cutoff frequency of 33.8 kHz. We program the second and third EPROMs with sine tables 120° and
240° out of phase from the first. Because the same counter indexes all three EPROMs they will retain these phase relationships over time. Three digital-to-analog converters following the EPROMs convert the 8-bit binary words to currents. Lastly, operational amplifiers convert these currents to voltages with impedances low enough to drive the current source. Because the inputs to the DACs are binary words from zero to 255, the output currents are unidirectional, selectable to positive or negative via a pin on the DAC. Therefore we use another set of op-amps to remove the DC component of the signal. The main drawback of this method is the limit of the maximum output frequency as a result of the settling time of the DAC output current; we present this effect later in this chapter.

5.2.1 Clock and Counter

A function generator drives the circuit initially. This allows us to troubleshoot the circuit by varying the frequency of the input to the counter, thereby varying the frequency of the sine wave output. To get a 5 kHz sine wave with 256 steps, the frequency of the input to the clock is $256 \times 5,000 \text{ Hz} = 1.28 \text{ MHz}$. We can slow this down to a few cycles per second to see the incremental steps in the output from the digital to analog converter. The final setup uses a dedicated 1.54 MHz clock, to give an output sine wave at 6.0 kHz.

The counter is an eight-bit, binary, up-down counter; the input is the leading edge of a 0-5 volt signal and the output is an 8-bit binary word written across eight output pins. According to the literature the counter can execute a 115 MHz count cycle, which corresponds to a 9 ns cycle time. This means it should not be a speed limiting factor in the throughput of the signal generator.

5.2.2 EPROM

Many systems use EPROMs as fixed memory. Their basic function is to output a digital word contained in the address corresponding to the word written across the input pins. They are available in many different sizes, arranged by input x output
dimensions. For this application, the size is not an issue; we only require an 8-bit word to implement a 256-step sine wave. Actually, finding a chip with such a small output word size was more of a challenge!

Programming a sine table into the EPROM is relatively simple. First we create a file which represents one period of a sine wave. The format is a column vector, with each entry corresponding to the magnitude of a $\frac{1}{256}$th step of the sine wave. To be valid 8-bit words for the DAC, the values must all be positive, so the sine wave goes from zero to 255 instead of -128 to 127. We write the file in hexadecimal format so that it can be converted to an Intel hex file for downloading into the EPROM. We use a Matlab code to construct the three files necessary to program three EPROMs with the proper sine tables; these files are simply a list of the hexadecimal numbers with the additional header line

```
# set.address = 0;
```

to tell the EPROM programming software where in the chip’s memory to place the numbers. In our case we begin at the first memory location so that the first step of the sine wave corresponds to the 8-bit binary input 00000000.

We use EPROM chips donated by the M.I.T. Electrical Engineering stock room from a supply of EPROMs no longer in use. The nominal access time from the application of an input word to the appearance of a stable output is 35 ns. At 1.54 MHz, each cycle takes 650 ns, so the chip is fast enough that it is not a limiting factor in the speed of the system. We connect the eight output pins from the EPROM to the input pins of the DAC.

### 5.2.3 Digital to Analog Converters

After the 8-bit digital word is output from the EPROM, the DAC converts it into an analog signal; we use the DAC0830 from National Semiconductor\(^1\). We run the EPROM "wide open". As the clock goes high, the counter output changes, writing

\(^1\text{National Semiconductor Corporation, Santa Clara, CA}\)
the new value across the EPROM input. The new value from the sine table appears across the output pins approximately 35 ns later. The DAC has a buffer which latches the data; the timing of this latch is important for the throughput of the signal. As the clock signal goes low, the DAC updates the output current: it releases the previous value and applies the new value. It then latches the next input value as the clock goes high. The new value should settle for at least 50 ns before the clock goes high to avoid latching erroneous data. For this reason we trigger the counter on the same edge as the DAC, to give the maximum amount of time for the data to settle before it is latched. After the new value is updated it must settle, which takes time. The data sheet gives a 1μs full scale settling time, but experimental output shows that the small relative amplitudes of the 256-step sine wave settle in about 600 ns. This limits the output of the signal generator to about 6.5 kHz (a 6.5 kHz output requires the system to run at $1.67 \text{ MHz} = \frac{1}{600 \text{ ns}}$). Thus the limiting factor of the signal generator throughput is the settling time of the DAC output current.

The output signal contains some additional high-frequency noise due to the settling of the current, but again the low-pass filter will attenuate this before it is fed into the current supply. Faster DAC’s are available, but because the actuator bandwidth will be less than 250 Hz, increasing the sensor excitation frequency is not a priority.

The output of the DAC is a current which feeds directly into the inverting pin of an operational amplifier, as shown in Figure 5-3. The direction of the current $i_{out1}$ depends on the sign of the reference voltage, $V_{REF}$ such that a negative voltage means the DAC sinks current through $i_{out1}$ as in the figure. An internal feedback resistor which matches the R-2R ladder inside the DAC connects to the output of the op-amp to close the feedback loop (see $R_{fb}$ in Fig. 5-3). Using this internal resistor ensures uniform temperature tracking of the resistor values. The output of $op \text{ amp 1}$ is now a voltage from zero to $-V_{REF}$: the first op amp is an inverter, so that a negative current signal from the DAC becomes a positive voltage at $V_{OUT}$ (recall that $V_{REF} = -10V$ in this case). To give a voltage which oscillates around zero we use $op \text{ amp 2}$ as a summing junction, removing the DC component of the signal by summing it with $\frac{1}{2}V_{REF}$. The output is now a 5 kHz sine wave with no DC component, ready to be
Figure 5-3: Detailed view of the output from the DAC. The two current outputs drive the first op-amp, and the second op-amp removes the DC component of the signal. As configured, the output has a range of ±5 V

used as a command signal to drive the current supply.

5.2.4 Board Layout

Figure 5-4 is a photograph of the signal generator. We begin with a two-sided, plated breadboard with pre-printed rows which we use as power and ground strips. First we arrange the components, then establish a ground plane by connecting the grounded pre-printed rows in a mesh across the board. The ground plane is important for removing high frequency oscillations which can arise as a result of having a long ground lead. When a device must sink a current along a long lead, the lead acts as an inductance and can make the signal “ring.” Additionally, we place by-pass capacitors as near as possible to the power supply input into the IC’s (with the other lead connected to the ground plane) to attenuate high frequency voltage noise input into the IC.
Figure 5-4: Photograph of the top and bottom of the three-phase signal generator circuit board. We use a combination of wire wrap and solder to build the circuit. The tape strips labeled “A”, “B” and “C” cover the UV window on the EPROMs so that the ambient light does not erase the memory over time.
Figure 5-5: Circuit diagram of the current amplifier. The inductive load is eight primary coils from eight sensors. Three identical current supplies drive the three phases separately.

5.3 Current Supply

As discussed earlier in Chapter 2, using a current supply instead of a voltage supply results in a lower order system. This section details the layout of the current amplifier and selection of the component values we use to set the characteristics of the control loop. Many thanks in advance to Dr. Mark Williams, who has unknowingly contributed to my understanding of the circuit through the careful description of a similar system in his PhD dissertation [14]. For the current supply hardware we use previously constructed circuit boards, also from Dr. Williams’ PhD work.

Three 5 kHz voltages from the signal generator each drive a current amplifier, with each signal 120° out of phase from the others. The current supplies then drive the primary coils of the sensor at the same frequency, with the gain determined by the components on the board. Three separate current supplies drive these three signals individually; the rest of the derivation follows only one of the sinusoidal signals.

Figure 5-5 shows the circuit diagram of the amplifier. The first stage, from $V_1$ to...
$V_2$ is a differential low-pass filter which takes the differential voltage across the input terminals and allows only those with a frequency lower than the cutoff frequency to pass unattenuated. Anything with a higher frequency is attenuated at a rate of 20 decibels per decade above the breakpoint. The second stage is the control loop, consisting of a summing junction at op-amp 2, a PA-12 power op-amp to drive the load, a sense resistor $R_s$ to give an output voltage proportional to the current through its terminals, and the feedback resistor $R_7$ which completes the control loop. We choose the resistance of $R_7$ a few orders of magnitude higher than that of $R_s$ to minimize current flow through $R_7$ so that the summing junction merely “picks off” the voltage across the sense resistor. We now examine each of these subsystems in order to derive a transfer function for the circuit. Using this transfer function we select component values to give the desired gain and frequency response.

5.3.1 Low-Pass Filter

Rather than connecting the non-inverting input on op-amp 1 directly to ground, we connect it to the other terminal of the voltage $V_1$, so that the input becomes a differential voltage. This means that any common-mode voltage across $V_1$ will be rejected; this is important for example in removing 60 Hz line noise. When $\frac{R_4}{R_1} = \frac{R_3}{R_2}$ and $R_4C_2 = R_3C_1$, the transfer function from the input voltage $V_1$ to the output voltage $V_2$ is

$$\frac{V_2}{V_1} = \frac{\frac{R_4}{R_2}}{R_4C_2 s + 1}.$$  \hspace{0.5cm} (5.1)

Setting $R_4 = R_3 = R_2 = R_1 = 10k\Omega$ and $C_1 = C_2 = 470$ pF gives a low pass filter with a breakpoint at 33.8 kHz and unity DC gain. We place the breakpoint as close as possible to the planned excitation frequency of 5 kHz without significantly attenuating the primary frequency. A lower breakpoint will attenuate more high-frequency noise, but will attenuate the primary signal as well. Because the important characteristics of the signal are the frequency and the relative phase between the three signals, phase lag is unimportant if it is the same for each channel.
5.3.2 Primary Coil Load

We represent the primary coil load as an inductive-resistive circuit. In our system, we use three current sources to drive all eight sensors such that each current supply drives eight coils, one from each sensor. To calculate the inductance $L$ of a single coil we use

$$L = \frac{\mu_0 N^2 A}{g},$$

(5.2)

with $N = 36$ turns; the pole face area $A = 3.175 \cdot 10^{-5} \text{m}^2$; and the distance from one pole face to another $g = .0267 \text{ m}$. This gives

$$L = 1.94 \mu\text{H}.$$  

(5.3)

We use a constant gap to calculate this inductance, but the gap is actually a function of the tube position. However the resulting change in the inductance is not great, so we expect this is a fair estimate regardless of the tube position. This is the ideal case, so we expect this estimate to be just an order-of-magnitude guess. Also, this is the inductance for one coil, whereas the load is all eight coils. The total inductance will therefore be $L_L = 15.5 \mu\text{H}$. To check this calculation we use a dynamic signal analyzer to determine the frequency response. We use five primary coils from five different sensors in series as the inductive load, with a 1 $\Omega$ sense resistor across which we take the output voltage. We see a breakpoint at 2.24 kHz, which with a resistance of 2.7 $\Omega$ gives an inductance of 191 $\mu\text{H}$ for the five coils or 38 $\mu\text{H}$ for each coil. Because the current source must drive these actual coils and not theoretical ones, we will use the experimental values for calculation. For eight coils, the inductance is therefore 307 $\mu\text{H}$.

To calculate the resistance we use

$$R = \frac{l}{\sigma A_{xe}},$$

(5.4)

with the wire length $l = 0.68 \text{ m}$; the conductivity of copper $\sigma = 7 \cdot 10^{7} \frac{1}{\Omega \text{m}}$ (at 20°C);
and the wire cross-sectional area $A_{xc} = 1.28 \cdot 10^{-7}$. This gives

$$R = 0.078 \, \Omega,$$

or $R_L = 0.61 \, \Omega$ for all eight coils. Again we check this experimentally and find that each coil is actually about 0.34 \, \Omega. This is likely because of the extra wire, molex connections and solder joints. We will use this value as the load resistance for each coil, giving a total resistance of 2.72 \, \Omega.

We now look at the current supply control loop. Including the sense resistor, (used to find the voltage for the summing junction in the feedback loop), the transfer function from $V_4$ to $V_5$ is

$$\frac{V_5}{V_4} = \frac{R_s}{L_L s + R_s + R_L}. \tag{5.6}$$

This has a breakpoint at 1.93 kHz when $R_s = 1 \, \Omega$ for the above experimental values of $R_L$ and $L_L$.

### 5.3.3 PA-12 Power Amplifier

The PA-12 is a bi-directional power amplifier which supplies the current to the load of 8 primary coils. Each coil has a resistance and an inductance as specified above, giving a total complex impedance of $Z_L = (2.72 + j307 \times 10^{-6} \omega) \, \Omega$ for all eight sensors. We connect the PA-12 as a non-inverting amplifier with a gain of

$$\frac{V_4}{V_3} = \frac{R_{11} + R_{12}}{R_{11}}. \tag{5.7}$$

The resistors $R_{cl+}$ and $R_{cl-}$ program a current limit to keep the PA-12 in the range of safe operation so that its output does not break down. The diodes create a flyback circuit to handle the inductive load.
5.3.4 Feedback Loop

To ensure proper signal-following we implement an analog proportional controller using an op-amp configured as a summing amplifier (Op-Amp 2, Figure 5-5). Because our system operates at a single, constant frequency, we choose to employ only a proportional controller. If we required the current supply to follow a DC command, we would include an integrator term in the controller (by including a capacitor in series with $R_p$). Because this is an inverting summing junction the transfer functions $\frac{V_3}{V_2}$ and $\frac{V_3}{V_5}$ are simply the negative of the ratios of the relevant resistors,

$$\frac{V_3}{V_2} = -\frac{R_p}{R_6}, \quad \frac{V_3}{V_5} = -\frac{R_p}{R_7}. \quad (5.8)$$

The control loop is now as seen in Figure 5-6, where the voltages noted on the block diagram correspond to those shown in Fig. 5-5.

![Current amplifier control block diagram.](figure)

To simplify the circuit we move the gain in the feedback path to the forward path. The negative loop transmission is now

$$-LT = \left( \frac{R_p}{R_7} \right) \left( \frac{R_{11} + R_{12}}{R_{11}} \right) \left( \frac{R_s}{L_s + R_s + R_L} \right). \quad (5.9)$$

The breakpoint of the loop transmission is from the load, (1.93 kHz), while the DC gain $K$ is

$$K = \left( \frac{R_p}{R_7} \right) \left( \frac{R_{11} + R_{12}}{R_{11}} \right) \left( \frac{R_s}{R_s + R_L} \right). \quad (5.10)$$
5.3.5 Choosing Component Values

We operate the sensor at about 5 kHz, and choose the bandwidth of the current amplifier to be 200 kHz. We initially designed the system to follow an input frequency of 10 kHz. We maintain this high crossover to allow for the use of a faster input signal if desired in the future. Rewriting equation (5.9) using the above definition for $K$ and with $\frac{L_L}{R_s+R_L} = \tau$ gives

$$-LT = \frac{K}{\tau s + 1}. \quad (5.11)$$

Solving this for $K$ to make $| -LT | = 0$ dB at 200 kHz gives $K = 104.7$. Letting $R_7 = 10$ kΩ, $R_s = 1$ Ω, $R_{11} = 10$ kΩ and $R_{12} = 40$ kΩ and solving for $R_p$ gives\(^2\)

$$R_p = 779\text{kΩ}. \quad (5.12)$$

---

\(^2\)We pick many of these for efficiency; they are the values already on the board.
We plot the loop transmission in Figure 5-7 using these component values. To find the gain of the closed loop transfer function at a frequency of 5 kHz we substitute the value \( s = j(5,000 \times 2\pi) \) into the transfer function

\[
\frac{-LT}{1 - LT}.
\] (5.13)

The magnitude at this frequency is 1.01. We desire a current of 0.1 A through the primary coils; we choose this value because it is low enough not to overheat the coils during operation. This means the voltage across the sense resistor will be 0.1 V. To get this output across the sense resistor given an input voltage of 5 V, the total gain must be 0.1/5. This means the gain \( \frac{R_7}{R_6} \) must be

\[
\frac{0.1}{5 \times 1.01} = .0198.
\] (5.14)

With \( R_7 = 10k\Omega \), \( R_6 \) must be 505k\Omega. Using these values we plot the response of the closed loop system in Figure 5-8.

### 5.4 Signal Processing Circuit

The magnetic field induced by the primary coils results in voltages across the secondary coil terminals as explored earlier in Chapter 4. We now examine the circuit board which processes these voltage signals. There are four steps to this signal processing stage: first the demodulation chips rectify the signals to obtain the magnitude of the signal; then operational amplifiers sum the signals to form \( x \)- and \( y \)-dependent voltages; next a fourth-order, low-pass filter filters the signals to allow only the DC component of the signal to pass; finally a gain stage amplifies the signals to a range most suitable for the control system. We show the circuit layout for the system in Figure 5-9.
Figure 5-8: Complete transfer function from the input to the output of the current supply

5.4.1 Rectification

As described in Chapter 4, we rectify the incoming signal to find the amplitude. The AD630 demodulation chip was used to demodulate the signals against themselves, which gives us the magnitude and sign of the signal. Before and after the demodulation chip we place voltage followers to buffer the signal. The demodulation circuit must not have a loading-effect on the voltage across the secondary coils, so we add the voltage followers to present the demodulation chip with a low impedance to minimize any back effect. For the same reason there are voltage followers between the demodulator and the summing junctions. Initially we omitted this second set of voltage followers, but we get a clearer signal with them in place. The output of the demodulation chip is a rectified sinusoid: a combination of a DC level and a 10 kHz primary frequency, along with higher frequencies from the discontinuity at the point where the original signal changes sign.
Figure 5.9: Demodulation circuit layout
5.4.2 Summing Junctions

We use summing junction to combine the three voltages in the proportions described in Chapter 4. This entails choosing the proper values of the resistors $R_1$ through $R_9$ to give the desired weights to the signals. We use the variable resistors $V_x$ offset and $V_y$ offset to tune the circuit and center the output voltages to the desired point. Because of possible inconsistencies in the manufacturing of the sensors, it may be that the $V_x = 0, V_y = 0$ point is not exactly where it should be, i.e., where the tube is in the center of the sensor aperture. We choose these variable resistors as small as possible, because potentiometers are more susceptible to variation due to temperature changes than standard film resistors. For example, the value of $R_6$ is nominally 10 kΩ, but we must use a 5 kΩ resistor in series with a 15 kΩ potentiometer to give a ±50% variability. Experience with the sensors shows this large variation is necessary. This difficulty in winding symmetric coils is also noted in [8], where the author recommends the use of precision winding machines as essential for achieving usefully symmetric differential flux coils. Apart from asymmetry in the secondary coils, asymmetry in the lamination section and primary coils will also affect the output. We assemble the lamination section from three separate thirds of the sensor. Even though we grind the mating surfaces flat, any irregularity in the air gap where two section meet will alter the flux paths and diminish the symmetry. In Chapter 6 we examine the construction of the sensor in detail.

5.4.3 Low-Pass Filter

To output a constant voltage we low-pass filter the summed signals from the previous stage using active, fourth order filters. We avoid using passive low-pass RC filters for this system because we desire a high bandwidth, and passive filters roll off too slowly for this application. The bandwidth of the sensor should be around 1 kHz, only $\frac{1}{10}$ of the primary frequency (recall that we’ve rectified the 5 kHz signal and now have the primary frequency at 10 kHz). This requires a higher order filter than can be implemented using a passive filter. First order passive filters are simple, but to
cascade them is not straightforward. Loading-effects require that cascaded passive filters be significantly different in impedance levels, but this would make a 1 kHz bandwidth difficult to achieve. An active filter can be of higher order than a passive filter without encountering the same loading-effects. We use the traditional Sallen-and-Key arrangement for the second order active filter as seen in Figure 5-9; this is the section consisting of resistors $R_{18}$ to $R_{21}$, including the op-amp and capacitors [4]. Each channel ($V_x$ and $V_y$) has two cascaded second-order active filters to give a fourth-order response.

The choice of component values is important, as different values for the components will result in different behaviors for the filters. For example the gain response of a Butterworth filter minimizes passband ripple and remains flat up until the break frequency, but has poor phase characteristics [4]. A Bessel filter sacrifices this uniform response in the passband in exchange for a slower phase loss with increase in frequency. In our initial testing of tube levitation we used a passive filter, and the excessive phase loss was the major factor in instability of the levitated system. For the active filter we choose the Bessel arrangement to minimize phase loss at crossover. The sacrifice of non-uniform gain in the passband is less important for our situation because we operate the sensor at a fixed frequency, so we will not see the effect of the gain variability. For each channel we implement two cascaded fourth-order Bessel filters with a 1 kHz cutoff frequency. The values for the resistive and capacitive components are listed in [4]. Table 5.1 contains the values of the components shown in Figure 5-9.

5.4.4 Output Gain

Because the output is more linear with respect to displacement while the tube is in the center of the opening, we use this as the desired operating point. We design the system to have a measured range of about $\pm 2$ mm in any direction and tune the sensor to have the greatest sensitivity while the tube is in this range. The output gain stage is a simple inverting amplifier with an adjustable gain given by the series
Values of components used in the demodulation circuit, as seen in Figure 5-8:

<table>
<thead>
<tr>
<th>Component</th>
<th>Value (kΩ)</th>
<th>Component</th>
<th>Value (kΩ)</th>
<th>Component</th>
<th>Value (kΩ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1$</td>
<td>5</td>
<td>$R_{13}$</td>
<td>51</td>
<td>$R_{25}$</td>
<td>47</td>
</tr>
<tr>
<td>$R_2$</td>
<td>20</td>
<td>$R_{14}$</td>
<td>47</td>
<td>$R_{26}$</td>
<td>10</td>
</tr>
<tr>
<td>$R_3$</td>
<td>20</td>
<td>$R_{15}$</td>
<td>47</td>
<td>$R_{27}$</td>
<td>920</td>
</tr>
<tr>
<td>$R_4$</td>
<td>20</td>
<td>$R_{16}$</td>
<td>33</td>
<td>$R_{28}$</td>
<td>10</td>
</tr>
<tr>
<td>$R_5$</td>
<td>20</td>
<td>$R_{17}$</td>
<td>47</td>
<td>$R_{29}$</td>
<td>1100</td>
</tr>
<tr>
<td>$R_6$</td>
<td>5</td>
<td>$R_{18}$</td>
<td>51</td>
<td>$V_x\text{ offset}$</td>
<td>0-10 (variable)</td>
</tr>
<tr>
<td>$R_7$</td>
<td>10</td>
<td>$R_{19}$</td>
<td>51</td>
<td>$V_y\text{ offset}$</td>
<td>0-10 (variable)</td>
</tr>
<tr>
<td>$R_8$</td>
<td>20</td>
<td>$R_{20}$</td>
<td>4.2</td>
<td>$V_x\text{ gain}$</td>
<td>0-50 (variable)</td>
</tr>
<tr>
<td>$R_9$</td>
<td>20</td>
<td>$R_{21}$</td>
<td>51</td>
<td>$V_y\text{ gain}$</td>
<td>0-50 (variable)</td>
</tr>
<tr>
<td>$R_{10}$</td>
<td>51</td>
<td>$R_{22}$</td>
<td>47</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_{11}$</td>
<td>51</td>
<td>$R_{23}$</td>
<td>47</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_{12}$</td>
<td>4.2</td>
<td>$R_{24}$</td>
<td>33</td>
<td>$C$</td>
<td>.0027μF</td>
</tr>
</tbody>
</table>

Table 5.1: Values of components used in the demodulation circuit, as seen in Figure 5-9.

resistance $R_{29}, V_x\text{Gain}$, and $R_{27}, V_y\text{Gain}$ for the $x$- and $y$-voltages respectively.

5.4.5 Circuit Layout

Because we need eight circuit boards we send our design out to a printed circuit board (PCB) manufacturer for fabrication. We design the boards using ACCEL layout software, a CAD program which incorporates layout rules when placing components and wires. The software also offers a vast library of pre-configured pinout patterns.

Figure 5-10 shows the final circuit layout. The board has two layers; we use the top mainly for signals and the bottom primarily as a power and ground layer. Placing some ground circuits on the signal layer (and vice-versa) greatly simplifies the routing of the wires. Contrary to the design of the signal generation board which was hand-soldered, we save the ground plane until the end. First we place the IC's and route the power bus to supply +15 V and -15 V to the chips. Then we lay out the resistors and capacitors, and route the connections keeping the signal lines on the top of the board.
Figure 5-10: Demodulation circuit wiring diagram. The top figure is the signal layer, complete with silk screen printing which includes the text and the component outlines; the bottom figure is the power and ground layer, the thinner wires will be absorbed in the copper pour.
board and the power and ground lines on the bottom. This consumes the majority of the design time. We choose a 12 mil (.012 inch) line width for the signal lines and a 25 mil line width for the power bus.

After all the components are in place and connected, we establish the ground plane. ACCEL allows a “copper pour” on a layer to create as large a ground plane as possible given the layout of the board. To do this we select an area, and also select all the wires we wish to incorporate into the ground plane (i.e., the ground wires). ACCEL then fills in the empty space with copper, absorbs the selected wires in the copper, and excludes all other connections and wires. This gives a ground plane, with the same benefits as described in Section 5.2.4. As seen in Fig 5-10, the bottom half has not yet had it’s copper poured. The thin lines are the ground circuit, and these will all be absorbed into the copper pour. Figure 5-11 is a photograph of the two sides of a circuit board. The top view is of a fully populated board, and the bottom is the underside of a board before installation of components. Here we see the copper pour in place.
Figure 5-11: Top view of a populated board and bottom view of a bare board. Except for a few signal wires, the bottom of the board is exclusively for power and ground circuits.
Chapter 6

Construction

6.1 Introduction

Except for the signal processing circuit boards which we send out to a fabrication company, we make the entire setup in our laboratory and machine shop. This chapter details our design choices, and the construction of the sensor and related components. We cut most of our aluminum pieces with a water jet machine, which has proven to be indispensable for the rapid cutting of complex shapes [9]. We wind all the coils with a machine in our lab.

6.2 E-pickup

The construction time for the E-pickup is minimal, consisting mainly of coil winding. We use an off-the-shelf E-core, with the dimensions as shown in Figure 3-8. We place electrical tape on the pole faces to avoid scoring the coils on the edges. Although the ferrite is essentially nonconducting, breaks in the insulation on the wire can still lead to wire turns shorting together. This will not only reduce the number of turns around the pole, but will create a conducting loop around the flux path, which will reduce the flux by creating an opposing current in the wire as described by Ampere’s law. We wind the coils on a winding machine in our laboratory, using magnet wire with thermal bonding insulation. For these first sets of coils, the geometry is such that the
coils stack uniformly. This allows us to pot the coils in their own insulation, which we do by passing a current of about one amp through the wire. This heats the wire enough to melt the insulation, but we must take care not to melt it completely to avoid short circuits. There is no precise current level at which we melt the insulation, rather we use a thermocouple to keep the temperature at about 250°C until the wire bonds to itself. To apply this variable current we use a Powerstat variable autotransformer.

### 6.3 Modified E-pickup

To modify the arrangement of the secondary poles we combine ferrite pieces cut from E-cores. We use a diamond-coated blade on a rotary chop saw because the ferrite is too brittle to cut on a band-saw. We join the pieces with epoxy to form the desired shape, and reinforce the bonds with tape to ensure rigidity. Figure 6-1 is a photograph of the modified E-pickup without shielding.

For the shielding we begin with a 1 mm thick copper sheet and fold this into the appropriate shape. We make the shielding in two separate pieces, both for ease of assembly and to make it easier to electrically isolate the two pieces. We again wind the coils on the winding machine and melt the wire insulation to pot the coils. Figure 6-1 is a photograph of the shielding separated from the sensor. The copper also serves as an electrostatic shield; we ground it at a single point by connecting it to earth ground as seen in the figure.

### 6.4 Three-Phase Sensor

For the three-phase sensor we design the pieces to facilitate mass production (mass production of eight sensors, in our case). We use the water jet abrasive cutting machine to make many of the pieces because of the speed, ease of use, and ability to make more complex shapes. The accuracy of the water jet is less than that of a traditional milling machine, so we must make any final cuts with a more precise

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1 The Superior Electric Company, Bristol, CT
machine if higher accuracy is necessary. Specifically we do this with the mating surfaces on the rail-mount bracket, seen in Figure 6-5.

6.4.1 Lamination Pieces

The major reason for using three e-shaped pieces to form the circular sensor backbone is the installation of the secondary coils. Traditional motor laminations are shaped similarly, but were not used here because they have a uniform spacing of poles filling the entire circumference. Such motors laminations are secured together in a stack before insertion of the coils, which encircle the poles but not the outer ring of the lamination. The coils are then fed into the gaps between the poles in the desired arrangement to create the magnetic field for driving the rotor. Conversely, in our design we wind all the coils before installation. We install the primary coils simply by slipping them over the poles, but the secondary coils must encircle the surrounding ring, so a continuous ring would make winding the secondary coils unnecessarily
tedious. Instead we cut the lamination sections into three e-shaped sections so that we may wind the coils separately then install them on the sensor. Figure 6-2 shows a stack of 20 laminations glued together. Also in the photograph is a set of three lamination stacks is wrapped in Teflon tape and arranged as in the sensor. We use the tape to prevent the sharp lamination edges from scoring the coil wire insulation. After securing the laminations together we file the edges smooth, but still use the Teflon tape as a preventative.

![Figure 6-2: Photograph of the lamination pieces. Also shown are three sections arranged in a circle, wrapped in Teflon tape to prevent the sharp edges from scoring the coil wire.](image)

**Material Choice**

The earlier versions use a ferrite core, which is nonconductive but more difficult to work with because it is brittle. The permeability of the ferrite we use is \( \mu = 2700\mu_0 \). For the final sensor we use 7-mil (0.007 inch) thick silicon-iron laminations. We choose these lamination for a variety of reasons: ease of machining, lower cost, and higher permeability. The main drawback is that we must cut many layers and secure them together.

Because ferrite is so brittle, machining it is difficult to do without incurring frac-
ture. Our benchtop-scale model requires small pole pieces, and cutting these out of ferrite would be problematic. The laminations are much more durable, and we can cut them to the proper size without breakage. To cut the laminations we first epoxy five lamination sheets together. Then we cut out the e-shape using the water jet machine. The choice of epoxy is important because if it does not hold well enough, the abrasive sand particles from the water jet will lodge between the lamination layers and separate them. We first used a widely available five-minute epoxy, but finally settled on thermally conductive epoxy CC3-300, a product of Cast Coat Inc.\(^2\) This epoxy is much harder when it cures, as opposed to the five-minute epoxy which remains slightly flexible. The thermal conductive property can help to dissipate heat, however our sensor consumes such little power that the heat created is negligible. After cutting enough e-sections five lamination sheets thick, we epoxy four of these sections together to make a 20 lamination section which is \(\frac{1}{4}\) inch thick. This is a third of the sensor; three of these form the complete circle with primary poles, as seen in Figure 6-2.

### 6.4.2 Coil Winding

The primary coils are relatively simple to wind. We machine a bobbin of the proper size and mount this to the coil winding machine. We wind the coil then pot it in five-minute epoxy to secure it against coming unwound. We choose to use this epoxy rather than to melt the wire insulation because the epoxy is more reliable and less risky. Heating a coil for too long can destroy it, whereas too much epoxy can simply be wiped off. We use 26 AWG for these coils, which have 35 turns each.

The secondary coils must fit the arc of the sensor back steel, so winding them is more difficult. We again build a bobbin on which we wind the coils, but this time the wire does not stack as uniformly as before. We choose not to melt the insulation on the wire to pot the coils both for the above mentioned reason and because the contact between wires is not uniform, and melting the epoxy would only lead to a

\(^2\)354 West Street, West Bridgewater, MA 02379
poorly adhered coil. We use 34 AWG wire for the secondary coils, because we wish to maximize the number of turns. Thinner wire breaks too easily, while thicker wire does not permit as many turns. Each secondary coil has 350 turns.

6.4.3 Shielding

We again use the water jet for the aluminum shielding. There are six pieces of shielding for each sensor: three interior pieces, two flat pieces to sandwich the sensor, and one outer ring which holds all the components in place. We cut the interior pieces and the outer ring as a single piece, along with the rail mounting bracket described below. In Figure 6-3 we show a photograph of the entire sensor, complete with coils and shielding. We also include a plastic ring shaped to hold the coils in place on the laminations, this can also be seen in the figure.

Figure 6-3: Photograph of the complete sensor, showing the coils, shielding, laminations and the plastic ring which positions the components.
6.5 Rail Mounting System

To align the components we choose to make an aluminum rail to which we mount the sensors and actuators. Figure 6-4 shows the partial setup with five actuators and five sensors. The higher the tube is above the rail, the more the flexibility of the brackets will come into play. For this reason we design the sensor and actuator mounts to position the tube close to the rail.

![Figure 6-4: Five sensor/actuator stations along the aluminum rail. We mount the feet of the sensor brackets flush along one side of the rail (the right side as seen in the photograph).](image)

6.5.1 Sensor Mount

The sensor mounting brackets, seen in Figure 6-5, use setscrews to position the sensor in the $x, y$ plane. We machine the mating feet flat to minimize the effect of crowning between the rail and the bracket. Toe clamps hold the feet to the rail and allow repositioning with minimum effort.
Figure 6-5: Photograph of the sensor mount and printed circuit board. We mill a slot out of the side of the mounting bracket for the wire to pass through. Three set-screws allow for some final positioning of the sensor as necessary.

### 6.5.2 PCB Holder

We choose to mount the sensor electronics close to the board. Plexiglass serves as an easy-to-use material for this purpose. This piece does not need to be strong, but we do not want to spend too much time in fabrication. For this reason we make them out of Plexiglass, and cut them on the water jet machine. Each piece takes approximately six minutes to cut out, with another five minutes of dressing on the mill required to make the ledge to house the board. The electronics board mounts with the same screw which holds the toe clamp to the rail.
Chapter 7

Conclusions and Suggestions for Further Work

In this thesis we detail the design, analysis and construction of a two-axis noncontact position sensor. We model the sensor with an analogous magnetic circuit which, after some parameter adjustment, represents the behavior of the sensor. Three circuit boards: the signal generator, the current supply and the signal processor, power the sensor and process the sensor output signal for use in the magnetic levitation stage.

The sensor has a bandwidth of about 1 kHz, and is sufficiently linear to levitate the tube. The major source of noise in the output is the residual 10 kHz signal (from the rectified 5 kHz signal) which is not completely attenuated by the low-pass filter. We use a fourth order filter with the current set of electronics, but this could easily be increased to give more attenuation of the 10 kHz frequency at the output. With a sensitivity set at two volts per millimeter of tube movement, the peak-to-peak amplitude of the 10 kHz noise is about 100 mV. There is some random noise at the output as well, with a peak-to-peak amplitude of about 20 mV.

7.0.3 Experimental Issues

As of the current date, we have two sensor-actuator stations working together to levitate a 60 cm section of steel tube. Figure 7-1 shows the benchtop-model in our
laboratory with the tube suspended.

![Image](image.png)

Figure 7-1: Tube levitated with two sensor-actuator stations. We add magnetic shielding to the sensor to reduce the effect of the field from the actuator on the sensor output.

A few issues which require attention for the effective implementation of the sensor-actuator pairs are sensor-actuator coupling and sensor-sensor coupling. After connecting two actuators and two sensors to the control computer and closing the loop, we find that the magnetic field from the actuators effects the output of the sensor. When the sensors are closer than about 20 cm, we see that the changing field from the actuator disrupts the output of the sensor. If we were to demodulate the signals from the sensor output coils, this effect would be minimized. However, because we only rectify the signals, the noise is still a factor.

To overcome this effect we add magnetic shielding to the sensor; we use ferrous steel plates to attract the field and prevent it from entering the sensor. This attenuates the signal enough to close the loop in a staple feedback setup. Currently we situate the sensor about 8 cm from the actuator, and achieve acceptable attenuation of this coupling effect. Future versions of the signal processing board will likely use a true demodulation setup to remove any signal which is different from the reference signal.

The other issue, which is at present unsolved, is the response of the sensor output of any given sensor to input from other sensors. The effect we observe is that the position reading of one sensor changes slightly according to the tube position in a completely distinct sensor. While this coupling effect does not prevent stable loop
closure in the two-station setup, we expect this may be a problem when adding more
sensor-actuator stations to the system.

7.1 Closing Thoughts

Two of the more notable insights I gained from this thesis are the utility of modeling
a lumped-parameter magnetic system with an analogous magnetic circuit, and the
cost-efficiency of mass-printed circuit boards.

Although we had to adjust the calculated reluctances to match the experimental
values more closely, the correspondence of the model predictions to the experimental
output was very close. Because the methods of analyzing electric circuits are so
straightforward, we can easily analyze complex magnetic circuit geometries using
these simple rules.

Though it took a while to learn how to use the layout software, designing the signal
processing boards on the computer drafting program and having them mass-produced
at a commercial manufacturer saved countless hours. Troubleshooting small, de-
tailed circuits can absorb valuable time, and invariably each hand-build circuit will
be slightly different. Wire-wrap joints can accidentally loosen, and laying out com-
ponents physically as a means of design can lead to more contorted geometries than
laying out components using the layout software. Printing the boards ensures unifor-
mity, and soldering to the plated holes is quite simple.

7.2 Suggestions for Further Work

Although we successfully used the sensors to levitate a tube, three important areas
need improvement. These are the signal processing method, the construction method.

The sensor offers six sources of information, i.e., the magnitude and phase of each
secondary coil voltage signal. We only use three of these, and even then only use
a linear combination to arrive at the voltages $V_x$ and $V_y$. This results in a position
sensor which only accurately detects position while the tube is near the center of the
sensor opening. Another method would be to use the same signal processing board we currently use, along with a lookup table, such that the entire range of the sensor is usable. Or, we could take all three outputs from the sensor and input them directly into the control computer, then use the results of the magnetic circuit analysis to create an algorithm for inverting the voltages to solve for the tube position. The extra inputs into the computer may be an expensive resource; for eight sensors this method will require twenty four analog to digital ports instead of the sixteen currently required, but this could result in a much more accurate position sensor.

The other method upon which we could improve is the construction of the sensor. From winding all the coils to cutting all the laminations, each sensor takes a significant amount of time to make. The water jet allows us to quickly fabricate odd shapes, but the number of parts should be reduced for faster construction. Casting the shielding in two pieces, having the coils professionally wound, and having the laminations stamped would reduce the fabrication time greatly. The circuit board is a good example of an efficient method of mass production. Apart from the learning curve of the software, having the boards made by a professional manufacturer saved countless hours of wire-wrapping and debugging.

In conclusion, the sensor design is successful for the present purposes and will allow us to test the eight actuator, eight-sensor system in the near future. The bandwidth of the sensor is high enough to observe all the relevant motion of the tube, and the air-gap is large enough to allow for the tube to be processed, i.e., painted, etc. Additionally, we find that even though the tube is seam-welded, the position reading is independent of the angle of rotation of the tube. In the previous E-pickup setup, we noticed that rotating the tube would change the observed output, as if the tube were in fact moving in the x- or y-direction. In the two-station setup, we can spin the tube and the position reading from the sensor remains unchanged. Thus, our sensor performs well enough to suspend the tube as required. With some improvements in the signal processing circuit, we expect an even more linear output and an increased linear range of measurement.
Appendix A

Matlab Code

We use Matlab to perform many of the simulations in this thesis; we present the relevant code below.

A.1 Magnetic Vector Potential and Field Lines

A.1.1 Case 1: Uniform Field

```matlab
% Program Vec_Poten_in_Sensor.m
% This program plots the vector potential inside the sensor air gap inside the and tube for the case where the imposed field is uniform.
% Robert Ritter, January 1999

clear % System Constants

%-----------------------------------------------------------% %
sigma=0.75e7; % conductivity of tube
w=2*pi*5e3; % excitation frequency
mu_0 = 4e-7 * pi; % permeability of air
mu=5000*mu_0; % tube permeability
a=.00215; % inner tube radius
```
c = 0.00315; % outer tube radius
e = 0.00795; % sensor opening
m = 1; % angular wave number
mr = mu_0/mu; % relative permeability (inverse of)
gamma = sqrt(i*mu*sigma*w);
Ao = 10; % applied vector potential magnitude
array_size = 200; % number of points "n" in the "nxn" array
extent = e; % outer limit of array

% First we define the matrices which have as their entries
% the x and y coordinate of each entry. Next we define radius
% and theta matrices which have as their entries the radial and
% angular coordinates, respectively

x = [-extent:2*extent/array_size:extent];
y = x;
for k = 1:array_size + 1,
    for l = 1:array_size + 1,
        r(k,l) = sqrt(x(k)^2 + y(l)^2);
        th(k,l) = atan2(y(l), x(k));
    end
end

% To ease analysis we define the field throughout the entire region,
% but the terms do not change with radius, only with theta.
% Therefore each term is only strictly correct at the corresponding
% boundary radius. The for-loop at the end computes the vector
% potential everywhere in the sensor opening.

He_r = -i*m*Ao*cos(th)/(e*mu_0);
Ae = -mu_0*e*He_r/(j*m);

Hd_r = He_r*e*gmxy(c,e)/(a*mr*gmxyg(c,a,gamma)*c*mr*gmxyg(a,c,gamma)).../(m-a*mr*fmxyg(c,a,gamma)) - c*fmxy(e,c) + c*mr*...*gmxyg(a,c,gamma);
Ad = -mu_0*c*Hd_r/(j*m);

Hc_r = mr*Hd_r;
Ac = -mu*c*Hc_r/(j*m);

Ha_r = Hd_r*c*mr*gmxyg(a,c,gamma)/(-m-a*mr*fmxyg(c,a,gamma));
Aa = -\mu_0 a H_a r / (j m);

Hb. r = mr H_a r;
Ab = -\mu a H_b r / (j m);

Hma = besselh(m, j*gamma*a);
Hmc = besselh(m, j*gamma*c);
Jma = besselj(m, j*gamma*a);
Jmc = besselj(m, j*gamma*c);
Hmr = besselh(m, j*gamma*r);
Jmr = besselj(m, j*gamma*r);

for k=1:array_size+1,
   for l=1:array_size+1,
      if (r(k,l)<e) & (r(k,l)>c),
         A(k,1) = Ae(k,1)*((c/r(k,l))^m - (r(k,l)/c)^m)/(c/e)^m - ...
                  (e/c)^m) + Ad(k,1)*((r(k,l)/e)^m - (e/r(k,l))^m)/(c/e)^m - ...
                  (e/c)^m);
      elseif (r(k,l)<c) & (r(k,l)>a),
         A(k,1) = (Ac(k,1)*(Hma*Jmr(k,1) - Jma*Hmr(k,1)) + ...
                  Ab(k,1)*(Jmc*Hmr(k,1) - Hmc*Jmr(k,1)))/(Hma*Jmc - Jma*Hmc);
      elseif (r(k,l)<a),
         A(k,1) = Aa(k,l)*(r(k,1)/a)^m;
      elseif (r(k,l)>e),
         A(k,1) = 0;
      end
   end
end

figure(1),clf,zoom on
contour(x,y,r,[a a],'k'); hold on; contour(x,y,r,[c c],'k');
contour(x,y,r,[e e],'k');
contour(x,y,real(A),50);
axis equal;
colormap gray;
colorbar;

%%%%% Following is a movie of the field changing with time
%%%%% Figure 3
figure(3)
skinmoviel=moviein(24);
set(gca,'NextPlot','replacechildren');
for k=1:24
   B=(cos(k*pi/12) + i*sin(k*pi/12))*A;
end
mesh(real(B));
axis([0 250 0 250 , -10,10]);
skinmoviel(:,k) = getframe;
end
movie(skinmoviel,3,5)

Functions Called in Code

function out=fmxy(x,y)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% computes the value of fm given x,y
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
m=1;
out=(m/y)*((x/y)^m + (y/x)^m)/((x/y)^m - (y/x)^m);

function out=fmxyg(x,y, gamma)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% computes the value of fm given x,y, gamma
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
m=1;
out= m/y + (j*gamma)*(besselh(m,j*gamma*x)*besselj(m-1,j*gamma*y)-... 
besselj(m,j*gamma*x)*besselh(m-1,j*gamma*y))/... 
(besselj(m,j*gamma*x)*besselh(m,j*gamma*y) - ... 
besselj(m,j*gamma*y)*besselh(m,j*gamma*x));

function out=gmxy(x,y)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% computes the value of gm given x,y
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
m=1;
out=(2*m/x)/((x/y)^m - (y/x)^m);

function out=gmxyg(x,y, gamma)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% computes the value of gm given x,y, gamma
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
m=1;
out=(-2*j/(pi*x))/(besselj(m,j*gamma*x)*besselh(m,j*gamma*y) - ... 
        besselj(m,j*gamma*y)*besselh(m,j*gamma*x));

A.1.2 Case 2: Fourier Series Field

% Program Vec_Poten_Fourier.m
%
% This program plots the vector potential inside the sensor air gap
% inside the and tube for the case where the imposed field is a
% Fourier series approximation of the field at the outer surface
%
% Robert Ritter, January 1999

clear

%System Constants

sigma=0.75e7;       % conductivity of tube
w=2*pi*5e3;         % excitation frequency
mu_0 = 4e-7 * pi;   % permeability of air
mu=5000*mu_0;       % tube permeability
a=.00215;           % inner tube radius
c=.00315;           % outer tube radius
e=.00795;           % sensor opening
m=1;                % angular wave number
mr=mu_0/mu;         % relative permeability (inverse of)
gamma = sqrt(i*mu*sigma*w);
array_size=50;       % number of points "n" in the "nxn" array
extent=e;           % outer limit of array

% First we define the matrices which have as their entries
% the x and y coordinate of each entry. Next we define radius
% and theta matrices which have as their entries the radial and
% angular coordinates, respectively

x=[-extent:2*extent/array_size:extent];
y=x;
for k=1:array_size+1,
    for l=1:array_size+1,
        r(k,l)=sqrt(x(k)^2 + y(l)^2);
        th(k,l)=atan2(y(l),x(k));
    end
end

% The following variables concern the Fourier representation of
% the three-phase field.

theta0=.157; % pole face width (angular, in radians)
terms=10; % number of terms in fourier series
HO=1e8; % magnitude of applied field
low_ext=-3*terms-2; % lower extent of Fourier series
up_ext=3*terms+1; % upper extent of Fourier series
He_r=0; % initializing the variable
A_fourier=zeros(size(r)); % initializing
A=A_fourier; % initializing

% We compute the field distribution for each value of m
% then sum them at the end to find the total field

for m=low_ext:3:up_ext;
    Hc_r = mr*Hd_r;
    Ac = -mu*c*Hc_r/(j*m);
    Ha_r = Hc_r*c*mr*gmxygm(a,c,gamma,m)/...
            (-m-a*mr*gmxygm(c,a,gamma,m));
    Aa = -mu*Ha_r/(j*m);
    Hb_r = mr*Ha_r;
    Ab = -mu*a*Hb_r/(j*m);
end
\begin{verbatim}
Hma = besselh(m,j*gamma*a);
Hmc = besselh(m,j*gamma*c);
Jma = besselj(m,j*gamma*a);
Jmc = besselj(m,j*gamma*c);
Hmr = besselh(m,j*gamma*r);
Jmr = besselj(m,j*gamma*r);

for k=1:array_size+1,
    for l=1:array_size+1,
        if (r(k,1)<e) & (r(k,1)>c),
            A(k,1) = A(k,1) + (c/r(k,1))^m - (r(k,1)/c)^m / (c/e)^m - (e/c)^m +
            Ad(k,l)*((r(k,l)/e)^m - (e/r(k,l))^m) / (c/e)^m - (e/c)^m;
        elseif (r(k,1)<c) & (r(k,l)>a),
            A(k,1) = (Ac(k,l)*(Hma*Jmr(k,1) - Jma*Hmr(k,1)) +
            Ab(k,l)*(Jmc*Hmr(k,l) - Hmc*Jmr(k,l))) / (Hma*Jmc - Jma*Hmc);
        elseif (r(k,l)<a),
            A(k,1) = Aa(k,1)*(r(k,1)/a)^m;
        elseif (r(k,1)>e),
            A(k,l) = 0;
        end
    end
end
A_fourier=A_fourier+(A);
end

figure(1), clf, zoom on
contour(x,y,r,[a a], 'k'); hold on; contour(x,y,r,[c c], 'k');
contour(x,y,r,[e e], 'k');
contour(x,y,real(A_fourier),50);
axis equal;
colormap gray;
colorbar;

%------------------------------------------
% Following is a movie of the field changing with time
%------------------------------------------

figure(2)
skinmovie1=moviein(24);
set(gca, 'NextPlot', 'replacechildren');
for k=1:24
    B=(cos(k*pi/12) + i*sin(k*pi/12))*A_fourier;
end
\end{verbatim}
mesh(real(B));
axis([0 array_size 0 array_size , -1,1]);
skinmovie1(:,k) = getframe;
end

movie(skinmovie1,5,10)

Functions Called in Code

function out=fmxym(x,y,m)
%-----------------------------------------------------------------x%
% computes the value of fm given x,y,m
%-----------------------------------------------------------------x%
out=(m/y)*((x/y)^m + (y/x)^m)/((x/y)^m - (y/x)^m);

function out=fmxygm(x,y,gamma,m)
%-----------------------------------------------------------------x%
% computes the value of fm given x,y,gamma,m
%-----------------------------------------------------------------x%
out= m/y + (j*gamma)* (besselh(m,j*gamma*x)*besselj(m-1,j*gamma*y)-...
    besselj(m,j*gamma*x)*besselh(m-1,j*gamma*y))/...
    (besselj(m,j*gamma*x)*besselh(m,j*gamma*y) - ... 
    besselj(m,j*gamma*y)*besselh(m,j*gamma*x));

function out=gmxym(x,y,m)
%-----------------------------------------------------------------x%
% computes the value of gm given x,y,m
%-----------------------------------------------------------------x%
out=(2*m/x)/((x/y)^m - (y/x)^m);

function out=gmxygm(x,y,gamma,m)
%-----------------------------------------------------------------x%
% computes the value of gm given x,y,gamma,m
%-----------------------------------------------------------------x%
out=(-2*j/(pi*x))/(besselj(m,j*gamma*x)*besselh(m,j*gamma*y) - ... 
    besselj(m,j*gamma*y)*besselh(m,j*gamma*x));
A.2 Predicted Output from Magnetic Circuit Analysis

% Program Output_Voltage.m
%
% This program plots the theoretical output voltage which we
% calculate using the Magnetic Circuit Analogy representation
% of the three-phase sensor.
%
% Robert Ritter, January 1999

clear

%------------------------------------------------------------------
% System Constants
%------------------------------------------------------------------

omega=2*pi*5e3;       % excitation frequency
Np=36;                 % primary coil turns
Ns=350;                % secondary coil turns
I0=1e33;               % current amplitude
rt=sqrt(3);            % to make things faster
depth = 0.5*.0254;     % thickness of sensor
mu_0 = 4e-7*pi;        % permeability of air
ri=.0032;              % tube radius
r0 = .0075;            % sensor opening radius
array_size=200;        % number of points "n" in the "nxn" array
wp=.0009;              % one-third the pole face width

%------------------------------------------------------------------
% First we define two vectors which have as their entries the x
% and y coordinate of each entry, respectivey. Next we define
% matrices which have as entries the x or y coordinate of the entry.
%------------------------------------------------------------------

x=[-r0:2*r0/array_size:r0];
y=x;
xx=ones(size(x))'*x;
yy=xx';
I=eye(size(xx));
for j=1:array-size+1,
    for k=1:array-size+1,
        radius = sqrt(x(j)^2 + y(k)^2);
        if radius<(rO-ri),
            xx(j,k)=0;
            yy(j,k)=0;
        end
    end
end

% We define matrices whose entries correspond to the values of the
% relevant parameters for each location inside the sensor.

A1=.5*(rt*yy + xx);
A2=.5*(rt*yy - xx);

fudge_1=7;  % to adjust the reluctances if necessary
La = (sqrt((rO^2 - A1.^2)) + .5*(-rt*xx + yy) - ri);
Lb = (sqrt((rO^2 - xx.^2)) - yy - ri);
Lc = (sqrt((rO^2 - A2.^2)) + .5(rt*xx + yy) - ri);
d1 = (sqrt((xx.^2 + (rO + yy).^2)) - ri)/fudge_1;
d2 = (sqrt(( (.5*rt*rO - xx).^2 + (.5*rO - yy).^2 )) - ri)/fudge_1;
d3 = (sqrt(( (.5*rt*rO + xx).^2 + (.5*rO - yy).^2 )) - ri)/fudge_1;

R1t = 2*d1./(muO*depth*(2*ri+wp));
R2t = 2*d2./(muO*depth*(2*ri+wp));
R3t = 2*d3./(muO*depth*(2*ri+wp));

RLA = 2*rt*rO./(muO*depth*(La+wp));
RLB = 2*rt*rO./(muO*depth*(Lb+wp));
RLC = 2*rt*rO./(muO*depth*(Lc+wp));

Rsum = R1t.*R2t + R2t.*R3t + R3t.*R1t;
DEN = RLA./((RLA.*R3t + Rsum) + RLB./((RLB.*R1t + Rsum) + ... + RLC./((RLC.*R2t + Rsum);

R1 = RLA.*RLC.*Rsum./DEN;
R2 = RLA.*RLB.*Rsum./DEN;
R3 = RLB.*RLC.*Rsum./DEN;
Raeq = (R1.*R2 + R2.*R3 + R3.*R1)./R3;
Rbeq = (R1.*R2 + R2.*R3 + R3.*R1)./R1;
Rceq = (R1.*R2 + R2.*R3 + R3.*R1)./R2;

\[
R = [1./Rbeq; 1./Rceq; 1./Raeq];
\]
A = [-I, exp(i*5*pi/3)*I, 2*exp(i*4*pi/3)*I; 2*I, exp(i*5*pi/3)*I, ...
   exp(i*pi/3)*I; -I, 2*exp(i*2*pi/3)*I, exp(i*pi/3)*I];
V = A*R;
V = V*omega*Np*Ns*I0/(rt);

Va = V(1:arraysize+1,:);
Vb = V(array_size+2:2*array_size+2,:);
Vc = V(2*array_size+3:3*array_size+3,:);

figure(1)
clf
contour(abs(Vb), 25), colormap gray, colorbar, grid on, axis equal

figure(2)
clf
contour(angle(Vb), 25), colormap gray, colorbar, grid on, axis equal

Vx = (2*ri*(abs(Va)-abs(Vc)))/(2*rt*rO);
Vy = (2*ri*(-2*abs(Vb)+abs(Vc)+abs(Va)))/(6*rO);

figure(3)
clf
hold on
for j = 1:10:arraysize+1,
    plot(Vx(:,j), Vy(:,j))
    plot(Vx(j,:), Vy(j,:))
end
axis equal

% We compare this to the output using the squared terms also:
\[ V_x = (\text{abs}(V_a)^2 - \text{abs}(V_c)^2 + \ldots \]
\[ 2*\text{ri}(\text{abs}(V_a) - \text{abs}(V_c))/(2*\text{rt}\times r_0); \]
\[ V_y = -(2*\text{abs}(V_b)^2 - \text{abs}(V_c)^2 - \text{abs}(V_a)^2 + \ldots \]
\[ 2*\text{ri}(2*\text{abs}(V_b) - \text{abs}(V_c) - \text{abs}(V_a))/(6*\text{r}_0); \]

figure(4)
clf
hold on
for j=1:10:array\_size+1,
    plot(Vx(:,j),Vy(:,j))
    plot(Vx(j,:),Vy(j,:))
end
axis equal

%----------------------------------------------------------- %
% We can also use the phase to find the position:
%----------------------------------------------------------- %

\[ V_x = (\text{angle}(V_b) - \text{angle}(V_c) - \text{angle}(V_a))/(2); \]
\[ V_y = -(\text{angle}(V_c) - \text{angle}(V_a))/(\text{rt}); \]

figure(5)
clf
hold on
for j=1:12:array\_size+1,
    plot(Vx(:,j),Vy(:,j))
    plot(Vx(j,:),Vy(j,:))
end
axis equal
Bibliography


