Flow of Grains in Channels: The Granular Jump

by

Bassam George Kassab

Submitted to the Department of Civil and Environmental Engineering on December 20, 1999, in partial fulfillment of the requirements for the degree of Master of Science

Abstract

This work focuses on an experimental study of granular flow in the vicinity of an abrupt jump in a channel formed upstream of a weir. The occurrence of this jump is controlled by the flow rate of the grains and the channel slope. Observations of the granular jump showed four zones: a stagnation zone, a zone of slowly moving grains, a zone of fast moving grains, and a zone of “flying” grains. The extent of the stagnation zone along the channel upstream of the weir decreases linearly with increasing channel inclination angle. When the flow rate in the channel is increased, the length of the stagnation zone along the channel bed and its depth along the weir decrease. Knowing the positions of the free surface profile and the stagnation zone upper boundary results in determining the changing depth along the channel and, consequently, the average velocity profile.

Thesis Supervisor: L. Mahadevan
Title: van Tassel Associate Professor of Mechanical Engineering
Acknowledgments

I cannot say enough in gratitude to my thesis advisor, Prof. L. Mahadevan, for his patience, honesty, and compassion. The directions he offered to me while I was dealing with a new subject and I was learning new experimental techniques were very helpful. But, of course, I assume full responsibility for any errors in this thesis.

Prof. C. C. Mei’s overall support, friendliness and encouragement during the last year are very appreciated too.

I would like also to acknowledge the following people who facilitated the work for my thesis by being there when I needed them: Prof. Arshad Kudrolli and his group at the Physics Department of Clark University, Dr. Jack Germaine at the Geotechnical Laboratory of MIT, Peter Morley at MIT Central Machine Shop, Tony Caloggero at the Edgerton Center, Cynthia Stewart, Ray Hardin, George La Bonte, Wynn Sanders, and particularly Prof. Patricia Culligan for her personal support throughout my journey at MIT.

I have been blessed with a number of people who had a strong impact on me during my presence at MIT. Those are, in a chronological order: Joe Travers, Amanda Trombley, Sanjay Pahuja, Hrund Andradottir, Father William Campbell, David Diamond, M.D., my skydiving team (Ibrahim Abou Faiçal and Saad Mneimneh), Jeffrey Merrill, M.D., Margaret Pazant (a.k.a. Beautiful), Steven Cadwell, Ph.D., Brian Sullivan, Will Coons, Jorge Otero-Pailos (and Cairo), Anthony Garcia, Maria Kamvysellis, Dean Milena Levak, Faisal Alam, Tanju Fuat Abdullah (a.k.a. T.J. Armand), Howard Heller, M.D., John Lloyd, M.D., George Helo, Jr., and Mouwafa Sidaoui.

I want to acknowledge many MIT students and staff in Ralph M. Parsons Laboratory, the Lebanese Club at MIT, the International Students Office, the Office of Residential Life and Students Life Programs, the GLB Graduate Student Coffeehouse, GaMIT, and La Maison Française. Also I would like to extend my gratitude to the members of KARAMA (New England Lavender Arab Society), especially Charbel Achkar, Natasha Iskander and George Helo, Sr.

This research was supported in part by the Mechanics and Materials group at
the Mechanical Engineering Department of MIT. I would not have been able to start working on this thesis or to graduate without the help of Raymond Karnabe who has co-signed my *Tech Loan*. The support of the Graduate Students Office is also acknowledged. I am extremely appreciative to my parents, George Kassab and Aida Kazan, for their unconditional love and support. Also I owe them a lot for respecting my decisions in life and financing my education. Last, but not least, I would like to thank my siblings Nicolas and Maria Kassab (and their significant others — Layla Saadé and Zaman Aoun) for their ability to cope with the results of my journey at MIT.
You ask me how I became a madman. It happened thus: One day, long before many gods were born, I woke from a deep sleep and found all my masks were stolen – the seven masks I had fashioned and worn in seven lives. I ran maskless through the crowded streets shouting, “Thieves, thieves, the cursed thieves.” Men and women laughed at me and some ran into their houses in fear of me. And when I reached the market place, a youth standing on a housetop cried, “He is a madman.” I looked up to behold him; the sun kissed my own naked face for the first time. For the first time the sun kissed my naked face and my soul was inflamed with love for the sun, and I wanted my masks no more. And as if in a trance I cried, “Blessed, blessed are the thieves who stole my masks.” Thus I became a madman. And I have found both freedom and safety in my madness; the freedom of loneliness and the safety from being understood, for those who understand us enslave something in us.

— Khalil Gibran, The Madman
To George...
# Contents

1 Introduction ............................................. 11
   1.1 Motivation ............................................. 11
   1.2 Description of the Problem ............................. 12
   1.3 Literature Review ...................................... 13
   1.4 Scope of Work .......................................... 16

2 Granular Jump Experiment ................................. 17
   2.1 Introduction ............................................. 17
   2.2 Experimental Setup ..................................... 18
   2.3 Experimental Measurements ............................. 18
      2.3.1 Properties of the granular materials ............... 19
      2.3.2 Phase diagram ...................................... 21
      2.3.3 Stagnation zone ..................................... 24
      2.3.4 Depth and velocity profiles ........................ 28

3 Analytical Models ......................................... 35
   3.1 Introduction ............................................. 35
   3.2 Haff’s Model ............................................ 36
   3.3 BCRE Model ............................................. 38
   3.4 Savage’s Model .......................................... 41
   3.5 Application to the Granular Jump ..................... 45

4 Conclusions and Recommendations ....................... 50
List of Figures

1-1 Schematic diagram of a granular jump ................................. 14

2-1 Experimental setup .......................................................... 19

2-2 Granular jump: abrupt transition from shallow and fast granular flow mode to deeper and slower — based on depth average — motion ... 22

2-3 Smooth transition .............................................................. 22

2-4 Phase diagram for weir height $h_w = 2.0cm$ ............................... 23

2-5 Phase diagram for $h_w = 1.4cm$ .......................................... 24

2-6 Schematic diagram of the various flow zones for a jump formed on an inclined channel upstream of a weir .................................. 25

2-7 Stagnation zone extent for $h_w = 2.0cm$ .................................. 26

2-8 Stagnation zone extent for $h_w = 1.4cm$ .................................. 27

2-9 Granular jump of parameters $\theta = 27^\circ$, $h_w = 2.0cm$ and $Q = 24.4cc/s$ ............................... 30

2-10 Depth and velocity profiles for the jump of parameters $\theta = 27^\circ$, $h_w = 2.0cm$ and $Q = 24.4cc/s$ ............................... 30

2-11 Granular jump of parameters $\theta = 27^\circ$, $h_w = 2.0cm$ and $Q = 106.9cc/s$ ................................................................. 31

2-12 Depth and velocity profiles for the jump of parameters $\theta = 27^\circ$, $h_w = 2.0cm$ and $Q = 106.9cc/s$ ................................................................. 31

2-13 Granular jump of parameters $\theta = 26^\circ$, $h_w = 1.4cm$ and $Q = 24.4cc/s$ ................................................................. 32

2-14 Depth and velocity profiles for the jump of parameters $\theta = 26^\circ$, $h_w = 1.4cm$ and $Q = 24.4cc/s$ ................................................................. 32

2-15 Granular jump of parameters $\theta = 26^\circ$, $h_w = 1.4cm$ and $Q = 106.9cc/s$ ................................................................. 33
2-16 Depth and velocity profiles for the jump of parameters $\theta = 26^\circ$, $h_w = 1.4\text{cm}$ and $Q = 106.9\text{cc/s}$.
List of Tables

<table>
<thead>
<tr>
<th></th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Physical properties of the granular materials</td>
<td>20</td>
</tr>
<tr>
<td>2.2</td>
<td>Best fit lines of $x_<em>^</em>$ versus $\theta$ data for different $h_w$ and $Q$</td>
<td>28</td>
</tr>
<tr>
<td>3.1</td>
<td>Notations used in Chapter 3</td>
<td>48</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

"How do we know that the creations of worlds are not determined by falling grains of sand?" — Victor Hugo, Les Misérables

1.1 Motivation

René Descartes wrote in “Principia Philosophiae” (1644): “A body is liquid if it is divided into many parts that move separately and it is solid if all its parts are in contact.” Descartes was not probably thinking of granular materials while writing this statement. Though granular materials are “divided into many parts that move separately,” they do not possess the classical properties of liquids. On the other hand, even if all the “parts are in contact,” they are not as solid as the great scientist was thinking. Descartes’ statement clearly illustrates the lack of knowledge about the behavior of granular media that still persists in spite of the fact that granular materials are widely spread in nature and they enter in many domains of the human activity. [6]

Granular materials are ubiquitously present in our daily lives: salt or sugar in a glass container on the kitchen counter, a pile of sand at a construction site, or a heap of stones at the bottom of a cliff. Who has not been fascinated by the precision of sand flow in an hourglass? Granular materials have an important industrial role: pharmaceutical products, construction, mining, and agriculture. Also they are
involved in geological phenomena, like landslides and erosion. [11]

Granular materials physics is still considered a new area of research in its infancy. The laws governing granular flows are still being questioned by different researchers. Even the experimental techniques are not standardized: every physicist tries to develop a new method which allows the measurement of the average density of the flowing grains [14], for instance. So granular physics is an interesting field to look into knowing that (i) it is crucial for daily life activities such as production, agriculture, and Nature’s movements and (ii) it is challenging for being mysterious, not fully discovered yet.

Pierre-Gilles de Gennes [6], a well established physicist, wrote: “Granular physics has great ancestors: Coulomb under Louis XVI, Faraday and Reynolds in the XIXth century, and that extraordinary British, Bagnold, taken by desert sands like T. E. Lawrence but searching to better understand the laws. Despite all these pioneers, despite all their efforts, powder mechanics remains, in its major part, misunderstood: some primary questions have not yet received clear answers.” Several decades after Bagnold, some physicists interested in sand, powder or grains performed simple experiments to better understand the laws of granular physics. “Or almost simple: granular media are tricky from this perspective,” added de Gennes. “A naive theoretician like me may think that with one kilogram of sand, a funnel, some pipes and glass containers one can almost do anything. In fact, a thousand technical details may disturb an experiment and lead to erroneous results. Granular physics is not expensive but it requires much carefulness.”

1.2 Description of the Problem

This thesis deals with the macroscale mechanics of particulate matter flow regimes. The laws describing the flow of dry grains, sand, and powder as a continuum are complex. The objective of the research is to get a better understanding of granular flows with free surfaces. The research focuses on the physics of a simple flow problem: flow down an inclined plane and the granular jump phenomenon. An experimental
approach will be used to quantify the granular jump. The purpose of the experiments is to develop constitutive models of simple flows by relying on the data acquisition of free surface profiles and velocities. Then a review of various theories governing free surface granular flow will be presented and their relative applicability to the granular jump case will be considered.

1.3 Literature Review

Bagnold was the first scientist to look thoroughly into the laws governing moving grains in the 1930s. He is credited with the development of modern research in granular material flows with his experiments and theories. However, it is not until the last decade that granular physics benefited from a significant global research effort that resulted into many experimental observations, numerical simulations, new concepts and theoretical models. [6]

In 1954, Bagnold [1] published an experimental and theoretical study on the dispersion of solid grains sheared in a Newtonian fluid. For the flow of dry sand under gravity, he concluded that the velocity is proportional to the $3/2$ power of the sand depth in the channel:

\[ V = \frac{2}{3} \cdot 0.165 \sqrt{g \sin \theta} \frac{h^{3/2}}{d} \]

where $V$ (cm/s) is the average flow velocity, $h$ (cm) is the flow depth normal to the plane of zero movement, $d$ (cm) is the particle diameter, $\theta$ is the channel slope, and $g$ (cm/s$^2$) is the gravitational acceleration. The theoretical velocity prediction (equation 1.1) overestimated the experimental measurements by a factor of about 50%. This discrepancy doesn’t question the validity of Bagnold’s theory; it is more likely to be attributed to the lack of reliable measurement techniques in granular physics. Even 55 years later, Mih [13] wrote: “It is difficult to make accurate granular flow measurements. The uncertainty of these measured data is large and estimated at ±50%.”

The interest in understanding the flow of granular materials has grown during the last two decades. Some researchers have focused on the granular jump phenomenon
because it offers the ability to look at a simple flow with a singularity. Experimental investigations of the granular jump in open channels for dry, cohesionless granular materials were conducted by Savage [15] in 1979 and then by Brennen et al. [3] in 1983. Their objectives were to find relations between the upstream and the downstream flows across the jump. The jump was created by placing a vertical discharge weir at the end of a flat, inclined channel bed (Figure 1-1). In both papers, the authors analyzed the granular jump using concepts from open channel flow hydraulics, like the Froude number or supercritical and subcritical flows that are elaborated upon below.

figure 1-1: Schematic diagram of a granular jump. The jump is formed on an inclined channel bed by placing an overflow weir, of height $h_w$, at its bottom end.

Savage [15] noted that the granular jump is very abrupt in comparison with the analogous hydraulic jump. Both can be characterized by the Froude number. The Froude number is

$$Fr = \frac{V}{\sqrt{gh \cos \theta}},$$

where the velocity $V$ is computed from the relation

$$V = \frac{\dot{m}}{\rho_s \nu_c bh}$$

using the experimentally measured values of $\dot{m}$ (mass flow rate of particles), $\rho_s$ (density of solid particles), $\nu_c$ (critical solids fraction), and $b$ (channel width). The term $\rho_s \nu_c$ represents the bulk density, $\rho_B$. So the average flow velocity is obtained by
dividing the volumetric flow rate \((Q = \dot{m}/\rho_B)\) by the flow area \((bh)\). Here, \(\nu_c\) is the minimum solids fraction from loose static packing tests. Brennen et al. [3] recommended that “it would be very valuable to devise experimental techniques for local solids fraction measurements in granular material flows” as they recognized this need.

Patton et al. [14] suggested and used a technique to measure the average density of flowing grains at different locations in a channel. A device formed of two plates connected by a handle is suddenly pushed into the channel to trap the flowing grains in the area between the plates. The local density, \(\rho_B(x)\), is found by knowledge of the weight of the trapped material, the measured depth of the flow, \(h(x)\), and the dimensions of the trap. Similar measurements at different locations allow the evaluation of density gradients. Therefore, meaningful average velocities can be computed by using a varying \(\rho_B(x)\) in equation (1.3) instead of a constant \(\rho_B = \rho_S \nu_c\).

Campbell et al. [5] experimentally studied subcritical and supercritical flows on a long inclined chute without a downstream discharge weir. Supercritical flow is swift and shallow with \(Fr > Fr_c = \sqrt{\cos \theta}\), while subcritical flow \((Fr < Fr_c)\) is slow and deep. Supercritical and subcritical flows represent conjugate states of open-channel flows — whether in hydraulics or granular media —, i.e., a given supercritical flow transitions via a jump into a unique subcritical flow with considerable energy loss. Supercritical flow rates are solely controlled by the height of the entrance gate opening. On the other hand, the rate of a subcritical flow is independent of the entrance gate, it depends only on the material properties — friction angle, particle diameter and particle specific gravity —, the channel inclination angle and the chute geometry (length to width ratio).

Campbell and Brennen [4] performed computer simulations on granular flow in inclined chutes to get solids fraction and velocity profiles. The results were compared with existing theories and limited experimental data. The rather poor comparisons between theoretical, numerical and experimental methods of analyses are suggestive of the complexity of research in granular physics.
1.4 Scope of Work

This thesis consists of two parts: (i) a description of the granular jump experiments — design of the experimental setup, measurement techniques, data collected, granular jump profiles, velocity profiles and analysis of the results — is presented in Chapter 2, and (ii) a critical review of the main analytical models describing free surface granular flows is presented in Chapter 3. Finally, Chapter 4 provides some concluding remarks and contributions as well as guidelines for future work.
Chapter 2

Granular Jump Experiment

"I think that a particle must have a separate reality independent of the measurements. That is an electron has spin, location and so forth even when it is not being measured. I like to think that the moon is there even if I am not looking at it." — Albert Einstein

2.1 Introduction

Research on granular flows has been severely restrained by the lack of reliable, nonintrusive instrumentation to measure precisely the velocity, solids fraction, and random particle motion within the flow [4]. In this chapter, the focus is on investigating the characteristics of a granular jump and looking into the velocity distribution along the channel. After we describe the experimental setup and the physical properties of the granular materials, we will present a phase diagram showing the ranges of granular flow rates and channel inclination angles for which a jump takes place upstream of a weir placed at the lower end of the chute. Then the extent of the stagnation zone length upstream of the weir is analyzed in terms of the flow rate and the inclination angle. Finally, the free surface profile of the flow is used to measure the velocity profile along the granular jump in the channel.
2.2 Experimental Setup

The experimental setup, shown in Figure 2-1, was designed for the purpose of studying the granular jump. Images of the flow were taken using a digital camera.

The experimental setup comprises a 77-cm-long, 6.1-cm-wide and 8-cm-deep channel made of 6-mm-thick plexiglass. The channel can be inclined from a horizontal position up to a maximum angle $\theta \approx 60^\circ$. The granular materials flow from a reservoir through an inlet sluice gate whose opening height is controlled. The maximum grain surface elevation in the reservoir above the gate opening is 61 cm. But unlike the case of fluid flow from a reservoir, the flow of granular materials does not depend on the the free surface level in the reservoir upstream of the sluice gate as long as it is higher than a minimum level, $h_{\text{min}}$. The value of $h_{\text{min}}$ was found to be around the larger dimension of the rectangular horizontal section of the feeding bin ($h_{\text{min}} \approx 10\text{cm}$).

At the bottom end of the channel, either of two different discharge weirs, of heights $h_w = 1.4\text{cm}$ and $2.0\text{cm}$, is used to induce a granular jump upstream. The distance between the sluice gate and the weir is 58.5 cm. The rough bed condition was obtained by gluing a felt sheet to the surface of the channel bottom.

Every flow experiment was recorded by a high performance CCD camera (manufactured by COHU). It takes movies at a rate of 30 frames per second. A 650-Watt-light (manufactured by Mole-Richardson Co.) supplied adequate illumination for proper filming of the experiments.

2.3 Experimental Measurements

A detailed analysis of the physical properties of the granular materials was carried out first. Then numerous experiments were performed for different combinations of channel bed inclination angles, granular flow rates, and discharge weir heights. Observations were recorded showing two types of flow transition in the channel: an abrupt granular jump or a smooth transition, both occurring from a shallow fast flow to a deeper slower flow. Attention was given to the shape and length of the stagnation
zone formed upstream of the weir and downstream of the jump. Finally, flow depth and velocity profiles are analyzed for the granular jump case.

### 2.3.1 Properties of the granular materials

The granular particles used in the experiments possess certain physical properties that need to be known: type of material, shape, size, material density, solids fraction, bulk density and angle of repose (Table 2.1).

First of all, the grains are spherical glass beads. The particle diameter is measured to be $d = 1.1 \pm 0.1 \text{mm}$.

A specific gravity analysis is performed on the glass grains in accordance to ASTM standards [7]. The measured specific gravity of the spheres is $G_s = 2.479$. This is an average value for 24 tests, with a standard deviation of $\sigma_s = 0.007$. The
Table 2.1: Physical properties of the granular materials.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diameter of glass grains, $d$ (mm)</td>
<td>$1.1 \pm 0.1$</td>
</tr>
<tr>
<td>Specific gravity, $G_s$</td>
<td>2.479</td>
</tr>
<tr>
<td>Material density, $\rho_s$ (g/cc)</td>
<td>2.475</td>
</tr>
<tr>
<td>Bulk density, $\rho_B$ (g/cc)</td>
<td>1.45</td>
</tr>
<tr>
<td>Solids fraction, $\nu$</td>
<td>58%</td>
</tr>
<tr>
<td>Angle of repose, $\theta_r$</td>
<td>23.5°</td>
</tr>
</tbody>
</table>

Density of the glass spheres at room temperature is found to be $\rho_s = 2.475g/cc$ after using the relation $G_s = \rho_s/\rho_w$, where $\rho_w$ is the density of water at the same temperature as the solid spheres.

The bulk density of the glass grains at rest is measured by filling glass tubes of volume 10cc (labeled with 0.1cc intervals) with grains and weighing them. Fourteen tests are performed resulting in an average value of $\rho_B = 1.45g/cc$, with a standard deviation of $\sigma_B = 0.033$. The solids fraction, defined as

$$\nu = \frac{\rho_B}{\rho_s}, \quad (2.1)$$

is therefore 58% when the grains are at rest. However, the value of the solids fraction and, consequently, the value of the bulk density decrease considerably when the grains are in motion.

The value of the angle of repose, $\theta_r$, of the aforementioned granular materials was experimentally found after using two different methods. The first one is based on observations of maximum static slope. Using a surface on which several layers of granular materials were placed, it should be tilted gradually and the first grain to start rolling should be observed. Recording the corresponding slope gives an estimate of a static angle of repose $\theta_r^s \approx 24^\circ$. In contrast with the first method that finds the angle of initiation of motion, the second method looks for the angle at which the flow of grains stops. A pile of grains was formed using an inverted conical funnel of large angle $\alpha (\alpha = 58^\circ \gg \theta_r)$, then the funnel was slowly removed allowing a transient flow on the heap surface till the motion reaches a halt. This dynamic angle of repose of
the pile is measured to be \( \theta_r^d \approx 23^\circ \). Finally, the arithmetic average of both \( \theta_r^s \) and \( \theta_r^d \) can be looked at as the angle of repose of the glass beads: \( \theta_r = 23.5^\circ \).

2.3.2 Phase diagram

We now turn to the granular jump. For each weir height \( (h_w) \), many granular flow experiments are carried out in the channel for various bed inclination angles \( (\theta) \) and flow rates \( (Q) \). The grains entering the channel upper end have a relatively high velocity and a small depth. According to the value of \( (\theta, Q) \), either an abrupt granular jump (Figure 2-2) or a smooth transition (Figure 2-3) forms. The flow then relaxes to one with a larger depth and smaller average velocity, until it flows over the weir in a thin layer and a relatively large discharge velocity at the lower end of the chute. Right behind the weir, there is a stagnant triangle. This is very different from the fluid case, where an eddy sits behind the weir.

The phase diagrams show many values of the pair \( (\theta, Q) \) for the weir heights \( h_w = 2.0cm \) (Figure 2-4) and \( h_w = 1.4cm \) (Figure 2-5). Each of these diagrams highlights two regions: a region where an abrupt granular jump can develop and another region where a jump gives way to a smooth transition over a longer reach allowing the flow to move from a shallow-fast mode to a deep-slow mode. The dividing line between the two distinct zones is marked in dashes on each of the two diagrams. The granular jump zone is marked by “circles,” each of them representing an experimental observation of a jump for a particular value of the angle \( \theta \). For every observation, the value of the flow \( Q \) is obtained via the procedure detailed at the end of the current section (2.3.2). The “no jump” or smooth transition zone is graphically marked by “stars.”

For \( \theta \leq 24^\circ \) and irrespective of the inlet gate opening height, there was no flow developed in the channel. This was another confirmation that the angle of maximum static slope is \( \theta_r^s = 24^\circ \). These “no-flow” observations were marked also by “stars” — because “no-flow” means also “no-jump” — on the \( Q = 0 \) line of both phase diagrams.

No granular jump was observed for any angle \( 24^\circ < \theta < 25.5^\circ \), even if the inlet gate opening was large enough to generate a flow \( Q \gg 100cc/s \) (such large flow values are
Figure 2-2: Granular jump: abrupt transition from shallow and fast granular flow mode to deeper and slower — based on depth average — motion. *This photograph corresponds to an experiment with the following parameters:* $\theta = 27^\circ$, $h_w = 2.0\text{cm}$ and $Q = 64.8\text{cc/s}$.

Figure 2-3: Smooth transition. *This photograph shows the passage from a shallow-fast-flow to a deep-slow-flow over a longer reach as opposed to the granular jump case. The parameters of the experiment are:* $\theta = 27^\circ$, $h_w = 2.0\text{cm}$ and $Q = 16.4\text{cc/s}$.

not shown in figures 2-4 and 2-5). The dashed-line in the phase diagrams demarcates the boundary between a granular jump region and a smooth transition region. The redundancy of any value falling on the boundary line and indicating the possibility of two different phases for a single pair $(\theta, Q)$ is simply explained by the sensitivity of the observations to the channel slope, $\theta$, which is not measured with high accuracy. The chute inclination angle is directly measured using a “Pitch and Angle Locator” (manufactured by *Johnson Co.*).

The flow rate, $Q$, was indirectly measured as detailed in the following procedure. For every experiment, we timed the collection of grains discharging from the chute
Figure 2-4: Phase diagram for weir height \( h_w = 2.0\, \text{cm} \). The dashed-line separates between the two phases: a granular jump phase marked by “circles” representing a single experiment each, and a smooth transition — no jump — phase marked by “stars”.

into \( n \) different buckets. The mass of granular materials (in grams) in each bucket was then measured: \( m_i, i = 1, 2, \ldots, n \). Dividing \( m_i \) by the time increment \( \Delta t_i \) (in seconds) gives the mass flow rate, \( \dot{m}_i \), as obtained from bucket number \( i \). Averaging over all the buckets results into a mean value of the mass flow rate, \( \bar{m} \) (g/s), for a single experiment:

\[
\bar{m} = \frac{1}{n} \sum_{i=1}^{n} \dot{m}_i . \tag{2.2}
\]

Then the volumetric flow rate \( Q \) (cc/s) is computed by using a constant bulk density value (\( \rho_B = 1.45\, \text{g/cc} \)) as obtained in Section 2.3.1 for the grains at rest:

\[
Q = \frac{\bar{m}}{\rho_B} . \tag{2.3}
\]
2.3.3 Stagnation zone

During a granular jump experiment, the grains are found in a number of different states (Figure 2-6), identified as follows:

1. **Zone of “flying” grains**: the grains are not in contact with each other most of the time, except during the short time of collision. This region has a negligible density ($\rho_1 \ll \rho_B = 1.45\text{g/cc}$). The velocity of the flying grains is very high. Flying grains are observed in experiments with steep channel bed, therefore, in cases of very abrupt jumps. These grains are scattered on the top of the free granular surface and they occupy a zone extending few centimeters downstream of the jump.

2. **Zone of fast moving grains**: the grain is almost always in contact with surrounding grains but while simultaneously undergoing relative motion. The zonal density $\rho_2$ is smaller than $\rho_B$ but not as small as $\rho_1$ ($\rho_1 \ll \rho_2 < \rho_B$). The velocity in this region is $v_2 \approx v_1$. The zone of fast moving grains, of depth $h_2(x)$, is
present along all the channel, from the inlet sluice gate, to the jump, and down to the discharge weir.

3. **Zone of slowly moving grains:** the grain is in contact with its surrounding grains for a relatively long period of time. The density of the grains in this zone is very close to the bulk density \((\rho_3 \approx \rho_B)\). The velocity \(v_3\) is such that \(0 < v_3 < v_2\). This zone starts underneath the jump and extends to the weir with a varying depth, \(h_3(x)\). It always forms a layer separating between the fast moving grains and the stagnant grains.

4. **Stagnation zone:** the grains are not moving at all. The density and the velocity in the stagnation zone are \(\rho_4 = \rho_B\) and \(v_4 = 0\), respectively. The stagnation zone always occupies the bottom of the channel, immediately behind the weir, for any combination of parameters \((\theta, Q)\). The extent of the stagnation triangle upstream of the weir, \(x_s\), varies with \(\theta\), \(Q\) and \(h_w\).

At any cross sectional plane \((y-z)\) perpendicular to the flow direction (i.e., normal to the \(x\)-axis), the mass flow rate contributions from the different zones should sum up to the total mass flow rate, \(\overline{m}\):

\[
\rho_1 h_1 v_1 + \rho_2 h_2 v_2 + \rho_3 h_3 v_3 + \rho_4 h_4 v_4 = b \sum_{i=1}^{4} \rho_i h_i v_i = \overline{m},
\]  
(2.4)
where $h_i$ is the depth of one of the flow zones measured along the z-axis perpendicular to the channel bed, $b$ is the channel width, $v_i$ is the average granular velocity in the $x$-direction for each zone, and $\rho_i$ is the zonal density such that $\rho_1 \ll \rho_2 \ll \rho_3 \approx \rho_4 = \rho_B$.

We first turn our attention to the stagnation zone which is prismatic, of width equal to the channel width $b$, and has a roughly triangular cross-section when viewed from the side. To determine the extent of the stagnation zone upstream of the weir for each granular jump experiment, we located the first mobile grain on the bed, when approached from the weir side, which defines $x_s$. Knowledge of $x_s$ and $h_4(x = 0)$ is sufficient to describe the geometry of the stagnation zone prism.

![Figure 2-7](image.png)

Figure 2-7: Stagnation zone extent for $h_w = 2.0\,\text{cm}$. Each data point on the graph gives the dimensionless length, $x^*_s$, of the stagnation zone formed between the granular jump and the weir for one experiment marked by the pair $(\theta, Q)$. The stagnation zone becomes shorter with increasing flow rate, $Q$, and increasing chute angle, $\theta$. The data is grouped graphically into four flow rate categories, $Q_i$, $i = 1, \cdots, 4$, labeled by their mean values and standard deviations: $Q_i = \overline{Q_i} \pm \sigma_i$. The error bar in reading $x^*_s$ is shown for two experiments in each flow range. The best fit curve for each flow range is a straight line.
Figures 2-7 and 2-8 present the stagnation zone extent for the weir heights $h_w = 2.0\text{cm}$ and $1.4\text{cm}$, respectively. These are plots of the nondimensional variable $x^*_s = x_s/h_w$ in function of the chute angle for various ranges of flow rate values. Data for $x_s$ was collected for all 50 granular jump experiments for the case of $h_w = 2.0\text{cm}$ and all 42 experiments for $h_w = 1.4\text{cm}$. These values were distributed into groups of flow rate values $Q_i$ (cc/s): for $h_w = 2.0\text{cm}$, $0 < Q_1 < 20 \leq Q_2 < 40 \leq Q_3 < 60 \leq Q_4$, and for $h_w = 1.4\text{cm}$, $0 < Q_1 < 30 \leq Q_2 < 50 \leq Q_3 < 60 \leq Q_4$. The legends in figures 2-7 and 2-8 show for every range of flow rate values ($Q_i$, $i = 1, \cdots, 4$) its mean ($\overline{Q_i}$) and its standard deviation ($\sigma_i$), presented in the following format: $Q_i = \overline{Q_i} \pm \sigma_i$.

The data points for the pairs $(x^*_s, \theta)$, arranged into four flow rate ranges for each weir, are best fitted to straight lines (Table 2.2) of the form $x^*_s = A_i \theta + B_i$, with $i = 1, \cdots, 4$.

As a first general observation, the best fit lines on the graphs 2-7 and 2-8 show that, for a constant flow rate range, $x^*_s$ decreases linearly as $\theta$ increases. Furthermore, for a constant chute inclination angle, $x^*_s$ decreases as $Q$ increases.

Figure 2-8: Stagnation zone extent for $h_w = 1.4\text{cm}$.
Table 2.2: Best fit lines of the dimensionless stagnation length ($x^*_s$) versus channel slope ($\theta$) data for different weir heights ($h_w$) and flow rates ($Q$).

<table>
<thead>
<tr>
<th>$h_w$ (cm)</th>
<th>$i$</th>
<th>$Q_i$ (cc/s)</th>
<th>$x^*_s = A_i \theta + B_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>1</td>
<td>9.9 ± 3.7</td>
<td>-0.96 $\theta + 35.6$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>28.0 ± 2.8</td>
<td>-0.87 $\theta + 31.6$</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>48.6 ± 6.1</td>
<td>-0.82 $\theta + 29.4$</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>70.5 ± 8.5</td>
<td>-0.53 $\theta + 19.3$</td>
</tr>
<tr>
<td>1.4</td>
<td>1</td>
<td>19.4 ± 8.1</td>
<td>-0.97 $\theta + 33.5$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>41.2 ± 5.4</td>
<td>-1.15 $\theta + 37.3$</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>54.2 ± 2.5</td>
<td>-1.20 $\theta + 37.8$</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>75.6 ± 10.1</td>
<td>-1.23 $\theta + 38.1$</td>
</tr>
</tbody>
</table>

The error bars around the values of $x^*_s$ are shown in figures 2-7 and 2-8. These dimensionless values of error constructed around the readings of $x^*_s$ varied between a minimum of 0.071 and a maximum of 0.153 for the 16 experimental points, thus for the 48 readings of $x^*_s$. These dimensionless errors are relatively small: the maximum is 0.3cm relative to a stagnation zone length of 3.5cm $< x_s < 15.5cm$. In 1999, Mih wrote about granular flow measurements: “The uncertainty of these measured data is large and estimated at ±50%” [13]. So a maximum error of 10% in reading the stagnation zone length seems acceptable.

2.3.4 Depth and velocity profiles

Granular jumps were produced in the channel by varying the parameters of the experiment: $h_w$, $\theta$ and $Q$. We will analyze below granular jump experiments that were conducted for four combinations of two different sets of channel parameters ($h_w = 2.0cm$, $\theta = 27^o$; and $h_w = 1.4cm$, $\theta = 26^o$) and two different flow rates ($Q = 24.4cc/s$ and 106.9cc/s).

The video camera and NIH Image software were used to take snapshots (Figures 2-9, 2-11, 2-13 and 2-15) of the jumps and to locate the following depth profiles:

- The depth of the stagnation zone — $h_4(x)$: it is measured normal to the chute bed (Figure 2-6). Its maximum height is located right next to the weir: $h_{4,\text{max}}$
\( h_4(x = 0) \leq h_w \). In fact, \( h_{4,\text{max}} \) is more likely to be strictly less than \( h_w \) because the difference should be at least equal to a grain diameter, \( d \). As an experimental fact, by observing jumps created by reasonable flow rates, there is always at least one layer of slowly moving grains being carried by the fast moving grains. \( h_{4,\text{max}} \) cannot always be safely assumed \( \approx h_w \). For large flow rate values, \( h_{4,\text{max}} \) can be as low as \( 0.6h_w \) (Figure 2-16).

- The height of the free surface — \( h(x) \): it is also measured perpendicular to the channel bottom. The free surface profile (Figure 2-6) is defined as the interface between the zone of fast moving grains and either the zone of “flying” grains — where it exists — or air.

Subtracting the stagnation zone depth from the total depth of the free surface gives the depth of the moving grains \( (h_m) \):

\[
h_m(x) = h(x) - h_4(x).
\]

(2.5)

The average velocity \( v(x) \) of the fast and slowly moving grains at any section along the \( x \)-axis is obtained by dividing the volumetric flow rate \( Q \) — based on a constant bulk density value \( \rho_B \) — by the flow area \( bh_m \):

\[
v(x) = \frac{Q}{bh_m(x)}.
\]

(2.6)

A diagram follows each of the photographs aforementioned representing one granular jump experiment. Each diagram (figures 2-10, 2-12, 2-14 and 2-16) represents the profiles of the following dimensionless variables:

\[
h^*(x^*) = \frac{h(x)}{h_w}, \quad h_4^*(x^*) = \frac{h_4(x)}{h_w}, \quad v^*(x^*) = \frac{v(x)}{\sqrt{gh_w}},
\]

(2.7)

where \( x^* = x/h_w \) and \( g \) is the gravitational acceleration \( (g = 980cm/s^2) \).

The maximum stagnant height at the weir, \( h_{4,\text{max}}^* \), decreases with increasing \( Q \). For weir height \( h_w = 2.0cm \), figures 2-10 and 2-12 show that it drops from a value of
Figure 2-9: Granular jump of parameters $\theta = 27^\circ$, $h_w = 2.0\text{cm}$ and $Q = 24.4\text{cc/s}$. The flow is from right to left. The x-axis runs along the ruler (which has readings in centimeters) from left to right, with its origin at the corner of the chute bed and the weir. The photograph shows the inclined channel in a horizontal position.

Figure 2-10: Depth and velocity profiles for the jump of parameters $\theta = 27^\circ$, $h_w = 2.0\text{cm}$ and $Q = 24.4\text{cc/s}$. The upper profile represents the total depth which is the dimensionless measurement of the free surface height, $h^*$, normal to the channel bed. The dash-dotted line is the stagnant depth ($h^*_s$) representing the boundary between the stagnation zone and the moving grains. The dimensionless velocity profile ($v^*$) is deduced from knowledge of the granular flow ($Q$) and the moving depth ($h_m = h - h^*_s$).
Figure 2-11: Granular jump of parameters $\theta = 27^\circ$, $h_w = 2.0cm$ and $Q = 106.9cc/s$.

Figure 2-12: Depth and velocity profiles for the jump of parameters $\theta = 27^\circ$, $h_w = 2.0cm$ and $Q = 106.9cc/s$. 
Figure 2-13: Granular jump of parameters $\theta = 26^\circ$, $h_w = 1.4cm$ and $Q = 24.4cc/s$.

Figure 2-14: Depth and velocity profiles for the jump of parameters $\theta = 26^\circ$, $h_w = 1.4cm$ and $Q = 24.4cc/s$. 

**Depth and average velocity along the channel bed starting from the weir, $x = x/h_w$**
Figure 2-15: Granular jump of parameters $\theta = 26^\circ$, $h_w = 1.4cm$ and $Q = 106.9cc/s$.

Figure 2-16: Depth and velocity profiles for the jump of parameters $\theta = 26^\circ$, $h_w = 1.4cm$ and $Q = 106.9cc/s$. 
0.98 to 0.84 as $Q$ is increased from 24.4 cc/s to 106.9 cc/s. Similarly, for $h_w = 1.4 cm$, figures 2-14 and 2-16 show that $h_{4,max}^*$ drops from 0.86 to 0.60 for the same increase in the flow value.

The different plots of $h^*$ and $v^*$ do not really give a clear insight about the variation of the length of the jump with flow rate or chute angle variation. Such measure of the jump steepness, $dh^*/dx^*$, was obstructed by the difficulty to differentiate between the “flying” grains and the fastly moving ones in locating the free surface.

In this chapter, an experimental channel design is presented to study the granular jump phenomenon. Depending on the flow rate and the chute inclination, either a smooth transition or a jump takes place upstream of an obstruction weir, thus allowing the flowing grains to move from a shallow, fast mode to a deeper flow regime which is slower when looked at from a depth average perspective. A prismatic stagnant zone is formed between the jump and the weir. The extent of the stagnation zone upstream of the weir and along the channel bed is analyzed: it drops with an increasing flow rate and it decreases linearly with increasing channel inclination angle. Also the height of the stagnant zone along the weir decreases with an increasing flow rate. The free surface profile — from upstream of the jump, passing by it, and continuing to the overflow weir — is analyzed. Different flow zones are identified, of particular interest are a zone of fast moving grains and another of slowly moving grains: the sum of their depths defines the moving depth, $h_m$. An average velocity profile is deduced from knowledge of the moving depth and the flow rate. The next chapter presents the major analytical models used to describe granular flows.
Chapter 3

Analytical Models

"The sciences do not try to explain, they hardly even try to interpret, they mainly make models. By a model is meant a mathematical construct which, with the addition of certain verbal interpretations, describes observed phenomena. The justification of such a mathematical construct is solely and precisely that it is expected to work." — John von Neumann

3.1 Introduction

During the last fifteen years, there has been a growing research interest in granular materials. Experimental studies, numerical simulations and analytical models were presented to explain the physics of granular media. Three analytical models are summarized in this chapter.

Haff (1983) constructed solutions to the nonlinear and coupled conservation equations (mass, momentum and energy) by assuming that the interparticle separation distance is negligible relatively to the granular diameter [9]. The model of Bouchaud et al. (1994) deals with the dynamics of sandpile surfaces by approaching the problem from a less classical angle. This was achieved by incorporating immobile grains and rolling ones in a set of coupled kinematic equations [2]. Mahadevan and Pomeau [12] propose modifications to the model of Bouchaud et al. to better capture singularities. Savage’s 1998 model [16], unlike Haff’s model, uses a new variable — the granular
temperature — in the conservation equations. Finally, the last section attempts to link each of these models to the experimental results of the granular jump in Chapter 2.

3.2 Haff’s Model

Haff’s 1983 model [9] offers steady-state solutions for the conservation equations. Their derivations are based on the following assumptions:

- the continuum hypothesis is assumed to be valid;
- the granular flow is assumed to be incompressible;
- the grains are assumed to be of the same size;
- the grains are very close to, but in general not touching, other neighbouring grains;
- grains are assumed not to exert long-range attraction on each other;
- the granular materials undergo inelastic collisions, thus there is kinetic energy loss;
- the pairwise collision hypothesis is assumed to be valid at all densities;
- grain-grain collisions are noncentral, thus frictional forces exist and grain spin occurs; and
- the effect of grain spin is neglected (due to the difficulty of obtaining an analytical solution).

Since the density $\rho$ is assumed to be constant, the continuity equation is written as

$$\nabla \cdot \mathbf{v} = 0,$$

(3.1)
where \( \mathbf{v} \) is the velocity vector. (All the variables and the symbols used in this chapter are summarized in Table 3.1.) The momentum equation is given by the Navier-Stokes form:

\[
\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial}{\partial x_k}[p\delta_{ik} + \rho v_i v_k - \eta \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right)] + \rho g_i,
\]

where \( p = p(x, t) \) is the pressure, \( \eta \) is the dynamic viscosity, and \( g_i \) is a component of the gravitational acceleration \( \mathbf{g} \). The hydrodynamic form of the energy equation is used in Haff’s model:

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho v^2 + \frac{1}{2} \rho \bar{v}^2 \right) = -\frac{\partial}{\partial x_k} \left[ \rho v_k \left( \frac{p}{\rho} + \frac{1}{2} v^2 + \frac{1}{2} \bar{v}^2 \right) - v_i \eta \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) - K \frac{\partial}{\partial x_k} \left( \frac{1}{2} \rho \bar{v}^2 \right) \right] + \rho v_i g_i - I,
\]

where \( \bar{v} \) is the mean random fluctuation velocity of an individual grain (analog of a thermal velocity). Some terms in the momentum and energy equations need to be related to the variables \( v \) and \( \bar{v} \) used in the conservation equations. The pressure \( p \) is given by

\[
p = C_1 d \rho \frac{\bar{v}^2}{s},
\]

where \( d \) is the grain particle diameter and \( s \) is the mean separation distance between neighboring grains. The coefficient of viscosity \( \eta \) is expressed in terms of the collision rate \( \bar{v}/s \), as follows:

\[
\eta = C_2 d^2 \rho \frac{\bar{v}}{s}.
\]

The viscosity depends upon \( \bar{v} \), which is the solution to the energy equation (3.3). Therefore, the coupling between the momentum and energy equations, (3.2) and (3.3), becomes explicit. The coefficient of thermal diffusivity \( K \) is defined as

\[
K = C_3 d^2 \frac{\bar{v}}{s}.
\]

The term in the energy equation involving \( K \) is just the divergence of the internal energy flux. This term in equation (3.3) does not involve the flow velocity \( v \) explicitly, and therefore in principle it is present even where there is no macroscopic motion of
the granular system [9]. $C_1$, $C_2$ and $C_3$ are dimensionless constants. The collisional energy sink $I$ is expressed as

$$I = C_4 \rho \frac{v^3}{s}. $$

(3.7)

It is the rate at which energy is lost through collisions by unit volume and time. $I$ represents the energy irretrievably lost to the system due to the fact that grain-grain collisions are inelastic. $C_4$ is a dimensionless factor proportional to $1 - e_r^2$, where $e_r$ is the coefficient of restitution describing the collision of two grains.

The equations of motion are nonlinear and coupled. In the limit $s \ll d$, the governing equations become linear and can be solved analytically. Haff's solutions are for cases of elastic grains in a gravitational field, or inelastic grains without gravity. But all laboratory experiments involve inelastic grains, and most experiments are performed in a gravitational field. Thus the analytical solutions in this model are not comparable to the results of flow experiments.

### 3.3 BCRE Model

Bouchaud, Cates, Ravi Prakash and Edwards published a model [2] in 1994 describing the dynamics of sandpile surfaces by recognizing two categories of grains: immobile and rolling. Their work is referred to, below, as the BCRE model. The rolling grains are carried down the slope with an advection (a drift) velocity $v$, assumed to be constant for simplicity. For a sandpile slope below the angle of repose ($S < S_c = \tan \theta_r$), random sticking of the rolling grains takes place; for greater slopes, there is dislodgement of the immobile grains by the rolling ones.

The BCRE model is used to describe granular movement on a two-dimensional sandpile. The height of a stack of immobile grains $h(x, t)$ — relative to a sandpile resting at the angle of repose $\theta_r$ — depends on one spatial dimension. A suitable hydrodynamical description should include, along with $h$, the local density of rolling grains $R(x, t)$, which is the effective height of a liquid-like mobile phase that is similar to a free surface shear band sliding on top of the solid-like phase [12]. The rolling
grains are governed by
\[ R_t - v R_x = \Gamma(R, h) = \gamma R h_x , \quad (3.8) \]

where, for gravity-driven flows, the advection velocity is \( v \approx \sqrt{gd} \) and the frequency is \( \gamma \approx \sqrt{g/d} \). Here, \( \langle \rangle_a = \partial / \partial a \). The term \( \Gamma \) accounts for the conversion of immobile grains into rolling grains, and \textit{vice versa}. The height of immobile grains is governed by the following equation:
\[ h_t = -\Gamma(R, h) = -\gamma R h_x . \quad (3.9) \]

The BCRE equations (3.8) and (3.9), as written here, are valid on surfaces with a positive slope (\textit{cf.} Figure 2-6). The total number of grains \((h+R)\) is conserved locally since \((h+R)_t\) can be written as the divergence of a current.

Equations (3.8) and (3.9) are homogeneous and quasi-linear. They comprise the basic phenomenological theory of BCRE to describe the surface evolution of sandpile surfaces [12]:

1. The surface does not evolve unless there are grains in motion (\textit{i.e.}, when \( R = 0 \), \( h \) is constant). When there is free surface evolution, the rate of change of the granular height is proportional to the difference in the slope from the slope at the angle of repose \((S_c = \tan \theta_r)\). Thus the simplified BCRE equations (3.8) and (3.9) are valid only when the free surface slope is close to \( S_c \), \textit{i.e.}, when \( R_{xx} \) and \( h_{xx} \) are negligible.

2. The equations (3.8) and (3.9) are valid when \( R \) is small relatively to \( h \).

3. All moving grains are assumed to be in contact with the immobile grains.

4. There is an assumed balance between gravity and inelastic collisions, resulting in a constant convection speed \( v \) for mobile grains in the negative \( x \)-direction. (There is no convection speed for immobile grains.)
5. The BCRE equations, when linearized about a steady state \( h(x, t) = h_0(x) \)
and \( \mathcal{R}(x, t) = \mathcal{R}_0 = \text{constant} \), become hyperbolic with two wave speeds \( v \) and 
\( \gamma \mathcal{R}_0 \). The former is the advection velocity of the mobile grains, while the latter 
corresponds to waves opposing the flow, i.e., traveling upwards in the positive 
\( x \)-direction.

The BCRE equations can be analyzed for deterministic examples as well as for 
cases of surface evolution with the presence of noise [2]. A simple physical determin-
istic situation — the evolution of a bump — is presented below. A small bump sits 
in the middle of an otherwise flat surface with a slope \( S_c \). The rolling grains have a 
constant density \( \mathcal{R}_0 \). Equation (3.9) implies that the bump propagates uphill with velocity 
\[
v_h = \gamma \mathcal{R}_0 . \tag{3.10}
\]
For a sandpile, this behavior reflects the fact that rolling grains deposit on the flatter 
(uphill) part of the bump and erode the steeper (downhill) part, resulting in an uphill 
translation [2]. So the model shows that rolling grains induce an upward convection 
of a bump. Equation (3.10) allows the extraction of the value of \( \gamma \) from an experiment 
where \( \mathcal{R}_0 \) and \( v_h \) are measured.

Mahadevan and Pomeau [12] looked at singular solutions that enabled them to 
criticize the strengths and weaknesses of the BCRE model and to suggest regimes of 
applicability of its equations. Also they looked for similarity solutions that allowed 
them to characterize certain regimes where the effects of boundaries and initial con-
ditions are unimportant. The quasi-linear hyperbolic system — equations (3.8) and 
(3.9) — is capable of forming singularities or shocks in finite time from smooth initial 
data beyond which the system is ill-posed. These shock-like solutions are presented 
as propagating discontinuities in the height of the free surface: A shock is an uphill-
propagating front of erosion corresponding to a thicker mobile region downstream 
invading a thinner mobile region upstream. A shock occurs at locations in the initial 
profile where there is a point of inflection, i.e., at \( \mathcal{R}_{xx}(x_0, 0) = 0 \).

The BCRE equations as they stand do not differentiate between thick and thin
layer flows. But experimental observations have shown that often $R(x, t)$ eventually saturates to a constant value, since all the grains do not interact with the solid phase. Then a model for thick avalanches is based on a layer with small vertical velocity gradients riding on the solid phase with a thin shear band separating the two [12]. Equations (3.8) and (3.9) can be modified by using a frequency dependent on the depth of mobile grains $\gamma(R)$. Thus we get the generalized BCRE equations:

$$R_t - vR_x = \gamma(R)Rh_x$$  \hspace{1cm} (3.11)

and

$$h_t = -\gamma(R)Rh_x.$$  \hspace{1cm} (3.12)

As $R$ increases, $\gamma(R)R$ saturates.

In simple cases for which analytical solutions of (3.11) and (3.12) are obtainable, such solutions $h(x, t)$ and $R(x, t)$ are functions of the parameters $v$ and $\gamma$. From an experimental point of view, this allows the estimation of $v$ and $\gamma$ by looking at spatial and temporal gradients of the flow.

### 3.4 Savage's Model

A summary of Savage's 1998 model [16] is presented below. Savage assumes in his theoretical model that the flowing grains are cohesionless and inelastic. Also the flow is dense and slow. His model is relaxed as it assumes a frictional, compressible flow. Savage's model focuses on flows involving large strains. It seeks a representation of the behavior for slow, concentrated granular flows having a form that can be merged smoothly with the kinetic theory results for rapid, collisional flows. The goal is to predict translational and rapid flows.

The instantaneous particle velocity, $v(r, t)$, is decomposed into a slowly varying mean, $u(r, t)$, and a fluctuating part, $c(r, t)$:

$$v(r, t) = u(r, t) + c(r, t).$$  \hspace{1cm} (3.13)
Below are the three conservation equations governing Savage’s model. The mass conservation equation is written as

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{3.14} \]

the linear momentum conservation equation is given by

\[ \rho \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = \rho g_i - \frac{\partial p_{ij}}{\partial x_j}, \tag{3.15} \]

and, finally, the translational fluctuation energy equation is

\[ \frac{3}{2} \rho \left( \frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} \right) = -p_{ij} \frac{\partial u_i}{\partial x_j} - \frac{\partial q_i}{\partial x_j} - \kappa, \tag{3.16} \]

in which \( T \) is the granular temperature. Equations (3.15) and (3.16) contain the pressure tensor \( \mathbf{p} \), the energy flux vector \( \mathbf{q} \), and the rate of energy dissipation per unit volume \( \kappa \). Explicit constitutive equations for \( \mathbf{p} \), \( \mathbf{q} \), and \( \kappa \) are derived below.

Let’s introduce first the elliptical yield function defined in two-dimensional flow in terms of the mean stress \( p \) and the maximum shear stress \( q \) as

\[ F(p, q) = (p + a)^2 + e^2 q^2 - a^2 = 0, \tag{3.17} \]

where \( e \) is the eccentricity. The stresses \( p \) and \( q \) are given by:

\[ p = \frac{\sigma_1 + \sigma_2}{2}, \quad q = \frac{\sigma_1 - \sigma_2}{2}, \tag{3.18} \]

where \( \sigma_1 \) and \( \sigma_2 \) are the major and minor principal stresses, respectively, defined by the intersection of Mohr’s circle of stress with the normal stress axis. \( a \) is the mean stress value at the centre of the elliptical yield curve.

\[ a = a(\nu, T) = -p = e|q| = \frac{|q|}{\sin \phi}, \tag{3.19} \]

where \( \phi \) is the internal friction angle and \( \nu \) is the solids fraction. The size of the
elliptical yield envelope $F$ — in $(p, q)$-space — grows with $T$ and $\nu$.

Expressing the elliptical yield function $F$ in terms of the stress components $\sigma_{xx}$, $\sigma_{yy}$ and $\sigma_{xy}$:

$$F(\sigma_{xx}, \sigma_{yy}, \sigma_{xy}) = \left[ \frac{\sigma_{xx} + \sigma_{yy}}{2} + a \right]^2 + e^2 \left[ \frac{\sigma_{xx} - \sigma_{yy}}{2} + \sigma_{xy}^2 \right] - \sigma^2 = 0, \quad (3.20)$$

the strain rate can be defined as

$$e_{ij} = \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] = \frac{\lambda}{\sigma_{ij}} \partial F,$$

where $\lambda$ is a function of the strain rate determined by combining equations (3.20) and (3.21).

Solving for the stress components in terms of the strain rates, we get

$$\sigma_{ij} = -\frac{a}{\Delta} \left[ \frac{2}{e^2} e_{ij} + \delta_{ij} \left( 1 - \frac{1}{e^2} \right) e_{kk} \right] - \delta_{ij} a, \quad (3.22)$$

where $\Delta$ is a function of $e$, $\sigma_{xx}$, $\sigma_{yy}$, and $\sigma_{xy}$. Following Hibler [10], the strain rate $e_{ij}$ fluctuates around its mean value $< e_{ij} >$. By assuming that the strain rate distribution function for the fluctuations $f(e_{ij})$ is Gaussian (in analogy with the Maxwellian velocity distribution function for gases, which is basically applicable for low densities and small collision frequencies, unlike granular materials), the mean stress value will be given by:

$$< \sigma_{ij} > = \frac{1}{\epsilon^3(2\pi)^3/2} \int \exp \left( -\frac{(e_{xx} - < e_{xx} >)^2 + (e_{yy} - < e_{yy} >)^2 + (e_{xy} - < e_{xy} >)^2}{2\epsilon^2} \right)$$

$$\sigma_{ij} \delta_{xx} \delta_{yy} \sigma_{ij}, \quad (3.23)$$

where $\epsilon$ is the standard deviation of the strain rate fluctuations ($\epsilon^2$ is analogous to $T$). The mean stress tensor $< \sigma_{ij} >$ to first order in $< e_{ij} >$ can be written as:

$$< \sigma_{ij} > = 2\mu < e_{ij} > + (\zeta - \mu) < e_{kk} > \delta_{ij} - \alpha \delta_{ij}. \quad (3.24)$$
By similarity with the stress-strain-rate relation for a Newtonian viscous fluid, \( \mu \) is the shear viscosity and \( \zeta \) is called the bulk viscosity [10]. The viscosities are related to \( \epsilon \) and \( a(\nu, T) \) through constants varying with \( \phi \):

\[
\mu = \frac{aA(\phi)}{\epsilon} ; \quad \zeta = \frac{aB(\phi)}{\epsilon} .
\] (3.25)

Note also that the rate of energy dissipation per unit volume, \( \kappa \), in equation (3.16), is similarly expressed as:

\[
\kappa = \frac{\langle \sigma_{ij}^r \varepsilon_{ij}^r \rangle}{a} = a\epsilon D(\phi) .
\] (3.26)

An explicit form of \( a(\nu, T) \) is needed. The yield ellipse grows in size with an increase in pressure, thus in granular temperature, \( T \), and with an increase in solids fraction, \( \nu \). Savage assumes that \( a \) is formed of two contributions:

\[
a(\nu, T) = a_\nu(\nu) + a_T(\nu, T) .
\] (3.27)

The first contribution is called the quasi-static contribution

\[
a_\nu(\nu) = a_0 \log \left[ \frac{\nu_\infty - \nu_0}{\nu_\infty - \nu} \right] ,
\] (3.28)

and the second one is the collisional stress contribution

\[
a_T(\nu, T) = \rho_s \nu \left( 1 + 2G \right) T ,
\] (3.29)

where \( a_0 \) is the reference value of \( a \), \( \nu_\infty \) is the solids fraction corresponding to closest packing, \( \nu_0 \) is the minimum solids fraction, \( \rho_s \) is the density of individual grains, and \( \rho = \rho_s \nu \) is the density of granular packing. The expression of \( G \) is given as:

\[
G = \nu g(\nu) = \frac{\nu(16 - 7\nu)}{16(1 - \nu/\nu_\infty)^2} ,
\] (3.30)

where \( g(\nu) \) is the radial distribution function at contact. Note that both \( a_\nu(\nu) \) and \( a_T(\nu, T) \) diverge as \( \nu \) approaches \( \nu_\infty \). For high deformation rates, the collisional term,
\( a_T(\nu, T) \), is dominant.

In granular flows, there is no wide difference between the microscale, i.e., the particle diameter, \( d \), and the macroscale, typically associated with the thickness of the shear layer, \( O(10d) \). Thus, a direct relation between the standard deviation of the strain-rate fluctuations, \( \epsilon \), and the granular temperature, \( T \), is proposed:

\[
\epsilon = \frac{\beta T^{1/2}}{d},
\]

where \( \beta \) is a constant of order unity (\( \beta \approx 1/4 \) for solids fraction \( 0.65 < \nu < 0.9 \)).

Substituting \( \epsilon \) (from equation 3.31) in the expression of \( \kappa \) (equation 3.26), we get the rate of energy dissipation:

\[
\kappa = \frac{a\beta T^{1/2}D(\phi)}{d}.
\]

In summary, explicit constitutive equations for \( p, q, \) and \( r \) were presented by equations (3.18) and (3.32). These are to be used in the conservation equations (3.14) to (3.16) to solve for the velocity and the granular temperature.

### 3.5 Application to the Granular Jump

A hydraulic jump is a singularity in open-channel flow that required special treatment by hydraulicians. Similarly, a jump in free surface granular flow needs adequate modeling. So far, the analytical models have not given the granular jump special attention. In this section, an attempt is made to link the existing analytical models to the granular jump experiments in Chapter 2.

The solutions of Haff's model [9] are better not to be compared to the experimental results of Chapter 2 because the latter are obtained in a gravitational field using inelastic grains, whereas Haff's equations cannot be solved for such conditions analytically.

Savage applied his approach to cases of simple shear flow and vertical channel flow [16]. He is still in the process of applying his model to free surface flow in rough inclined channels. The goal of Savage's model is to predict translational and rapid
flows. The granular jump of Chapter 2 is an example of a rapid flow in an inclined channel. The velocity profiles obtained experimentally for the granular jump can be used for comparison with a numerical solution of Savage's equations applied to free surface flow in channels only if appropriate boundary conditions are imposed to describe the granular jump singularity taking place in the midst of the channel due to a downstream obstacle in the flow, a task which is not straightforward. An analytical solution of Savage's conservation equations is not obtainable knowing the complexity of the granular jump case.

The BCRE model [2] has room to describe flow singularities. For certain initial conditions, the BCRE equations predict the formation of shocks which correspond to upward-moving erosion fronts or, equivalently, upstream granular jumps induced by an obstacle in the flow. The limitation of the BCRE equations (3.8) and (3.9) is that they do not apply to the region upstream of the jump, which violates two of the BCRE assumptions: the mobile grains are not in contact with the immobile layer (Figure 2-6) as they are directly in contact with the channel bed; and, second, the free surface slope of the fast moving grains upstream of the jump is parallel to the channel bed, thus it is considerably larger than the slope at the angle of repose $S_r = \tan \theta_r$.

The generalized BCRE equations (3.11) and (3.12), as presented by Mahadevan and Pomeau [12], can be applied to describe a singular phenomenon that is not affected by initial conditions of the initiation of the flow, like in the case of a steady granular jump. While the BCRE equations have been successful in predicting some features of simple granular flow scenarios, they have been experimentally probed with care only in very simple situations. Further experiments are necessary to quantitatively probe the predictions. Furthermore, new detailed experiments will allow the modification of the BCRE equations to include (i) momentum balance to determine the average velocity $v$ of the mobile layer and (ii) a depth-dependent density for abrupt flow profiles around singularities [12].

Finally, the BCRE model seems to be the most suitable to describe the granular jump. Its analytical solutions $h(x, t)$ and $R(x, t)$, once obtained by the method of
characteristics, can be fitted to the steady experimental results of the profiles of the stagnation zone $h_4(x)$ and the moving zone $h_m(x) = h_2 + h_3$ (cf. Figure 2-6), respectively, in order to obtain the model parameters.
### Table 3.1: Notations used in Chapter 3.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a(v, T) $</td>
<td>mean stress value at the centre of the elliptical yield curve</td>
</tr>
<tr>
<td>$a_0$</td>
<td>reference value of $a$</td>
</tr>
<tr>
<td>$a_v(v)$</td>
<td>quasi-static contribution</td>
</tr>
<tr>
<td>$a_T(v, T)$</td>
<td>collisional stress contribution</td>
</tr>
<tr>
<td>$c$</td>
<td>velocity fluctuation from the mean</td>
</tr>
<tr>
<td>$d$</td>
<td>grain particle diameter</td>
</tr>
<tr>
<td>$e$</td>
<td>eccentricity of the elliptical yield curve</td>
</tr>
<tr>
<td>$e_{ij}$</td>
<td>strain rate</td>
</tr>
<tr>
<td>$e_r$</td>
<td>coefficient of restitution describing the collision of two grains</td>
</tr>
<tr>
<td>$g$</td>
<td>gravitational acceleration</td>
</tr>
<tr>
<td>$g(v)$</td>
<td>radial distribution function at contact</td>
</tr>
<tr>
<td>$h$</td>
<td>depth of granular materials</td>
</tr>
<tr>
<td>$p$</td>
<td>pressure tensor</td>
</tr>
<tr>
<td>$p$</td>
<td>mean stress, $p = (\sigma_1 + \sigma_2)/2$</td>
</tr>
<tr>
<td>$q$</td>
<td>energy flux</td>
</tr>
<tr>
<td>$q$</td>
<td>maximum shear stress, $q = (\sigma_1 - \sigma_2)/2$</td>
</tr>
<tr>
<td>$r$</td>
<td>spatial coordinate</td>
</tr>
<tr>
<td>$s$</td>
<td>mean separation distance between neighboring grains</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
</tr>
<tr>
<td>$u$</td>
<td>mean velocity ($u = \langle v(r, t) \rangle$)</td>
</tr>
<tr>
<td>$v$</td>
<td>instantaneous velocity vector</td>
</tr>
<tr>
<td>$v$</td>
<td>advection or drift velocity</td>
</tr>
<tr>
<td>$\bar{v}$</td>
<td>mean random fluctuation velocity</td>
</tr>
<tr>
<td>$v_h$</td>
<td>uphill propagation velocity of a bump</td>
</tr>
<tr>
<td>$x, y, z$</td>
<td>linear spatial coordinates</td>
</tr>
<tr>
<td>$A(\phi), B(\phi), D(\phi)$</td>
<td>constants varying with $\phi$ only</td>
</tr>
<tr>
<td>$C_1, \ldots, C_4$</td>
<td>dimensionless constants</td>
</tr>
<tr>
<td>$F$</td>
<td>elliptical yield function</td>
</tr>
<tr>
<td>$G$</td>
<td>expression defined in equation (3.30)</td>
</tr>
<tr>
<td>$I$</td>
<td>collisional energy sink</td>
</tr>
<tr>
<td>$K$</td>
<td>coefficient of thermal diffusivity</td>
</tr>
<tr>
<td>$\mathcal{R}$</td>
<td>density of rolling grains</td>
</tr>
<tr>
<td>$\mathcal{R}_0$</td>
<td>constant density of rolling grains</td>
</tr>
<tr>
<td>$S, S_0$</td>
<td>sandpile slope</td>
</tr>
<tr>
<td>$S_c$</td>
<td>critical slope corresponding to the angle of repose ($S_c = \tan \theta_r$)</td>
</tr>
<tr>
<td>$T$</td>
<td>granular temperature</td>
</tr>
<tr>
<td>Symbol</td>
<td>Explanation</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>$\beta$</td>
<td>constant of order unity</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>constant frequency</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>standard deviation of the strain rate fluctuation</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>bulk viscosity</td>
</tr>
<tr>
<td>$\eta$</td>
<td>dynamic viscosity</td>
</tr>
<tr>
<td>$\theta_r$</td>
<td>angle of repose</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>rate of energy dissipation per unit volume</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>function of the strain rate deduced from equations (3.20) and (3.21)</td>
</tr>
<tr>
<td>$\mu$</td>
<td>shear viscosity</td>
</tr>
<tr>
<td>$\nu$</td>
<td>solids fraction</td>
</tr>
<tr>
<td>$\nu_0$</td>
<td>minimum solids fraction</td>
</tr>
<tr>
<td>$\nu_\infty$</td>
<td>solids fraction corresponding to closest packing</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density of granular packing, $\rho = \rho_s \nu$</td>
</tr>
<tr>
<td>$\rho_s$</td>
<td>density of individual grains</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>normal stress</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>major principal stress</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>minor principal stress</td>
</tr>
<tr>
<td>$\phi$</td>
<td>internal friction angle</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>conversion term between rolling and immobile grains</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>function of the normal stresses, $\sigma_{ij}$, and the eccentricity, $e$</td>
</tr>
<tr>
<td>$(\cdot)_a$</td>
<td>denotes a partial derivative with respect to $a$: $(\cdot)_a = \partial / \partial a$</td>
</tr>
<tr>
<td>$&lt;&gt;$</td>
<td>angle brackets denoting averages</td>
</tr>
</tbody>
</table>
Chapter 4

Conclusions and Recommendations

"Say not, 'I have found the truth,' but rather say, 'I have found a truth'."

— Khalil Gibran, The Prophet

Chapter 1 presented the motivation behind studying granular materials flow and it offered an overview of the experimental work that has been done in the topic of granular jump.

Chapter 2 focused on the presentation and analysis of the results of the granular jump experiment. First of all, the physical properties of the glass beads were measured. Then, phase diagrams were experimentally developed to highlight the sensitivity of the occurrence of a granular jump depending on a combined value of the channel inclination angle and the flow rate.

Observations of the granular jump showed different zones: stagnation region, zones of slowly and fast moving grains, and zone of “flying” grains. The extent of the stagnation zone along the channel upstream of the weir was analyzed: the results show that it decreases linearly with increasing channel inclination angle. When the flow rate in the channel is increased, the length of the stagnation zone along the channel bed and its depth along the weir decrease. That is, higher flow rates tend to flush more grains from the immobile region.

The fast and slowly moving zones constitute the bulk of the moving depth, $h_m$. Knowledge of the positions of the free surface profile and the stagnation zone gives
the value of $h_m$ which is useful in determining an average velocity profile along the channel.

In Chapter 3, analytical models for granular flow analysis were described. Haff [9] and Savage [16] offered solutions to the coupled mass, momentum and conservation equations. Bouchaud et al. [2] developed and solved phenomenological equations based on approaching the granular flow problem from a different perspective: there is continuous granular exchange between an immobile zone and a zone of rolling grains. BCRE equations [12] are the most promising to describe a singularity like the granular jump.

For future work in the topic of granular jump, it is recommended to perform experimental studies on the velocity distribution along the depth of the moving grains. Velocity contours can be useful in showing the high variability in velocity along the depth.
Bibliography


