Essays on Finance and Macroeconomics

by

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Submitted to the Department of Economics
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy
at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2013

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Abstract

This thesis studies the role of the financial system in the amplification and propagation of business cycles.

Chapter 1 studies the origin and propagation of balance sheet recessions. I first show that in standard models driven by TFP shocks, the balance sheet channel disappears when agents are allowed to write contracts on the aggregate state of the economy. In contrast, I show how uncertainty shocks can drive balance sheet recessions with depressed asset prices and growth, and trigger a “flight to quality” event with low interest rates and high risk-premia. Uncertainty shocks create an endogenous hedging motive that induces financial intermediaries to take on a disproportionate fraction of aggregate risk, even when contracts can be written on the aggregate state of the economy. Finally, I explore some implications for financial regulation.

Chapter 2 studies a tractable model of dynamic moral hazard with purely pecuniary private benefits. The agent can trade a productive asset and secretly divert funds to a private account and use them to “recontract”: at any time he can offer a new continuation contract to the principal, who accepts if the new contract is attractive. The main result is that the optimal contract can be characterized as the solution to a standard portfolio problem with a simple “skin in the game” constraint. The setting places few restrictions on preferences and the distribution of shocks, distinguishes between (observable) aggregate shocks and (unobservable) idiosyncratic shocks, and takes arbitrary general equilibrium prices as given. This makes the results easily applicable to many macro and financial applications.

Chapter 3 explores under what conditions the presence of moral hazard can create a balance sheet amplification channel. If the private action of the agent exposes him to aggregate risk through his unobserved private benefit, the optimal contract will try to over-expose him to aggregate risk to deter him from misbehaving. This creates a tradeoff between aggregate and idiosyncratic risk-sharing. More productive agents naturally want to leverage more and therefore have larger incentives to distort their aggregate risk-sharing in order to reduce their exposure to idiosyncratic risk. In equilibrium, therefore, more productive agents take on a disproportionate fraction of aggregate risk, creating a balance sheet channel.
Acknowledgments

I am deeply indebted to Ivan Werning, who has been an outstanding teacher and mentor. His advice and comments were always profound and insightful; our many conversations have shaped the way I think about economics and research. And more than this, he has been generous with encouragement, guidance, and support at all times. Learning from, and working with him over these past five years has been a true privilege.

I am extremely grateful towards Daron Acemoglu, Guido Lorenzoni, and George-Marios Angeletos, who provided invaluable advice. They always pushed me to explore ideas at a deeper level and focus on the big questions. I have also benefited from discussing my ideas with Rob Townsend, Ricardo Caballero, Veronica Guerrieri, and Leonid Kogan, and from feedback from participants in the MIT Macroeconomics Lunch. I cannot fail to mention my professors at UTDT who encouraged me to embark on a PhD, especially Andrea Rotnitzky, Juan Dubra, Leandro Arozamena, and Hugo Hopenhayn.

I have had the great fortune of sharing this experience with a great group of friends and classmates. I am very grateful to Xiao Yu Wang, Juan Passadore, Marco Di Maggio, Will Mullins, Felipe Severino, Luis Zermeño, Nicolas Perez Truglia, Vladimir Asriyan, and especially to Juan Pablo Xandri and Victoria Vanasco. Their help, support, and friendship were priceless, and I know will continue into the future.

I have always had the love and support of my family. I want to thank my father for his wise words, my mother for her encouragement and always believing in me, and my sister for her friendship. Last, but certainly not least, I would like to thank my wife Meche, who stood by my side every step of the way, and helped make these last five years a wonderful adventure filled with discovery, hard work, and fun. For her unwavering support and love, I am truly grateful.

Cambridge, MA
May 15th, 2013
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Chapter 1

Uncertainty Shocks and Balance Sheet Recessions

1.1 Introduction

The recent financial crisis has underscored the importance of the financial system in the transmission and amplification of aggregate shocks. During normal times, the financial system helps allocate resources to their most productive use, and provides liquidity and risk sharing services to the economy. During crises, however, excessive exposure to aggregate risk by leveraged agents can lead to balance sheet recessions. Small shocks will be amplified when these leveraged agents lose net worth and become less willing or able to hold assets, depressing asset prices and growth. And since it takes time for balance sheets to be rebuilt, transitory shocks can become persistent slumps. While we have a good understanding of why balance sheets matter in an economy with financial frictions, we don’t have a good explanation for why agents are so exposed to aggregate risk in the first place. The answer to this question is important not only for understanding the balance sheet channel, but also for the design of effective financial regulation. In this paper I show that uncertainty shocks can help explain the apparently excessive exposure to aggregate risk that drives balance sheet recessions.

In order to understand agents’ aggregate risk-sharing decisions, I derive financial frictions from a standard moral hazard problem. I allow them to write optimal contracts on all observable variables, and I find that the type of structural shock hitting the economy takes on a prominent role. The first contribution of this paper is to show that in standard models of balance sheet recessions driven by TFP shocks, the balance sheet channel completely disappears when agents are allowed to write

---

1The idea of balance sheet recessions goes back to Fisher (1933) and, more recently, Bernanke and Gertler (1989) and Kiyotaki and Moore (1997). Several papers make the empirical case for balance sheet effects, such as Sraer, Chaney, and Thesmar (2011), Adrian, et al. (2011) and Gabaix, Krishnamurthy, and Vigneron (2007).

2In standard models of balance sheet recessions such as Bernanke and Gertler (1989) and Kiyotaki and Moore (1997), or more recently Brunnermeier and Sannikov (2012), He and Krishnamurthy (2011), or Kiyotaki, Gertler, and Queralto (2011) agents face ad-hoc constraints on their ability to share aggregate risk. Krishnamurthy (2003) and Rampini, Sufi, and Viswanathan (2012) also tackle the question of why agents don’t insure against aggregate risk.
contracts contingent on the observable aggregate state of the economy. Optimal contracts break the link between leverage and aggregate risk sharing, and eliminate the excessive exposure to aggregate risk that drives balance sheet recessions. As a result, balance sheets play no role in the transmission and amplification of aggregate shocks. Furthermore, these contracts are simple to implement using standard financial instruments such as equity and a market index. In fact, the balance sheet channel vanishes as long as agents can trade a simple market index. The intuition behind this result goes beyond the particular environment in this model.

The second contribution is to show that, in contrast to standard TFP shocks, uncertainty shocks can create balance sheet recessions, even when contracts can be written on the aggregate state of the economy. I introduce an aggregate uncertainty shock that increases idiosyncratic risk in the economy, and show it generates an endogenous hedging motive that induces agents to take on aggregate risk. Balance sheets therefore amplify the effects of the initial shock, depressing growth and asset prices. The balance sheet channel in turn amplifies the hedging motive, inducing agents to take even more aggregate risk in a two-way feedback loop. In addition, an increase in idiosyncratic risk leads to an endogenous increase in aggregate risk, and triggers a “flight to quality” event with low interest rates and high risk premia.

I use a continuous-time growth model similar to the Brunnermeier and Sannikov (2012) and He and Krishnamurthy (2011) models of financial crises (BS and HK respectively). I derive financial frictions from a moral hazard problem, and I allow agents write contracts on all observable variables. The continuous-time setting makes the problem tractable and allows for clean solutions. The results of this paper are driven by the interaction between optimal contracts and the general equilibrium. There are two types of agents: experts who can trade and use capital to produce, and consumers who finance them. Experts can continuously trade capital, which is exposed to both aggregate and (expert-specific) idiosyncratic Brownian TFP shocks. They want to raise funds from consumers and share risk with them, but they face a moral hazard problem that imposes an “equity” constraint: experts must keep a fraction of their equity to deter them from diverting funds to a private account. This limits their ability to share idiosyncratic risk, and imposes a cost to leverage. The more capital an expert buys, the more idiosyncratic risk he must carry on his balance sheet. Experts will therefore require a higher excess return on capital when their balance sheets are weak, and thus their balance sheets will affect the economy.

When contracts are constrained and cannot be written on the aggregate state of the economy, experts are mechanically exposed to aggregate risk through the capital they hold, and any aggregate shock that depresses the value of assets will have a large impact on their balance sheets. In contrast, when contracts can be written on the aggregate state of the economy, the decision of how much capital to buy (leverage) is separated from aggregate risk sharing, and optimal contracts hedge the (endogenously) stochastic investment possibility sets provided by the market. In equilibrium,

---

3BS and HK also derive financial frictions from a similar contracting problem, but they impose constraints on the contract space that limit agents ability to share aggregate risk.
aggregate risk sharing is governed by the hedging motive of experts relative to consumers. Brownian TFP shocks don't have a direct effect on investment possibility sets and hence don't generate a hedging motive. Experts and consumers therefore share aggregate risk proportionally to their wealth, so balance sheets don't induce a hedging motive either. In equilibrium, TFP shocks have only a direct impact on output, but are not amplified through balance sheets and do not affect the price of capital, growth rate of the economy or the financial market.

In addition to TFP shocks, the economy is hit by an aggregate uncertainty shock that increases idiosyncratic risk for all experts. In contrast to Brownian TFP shocks, uncertainty shocks create an endogenous hedging motive that induces experts to choose a large exposure to aggregate risk, and underlies the balance sheet amplification channel. The intuition is as follows. Downturns are periods of high uncertainty, with endogenously depressed asset prices and high risk premia. During downturns, experts who invest in these assets and receive the risk premia are relatively better off, in comparison to consumers. On the one hand, this creates a substitution effect: if experts are risk-neutral, they will prefer to have more net worth during downturns in order to get more bang for the buck. This effect works against the balance sheet channel, since it induces experts to insure against aggregate risk. On the other hand, experts require more net worth during booms in order to achieve a given level of utility. This creates a wealth effect: risk averse experts will prefer to have more net worth during booms. I argue the empirically relevant case is the one in which the wealth effect dominates the substitution effect, and drives the balance sheet amplification channel.

Asset prices and risk premia, however, are endogenous and depend, among other things, on experts' balance sheets. After an uncertainty shock, experts' balance sheets will be weak, reducing their willingness to hold capital and further depressing asset prices and growth. This amplifies the hedging motive, inducing experts to take even more aggregate risk ex-ante. Equilibrium is a fixed point of this two-way feedback between aggregate risk sharing and endogenous hedging motives. The continuous-time setup allows me to characterize the equilibrium as the solution to a system of partial differential equations, and analyze the full equilibrium dynamics instead of linearizing around a steady state. It also makes results comparable to the asset pricing literature.

Furthermore, I show that these uncertainty shocks are equivalent to an exogenous shock to the degree of moral hazard (how efficient experts are at stealing capital). this translates directly into a tightening of the financial constraints. I will call these shocks "financial shocks" for short. The intuition for this result is as follows. In an economy without financial frictions idiosyncratic risk shouldn't matter, since it can be aggregated away. Moral hazard, however, forces agents to keep a fraction of the idiosyncratic risk in their capital. It is immaterial to them whether they must keep a constant fraction of more idiosyncratic risk, or a larger fraction of a constant idiosyncratic risk. Although uncertainty shocks and financial shocks are isomorphic within the model, they might follow different stochastic processes. I also solve the model with a stochastic process more appropriate for

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4The wealth effect dominates when the coefficient of relative risk aversion is larger than 1 (agents are more risk averse than log).
financial shocks.

A possible concern with an optimal contracts approach is that they might require very complex and unrealistic financial arrangements. I show this is not the case. Optimal contracts can be implemented in a complete financial market with minimal informational requirements. Experts can be allowed to invest, consume, and manage their portfolios, subject only to an equity constraint. In fact, the TFP-neutrality result does not require the financial market to be complete. It is enough that it spans the aggregate return to capital. A market index of experts’ equity accomplishes precisely this. As long as experts and consumers can trade a market index, the balance sheet channel disappears in the Brownian TFP benchmark.

There are several lessons for financial regulation. First, the results in this paper suggest regulation should focus on experts’ exposure to aggregate risk instead of their leverage. Limiting leverage is costly because it prevents capital from going to the most efficient users, and is related to idiosyncratic risk taking. There may be good reasons to regulate idiosyncratic risk taking by, for example, financial institutions that are considered too big to fail. But inasmuch as we care about the health of the aggregate financial sector, it is useful to keep in mind the distinction between leverage and aggregate risk sharing. Second, experts may actually have good reasons to take on aggregate risk. I show that, while the competitive equilibrium is not constrained efficient, a policy of regulating experts’ exposure to aggregate risk in order to completely eliminate the balance sheet channel is not optimal either. The same uncertainty shocks that help explain agents’ aggregate risk sharing behavior also affect the planner’s problem. Optimal financial regulation must take into account how optimal contracts endogenously respond to policy, not only through direct regulation, but also through the indirect dependence of optimal contracts on the general equilibrium.

Literature Review. This paper fits within the literature on the balance sheet channel going back to the seminal works of Bernanke and Gertler (1989), Kiyotaki and Moore (1997), and Bernanke, Gertler, and Gilchrist (1999). It is most closely related to the more recent Brunnermeier and Sannikov (2012) and He and Krishnamurthy (2011). The main difference with these papers is that I allow agents to write contracts on all observable variables, including the aggregate state of the economy. Krishnamurthy (2003) is the first paper to point out the importance of risk-sharing for the balance sheet channel. He finds that when agents are able to trade state-contingent assets, the amplification mechanism through asset prices in KM disappears. He then shows the feedback from asset prices to experts’ balance sheets reappears when limited commitment on consumers’ side is introduced: if consumers also need collateral to credibly promise to make payments during downturns, they might be constrained in their ability to share aggregate risk with experts. This mechanism is similar to the one in Holmstrom and Tirole (1996). The limited commitment on

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5 Even if we care about the aggregate financial sector, there may be practical reason to focus on leverage, as Haldane (2012), for example, argues.

6 Experts might still be overexposed to aggregate risk in his model because they must chose between sharing aggregate risk and raising funds up front. A similar tradeoff is explored by Rampini, Sufi, and Viswanathan (2012).
the consumers' side is only binding, however, when experts as a whole need fresh cash infusions from consumers. Typically, debt reductions are enough to provide the necessary aggregate risk sharing, and evade consumers' limited commitment (experts' debt can play the role of collateral for consumers). Cooley, Quadrini, and Marimon (2004) limit aggregate insurance with limited contract enforceability. After a positive shock raises entrepreneurs' outside option, their continuation utility must also go up to keep them from walking away. This relaxes the contractual problem going forward and propagates even transitory aggregate shocks. The effect, however, is asymmetric: negative shocks to entrepreneurs' outside option are not amplified in the same way as positive ones. In contrast to these papers, I don't constrain agents' ability to share aggregate risk. The continuous-time setting allows agents to leverage and share aggregate risk freely as long as they are not up against their solvency constraint.

Kiyotaki, Gertler, and Queralto (2011) also tackle the question of why banks' balance sheets are so highly exposed to aggregate risk, and focus on the debt vs. equity tradeoff for banks, while Adrian and Boyarchenko (2012) build a model of financial crises where experts use long-term debt and face a time-varying leverage constraint. Here, instead, I don't impose an asset structure on agents. Geanakoplos (2009) emphasizes the role of heterogeneous beliefs. More optimistic agents place a higher value on assets and are naturally more exposed to aggregate risk. The balance sheet channel in my model, in contrast, does not rely on heterogenous beliefs. Experts take on more aggregate risk in order to take advantage of endogenous investment possibility sets. Myerson (2012), on the other hand, builds a model of credit cycles allowing long-term contracts with a similar moral hazard problem to the one in this paper. In his model the interaction of different generations of bankers can generate endogenous credit cycles, even without aggregate shocks. Shleifer and Vishny (1992) and Diamond and Rajan (2011) look at the liquidation value of assets during fire sales, and Brunnermeier and Pedersen (2009) focus on the endogenous determination of margin constraints.

Several papers make the empirical case for the balance sheet amplification channel. Sraer, Chaney, and Thesmar (2011), for example, use local variation in real estate prices to identify the impact of firm collateral on investment. They find each extra dollar of collateral increases investment by $0.06. Gabaix, Krishnamurthy, and Vigneron (2007) provide evidence for balance sheet effects in asset pricing. They show that the marginal investor in mortgage-backed securities is a specialized intermediary, instead of a diversified representative agent. Adrian, Etula, and Muir (2011) use shocks to the leverage of securities broker-dealers to construct an “intermediary SDF” and use it to explain asset returns. At the same time, there has been a lot of recent interest in the role of uncertainty shocks in business cycles. Bloom (2009) and, more recently, Bloom, Floetotto, Jaimovich, Saporta-Eksten, and Terry (2012) build a model where higher volatility leads to the postponement of investment and hiring decisions, and show that uncertainty shocks can do a good job matching the empirical time series. More related to the model in this paper, Christiano, Motto, 

\footnote{In Adrian and Boyarchenko (2012) experts are also risk-neutral and trade with risk-averse consumers.}

\footnote{On the other hand, Bachmann and Moscarini (2011) argue that causation may run in the opposite direction,}
and Rostagno (2012) introduce shocks to idiosyncratic risk in a model with financial frictions and incomplete contracts. They fit the model to U.S. data and find this uncertainty shock to be the most important factor driving business cycles\(^9\). Angeletos (2006) studies the effects of uninsurable idiosyncratic capital risk on aggregate savings. In the asset pricing literature, Campbell, Giglio, Polk, and Turley (2012) introduce a volatility factor into an ICAPM asset pricing model. They find this volatility factor can help explain the growth-value spread in expected returns. Bansal and Yaron (2004) study aggregate shocks to the growth rate and volatility of the economy, and Bansal, Kiku, Shaliastovich, and Yaron (2012) study a dynamic asset pricing model with cash flow, discount rate and volatility shocks. Idiosyncratic risk, in particular, is studied by Campbell, Lettau, Malkiel, and Xu (2001). Eggertsson and Krugman (2010), Guerrieri and Lorenzoni (2011) and Buera and Moll (2012) also consider exogenous shocks to financial frictions.

I embed optimal contracts in a general equilibrium setup. Sannikov (2008) studies optimal dynamic contracts in a continuous-time setting. With a purely pecuniary private benefit, however, if agents could commit to long-term contracts that control their consumption, the moral hazard problem would vanish. DeMarzo and Sannikov (2006) make the agent's consumption unobservable and obtain an “equity” constraint for the linear preferences case. Instead, I introduce re-contracting. Brunnermeier and Sannikov (2012) consider a similar environment and extend it to consider intermediaries and Poison processes. He and Krishnamurthy (2011) use a discrete-time limit and obtain the same optimal contract. Both rule out contracts on the aggregate state of the economy, however. Strulovici (2011) also considers renegotiation-proof contracts in a continuous-time setting. DeMarzo, He, Fishman, and Wang (2012) show how optimal contracts can punish agents for outcomes for which they are not responsible. As in this paper, moral hazard itself does not limit aggregate risk sharing, but introduces a “hedging motive” that wouldn't exist in the absence of moral hazard.

The intuition behind experts' exposure to uncertainty shocks is related to ICAPM models going back to Merton (1973). Even though moral hazard does not restrict agents' ability to share aggregate risk, it makes investment possibility sets endogenously stochastic and introduces a relative hedging motive for aggregate risk sharing that wouldn't be present without moral hazard. In contrast to ICAPM models, here there's a feedback from agents' risk sharing back into investment possibility sets, through their balance sheets.

**Layout.** The rest of the paper is organized as follows. In Section 1.2 I introduce the setup of the model and the contractual environment. In Section 1.3 I characterize the equilibrium using a recursive formulation, and study the different effects of TFP and uncertainty shocks. I also solve the model with financial shocks. Section 1.4 looks at financial regulation. Section 1.5 explores several

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1.2 The model

The model purposefully builds on Brunnermeier and Sannikov (2012) and He and Krishnamurthy (2011), adding idiosyncratic risk and general EZ preferences to their framework. As in those papers, I derive financial frictions endogenously from a moral hazard problem. In contrast to those papers, however, contracts can be written on all observable variables.

1.2.1 Setup

**Technology.** Consider an economy populated by two types of agents: “experts” and “consumers”, identical in every respect except that experts are able to use capital\(^1\). There are two goods, consumption and capital. Denote by \(k_t\) the aggregate “efficiency units” of capital in the economy, and by \(k_{i,t}\) the individual holdings of an expert \(i \in [0, 1]\), where \(t \in [0, \infty)\) is time. An expert can use capital to produce a flow of consumption goods

\[
y_{i,t} = (a - \iota (g_{i,t})) k_{i,t}
\]

The function \(\iota\) with \(\iota' > 0, \iota'' > 0\) represents a standard investment technology with adjustment costs: in order to achieve a growth rate \(g\) for his capital stock, the expert must invest a flow of \(\iota (g)\) consumption goods. The capital he holds evolves\(^1\)

\[
\frac{dk_{i,t}}{k_{i,t}} = g_{i,t} dt + \sigma dZ_t + \nu_t dW_{i,t}
\]

where \(Z = \{Z_t \in \mathbb{R}^d; \mathcal{F}_t, t \geq 0\}\) is an aggregate brownian motion, and \(W_i = \{W_{i,t}; \mathcal{F}_t, t \geq 0\}\) an idiosyncratic brownian motion for expert \(i\), in a probability space \((\Omega, P, \mathcal{F})\) equipped with a filtration \(\{\mathcal{F}_t\}\) with the usual conditions. The aggregate shock can be multidimensional, \(d \geq 1\), so the economy could be hit by many aggregate shocks. For most results, however, there is no loss from taking \(d = 1\) and focusing on a single aggregate shock\(^2\). While the exposure of capital to aggregate risk \(\sigma \geq 0\) is constant\(^3\), its exposure to idiosyncratic risk \(\nu_t > 0\) follows an exogenous stochastic

\(^1\)We could allow consumers to use capital less productively, as in Brunnermeier and Sannikov (2012) or Kiyotaki and Moore (1997). I explore this setup in section 1.5.

\(^2\)This formulation where capital is exposed to aggregate risk is equivalent to a standard growth model where TFP \(a_t\) follows a Brownian Motion. Then if \(\kappa_t\) is physical capital, \(k_t = a_t \kappa_t\) is “effective capital”. To preserve scale invariance we must also have investment costs proportional to effective capital \(k_t\), which makes sense if we think investment requires diverting capital from consumption to investment (or in a richer model with labor).

\(^3\)This is in fact the approach I take when computing numerical solutions.
process

\[ d\nu_t = \lambda (\bar{\nu} - \nu_t) dt + \sigma_\nu \sqrt{\nu_t} dZ_t \]

(1.2)

where \( \bar{\nu} \) is the long-run mean and \( \lambda \) the mean reversion parameter. The loading of the idiosyncratic volatility of capital on aggregate risk \( \sigma_\nu \leq 0 \) so that we may think of \( Z \) as a "good" aggregate shock that increases the effective capital stock and reduces idiosyncratic risk. This is just a naming convention. With multiple aggregate shocks, \( d > 1 \), we may take some shocks to be pure TFP shocks with \( \sigma^{(i)}_\nu = 0 \), other pure uncertainty shocks with \( \sigma^{(i)} = 0 \), and yet other mixed shocks.

Preferences. Both experts and consumers have Epstein-Zin preferences with the same discount rate \( \rho \), risk aversion \( \gamma \) and elasticity of intertemporal substitution (EIS) \( \psi^{-1} \). If we let \( \gamma = \psi \) we get the standard CRRA utility case as a special case.

\[ U_t = \mathbb{E}_t \left[ \int_t^\infty e^{-\rho u} \frac{c_u^{1-\gamma}}{1-\gamma} du \right] \]

Epstein-Zin preferences separate risk-aversion from the EIS, which play different roles in the balance sheet amplification channel. They are defined recursively (see Duffie and Epstein (1992)):

\[ U_t = \mathbb{E}_t \left[ \int_t^\infty f (c_u, U_u) du \right] \]

(1.3)

where

\[ f (c, U) = \frac{1}{1-\psi} \left\{ \frac{\rho c^{1-\psi}}{[(1-\gamma) U]^{1-\psi}} - \rho (1-\gamma) U \right\} \]

I will later also introduce turnover among experts in order to obtain a non-degenerate stationary distribution for the economy. Experts will retire with independent Poisson arrival rate \( \tau \) and become consumers. There is no loss in intuition from taking \( \tau = 0 \) for most of the results, however.

Markets. Experts can trade capital continuously at a competitive price \( p_t > 0 \), which we conjecture follows an Ito process:

\[ \frac{dp_t}{p_t} = \mu_{p,t} dt + \sigma_{p,t} dZ_t \]

The price of capital depends on the aggregate shock \( Z \) but not on the idiosyncratic shocks \( \{W_i\}_{i \in [0,1]} \), and it's determined endogenously in equilibrium. The total value of the aggregate capital stock is \( p_t k_t \) and it constitutes the total wealth of the economy, since this is the only real asset.

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\(^{14}\)If \( 2\lambda \bar{\nu} \geq \sigma_\nu^2 \), this Cox-Ingersoll-Ross process is always strictly positive and has a long-run distribution with mean \( \bar{\nu} \).
There is also a complete financial market\(^\dagger\) with SDF \(\eta_t\):

\[
\frac{d\eta_t}{\eta_t} = -r_t dt - \pi_t dZ_t
\]

Here \(r_t\) is the risk-free interest rate and \(\pi_t\) the price of aggregate risk \(Z\). Both are determined endogenously in equilibrium. I am already using the fact that idiosyncratic risks \(\{W_t\}_i\in[0,1]\) have price zero in equilibrium because they can be aggregated away.

**Consumers' Problem.** Consumers face a standard portfolio problem. They cannot hold capital but they have access to a complete financial market. They start with wealth \(w_0\) derived from ownership of a fraction of aggregate capital (which they immediately sell to experts). Taking the aggregate process \(\eta\) as given, they solve the following problem.

**Definition 1.** Consumers' problem:

\[
U_0 = \max_{(c_t, \sigma_t)} \mathbb{E} \left[ \int_0^\infty f(c_t, U_t) \, dt \right]
\]

\[
st : \quad \frac{dw_t}{w_t} = (r_t + \sigma_{w,t} \pi_t - \hat{\pi}_t) \, dt + \sigma_{w,t} dZ_t
\]

where \(U_t\) is defined recursively as in 1.3, and a solvency constraint \(w_t \geq 0\).

I use \(w\) for the wealth of consumers, and reserve \(n\) for experts', which I will call “net worth”. Consumers get the risk-free interest rate on their wealth, plus a premium \(\pi_t\) for the exposure to aggregate risk \(\sigma_{w,t}\) they chose to take. The hat on \(\hat{\cdot}\) denotes the variable is normalized by wealth. Since the price of expert-specific idiosyncratic risks \(\{W_t\}\) is zero in equilibrium, consumers will never buy idiosyncratic risk. This is already baked into consumers' dynamic budget constraint.

### 1.2.2 Contracting Environment

Experts face a more complex problem. Similarly to consumers, experts can participate in the financial market. In addition, they can continuously trade and use capital for production. However, they face a moral hazard problem that creates a financial friction. In the interest of brevity, here I provide an overview of the contractual environment and the main characterization results. Appendix A develops the contracting environment in more detail. Throughout this section, the general equilibrium prices \(p\) and \(\eta\) are taken as given. To simplify notation, I suppress the reference \(i\) to the expert.

Each expert has a bank balance \(b_t\), with initial net worth \(n_0 > 0\). He wants to raise funds to buy capital and share risk, but he faces a moral hazard problem. He can divert capital to a private

\(^\dagger\) A complete financial market could be implemented with different asset structures. For example, a natural asset structure would include risk-free debt, equity in each expert’s investments and \(d\) market indices to span \(Z\). If \(d = 1\) we can do with only one market index.
account at a rate \( s = \{s_t; t \geq 0\} \) and obtain \( \phi \in (0, 1) \) units of capital per unit diverted, so stealing is inefficient and we always want to implement no stealing \( s = 0 \). His cumulative hidden savings \( S_t \) then follow\(^{16}\)

\[
dS_t = \phi p_t k_t s_t dt + S_t r_t dt.
\]

The parameter \( \phi \) captures the severity of the moral hazard: the more efficient stealing is for the expert, the tighter the moral hazard problem becomes. With \( \phi = 0 \) moral hazard disappears. The observable cumulative return from investing a dollar in capital is \( R^k = \{R^k_t; t \geq 0\} \), with

\[
dR^k_t = \left( g_t + \mu_{p,t} + \sigma_{p,t} \frac{a - \iota(g_t)}{p_t} - s_t \right) dt + (\sigma_{p,t}) dZ_t + \nu_t dW_t
\]

The expert expects to gain from growth in the value of his capital\(^{17}\) plus the output flow \( a - \iota(g_t) \). When he steals \( s_t \) he reduces the observable return of his investment. Of course, he gains \( \phi s_t dt \) in hidden savings per dollar invested in capital. His observable return is exposed to aggregate risk both exogenously from his capital \( \sigma \) and endogenously from its market price \( \sigma_{p,t} \), and to his own idiosyncratic risk \( W_t \) only through capital, with loading \( \nu_t \).

He designs a contract \((e, g, k, F)\) that specifies his consumption \( e_t \geq 0 \), investment \( g_t \), capital \( k_t \geq 0 \), and a cumulative cash flow\(^{18}\) \( F_t \) he will sell on the financial market, all adapted to the filtration \( \{\mathcal{H}_t\} \) generated by the observable variables\(^{19}\) \( Z \) and \( \{R^k_t\}_{t \in [0, T]} \). For now, let \((e, g, k, F)\) be all contractible, as well as \( b \). I will later relax this assumption. Under this contract the bank balance of the expert evolves:

\[
db_t = r_t b_t dt + p_t k_t \left( dR^k_t - r_t dt \right) - dF_t - e_t dt
\]

The expert is interested in maximizing his expected utility \( U_0 \), given by (1.3), and the market prices the cash flow \( F \). At time \( t \), the continuation of the contract has present value

\[
J_t = \mathbb{E}_t^Q \left[ \int_t^\infty \frac{B_{0,u}}{B_{0,t}} dF_u \right]
\]

where \( B_{0,t} = \exp \left( \int_0^t r_u du \right) \) is the value of risk free bond, the expectation is taken under the equivalent martingale measure \( Q \), and under no stealing, \( s = 0 \). The contract can only be sold if \( J_0 \geq 0 \). To make the problem well defined, we impose a solvency constraint. The expert must have

\(^{16}\) Hidden savings are invested in the risk-free asset.

\(^{17}\) The term \( \sigma_{p,t} \) captures the covariance between the expert's capital stock and the market price of capital.

\(^{18}\) \( F \) is an \( \mathcal{H}_t \)-adapted semimartingale, and is the most general process the market can price.

\(^{19}\) The general equilibrium processes \( p \) and \( \eta \) are also adapted to \( \mathcal{H}_t \) because they only depend on the aggregate shocks \( Z \). Also, even though the contract could in principle depend on other experts' returns, it is wlog to write it only on each expert's own observable return.

\(^{20}\) There is a one-to-one relationship between the state price density \( \eta_t \) and the equivalent martingale measure \( Q \). See Karatzas and Shreve (1998), pages 17-19.
in his bank account enough funds to cover the continuation value of his contract\textsuperscript{21}: \( b_t - J_t \geq 0 \). Define \( n_t = b_t - J_t + S_t \) as the expert's "net worth". In equilibrium, where no stealing occurs and \( S_t = 0 \), we have \( n_t = b_t - J_t \).

**Re-contracting.** As in Brunnermeier and Sannikov (2012) and He and Krishnamurthy (2011), I introduce re-contracting\textsuperscript{22}. At any time \( t \), the expert is committed to a single contract. However, he cannot commit to long term contracts. The expert can at any time settle his obligations and roll-over the contract. The expert cannot "run away" without paying, as in Kiyotaki and Moore (1997), but he can offer a new contract \((c', g', k', F')\) with present value \( J'_t \geq J_t \), using his hidden savings \( S_t \). Re-contracting allows the expert to use diverted funds \( S_t \) optimally, not only for hidden consumption but also for investment and risk sharing, and leads to a tractable "equity constraint" implementation.

**Characterization.** We can characterize the optimal contract with re-contracting using the solution to a constrained portfolio problem, in the following sense. If we solve the constrained portfolio problem, we can use the solution to build an optimal contract.

**Proposition 2.** The optimal contract with moral hazard and re-contracting can be characterized with the constrained portfolio problem of Definition 3.

**Definition 3.** The expert's problem:

\[
V_0 = \max_{(e,g,k,\theta,\phi) \geq \phi} E_{s=0} \left[ \int_0^\infty f(e_t, V_t) \, dt \right]
\]

where \( V_t \) is defined recursively as in 1.3, subject to the dynamic budget constraint for his net worth \( n_t \)

\[
\frac{dn_t}{n_t} = [\mu_{n,t} - \dot{e}_t] \, dt + \sigma_{n,t} dZ_t + \phi_{n,t} dW_t
\]  \hfill (1.5)

\[
\begin{align*}
\mu_{n,t} &= r_t + pt \hat{k}_t \left( E_{s=0} [dR_t^k] - r_t \right) - \left( 1 - \phi_t \right) pt \hat{k}_t (\sigma + \sigma_{p,t}) \pi_t + \theta_t \pi_t \\
\sigma_{n,t} &= \phi_t pt \hat{k}_t (\sigma + \sigma_{p,t}) + \theta_t \\
\phi_{n,t} &= \phi_t pt \hat{k}_t \nu_t
\end{align*}
\]  \hfill (1.6)

and a solvency constraint \( n_t \geq 0 \).

\textsuperscript{21}A lower bound for \( b_t - J_t \) is required to make the problem well defined.

\textsuperscript{22}With commitment to long-term contracts, it is possible to implement the first best contract without moral hazard. An alternative to re-contracting is to make consumption non-contractible, as in DeMarzo and Sannikov (2006) or Myerson (2012).
The hat denotes the variable is divided by \( n_t \), i.e. \( \hat{k}_t = \frac{k_t}{n_t} \). The expert chooses his consumption \( \hat{c} \), investment \( g \), capital holdings \( \hat{k} \), and a free process \( \theta_t \) which plays an important role and will be explained below. The expert’s net worth \( n_t \) has an expected growth rate \( \mu_{n,t} \) before consumption, and is exposed to both aggregate risk through \( \sigma_{n,t} \), and idiosyncratic risk through \( \tilde{\sigma}_{n,t} \). Due to re-contracting, the expert’s continuation utility depends on his net worth \( n_t \). To provide incentives for the expert to not steal, his net worth must be exposed to his observable return \( R^k_t \) with a loading \( \hat{\phi}_t \) of at least \( \phi \). That is, he must have “skin in the game”. The intuition for this local IC constraint is that if the expert steals one dollar, he gets \( \phi \) dollars into his private account, but loses \( \hat{\phi}_t \geq \phi \) dollars because his net worth is exposed to his observable return \( R^k_t \), which diminishes when he steals. This constraint is always binding, so we may take \( \hat{\phi}_t = \phi \).

The expression for \( \tilde{\sigma}_{n,t} \) says that the expert’s net worth is exposed to his own idiosyncratic risk \( W \) through the fraction \( \hat{\phi}_t \geq \phi \) of his return \( p_t \hat{k}_t dR^k_t \) he keeps, which has a loading of \( \nu_t \) on \( W \). The expert’s investment in capital also exposes him to aggregate risk, \( \hat{\phi}_t p_t \hat{k}_t (\sigma + \sigma_{p,t}) \). However, he gets an additional free process \( \{ \theta_t \in \mathbb{R}^d, t \geq 0 \} \) which allows him to separate his leverage \( p_t \hat{k}_t \) from his aggregate risk sharing \( \sigma_{n,t} \):

\[
\sigma_{n,t} = \hat{\phi}_t p_t \hat{k}_t (\sigma + \sigma_{p,t}) + \theta_t
\]

We can interpret \( \theta_t \) as the portfolio investment in a set of market indices with gain \( M_t \in \mathbb{R}^d \), normalized for simplicity with identity loading on the aggregate shock \( Z \):

\[
\frac{dM_t^{(j)}}{M_t^{(j)}} = \left( r_t + \pi_t^{(j)} \right) dt + dz_t^{(j)} \quad j = 1 \ldots d
\]

The equation for \( \mu_{n,t} \) says he gets a risk free return \( r_t \) on his net worth plus the excess return of capital for his investment \( p_t \hat{k}_t \). In addition, since he is keeping exposure \( \hat{\phi}_t \) to his own observable return, he is offloading an exposure \( 1 - \hat{\phi}_t \) onto the market. His observable return is exposed to both aggregate and idiosyncratic risk. The market doesn’t price idiosyncratic risk, but it does demand a premium for the aggregate risk, so the expert must pay \( \left( 1 - \hat{\phi}_t \right) p_t \hat{k}_t (\sigma + \sigma_{p,t}) \pi_t \). Finally, because he also takes on aggregate risk through \( \theta_t \), he gets a premium \( \theta_t \pi_t \).

**Implementation.** The characterization as a constrained portfolio problem suggests we can actually implement the optimal contract by allowing the expert to chose his own consumption \( \hat{c} \), investment \( g \), capital holdings \( \hat{k} \), and portfolio investments \( \theta \), subject only to an “equity constraint” that guarantees he does not find it optimal to steal. Under this implementation, the expert keeps a fraction \( \phi \) of the equity of a firm that holds the capital. The rest is held by outside investors. The expert also borrows using risk-free debt and trades a market index that allows him to make the liabilities of his firm contingent on the aggregate state of the economy. This implementation is not
the only possibility, of course, and depends on the available instruments in the financial market.\footnote{Notice that the expert does not necessarily invest all of his net worth into the equity of the firm. His net worth could be more valuable than a fraction $\phi$ of the equity of his firm.}.

Proposition 4. The optimal contract can be implemented as a portfolio problem in a complete financial market, allowing the expert to freely trade capital, risk-free debt, market indices, and his own equity subject to an equity constraint.

![Balance sheet diagram]

The informational requirements of this implementation are very low. The expert’s net worth must be minimally monitored to enforce the solvency constraint $n_t \geq 0$, but not more so than in any financial market. The equity constraint requires observing the expert’s portfolio, but only regarding the equity of his own project. It is not necessary to monitor the expert’s consumption or investment choices, or his portfolio choices beyond his own equity.

Constrained contracts. The free process $\theta$ plays a crucial role, allowing the expert to separate leverage $p_t \hat{k}_t$ from aggregate risk sharing. Brunnermeier and Sannikov (2012) and He and Krishnamurthy (2011) consider a similar contracting environment. Both, however, impose constraints on contracts that prevent experts from sharing aggregate risk, forcing $\theta_t = 0$ in (1.6). With this constraint, leverage and aggregate risk sharing become entangled. If the expert wants to buy capital he needs to accept a high exposure to aggregate risk.

$$\sigma_{n,t} = \hat{\phi}_t p_t \hat{k}_t (\sigma + \sigma_{p,t}) + \underbrace{\theta_t}_{=0}$$

In contrast, I allow agents to write contracts on all observable variables, including aggregate shocks $Z$ hitting the economy. The optimal contract then uses $\theta_t$ to separate the decision of how much to leverage $\phi p_t \hat{k}_t$, from the decision of how much aggregate risk to carry $\sigma_{n,t}$. Intuitively,
since the expert’s private action does not interact with aggregate risk, there is no need to restrict aggregate risk sharing for incentive provision purposes. Moral hazard limits experts’ ability to share idiosyncratic risk only.\textsuperscript{24}

In general, we may consider contracts that can be implemented in an incomplete financial market that spans a linear subspace of aggregate shocks $\tilde{Z}_t = BZ_t$ where the matrix $B \in \mathbb{R}^{d'_t \times d}$ has full rank lower than the number of aggregate shocks (i.e. $d'_t < d$) in addition to the experts’ observable returns $\{r_{i,t}^k\}_{i \in \{0,1\}}$. For example, instead of the full set of $d$ orthogonal normalized market indices $M^{(d)}$, we only have a linear combination $BM$. We still solve the same constrained portfolio problem 3, with the added constraint\textsuperscript{25} that $\theta_t = \tilde{\theta}_t B$ for some $\tilde{\theta} \in \mathbb{R}^{d'_t}$: the portfolio investment in the market indices must be in the row space of $B$. Now, however, the “skin in the game” constraint $\tilde{\phi}_t \geq \phi$ might not be always binding\textsuperscript{26}

1.2.3 Equilibrium

Denote the set of experts $I = [0, 1]$ and the set of consumers $J = \{1, 2\}$. We take the initial capital stock $K_0$ and its distribution among agents $\{k^i_0\}_{i \in I}$, $\{k^0_j\}_{j \in J}$ as given, with $\int_I k^i_0 di + \int_j k^0_j dj = k_0$. Let $k_{i,0} > 0$ and $k_{j,0} > 0$ so that all agents start with strictly positive net worth.

Definition 5. An equilibrium is a set of aggregate stochastic processes adapted to the filtration generated by $Z$: the price of capital $\{p_t\}$, the state price density $\{\eta_t\}$, and the aggregate capital stock $\{k_t\}$, and a set of stochastic processes for each expert $i \in I$ and each consumer $j \in J$ (each adapted to their information\textsuperscript{27}): net worth and wealth $\{n_{i,t}, w_{i,t}\}$, consumption $\{e_{i,t} \geq 0, c_{j,t} \geq 0\}$, capital holdings $\{k_{i,t}\}$, investment $\{g_{i,t}\}$, and aggregate risk sharing, $\{\sigma_{i,n,t}, \sigma_{j,w,t}\}$, such that:

1. Initial net worth satisfies $n^i_0 = p^0_0 k^i_0$ and wealth $w^i_0 = p^0_0 k^i_0$.

2. Each expert and consumer solves his problem taking aggregate conditions as given.

3. Market Clearing:

   \[
   \int_I e_{i,t} di + \int_j c_{i,t} dj + \int_I \ell (g_{i,t}) k_{i,t} di = \int_I ak_{i,t} di
   \]

   \[
   \int_I k_{i,t} di = k_t
   \]

   \[
   \int_I \sigma_{i,n,t} n_{i,t} di + \int_j \sigma_{j,w,t} w_{j,t} dj = \int_I p_t k_{i,t} (\sigma + \sigma_{p,t}) di
   \]

\textsuperscript{24}This doesn’t mean optimal contracts will not distort aggregate risk sharing. See DeMarzo, He, Fishman, and Wang (2012) for a model where the contract punishes the agent for aggregate outcomes out of his control.

\textsuperscript{25}The setup in Brunnermeier and Sannikov (2012) or He and Krishnamurthy (2011) has $B = 0$.

\textsuperscript{26}For example, keeping exposure to his own return might be the only way for the expert to get exposure to aggregate risk, even if it comes at the cost of some idiosyncratic risk.

\textsuperscript{27}The filtration generated by $Z$ and $W_t$ in the case of experts and only $Z$ in the case of consumers
4. Law of motion of aggregate capital:

\[ dk_t = \left( \int g_{t,t}k_{t,t}di \right) dt + k_t \sigma_dZ_t \]

The market clearing conditions for the consumer goods and capital market are standard. The condition for market clearing in the financial market is derived as follows. We already know each expert keeps a fraction \( \phi \) of his own equity. If we aggregate the equity sold on the market into indices with identity loading on \( Z \), there is a total supply of these indices \( (1 - \phi) p_t k_t (\sigma + \sigma_{t,p}) \). Consumers absorb \( \int_j \sigma_{i,m,t}w_{j,t}dj \) and experts \( \int_j \theta_{t,i}n_{i,t}di \) of these indices. Rearranging we obtain the expression above. By Walras' law, the market for risk-free debt clears automatically.

1.2.4 Benchmark without moral hazard

Without any financial frictions this is a standard AK growth model where balance sheets don't play any role. Experts share all of their idiosyncratic risk, so the volatility of idiosyncratic shocks \( \nu_t \) is irrelevant and the economy settles on a stationary growth path. The price of capital and the growth rate of the economy do not depend on experts' net worth: balance sheets are only relevant to determine consumption of experts and consumers.

**Proposition 6** (No moral hazard benchmark). If \( \rho - (1 - \psi)g^* + (1 - \psi) \frac{3}{2} \sigma^2 > 0 \) and without any financial frictions, there is a stationary growth equilibrium, where the price of capital is \( p^* \) and the growth rate \( g^* \), given by:

\[ \nu'(g^*) = p^* \]  

\[ p^* = \frac{a - \nu(g^*)}{\rho - (1 - \psi)g^* + (1 - \psi) \frac{3}{2} \sigma^2} \]

1.3 Solving for the equilibrium

Experts and consumers face a dynamic problem, where their optimal decisions depend on the stochastic investment possibility sets they face, captured by the price of capital \( p \) and the SDF \( \eta \). The equilibrium is driven by the exogenous stochastic process for \( \nu_t \) and by the endogenous distribution of wealth between experts and consumers. The recursive EZ preferences generate optimal strategies which are linear in net worth, and allow us to simplify the state-space: we only need to keep track of the net worth of experts relative to the total value of assets which they must hold in equilibrium, \( x_t = \frac{\text{net worth}}{\text{total value}} \). The distribution of net worth across experts, and of wealth across consumers, is not important. The strategy is to use a recursive formulation of the problem and look for a Markov equilibrium in \( (\nu_t, x_t) \), taking advantage of the scale invariance property of the economy which allows us to abstract from the level of the capital stock. The layout of this section is as follows. First I recast the equilibrium in recursive form and characterize agents' optimal plans.
I then study the effects of TFP shocks under different contractual environments. Finally, I solve the full model with uncertainty shocks and a balance sheet channel.

1.3.1 Recursive Formulation

First, conjecture that the value function for an expert with net worth $n$ takes the following power form:

$$ V(t, n) = \frac{(\xi_t n)^{1-\gamma}}{1-\gamma} $$

I call process $\xi = \{\xi_t > 0; t \geq 0\}$ the “net worth multiplier”. It captures the stochastic, general equilibrium investment possibility set the expert faces (i.e.: does not depend on his own net worth $n_t$). When $\xi_t$ is high the expert is able to obtain a large amount of utility from a given amount of net worth, as if his actual net worth was $\xi_t n_t$. Conjecture that it follows an Itô process

$$ \frac{d\xi_t}{\xi_t} = \mu_{\xi,t} dt + \sigma_{\xi,t} dZ_t $$

where $\mu_{\xi,t}$ and $\sigma_{\xi,t}$ must be determined in equilibrium. For consumers, the utility function takes the same form, but instead of $\xi_t$, we have $\zeta_t$ as the “wealth multiplier”, which follows $\frac{d\zeta_t}{\zeta_t} = \mu_{\zeta,t} dt + \sigma_{\zeta,t} dZ_t$, also determined in equilibrium.

I use a dynamic programming approach to solve agents’ problem. For experts, we have the Hamilton-Jacobi-Bellman equation after some algebra:

$$ \rho = \max_{\hat{e}, \hat{g}, \hat{k}, \theta} \left\{ \frac{\hat{e}^{1-\psi}}{1-\psi} \rho \xi_t^{\psi-1} + \mu_n - \hat{e} + \mu_\xi - \frac{\gamma}{2} (\sigma_n^2 + \sigma_\xi^2 - 2(1-\gamma)\sigma_n \sigma_\xi + \sigma_\xi^2) \right\} $$

subject to the dynamic budget constraint (1.6), and a transversality condition. Consumers have an analogous HJB equation.

Proposition 7. [Linearity] Growth is determined by a static FOC

$$ \tau'(g_t) = p_t $$

In addition, since net worth $n$ (or wealth $w$ in consumers’ case) drops out of the HJB, all experts chose the same $\hat{e}_t$, $g_t$, $\hat{k}_t$ and $\theta_t$, and all consumers the same $\zeta_t$ and $\sigma_{w,t}$.

Proposition 7 tells us two important things. The first is that the growth rate of the economy is linked to asset prices in a straightforward way. Anything that depresses asset prices will have a real effect on the growth rate of the economy. For example, with a quadratic adjustment cost function $\tau(g) = \Lambda g^2$, the growth rate of the economy is simply $g_t = \frac{p_t}{\Lambda}$. Proposition 7 also tells us that policy functions are linear in net worth. This is a useful property of EZ preferences and allows us to abstract from the distribution of wealth across experts and across consumers, and simplifies the state space of the equilibrium. We only need to keep track of
the fraction of aggregate wealth that belongs to experts: \( x_t = \frac{\eta_t}{p_t k_t} \in [0, 1] \). I look for a Markov equilibrium with two state variables: the volatility of idiosyncratic shocks \( \nu_t \), and \( x_t \):

\[
p_t = p(\nu_t, x_t), \quad \xi_t = \xi(\nu_t, x_t), \quad \zeta_t = \zeta(\nu_t, x_t), \quad r_t = r(\nu_t, x_t), \quad \pi_t = \pi(\nu_t, x_t)
\]

where \( p, \xi \) and \( \zeta \) are conjectured to be twice continuously differentiable. The first state variable \( \nu_t \) evolves exogenously according to (1.2). The state variable \( x_t \) is endogenous, and has an interpretation in terms of experts’ balance sheets. Since experts must hold all the capital in the economy, the denominator captures assets on experts’ balance sheets. The numerator is the net worth of experts. I will sometimes abuse notation and refer to \( x_t \) as “experts’ balance sheets”.

We know from Proposition 6 that without moral hazard, experts would be able to offload all of their idiosyncratic risk onto the market and hence neither \( \nu_t \) nor \( x_t \) would play any role in equilibrium. In contrast, in an economy with financial frictions, \( \phi > 0 \), experts’ balance sheets will play an important role. We say balance sheets matter if equilibrium objects depend on \( x \). In order for balance sheets to play a role in the transmission and amplification of aggregate shocks, we also need them to be exposed to aggregate shocks. In principle, \( x \) could be exposed to aggregate risk \( Z \) through its volatility term \( \sigma_{x,t} \), or through a stochastic drift \( \mu_{x,t} \). In practice, what we usually mean when we talk about a balance sheet channel is that experts’ balance sheets are disproportionally hit by aggregate shocks, so we want to focus on \( \sigma_{x,t} > 0 \). We say there is a balance sheet amplification channel if balance sheets matter and, in addition, \( \sigma_{x,t} > 0 \). For example, the total wealth in the economy is \( p_t k_t \), and it has an exposure to aggregate risk \( Z \) given by

\[
\text{vol} (p_t k_t) = \underbrace{\sigma p k_t}_{\text{TFP}} + \underbrace{p v_t \nu_t}_{\text{id. risk}} + \underbrace{p x_t \sigma_{x,t}}_{\text{B.S. channel}}
\]

So for a balance sheet channel we need both 1) \( p_x \neq 0 \) and 2) \( \sigma_{x,t} > 0 \). The first condition says balance sheets matter for the price of capital. The second conditions says balance sheets are exposed to aggregate risk. Together they create a balance sheet channel. We can now give a definition for a recursive equilibrium.

**Definition 8.** A Recursive Markov Equilibrium is a set of aggregate functions \((p, \xi, \zeta, r, \pi)(\nu, x)\) and policy functions \((\hat{e}, g, \hat{k}, \theta)(\nu, x)\) for experts and \((\hat{e}, \sigma_{w,t})(\nu, x)\) for consumers, and a law of motion for the endogenous aggregate state variable \( dx_t = \mu_x (\nu, x) dt + \sigma_x (\nu, x) dZ_t \) such that:

1. \( \xi \) and \( \zeta \) solve the experts’ and consumers’ HJB equations (1.9), and \((\hat{e}, g, \hat{k}, \theta)\) and \((\hat{e}, \sigma_{w,t})\) are the corresponding policy functions, taking \((p, r, \pi)\) and the laws of motion of \( \nu_t \) and \( x_t \) as given.

2. Market clearing:

\[
\dot{p}x + \dot{p}(1 - x) = a - \ell (g)
\]

\[
p k x = 1
\]
\[ \sigma_n x + \sigma_w (1 - x) = \sigma + \sigma_p \]

3. \( x \) follows the law of motion (1.10) derived using Ito’s lemma:

\[ dx_t = \mu_{x,t} dt + \sigma_{x,t} dZ_t \] (1.10)

\[ \mu_{x,t} = x_t \left( \mu_{n,t} - \dot{\epsilon}_t - g_t - \mu_{p,t} - \sigma \dot{\sigma}_{p,t} + (\sigma + \sigma_{p,t})^2 - \sigma_n (\sigma + \sigma_{p,t})' \right) \]

\[ \sigma_{x,t} = x_t (\sigma_n - \sigma - \sigma_{p,t}) \]

This recursive definition abstracts from the absolute level of the aggregate capital stock, which we can recover using \( \frac{dk_t}{k_t} = g_t dt + \sigma dZ_t \).

**Capital holdings.** Experts demand for capital is pinned down by the FOC from the HJB equation. After some algebra we obtain an expression that pins down the demand for capital \( \tilde{k} \):

\[ g_t + \mu_{p,t} + \sigma \dot{\sigma}_{p,t} + \frac{a - \ddot{\epsilon}_t}{p_t} - r_t \leq (\sigma + \sigma_{p,t}) \pi_t + \gamma \rho_t \tilde{k}_t (\phi \nu_t)^2 \]

Idiosyncratic risk is not priced in the financial market, because it can be aggregated away. However, because experts face an equity constraint that forces them to keep an exposure \( \phi \) to the return of their capital, they know that the more capital they hold, the more idiosyncratic risk they must bear on their balance sheets \( \tilde{\sigma}_{n,t} = \phi \rho_t \tilde{k}_t \nu_t \). They consequently demand a premium on capital for the fraction \( \phi \) of idiosyncratic risk that cannot be shared. Using the equilibrium condition \( p \tilde{k} x = 1 \) we obtain an equilibrium pricing equation for capital:

\[ \frac{g_t + \mu_{p,t} + \sigma \dot{\sigma}_{p,t} + \frac{a - \ddot{\epsilon}_t}{p_t} - r_t}{x_t} = \frac{(\sigma + \sigma_{p,t}) \pi_t}{x_t} + \frac{1}{x_t} (\phi \nu_t)^2 \] (1.11)

The left hand side is the excess return of capital. The right hand side is made up of the risk premium corresponding to the aggregate risk capital carries, and a risk premium for the idiosyncratic risk it carries. When experts balance sheets are weak (low \( x_t \)) and idiosyncratic risk \( \nu_t \) high, experts demand a high premium on capital. This is how \( x_t \) and \( \nu_t \) affect the economy, and we can see that without moral hazard, \( \phi = 0 \), neither \( x_t \) nor \( \nu_t \) would play any role, and experts would be indifferent about how much capital to hold, as long as it was properly priced. With moral hazard, instead, they have a well defined demand for capital, proportional to their net worth.

It is useful to obtain an expression for a “fictitious” price of idiosyncratic risk\(^{28}\)

\[ \omega_t = \frac{\phi \nu_t}{x_t} \]

Under this formulation, each expert faces a complete financial market without the equity constraint,\(^{28}\) See Cox and Huang (1989).
and where his own idiosyncratic risk $W_t$ pays a premium $\alpha_t$. Capital is priced as an asset with exposure $\phi_t \nu_t$ to this idiosyncratic risk, and can be abstracted from. We can verify that the expert will chose an exposure to his own idiosyncratic risk $\bar{\sigma}_{n,t} = \frac{\alpha_t}{\gamma} = \phi_t \frac{1}{\gamma} \nu_t$ as required in equilibrium. In this sense the fictitious price of idiosyncratic risk $\alpha_t$ is “right”. An advantage of the fictitious market formulation is that the only difference between experts’ and consumers’ problem is that experts perceive a positive price for idiosyncratic risk $\alpha_t > 0$, while consumers perceive a price of zero.

**Aggregate risk sharing.** Optimal contracts allow experts to share aggregate risk using the free process $\theta_t$. The optimal contract effectively separates the decision of how much capital to hold from the decision of how much aggregate risk to hold. The FOC for aggregate risk sharing for experts are:

$$\sigma_{n,t} = \frac{\pi_t'}{\gamma} - \frac{\gamma - 1}{\gamma} \sigma_{\xi,t}$$

(1.12)

Experts’ optimal aggregate risk exposure depend on a myopic risk-taking given by the price of risk $\gamma$ and the risk-aversion parameter, $\frac{\pi_t'}{\gamma}$, and a hedging motive driven by the stochastic investment possibility sets, $\frac{\gamma - 1}{\gamma} \sigma_{\xi,t}$. This hedging motive is standard in Intertemporal CAPM models, going back to Merton (1973), and it will play a crucial role in the amplification and propagation of aggregate shocks through experts’ balance sheets. Recall the “net worth multiplier” $\xi_t$ captures the stochastic general equilibrium conditions the expert faces

$$V_t(n) = \frac{(\xi_t n)^{1-\gamma}}{1-\gamma}$$

If the expert is risk neutral, he will prefer to have more net worth when $\xi_t$ is high, since he can obtain a lot of long-term utility out of each unit of net worth. This is a “substitution effect”, and works against the balance sheet channel, since $\xi_t$ is relatively higher during downturns. On the other hand, when $\xi_t$ is low he requires more net worth to achieve any given level of utility. If the expert is risk averse, he will prefer to have more net worth when $\xi_t$ is low. This is a “wealth effect”, and will underlie the balance sheet amplification channel for uncertainty shocks, inducing experts

---

20 We can use (1.11) to rewrite experts’ dynamic budget constraint

$$\frac{dn_t}{n_t} = (r_t + \pi_t \sigma_{n,t} + \alpha_t \bar{\sigma}_{n,t}) dt + \sigma_{n,t} dZ_t + \bar{\sigma}_{n,t} dW_{i,t}$$

where the expert can freely choose $\sigma_{n,t}$ and $\bar{\sigma}_{n,t}$. Experts problem then is to maximize their objective function, subject to an intertemporal budget constraint

$$\mathbb{E} \left[ \int_0^\infty \tilde{\eta} u e^u du \right] = n_0$$

where the fictitious SPD $\tilde{\eta}$ follows: $\frac{d\tilde{\eta}}{\tilde{u}} = -r_t dt - \pi_t dZ_t - \alpha_t dW_{i,t}$ for expert $i$.  

30 $\pi_t$ is a column vector and must be transposed, hence $\pi_t'$. 

25
to take aggregate risk. Which effect dominates depends on the risk aversion parameter\(^{31}\). When \(\gamma < 1\), equation (1.12) tells us the expert wants his net worth to be positively correlated with \(\xi_t\): the substitution effect dominates. When \(\gamma > 1\), instead, the wealth effect dominates. I focus on the case where the wealth effect dominates, \(\gamma > 1\).

Consumers have analogous FOC conditions for aggregate risk sharing

\[
\sigma_{w,t} = \frac{\pi'_t}{\gamma} - \frac{\gamma - 1}{\gamma} \sigma_{\xi,t}
\]

(1.13)

where the only difference is that consumers' investment possibility sets are captured by \(\xi_t\) instead of \(\xi_t\). Since consumers cannot buy capital, its price and idiosyncratic risk-premium does not affect them, but they still face a stochastic investment possibility set from interest rates \(r_t\) and price of aggregate risk \(\pi_t\).

The volatility of balance sheets \(\sigma_{x,t}\) arises from the interaction of experts' and consumers' risk-taking decisions. Using the equilibrium condition \(\sigma_nx + \sigma_w(1-x) = (\sigma + \sigma_p)\) we obtain the following expression for the exposure of experts' balance sheets to aggregate risk:

\[
\sigma_{x,t} = (1-x_t)x_t\frac{\gamma - 1}{\gamma} \left(\sigma_{\xi,t} - \sigma_{\xi,t}\right)
\]

(1.14)

Since experts and consumers cannot both hedge in the same direction in equilibrium, it is the difference in their hedging motives which will cause experts balance sheets to be overexposed to aggregate risk. The term \(\sigma_{\xi,t} - \sigma_{\xi,t}\) captures experts' and consumers' relative hedging motives. The terms \((1-x_t)\) and \(x_t\) arise because experts are able to hedge their investment possibility sets only to the extent that consumers as a whole are willing to take the other side of the hedge. When \(x_t\) is close to 1, the typical expert is dealing mostly with other experts in the financial market. On the other hand, when \(x_t\) is close to 0, even if experts want to take a very large risky position, they have very little net worth and are not able to affect the distribution of aggregate wealth very much. The \(\frac{\gamma - 1}{\gamma}\) term captures the "substitution" and "wealth" effects.

Experts' and consumers' investment possibility sets depend on balance sheets \(x_t\), and so are endogenously determined in equilibrium in a two way feedback: experts balance sheets are exposed to aggregate risk to hedge stochastic investment possibility sets, but the volatility of investment possibility sets actually depends on the exposure of experts' balance sheets to aggregate risk. We can use the Markov equilibrium characterization and Itô's lemma to obtain a simple expression for

\(^{31}\)EZ preferences separate risk aversion \(\gamma\) from the EIS \(\psi^{-1}\).
the relative hedging motive

\[ \sigma_{\zeta,t} - \sigma_{\xi,t} = \left( \frac{\xi_{\zeta}}{\zeta} - \frac{\xi_{\nu}}{\xi} \right) \sigma_{\nu} \sqrt{\nu_t} + \left( \frac{\xi_{\zeta}}{\zeta} - \frac{\xi_{\xi}}{\xi} \right) \sigma_{x,t} \]  

(1.15)

where the functions are evaluated at \((\nu_t, x_t)\). The locally linear representation allows a neat decomposition into an exogenous source of relative hedging motive, driven by the uncertainty shock to \(\nu_t\), and an endogenous source from optimal contracts' aggregate risk sharing \(\sigma_{x,t}\). We can solve for the fixed point of this two-way feedback:

\[ \sigma_{x,t} = \frac{(1 - x_t) x_t^{\gamma - 1}}{1 - (1 - x_t) x_t^{\gamma - 1}} \left( \frac{\xi_{\zeta}}{\zeta} - \frac{\xi_{\xi}}{\xi} \right) \sqrt{\nu_t} \sigma_{\nu} \]  

(1.16)

Only uncertainty shocks will create a balance sheet channel. This suggests that in the benchmark case with only Brownian TFP shocks there won't be a balance sheet channel. Notice that even though the presence of moral hazard does not directly restrict experts' ability to share aggregate risk, it introduces hedging motives through the general equilibrium which would not be present without moral hazard, as shown by Proposition 6.

### 1.3.2 Brownian TFP benchmark

When aggregate shocks come only in the form of Brownian TFP shocks to capital and we allow agents to write contracts on all observable variables, there is no balance sheet channel. After a negative TFP shock, the value of all assets \(p_t k_t\) falls and everyone, experts and consumers alike, loses net worth proportionally, so \(\sigma_{x,t} = 0\). Experts then have lower net worth, but the value of capital they must hold in equilibrium is also lower, so the idiosyncratic risk they must carry is constant as a proportion of their net worth. Investment possibility sets then are not affected by aggregate shocks, and consequently there is no relative hedging motive, \(\sigma_{\zeta,t} - \sigma_{\xi,t} = 0\). Balance sheets \(x_t\) may still affect the economy, due to the presence of financial frictions derived from the moral hazard problem, but they won't be exposed to aggregate risk and hence won't play any role in the amplification of aggregate TFP shocks. In fact, the equilibrium is completely deterministic, up to the direct effect of TFP shocks on the aggregate capital stock.

**Proposition 9.** With only Brownian TFP shocks, i.e. \(\sigma_{\nu} = 0\), if agents can write contracts on the aggregate state of the economy, the balance sheet channel disappears: the state variable \(x_t\), the price of capital \(p_t\), the growth rate of the economy \(y_t\), the interest rate \(r_t\), and the price of risk \(\pi_t\) all follow deterministic paths and are not affected by aggregate shocks.
The negative result of Proposition 9 has two ingredients: 1) optimal contracts separate the decision of how much capital to buy (leverage) from the decision of how much aggregate risk to hold (risk sharing). Risk sharing between experts and consumers will depend only on their relative hedging motives. The difference in hedging motives is ultimately traced to the fact that experts can trade and use capital. This gives us expression (1.14):

$$
\sigma_{x,t} = (1 - x_t) x_t \gamma - \frac{1}{\gamma} (\sigma_{\zeta,t} - \sigma_{\xi,t})
$$

And 2) aggregate Brownian TFP shocks don’t affect investment possibility sets directly and so don’t create a relative hedging motive by themselves. The exogenous source of relative hedging motive disappears, so we are left with only the endogenous component in expression (1.15):

$$
\sigma_{\zeta,t} - \sigma_{\xi,t} = \underbrace{\left( \frac{\zeta_x - \zeta_x}{\xi_x} - \xi_x \right) \sigma_x \sqrt{t}}_{\text{exogenous}} + \underbrace{\left( \frac{\zeta_x - \zeta_x}{\xi_x} - \xi_x \right) \sigma_x}_{\text{endogenous}}
$$

With no exogenous source, however, the unique Markov equilibrium has deterministic investment possibility sets, no relative hedging motive, and hence no overexposure to aggregate risk which could endogenously affect investment possibility sets. The continuous-time setting provides a locally linear relationship which guarantees this is the unique Markov equilibrium, given by equation (1.16). Without any source of aggregate volatility, the economy then follows a deterministic path.

**Implementation.** When contracts can be written on all aggregate observable shocks, leverage and risk sharing are two independent decisions. How realistic are these contracts? Proposition 4 shows the optimal contract can be implemented with simple financial instruments: an expert buys capital $p_t k_t$ and sells a fraction $1 - \phi$ of his equity. He then buys a market index, or shorts it, to obtain the right exposure to aggregate risk. Even though the ability to short the market index is important for deriving Proposition 9 in general, experts might typically be going long on market...
indices. We can compute their investment in the normalized market index $\vartheta_t$ explicitly:

$$\vartheta_t = (\sigma + \sigma_{p,t}) \frac{x_t - \phi}{x_t} + \frac{\sigma_{x,t}}{x_t} \frac{x_t - \phi}{x_t}$$

Their portfolio position on the market indices will be positive or negative depending on whether $x_t \geq \phi$. Without any relative hedging motives ($\sigma_{x,t} = \sigma_{\xi,t} = 0$), experts and consumers will hold a fraction of aggregate wealth proportional to their net worth or wealth, respectively: $\sigma_{n,t} = \sigma_{w,t} = \sigma + \sigma_{p,t} = \sigma$. Experts are required to hold a fraction $\phi$ of their equity, which already exposes them to a fraction $\phi$ of aggregate risk. If their net worth represents more than fraction $\phi$ of aggregate wealth $x_t > \phi$, they will want to further buy more aggregate risk by going long on a market index, or their competitors equity. On the other hand, if $x_t < \phi$, they will short the market index to get rid of some of the aggregate risk contained in their equity.

**Constrained Contracts.** The economy may be hit by a large number of orthogonal aggregate shocks, i.e. $d > 1$. The negative result in Proposition 9 doesn’t require complete markets, only that leverage and risk sharing be separated. In terms of implementation in a financial market, we need the financial market to span the exposure to aggregate risk of the return to capital $\sigma dZ$. In this case, an expert can buy capital and immediately get rid of the aggregate risk using financial instruments. He can then share aggregate risk with consumers using any available financial instruments. Without any endogenous hedging motives both experts and consumers will choose the same exposure to aggregate risk and eliminate the balance sheet channel.

**Proposition 10.** Even if the financial market spans only a linear combination of aggregate shocks $\hat{Z}_t = BZ_t$, for a matrix $B$ with full rank $d' < d$, then as long as the exposure of capital to aggregate risk $\sigma$ is in the row space of $B$, the result of Proposition 9 holds.

If experts can short the equity of their competitors, who have a similar exposure to aggregate risk as they do, they can get rid of the aggregate risk in their capital. In a competitive market, there is a large number of competitors so their idiosyncratic risks can be aggregated away. In other words, an index made up competitors’ equity is exactly the instrument required to separate leverage from risk sharing and obtain the negative result.

**Corollary 11.** As long as a market index of experts’ equity can be traded, the balance sheet channel disappears.

In contrast to Proposition 9, when we rule out contracts on aggregate shocks, i.e. $B = 0$, experts’ leverage and aggregate risk sharing become entangled. In the simplest case with $\phi = 1$ as in the baseline setting in BS, if experts are leveraged $p_t k_t > n_t$, then when a negative aggregate shock...
reduces the value of capital experts will lose net worth more than proportionally:

\[ \sigma_{x,t} = x_t (\sigma_n - \sigma - \sigma_{p,t}) = x_t \left( p_t \dot{k}_t - 1 \right) (\sigma + \sigma_{p,t}) > 0 \]

This reduces their ability to hold capital and lowers asset prices, further hurting their balance sheets, and amplifying and propagating the initial shock. This is precisely the mechanism behind the balance sheet channel in Brunnermeier and Sannikov (2012) and He and Krishnamurthy (2011).32

1.3.3 Solving the model with uncertainty shocks

The negative results in Propositions 9 and 10, and especially Corollary 11 should be interpreted as a TFP-neutrality benchmark. Allowing agents to write contracts contingent on the aggregate state of the economy creates a theoretical puzzle: what explains the disproportionate aggregate risk taking that drives financial crises? The literature has typically imposed constraints on the contract space that limit agents ability to share aggregate risk, or explored different contractual environments that micro-found these constraints. Here, however, I propose a different approach and look at the kind of structural shocks hitting the economy. When contracts cannot be written on the aggregate state of the economy, the type of aggregate shock hitting the economy is not relevant for the purposes of the balance sheet channel. As long as the aggregate shock depresses asset values and experts are leveraged, their balance sheets will be disproportionately affected and create a balance sheet recession. When we allow agents to write contracts on the aggregate state of the economy, on the other hand, the type of aggregate shock that hits the economy takes on a prominent role.

Uncertainty shocks will have a direct effect on the the price of capital and the growth rate of the economy, due to the presence of financial frictions. When \( \nu_t \) is high, experts demand a high idiosyncratic risk premium on capital. In contrast to Brownian TFP shocks, uncertainty shocks are also amplified through a balance sheet channel. Uncertainty shocks affect experts' balance sheets disproportionately, even though they are able to freely share aggregate risk. Experts therefore become less willing to hold capital, driving down its price and amplifying the effects of the uncertainty shock. This balance sheet channel is the result of an endogenous hedging motive which I will explain in detail below.

The strategy to solve for the equilibrium with uncertainty shocks is to first use optimality and market clearing conditions to obtain expressions for equilibrium objects in terms of the stochastic processes for \( p, \xi, \) and \( \zeta, \) and then use Ito's lemma to map the problem into a system of partial differential equations for the price of capital \( p(\nu, x) \) and the multipliers \( \xi(\nu, x) \) and \( \zeta(\nu, x). \) The pricing equation for capital (1.11), experts' HJB (1.9) and market clearing for consumption goods

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32In He and Krishnamurthy (2011) a similar mechanism underlies the volatility of experts' net worth (specialists in their model), but the price of capital falls because consumers are more impatient and interest rates must rise for consumption-goods markets to clear.
provide three functional equations\(^{33}\). Because neither experts nor consumers want to ever hit their solvency constraint and have zero wealth\(^{34}\), we look for an equilibrium with \(x \in (0, 1)\). In the interest of simplicity, I consider a one dimensional exogenous Brownian shock, i.e. \(d = 1\). This shock will affect both capital directly through \(\sigma > 0\) and the volatility of idiosyncratic risk \(\nu_t\) through \(\sigma \nu \nu_t < 0\).\(^{35}\) I show the procedure for solving the equilibrium in detail in Appendix C.

**Balance sheet channel.** Figure 1-1 shows the price of capital and the growth rate of the economy as functions of \(\nu\) and \(x\). Both higher idiosyncratic risk \(\nu\), and weaker balance sheets \(x\) increase the idiosyncratic risk premium on capital and reduce its price. This, in turn, depresses the growth rate of the economy \(g\) through the static FOC for investment

\[
I'(g) = p
\]

\(^{33}\)Two second order partial differential equations and an algebraic constraint.

\(^{34}\)With intertemporally linear preferences, \(\psi = 0\), agents might actually accept to reach their solvency constraint.

I show how to deal with this case in section 1.5.

\(^{35}\)Additional TFP shocks will only have a direct impact on the level of the effective capital stock, but will have no further effects on the economy.
An uncertainty shock has a direct effect on the price of capital because its idiosyncratic risk premium must rise,

\[
\begin{align*}
&\Delta t + \mu_{p,t} + \sigma' p_t + \frac{a - \mu_t}{\mu_t} - \pi_t = \\
&\quad (\sigma + \sigma_{p,t}) \pi_t + \frac{1}{x_t} (\phi \nu_t)^2 + \\
&\text{excess return} \quad \text{agg. risk premium} \quad \text{id. risk premium}
\end{align*}
\]

In addition, the effects of the uncertainty shock are amplified through a balance sheet channel. Uncertainty shocks create an endogenous hedging motive that induces experts to take aggregate risk, so \( x \) falls when an uncertainty shock hits the economy: \( \sigma_x > 0 \), as figure 1-2 shows. The intuition is the following. Downturns are periods of high idiosyncratic risk \( \nu_t \). Since experts cannot offload all of the idiosyncratic risk in their capital due to the moral hazard problem, in equilibrium they perceive a high fictitious price for their own idiosyncratic risk \( W_t \):

\[
\alpha_t = \gamma \frac{\phi \nu_t}{x_t} > 0
\]

Since in equilibrium they go long on idiosyncratic risk \( \delta_{x,t} = \frac{\alpha_t}{x_t} = \frac{\phi \nu_t}{x_t} > 0 \), experts benefit from an increase in \( \alpha_t \). Consequently, conditional on their net worth, they are better off during downturns, relative to consumers for whom expert-idiosyncratic risk is not priced, i.e.: \( \frac{\xi_x}{\zeta_x} - \frac{\xi_x}{\zeta_x} < 0 \). Figure 1-2 shows \( \log(\zeta) - \log(\xi) \) is a decreasing function of \( \nu \). The derivative with respect to \( \nu \) of the log difference is precisely \( \frac{\xi_x}{\zeta_x} - \frac{\xi_x}{\zeta_x} \).

This creates a relative hedging motive between experts and consumers

\[
\sigma_{\xi,t} - \sigma_{\zeta,t} = \left(\frac{\xi_x}{\zeta_x} - \frac{\xi_x}{\zeta_x}\right) \sigma_{\nu} \sqrt{\nu_t} + \left(\frac{\xi_x}{\zeta_x} - \frac{\xi_x}{\zeta_x}\right) \sigma_{x,t}
\]

(exogenous > 0)

(recall this relative hedging motive drives the volatility of balance sheets, according to equation (1.14) \( \sigma_{x,t} = (1 - x_t) x_t \frac{\gamma - 1}{\gamma} (\sigma_{\zeta,t} - \sigma_{\xi,t}) \)). If agents are risk neutral, the "substitution effect" dominates, and this relative hedging motive induces experts to insure against aggregate risk, in order to have more net worth during downturns, when each dollar of net worth gets them more utility relative to consumers. On the other hand, if agents are risk averse, with \( \gamma > 1 \), the "wealth effect" dominates. During economic booms, when idiosyncratic risk \( \nu_t \) and its price \( \alpha_t \) are low, experts need more net worth to obtain any given level of utility, relative to consumers, so the relative hedging motive induces experts to take aggregate risk and have more net worth during economic booms. This hedging motive drives the balance sheet channel if \( \gamma > 1 \):

\[
\sigma_{x,t} = (1 - x_t) x_t \frac{\gamma - 1}{\gamma} (\sigma_{\zeta,t} - \sigma_{\xi,t}) > 0
\]

\( >0 \)
Figure 1-2: The volatility of balance sheets $\sigma_x$ and $\log(\zeta) - \log(\xi)$ as functions of $\nu$ (left) $x = 0.25$ (solid), $x = 0.5$ (dotted), and $x = 0.75$ (dashed), and as a function of $x$ (right) for $\nu = 0.25$ (solid), $\nu = 0.5$ (dotted), and $\nu = 0.75$ (dashed). For parameter values see Numerical Solution below.

When an uncertainty shock hits the economy and idiosyncratic risk rises, experts' balance sheets are disproportionately hit, so their share of aggregate wealth $x_t$ falls ($\sigma_{x,t} > 0$). With weak balance sheets (low $x_t$), holding all the effective capital stock in the economy requires experts to leverage up and accept a large amount of idiosyncratic risk, so the fictitious price of idiosyncratic risk must go up:

$$\uparrow \uparrow \alpha_t = \gamma \frac{\phi \nu_t}{x_t} \uparrow \downarrow$$

Conditional on net worth, experts are better off (relative to consumers) when balance sheets are weak, i.e. $\frac{\zeta_x}{\zeta} - \frac{\xi_x}{\xi} > 0$. Figure 1-2 shows $\log(\zeta) - \log(\xi)$ is an increasing function of $x$. Thus weak balance sheets endogenously feed back to amplify the relative hedging motive:

$$\sigma_{\zeta,t} - \sigma_{\xi,t} = \begin{cases} <0 & \text{exogenous} > 0 \\ \left(\frac{\zeta_x - \xi_x}{\xi}\right) \sigma_\nu \sqrt{\nu_t} + \left(\frac{\zeta_x - \xi_x}{\xi}\right) \sigma_{x,t} & \text{endogenous} > 0 \end{cases}$$

which in turn amplifies experts’ aggregate risk taking according to equation (1.18). Equilibrium is

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36 Recall in equilibrium experts exposure to aggregate risk is $\sigma_{w,t} = \sigma + \sigma_{p,t} + \frac{\sigma_{x,t}}{x_t} \uparrow \sigma + \sigma_{p,t}$. 

33
a fixed point of this two-way feedback loop:

![Hedging motif](image)

![Balance sheet channel](image)

Figure 1-2 also shows the volatility of balance sheets, $\sigma_x$. It's positive throughout and larger during periods of high idiosyncratic risk $\nu$. This means uncertainty shocks that increase idiosyncratic risk $\nu$ in the economy hit experts' balance sheets disproportionately and reduce the share of aggregate wealth that belongs to experts $x$. This will create a balance sheet recession. Seen as a function of balance sheets $x$, their volatility $\sigma_x$ has an inverted U-shape. It's larger in the interior of the domain, and vanishes as it approaches the boundaries. This behavior is due to the term $(1-x)x$ in formula (1.14). When $x$ is close to either boundary, one type of agent has almost dropped from the economy, so the relative hedging between experts and consumers that underlies the volatility of balance sheets vanishes.

**Aggregate risk and flight to safety.** After an uncertainty shock exogenously raises idiosyncratic risk $\nu_t$ in the economy, the economy experiences also endogenously high aggregate risk $\sigma + \sigma_{p,t}$ and a flight to safety event with low interest rates $r_t$ and high risk premia $\pi_t$. Figure 1-3 shows aggregate risk rising when idiosyncratic risk $\nu_t$ is high and balance sheets $x_t$ are weak. The demand for aggregate risk from agents for hedging purposes also falls during downturns

$$\frac{\pi_t}{\gamma} + \frac{1-\gamma}{\gamma}(\sigma_{\xi,t}x_t + \sigma_{\xi,t}(1-x_t)) = \sigma + \sigma_p$$

This combination of reduced appetite for aggregate risk just when assets become more risky drives the price of aggregate risk $\pi_t$ up and create a flight to the safety of risk free bonds that depresses the interest rate $r_t$. In fact, uncertainty shocks may drive the risk-free interest rate below zero. In a richer model with sticky prices this could lead to a "liquidity trap".
Figure 1-3: Aggregate risk $\sigma + \sigma_p$, the price of risk $\pi$, and risk free interest rate $r$ as functions of $\nu$ (above) $x = 0.25$ (solid), $x = 0.5$ (dotted), and $x = 0.75$ (dashed), and as a function of $x$ (below) for $\nu = 0.25$ (solid), $\nu = 0.5$ (dotted), and $\nu = 0.75$ (dashed). For parameter values see Numerical Solution below.

Stochastic risk premia have been extensively studied in the asset pricing literature. Campbell and Cochrane (1999), for example, introduce habit and obtain stochastic risk-premia. Here, instead, risk premia respond to aggregate shocks due to the presence of financial frictions. In the benchmark without moral hazard, risk premia are constant. He and Krishnamurthy (2011) obtain stochastic risk premia in a similar model where balance sheets play an important role. In their model, agents cannot write contracts on the aggregate state of the economy, so risk premia must induce experts to take aggregate risk. Here instead, agents can share aggregate risk freely, since optimal contracts separate leverage from aggregate risk sharing.

Implementation Recall that optimal contracts can be implemented using standard financial instruments such as equity and a market index. The expert must keep a fraction $\phi$ of his own equity and this forces him to keep a fraction $\phi$ of his idiosyncratic risk, but he can adjust his exposure to aggregate risk using a market index. Recall also that $\theta_t$ can be interpreted as his portfolio investment in a market index normalized with identity loading on the aggregate risk

$$\theta_t = (\sigma + \sigma_{p,t}) \left( \frac{x_t - \phi}{x_t} \right) + \frac{\sigma_{x,t}}{x_t}$$

The first term corresponds to the myopic risk sharing motive. Since aggregate risk pays a premium, both experts and consumers want to buy some of it. They face the same price of risk $\pi_t$, so they have the same myopic incentives and they should share aggregate risk proportionally to their wealth. Experts must keep a fraction $\phi$ of their equity, which already exposes them to a fraction $\phi$ of aggregate risk. The first use of the market index is to adjust experts' exposure to aggregate risk to achieve proportional risk sharing. The second term captures the hedging motive.
Figure 1-4: Experts' investment in the market index, $\theta$, as a function of $\nu$ (left) $x = 0.25$ (solid), $x = 0.5$ (dotted), and $x = 0.75$ (dashed), and as a function of $x$ (right) for $\nu = 0.25$ (solid), $\nu = 0.5$ (dotted), and $\nu = 0.75$ (dashed). For parameter values see Numerical Solution below.

$\sigma_x \frac{\dot{i}}{x} > 0$. Experts want to increase their exposure to aggregate risk in order to take advantage of stochastic investment possibility sets. When the economy is hit only by a Brownian TFP shock, the hedging motive disappears and we are left with only the first term.

The most striking feature of Figure 1-4 is that $\theta$ is positive for most values of $(\nu, x)$. Not only are experts not getting insurance against aggregate risk, they are using the financial instruments at their disposal to buy even more aggregate risk than what the equity constraint forces them to.

**Long-run dynamics.** The volatility $\sigma_x$ vanishes near the boundaries, so $x$ never reaches them. In the short-run, an uncertainty shock that increases idiosyncratic risk weakens balance sheets because experts are overexposed to it, $\sigma_x > 0$. In the medium-run, however, higher idiosyncratic risk leads to stronger balance sheets, as experts obtain greater profits and postpone consumption, relative to consumers. Figure 1-5 shows the drift of balance sheets $\mu_x$ is increasing in $\nu$ and decreasing in $x$. In the long-run idiosyncratic risk goes back to its long-run mean, and so do balance sheets. Figure 1-5 also shows the drift of balance sheets $\mu_x$ is positive and diverging to $\infty$ when $x \to 0$, and becomes negative as $x \to 1$. We can verify that boundaries are never reached, and the equilibrium has a non-degenerate stationary distribution with a long-run steady state. This can be computed using the forward Kolmogorov equation.

**Numerical solution.** I use the following parameter values. *Preferences:* The discount rate is $\rho = 0.05$, the risk aversion is $\gamma = 5$, while the EIS$^{37}$ is set at 1.5 (i.e.: $\psi = 0.66$). Experts retire with Poisson arrival rate $\tau = 0.4$. *Technology:* capital productivity is normalized to $\alpha = 1$. Capital exposure to aggregate risk is $\sigma = 0.03$ in line with the observed volatility of output. *Moral Hazard:* Hedge funds typically keep 20% of returns above a threshold, so I set $\phi = 0.2$. The investment function takes the simple form $i(g) = 200g^2$, so $g(p) = \frac{P}{400}$ and $i(p) = \frac{p^2}{800}$, so that the growth rate

---

$^{37}$Campbell and Beeler (2009) uses an EIS of 1.5. Gruber (2006) estimates an EIS of 2 based on variation across individuals in the capital income tax rate.
Figure 1-5: The drift of balance sheets $\mu_x$ as a function of $\nu$ (left) $x = 0.25$ (solid), $x = 0.5$ (dotted), and $x = 0.75$ (dashed), and as a function of $x$ (right) for $\nu = 0.25$ (solid), $\nu = 0.5$ (dotted), and $\nu = 0.75$ (dashed). For parameter values see Numerical Solution below.

near the steady state is reasonable. *Idiosyncratic volatility* is set with a long-run mean of $\tilde{\nu} = 0.24$, mean reversion parameter $\lambda = 0.22$, and a loading on the aggregate shock $\sigma_\nu = -0.13$. I use data on idiosyncratic volatility from Campbell, Lettau, Malkiel, and Xu (2001), and fit the monthly idiosyncratic standard deviation to a discretized version of (1.2):

$$\nu_{t+1} - \nu_t = \lambda (\tilde{\nu} - \nu_t) + \sigma_\nu \sqrt{\nu_t} \epsilon_{t+1}$$

where the error terms $\epsilon_t$ are i.i.d. I then obtain the OLS estimators for the parameters $\lambda$, $\tilde{\nu}$, and $\sigma_\nu$.

### 1.3.4 Financial shocks

The economy I study above is driven by the exogenous process for the volatility of idiosyncratic risk $\nu_t$. A fraction $1 - \phi$ of idiosyncratic risk is shared and aggregated away, as if the volatility of idiosyncratic risk was actually $\phi \nu_t$. We may then take $\phi \nu_t$ as the exogenous state variable. Intuitively, it makes no difference to experts whether they must keep a fixed fraction $\phi$ of more idiosyncratic risk $\nu'$ or the same underlying amount of idiosyncratic risk $\nu$. A mathematically equivalent setup for the model takes $\nu$ fixed and lets the parameter $\phi_t$ follow a stochastic process.\(^{38}\) The only variable affected is $\theta_t$, experts' portfolio share in the market index. Otherwise, the equilibrium is characterized by the same set of equations.

**Proposition 12.** Aggregate shocks to idiosyncratic risk $\nu$ are equivalent to aggregate shocks to the moral hazard parameter $\phi$, up to implementation.

\(^{38}\)He and Krishnamurthy (2011) consider a stochastic moral hazard setting in the model. Because they restrict agents' ability to share aggregate risk and assume log preferences, the stochastic moral hazard does not change agents' choices nor does it create a balance sheet channel (which already exists in their model due to the contracting constraints they assume).
Since moral hazard micro-founds the financial friction in the model, this stochastic moral hazard specification has the interpretation of aggregate shocks to financial frictions, or financial shocks, since they induce experts to try to deleverage (although in equilibrium they can’t). In general, both \( \phi \) and \( \nu \) may be stochastic and help drive balance sheet recessions. Although the economic phenomena behind shocks to \( \phi \) or \( \nu \) might be very different, it’s possible to imagine a single structural shock, such as a deep loss of trust in the financial system, that both increases idiosyncratic risk \( \nu \) and tightens financial frictions \( \phi \). Eggertsson and Krugman (2010), Guerrieri and Lorenzoni (2011) and Buera and Moll (2012) also consider exogenous shocks to the financial friction in their models.

Although exogenous shocks to financial conditions, as captured by \( \phi_t \), are equivalent from a mathematical point of view to uncertainty shocks which increase the volatility of idiosyncratic risk in the economy, it is natural to assume they follow different stochastic processes. Here I explore an alternative specification of the model, where idiosyncratic volatility is constant, i.e.: \( \nu_t = \nu \), and the parameter for moral hazard follows an exogenous stochastic process

\[
d\phi_t = \sigma_\phi dZ_t
\]

Since \( \phi_t \in [0, 1] \) is a constraint, I set up two reflective boundaries, at \( \phi = 0 \) and \( \phi = 1 \), which impose the following boundary conditions on the system of partial differential equations:

\[
p_{\phi}(0, x) = p_{\phi}(1, x) = \xi_{\phi}(0, x) = \xi_{\phi}(1, x) = 0
\]

As we approach \( \phi \to 0 \) or \( \phi \to 1 \), the system becomes less sensitive to financial conditions \( \phi \). The intuition is as follows. A higher \( \phi \) directly constraints experts’ idiosyncratic risk sharing and hence reduces their demand for capital. In addition, because shocks to \( \phi \) are persistent, it affects agents’ expectations about the future path of \( \phi_t \) (and, endogenously, of \( x \)). As we approach \( \phi \to 1 \), however, this second, indirect effect vanishes. Agents know that financial conditions cannot get worse, so their expectations about the future path of \( \phi \) are less sensitive to further increases in \( \phi \). At the other boundary, \( \phi = 0 \) there is an analogous situation.

In addition, in order to show how the model is solved when the \( x = 1 \) boundary can be reached, I use intertemporally linear preferences \( \psi = 0 \), and I remove turnover \( \tau = 0 \). Now \( x = 1 \) is an absorbing state and is reached in finite time. When this happens consumers drop out and we have simple economy with identical agents, and therefore no balance sheet effects. I show how to solve this model in Appendix D, including how to handle the new boundary condition at \( x = 1 \).

As Figure 1-6 shows, the main results don’t change. The price of capital falls when financial conditions worsen, and experts’ overexposure to this source of aggregate risk, \( \sigma_x > 0 \), creates a balance sheet channel and amplify the effects of the shock. On the other hand, the balance sheet channel disappears near the boundaries \( \phi = 0 \) and \( \phi = 1 \), as shocks to \( \phi \) have a vanishing effect on agents’ value functions and hedging motives. This impacts the behavior of total volatility \( \sigma + \sigma_p \), risk-free interest rates \( r \) and the price of aggregate “financial” risk \( \pi \).
Numerical solution. I use the same parameter values as before, except \( \psi = 0 \), the idiosyncratic volatility \( \nu = 0.2 \) and moral hazard follows an arithmetic Brownian motion \( d\phi_t = -0.13dZ_t \).

1.4 Optimal policy

The model has several lessons for optimal policy. In standard models of balance sheet recessions driven by TFP shocks, where contracts cannot be written on the aggregate state of the economy, providing aggregate insurance to experts is a Pareto improving policy. For example, by promising to “bail out” experts, the government is implicitly forcing consumers to share aggregate risk with them, and this can help avoid balance sheet recessions. BS, for example, show how a social planner can achieve first best allocations if he has enough controls over the economy.

In contrast, when we allow agents to write contracts on the aggregate state of the economy, two new issues arise. First, experts may react to the policy intervention by taking more risk. They were sharing aggregate risk optimally from an individual point of view, so if government policy changes their underlying exposure to aggregate risk, they will simply try to undo its effects by taking more risk in order to achieve the same target exposure to aggregate risk. This is a separate issue from moral hazard and “too big to fail”, since experts incentives are not changing. They have the same target exposure to aggregate risk and are forced to achieve it through other means if the government provides them with some insurance.\(^\text{39}\)

Second, understanding agents’ incentives for taking aggregate risk is important not only on theoretical grounds, but also for the design of financial regulation. Once we introduce uncertainty

\(^{39}\)Even if agents are unable to share aggregate risk on their own, they might increase their consumption and dynamically weaken their balance sheets. This can lead to a “volatility paradox”, as in BS.
shocks and obtain a balance sheet channel, we also realize that experts may actually have good reasons to be taking so much aggregate risk. It's not clear anymore that eliminating the balance sheet channel is optimal. The following policy experiment illustrates this point.

**Financial regulation.** Consider the following reasonable policy intervention: the government regulates experts’ aggregate risk sharing, forcing them to take aggregate risk proportionally to their net worth

\[ \sigma_{n,t} = \sigma + \sigma_{p,t} \]

\[ \Rightarrow \sigma_{x,t} = 0 \]

so the balance sheet channel disappears. This can be achieved by forcing experts to short a market index to offset the aggregate risk contained in their capital:

\[ \theta_t = (\sigma + \sigma_{p,t}) \frac{x_t - \phi}{x_t} \]

The government also carries out a one time wealth transfer between experts and consumers in order to keep consumers indifferent. Normalize the capital stock to \( k_0 = 1 \), and let \( U^{reg}(\nu, x) = \left( \frac{e^{\mu x (1 - x) p}}{1 - \gamma} \right) \) be the present value of utility for consumers under this policy of financial regulation, and \( U^{eq}(\nu, x) \) their value in the unregulated equilibrium. Likewise, \( V^{reg}(\nu, x) \) and \( V^{eq}(\nu, x) \) are the corresponding value functions for experts. Then the government changes the distribution of wealth from \( x_0 \) to \( x_1 \) such that:

\[ U^{reg}(\nu, x_1) = U^{eq}(\nu, x_0) \]

We can then look at the utility of experts after the policy intervention: \( V^{reg}(\nu, x_1) - V^{eq}(\nu, x_0) \).

Figure 1-7 captures the results for \( x_0 = \frac{1}{3} \). The policy is Pareto improving if enacted during periods of low idiosyncratic risk, but becomes counterproductive when \( \nu \) is high. This allows us to draw two conclusions: 1) the competitive equilibrium is not efficient and can be improved upon by financial regulation, and 2) the naive policy of eliminating the balance sheet channel is not optimal.

**Subsidies to capital.** Another policy often proposed is for the government to intervene in asset markets in order to prop up their value. Imagine the government institutes a subsidy on capital \( b \). This changes the expected return of capital to

\[ \mathbb{E}_t \left[ dR_t^k \right] = \mu_{p,t} + \gamma_t + \sigma_{p,t} \gamma + \frac{a + b - \nu_t}{p_t} \]

The government then taxes experts on their wealth, so that in equilibrium there is no wealth transfer. Consumers are left altogether out of the scheme. This policy will increase the price of capital. As before, the government can carry out a one time transfer between experts and consumers in order to leave consumers indifferent \( U^{sub}(\nu, x_1) = U^{eq}(\nu, x_0) \), and we can look at experts’ value functions
Figure 1-7: The difference in experts' utility after the policy intervention $V^{reg}(\nu, x_1) - V^{eq}(\nu, x_0)$, for $x_0 = \frac{1}{3}$, as a function of $\nu$.

Figure 1-8: The difference in experts' utility $V^{sub}(\nu, x_1) - V^{eq}(\nu, x_0)$, as a function of $\nu$ for $x_0 = \frac{1}{3}$, after a subsidy $b = 0.1$ to capital is introduced.

$V^{sub}(\nu, x_1) - V^{eq}(\nu, x_0)$. Figure 1-8 shows a subsidy to capital of $b = 0.1$ increases experts' welfare. To some extent, the low price of capital was depressing investment and growth. But even with an exogenous growth rate $g$, increasing the value of capital improves idiosyncratic risk sharing in equilibrium. With higher price for capital, if consumers wealth is kept constant, experts will own a higher fraction of aggregate wealth, and this will allow them to reduce their leverage, and hence their exposure to idiosyncratic risk. Of course, since growth depends on the price of capital, subsidies that are too high could be inefficient.

1.5 Extensions and variants

Long-term contracts. In the contractual environment introduced in Section 1.2, experts' lack commitment to long-term contracts. Since their private benefit from stealing is purely pecuniary, if they could commit to long-term contracts that control their consumption, the first best contract will
full insurance against idiosyncratic risk could be achieved. Intuitively, the expert can divert funds
to a private hidden account, but can't really do anything useful with those funds. The only way
stealing can affect the experts' objective function is by changing the continuation of his consumption
stream $c$. But in the optimal contract without moral hazard the expert is completely insured from
the observable outcome $R_k$ which is the only thing he affects by stealing. Consequently, stealing
has no effects on the expert's utility and the optimal contract without moral hazard is also the optimal
contract in this case.

**Proposition 13.** *The optimal contract without moral hazard is also the optimal contract with moral
hazard if experts can commit to long-term contracts.*

**Corollary.** *The First Best equilibrium from Proposition 6 is also the equilibrium when experts have
full commitment to long-term contracts.*

One way to make the moral hazard problem binding is to make consumption not contractible.
This is the approach taken in Myerson (2012) or DeMarzo and Sannikov (2006). Here, instead, I
take the same approach as in Brunnermeier and Sannikov (2012) and He and Krishnamurthy (2011)
and introduce re-contracting. This yields a tractable "equity constraint" formulation for the optimal
contract and allows for a straightforward comparison with those papers. Alternatively, we could
explore a setting where the private action yields a private benefit in utility terms, such as Sannikov
(2008). This creates a moral hazard problem even with commitment to long-term contracts.

**Generalized moral hazard.** In the contracting environment developed in Section 1.2 the
agent's private action does not interact with aggregate shocks, so aggregate risk sharing does not
enter into the IC constraint. This is a standard result in the contracts literature, and leads to the
separation of experts' leverage and aggregate risk sharing which underlies the negative result of
Proposition 9.

An alternative to introducing ad-hoc constraints on the contract space is to explore different
contractual environments that may limit experts' ability to share aggregate risk. In Di Tella (2012)
I introduce pecuniary moral hazard in a more general way into a similar model with only Brownian
TFP shocks. I find that moral hazard will distort aggregate risk sharing for incentive-provision
reasons when the private action affects the exposure to aggregate risk of the private pecuniary
benefit.

Consider a CEO who must choose a supplier for his firm. He can choose an inefficient supplier
on which he has previously secretly invested. This has a negative impact on the firm's profits, but
he benefits through his secret investment in the supplier. Obtaining this private benefit, however,
required him to expose his net worth to the aggregate risk to which the supplier is exposed.

$$
\frac{dn_t}{n_t} = (r_t + \phi_t k_t s_t + \pi_t \sigma_n s_t) \, dt + (\sigma_n s_t - \lambda_t p_t k_t (\sigma + \sigma_{p,s}) + \beta_t p_t k_t s_t) \, dZ_t + \lambda_t s_t \left(dR^k_t - \mathbb{E} \left[ dR^k_t \right] \right)
$$
Here $\lambda_t$ is the exposure of the agent's net worth to the return of his firm (equity), $\phi$ is the private benefit he obtain per dollar diverted, and $\beta$ the impact of diversion on the exposure to aggregate risk of his private benefit. The optimal contract will then overexpose the agent to aggregate risk, to deter him from taking the private action that further increases his exposure. The local incentive compatibility constraint becomes:

$$\lambda_t \geq \phi - \sigma_{n,t} \beta$$

The more exposed to aggregate risk the agent is (higher $\sigma_{n,t}$) the less exposed to his firm's return he must be in order to implement any given $s_t$, and hence less exposed to idiosyncratic risk. The optimal contract takes advantage of this tradeoff between aggregate and idiosyncratic risk sharing. The resulting overexposure to aggregate risk will create a balance sheet channel.

The optimal contract in this case can also be implemented using equity and market indices, but the informational requirements are bigger. The principal must be able to monitor the agent's financial portfolio, not only the trading of his firm's equity. Otherwise, the agent would be able to secretly undo the incentive scheme.

This result suggests that if the agent's private action affects the exposure to aggregate risk of the firm's returns, the same tradeoff between aggregate and idiosyncratic risk sharing would arise. This is not the case, however, because non-linear contracts can provide better incentives. In continuous-time, the volatility of a stochastic process is observable, so the contract is able to severely punish the CEO if he takes an action that creates an unexpected aggregate volatility of observable returns.40 This is just an extreme case of a more general principle. In a discrete-time approximation, by using non-linear contracts that make the agent's net worth a concave function of the firm's return, the principal can induce an aversion to aggregate risk on the agent without need to overexpose him to it in equilibrium. The continuous-time contract can be understood as an extremely non-linear response to out of equilibrium behavior.

**Elasticity of intertemporal substitution.** The consumption goods market can play an important role. While aggregate risk sharing, and hence the balance sheet channel, depends on agents' relative risk aversion $\gamma$, the impact of higher idiosyncratic risk or weaker balance sheets on the price of capital depends on the EIS $\psi^{-1}$. I solve the model for empirically reasonable EIS $>1$ ($\psi < 1$). A special case arises if we let $\psi = 1$. This corresponds to preferences that are intertemporally log. Agents will consume a constant fraction $\rho$ of their wealth. Market clearing for consumption goods then pins down the price of capital

$$p = \frac{a - l}{\rho}$$

Neither idiosyncratic risk $\nu_t$, nor experts' balance sheets $x_t$ will affect the price of capital. This doesn't mean balance sheets $x_t$ are not important, and there will still be a balance sheet channel in

---

40 Technically, the private action does not induce an absolutely continuous change in the probability distribution of the observable firm's return $R^k$, so Girsanov's theorem doesn't apply.
the economy, $\sigma_x > 0$. But it will affect other aggregate prices, such as the risk-free interest rate $r_t$ and the price of aggregate risk $\pi_t$, which must endogenously adjust to keep the price of capital $p$ at a constant value consistent with equilibrium in the consumption goods market.

This is well a known result that rests on some simplifying assumptions which make the model more tractable. If consumers were able to use capital less efficiently than experts (instead of not at all) then the price of capital would not be constant. The market clearing condition in the consumption goods market would become

$$p = \frac{a_\kappa + a^c(1 - \kappa) - \iota}{\rho}$$

where $a^c < a$ the output of capital when controlled by consumers and $\kappa$ is the fraction of aggregate capital controlled by experts, which will generally depend on both $\nu_t$ and $x_t$.

In contrast, He and Krishnamurthy (2011) keep the simple limited participation setup, and use log preferences ($\gamma = \psi = 1$), but have consumers be more impatient than experts: $\rho^c > \rho^e$. The equation for market clearing in the consumption goods market becomes:

$$p(\rho^e x + \rho^c(1 - x)) = a - \iota$$

$$\Rightarrow p = \frac{a - \iota}{(\rho^e x + \rho^c(1 - x))}$$

When experts lose net worth ($x$ falls), the average discount rate in the market goes up: experts want to consume more and need to be convinced otherwise by a higher interest rate. This drives the price of capital down.

The model can be extended to deal with these cases in a straightforward manner. I focus on the simpler case with limited participation and homogenous preferences with $\psi < 1$.

**Structural Shocks.** I have focused on uncertainty shocks to $\nu_t$ (equivalent to financial shocks to $\phi$) but these are not the only possibilities. The broader lesson from this paper is that the type of structural shock hitting the economy can play an important role in generating balance sheet recessions. The same tools developed here can be used to study other structural shocks. Here I mention some possibilities.

First, aggregate shocks to the long-run growth rate of the economy are a natural place to look. After financial crises, expectations are often said to have been unduly optimistic before the crash, and the economy to have been “living beyond its means”. This is sometimes interpreted as evidence for financial bubbles, but it could also be the result of negative aggregate shocks to the growth rate of the economy, possibly as a result of structural change. The rosy future which was rationally expected in the past and was driving the high asset values and agents’ consumption decisions now seem unrealistic. Since balance sheet recessions seem to happen after financial crises, it would be interesting to explore to what extent long-run growth risk can help explain the excessive exposure...
Second, demand shocks play a prominent role in business cycle theory. In particular, during liquidity traps, the central bank is unable to stabilize the economy. The liquidity traps seem to occur after big financial crises, so it is natural to ask if the two phenomena are related. On the one hand, the balance sheet channel can help explain how small shocks get amplified and drive the natural rate of interest into negative territory (as it happens in this model). On the other hand, the depressed output during liquidity traps exacerbates balance sheet problems, and their particular properties might help explain why the balance sheets of experts are so exposed to aggregate risk.

1.6 Conclusions

In this paper I have shown how uncertainty shocks can help explain the apparently excessive exposure to aggregate risk that drives balance sheet recessions. While we have a good understanding of why the balance sheets of more productive agents matter in an economy with financial frictions, we don’t have a good explanation for why they are so exposed to aggregate risk. Even if agents face a moral hazard problem that limits their ability to issue equity, there is nothing preventing them from sharing aggregate risk, which can be accomplished by trading a simple market index. In fact, I show that in standard models of balance sheet recessions driven by TFP shocks, the balance sheet channel completely vanishes when agents are allowed to write contracts contingent on the aggregate state of the economy.

In contrast to TFP shocks, uncertainty shocks can generate an endogenous relative hedging motive that induces more productive agents to take on aggregate risk. Uncertainty shocks are therefore amplified through a balance sheet channel, depressing asset prices and growth, and triggering a “flight to quality” event with low interest rates and high risk premia. I also show that uncertainty shocks are isomorphic to financial shocks that tighten financial constraints. Finally, the model has lessons for the design of financial regulation. Most importantly, once we understand agents’ aggregate risk sharing behavior, we realize they might have good reasons to be highly exposed to aggregate risk. I show how a reasonable policy of regulating agents’ exposure to aggregate risk to eliminate the balance sheet channel can be counterproductive.

These results suggest three avenues for future research. The first is to think about optimal financial regulation more carefully. While completely eliminating the balance sheet channel is not optimal, neither is the competitive equilibrium. This suggests the question: how much exposure to aggregate risk is “right”? The second is to consider alternative structural shocks. While I have focused on uncertainty (and financial) shocks, the same tools developed in this paper can be used to study other kind of aggregate shocks. For example, shocks to the long-run growth possibilities of the economy can capture some features of financial crises. Indeed, pre-crisis growth projections are often judged unduly optimistic with hindsight. This could be the result of negative aggregate shocks to the growth rate of the economy. Third, liquidity traps seem to happen after big financial
crisis. During liquidity traps, monetary policy is unable to stabilize the economy, so balance sheet problems become more severe. Integrating nominal rigidities in models of balance sheet recessions would allow us to study the interaction of balance sheet recessions and liquidity traps.
1.7 Appendix A

In this Appendix I develop the contractual environment in more detail. The general equilibrium prices \( p \) and \( \eta \) are taken as given throughout, and I suppress the reference \( i \) to the expert to simplify notation. In addition, I repeat part of the setup introduced in Section 1.2 to make this appendix as self-contained as possible.

Setup. Each expert has an observable bank balance \( b_t \), with initial net worth \( n_0 > 0 \). He can continuously trade and use capital. The observable cumulative return from investing a dollar in capital is \( R^k = \{ R^k_t; t \geq 0 \} \), with

\[
dR^k_t = \left( g_t + \mu_{p,t} + \sigma_{p,t} + \frac{a - I(g_t)}{p_t} - s_t \right) dt + (\sigma + \sigma_{p,t}) dZ_t + \nu_t dW_t
\]

The expert can divert capital to a private account at a rate \( s = \{ s_t; t \geq 0 \} \) and reduce the observable return of his investment. He immediately sells it, and obtains \( \phi \in (0,1) \) units of capital per unit diverted, so his hidden savings\(^4\) from following a stealing plan \( s \) evolve:

\[
dS_t = \phi p_t k_t s_t dt + S_t r_t dt
\]

where \( \phi \) parametrizes the severity of the moral hazard.

He designs a contract \((e, g, k, F)\) that specifies his consumption \( e_t \geq 0 \), investment \( g_t \), capital \( k_t \geq 0 \), and a cumulative cash flow \( F_t \) he will sell on the financial market, all adapted to the filtration \( \mathcal{H}_t \) generated by the observable variables\(^2\) \( Z \) and \( \{ R^k_t \}_{t \in [0,1]} \). The cash flow \( F \) is an \( \mathcal{H}_t \)-adapted semimartingale, i.e.: the sum of a finite variation process and a local martingale, which we may write

\[
dF_t = d\mu_{F,t} + \sigma_{F,t} dZ_t + \tilde{\sigma}_{F,t} dR^k_t
\]

where \( \mu_{F,t} \) is a process of finite variation. For now, let \((e, g, k, F)\) be all contractible, as well as \( b \). I will later relax this assumption. We want to implement \( s_t = 0 \) always, and we can without loss of generality ignore the observable returns of other experts and focus only on \( Z \) and \( R^k_t \) for expert \( i \).

Under this contract the bank balance of the expert evolves:

\[
\db_t = r_t b_t dt + p_t k_t \left( dR^k_t - r_t dt \right) - dF_t - e_t dt
\]

The expert is interested in maximizing his expected utility \( U_0 \), given by (1.3), and the market prices

\(^4\)This specification of hidden savings assumes they are invested at the risk-free interest rate only.

\(^2\)The general equilibrium processes \( p \) and \( \eta \) are also adapted to \( \mathcal{H}_t \) because they only depend on the aggregate shocks \( Z \). Also, even though the contract could in principle depend on other experts' returns, it is wlog to write it only on each expert's own observable return.
the cash flow $F$. At time $t$, the continuation of the contract has present value

$$J_t = E^Q_t \left[ \int_t^\infty \frac{B_{0,t}}{B_{0,u}} dF_u \right]$$

where $B_{0,t} = \exp \left( \int_0^t r_u du \right)$ is the value of risk free bond, the expectation is taken under the equivalent martingale measure $Q$, and under no stealing, $s = 0$. To make the problem well defined, we impose a solvency constraint. The expert must have in his bank account enough funds to cover the continuation value of his contract$^{43}$: $b_t - J_t \geq 0$.

The price of idiosyncratic risk in the financial market is zero, and experts are risk averse, so in the first best contract without moral hazard, there is full insurance against idiosyncratic risk.

**Proposition 14.** The first best contract without moral hazard has full insurance against idiosyncratic risk.

As Proposition 13 shows, if agents could commit to long-term contracts that control their consumption, the first best would be implementable. To make the moral hazard problem binding, I introduce re-contracting as in Brunnermeier and Sannikov (2012) and He and Krishnamurthy (2011). This leads to a tractable characterization of optimal contracts and facilitates comparisons with the literature.

**Re-contracting.** At any time $t$, the expert is committed to a single contract. However, he cannot commit to long term contracts. The expert can at any time settle his obligations and roll-over the contract. The expert cannot “run away” without paying, as in Kiyotaki and Moore (1997), but he can offer a new contract $(c', g', k', F')$ with present value $J'_t \geq J_t$, and the market accepts. Define the expert’s net worth $n_t = b_t - J_t + S_t$: these are the funds he has when rolling over the contract. He adds his hidden savings to his legitimate bank account $b_t = b_{t-} + S_{t-}$ and $S_t = 0$, so recontracting doesn’t change his net worth, but allows him to “launder” his hidden savings to use it to optimally consume, invest, etc. in a new continuation contract. The set of available contracts at time $t$ for this expert depends on his net worth$^{44}$ $n_t$ and on aggregate general equilibrium conditions. Let $V(\omega, t, n)$ be the value of the best continuation contract the expert can get if he recontracts at $(\omega, t)$ with net worth $n$.

Before moving on, we can characterize the continuation value of the principal $J_t$ as follows:

**Lemma 15.** For a given feasible contract $(e, g, k, F)$ the continuation value of the principal is a semimartingale and follows under $s = 0$:

$$dJ_t = d\mu_{J,t} + \sigma_{J,t} dZ_t + \delta_{J,t} \left( dR^k_t - E_t \left[ dR^k_t \right] \right)$$

(1.21)

$^{43}$A lower bound for $b_t - J_t$ is required to make the problem well defined.

$^{44}$because of the solvency constraint $b_t - J_t \geq 0$. 
\[ d\mu_{t,t} = -d\mu_{F,t} + J_t r_t dt - (\sigma_{J,t} + \sigma_{F,t} + (\tilde{\sigma}_{J,t} + \tilde{\sigma}_{F,t})(\sigma + \sigma_{p,t})) \pi_t dt \] (1.22)

for some finite variation process \( \mu_{t,t} \) and progressively-measurable \( \sigma_{J,t}, \tilde{\sigma}_{J,t} \in M \), and the following transversality condition holds:

\[ \lim_{u \to \infty} \mathbb{E}_t^Q \left[ \int_t^\infty \frac{B_{0,t}}{B_{0,u}} J_u \right] = 0 \quad \text{a.e.}(\omega, t) \] (1.23)

**Proof.** The continuation value of the principal is

\[ J_t = B_{0,t} \mathbb{E}_t^Q \left[ \int_t^\infty \frac{1}{B_{0,u}} dF_u \right] \]

We can work with the following Q-martingale:

\[ Y_t = \int_0^t \frac{1}{B_{0,u}} dF_u + \frac{1}{B_{0,t}} J_t = \mathbb{E}_t^Q \left[ \int_0^\infty \frac{1}{B_{0,u}} dF_u \right] \]

and using the martingale representation theorem we get after some manipulation:

\[ dY_t = \frac{1}{B_{0,t}} dF_t + \left( \frac{1}{B_{0,t}} J_t \right) = \frac{1}{B_{0,t}} \sigma_{Y,t} dZ_t^Q + \frac{1}{B_{0,t}} \tilde{\sigma}_{Y,t} \left( dR_t^Q - \mathbb{E}_t^Q \left[ dR_t^Q \right] \right) \]

for some progressively-measurable \( \sigma_{Y,t} \) and \( \tilde{\sigma}_{Y,t} \) in \( M^Q \) where

\[ dZ_t^Q = dZ_t + \pi_t dt \]

\[ dR_t^Q - \mathbb{E}_t^Q \left[ dR_t^Q \right] = \left( dR_t^k - \mathbb{E}_t \left[ dR_t^k \right] \right) + (\sigma + \sigma_{p,t}) \pi_t dt \]

are Q-brownian motions \( dR_t^Q - \mathbb{E}_t^Q \left[ dR_t^Q \right] \) has exposure \( (\sigma + \sigma_{p,t}) \) to \( Z_t^Q \) and \( \nu_t \) to \( W_t \). Re-arranging we get:

\[ d \left( \frac{1}{B_{0,t}} J_t \right) = -\frac{1}{B_{0,t}} dF_t + \frac{1}{B_{0,t}} \sigma_{Y,t} dZ_t^Q + \frac{1}{B_{0,t}} \tilde{\sigma}_{Y,t} \left( dR_t^Q - \mathbb{E}_t^Q \left[ dR_t^Q \right] \right) \]

which tells us that \( J_t \) is a Q-semimartingale, so we can write:

\[ dJ_t = d\hat{\mu}_{J,t} + \sigma_{J,t} dZ_t^Q + \tilde{\sigma}_{J,t} \left( dR_t^Q - \mathbb{E}_t^Q \left[ dR_t^Q \right] \right) \]

where \( \hat{\mu}_{J,t} \) is a finite variation process and progressively measurable \( \sigma_{J,t} \) and \( \tilde{\sigma}_{J,t} \) in \( M^Q \). Using

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45 Define \( M \) as the set of processes \( \{x, y\} \) for which \( \mathbb{E} \left[ \int_0^T x_t^2 dt + \int_0^T y_t^2 (\sigma + \sigma_{p,t})^2 dt + \int_0^T y_t^2 \nu_t^2 dt \right] < \infty \) for all \( T \).

46 \( M^Q \) is defined as \( M \) but with the expectation taken under \( Q \).
Ito’s lemma on $\frac{1}{B_{0,t}}J_t$ we get an expression for $d\left(\frac{1}{B_{0,t}}J_t\right)$ and matching terms we get the following:

$$d\tilde{J}_t = -d\mu_{F_t} + J_t r_t dt + (\sigma_{F_t} + \tilde{\sigma}_{F_t} (\sigma + \sigma_{p,t})) \pi_t dt$$

$$\sigma_{J_t} = -\sigma_{F_t} + \sigma_{Y_t}$$

$$\tilde{\sigma}_{J_t} = -\tilde{\sigma}_{F_t} + \tilde{\sigma}_{Y_t}$$

changing the probability back to $P$ we get

$$dJ_t = -d\mu_{F_t} + J_t r_t dt + (\sigma_{J_t} + \sigma_{F_t} + (\tilde{\sigma}_{J_t} + \tilde{\sigma}_{F_t}) (\sigma + \sigma_{p,t})) \pi_t dt + \sigma_{J_t} dZ_t + \tilde{\sigma}_{J_t} \left(dR_t^k - \mathbb{E}_t \left[dR_t^k\right]\right)$$

For the transversality condition notice that since $Y_t$ is a $Q$-martingale

$$J_0 = Y_0 = \mathbb{E}^Q [Y_t] = \mathbb{E}^Q \left[\int_0^t \frac{1}{B_{0,u}} dF_u\right] + \mathbb{E}^Q \left[\frac{1}{B_{0,t}} J_t\right]$$

and taking the limit $t \to \infty$ we see we need $\lim_{t \to \infty} \mathbb{E}^Q \left[\frac{1}{B_{0,t}} J_t\right] = 0$, and a similar argument shows this must hold for almost all $(\omega, t)$ ($P$ and $Q$ are equivalent).

We have defined the agent’s net worth $n_t = b_t + S_t - J_t$, so we can use Lemma (15) to compute the law of motion of $n_t$ under no-stealing:

$$d n_t = r_t n_t dt + p_t k_t \left(dR_t^k - r_t dt\right) - (\sigma_{J_t} + \sigma_{F_t} + (\tilde{\sigma}_{J_t} + \tilde{\sigma}_{F_t}) (\sigma + \sigma_{p,t})) \pi_t dt$$

$$- (\sigma_{F_t} + \sigma_{J_t}) dZ_t - (\tilde{\sigma}_{F_t} + \tilde{\sigma}_{J_t}) \left(dR_t^k - \mathbb{E}_t^{s=0} \left[dR_t^k\right]\right) - e_t dt$$

(1.24)

If the agent steals, we add a drift using Girsanov’s theorem and the definition of stealing:

$$[\phi p_t k_t - p_t k_t + (\sigma_{F_t} + \tilde{\sigma}_{J_t})] s_t dt$$

Stealing yields a flow $\phi p_t k_t s_t dt$ to the private account $S_t$, and changes the probability distribution of $R_t^k$ affecting the drift of $b_t$ and $J_t$. Notice we have superfluous instruments to control $n_t$, so we can without loss of generality set $J_t = 0$ always and use $\tilde{\sigma}_{F_t}$ and $\sigma_{F_t}$. The transversality condition $\lim_{t \to \infty} \mathbb{E} [n_t J_t] = 0$ is then satisfied automatically. We just need to make sure, at the end, that $\mathbb{E}_t^Q \left[\int_0^\infty \frac{B_{0,t}}{B_{0,u}} dF_u\right] = 0$ for all $(\omega, t)$. Define $\tilde{\sigma}_{F_t} = p_t k_t (1 - \tilde{\phi})$ and $\theta_t = -\frac{\sigma_{F_t}}{n_t}$ and for $s = 0$ we get the dynamic budget constraint (1.5) and (1.6). We may then identify the contract $(\epsilon, g, k, F)$ with the corresponding $(\epsilon, g, k, \tilde{\phi}, \theta)$.

Since the general equilibrium is Markov in the state variables $X = (\nu, x)$ that follow Ito processes, we will look for $V(\omega, t, n) = V(X, n)$, and define it recursively as the viscosity solution to the following HJB equation:

$$0 = \max_{(\epsilon, g, k, \tilde{\phi}, \theta)} f(\epsilon_t, V_t) + \mathbb{E}^{s=0} [dV_t]$$

50
where $dV_t$ is computed with Ito's lemma, and subject to (1.5) and (1.6), and an IC constraint:

$$0 \in \arg \max_{s \geq 0} f(e_t, V_t) + \mathbb{E}_t^s [dV_t]$$

and an appropriate transversality condition. Using Ito's lemma and the law of motion for $n$, we write the IC constraint as:

$$V_n'(X_t, n_t) p_t k_t \left( \phi - 1 + (1 - \phi) \right) \leq 0$$

which for $V_n'(X_t, n_t) > 0$ boils down to

$$\tilde{\phi} \geq \phi$$

This is the HJB equation that characterizes the portfolio problem with the equity constraint in Definition 3. So if we solve that problem, we can build an optimal contract using the solution, as follows. Let $H(X, n)$ be the value function of the portfolio problem. We will use it for $V$. Let $(\hat{e}, g, \hat{k}, \hat{\theta})$ be the optimal policy functions. Consumption and capital holdings are built $e_t = \hat{e}_t n_t$ and $k_t = \hat{k}_t n_t$, while $g_t$ requires no transformation, and we define the cash flow $F$ as

$$dF_t = n_t \left( r_t dt + p_t k_t \left( dR^k_t - r_t dt \right) - \hat{e}_t dt - \frac{dn_t}{n_t} \right)$$

Finally, under no stealing, we can identify a history of $Z$ and $W_t$ with a history of $Z$ and $R^k_t$, so $(\hat{e}, g, \hat{k}, F)$ can be taken as functions of the histories of observables $Z$ and $R^k$, and hence $\mathcal{H}_t$-adapted. Because in the portfolio problem $n_t \geq 0$ and there is no stealing, $b_t - J_t = b_t = n_t \geq 0$ is satisfied in the contract. The value function of the portfolio problem is in fact twice continuously differentiable, so we have a classical solution to the HJB. Finally, we can use the fact that this is the solution to a portfolio problem to verify that $\mathbb{E}_t^Q \left[ \int_t^{\infty} \frac{B_{0,t}}{B_{0,u}} dF_u \right] = 0$ for all $(\omega, t)$, so the present value of the contract $J_t$ is in fact 0 at all times.

### 1.7.1 Proof of Proposition 4

**Proof.** Extend the expert's problem to allow him to chose his stealing $s$. If he steals, he adds a flow $\phi p_t k_t s_t dt$ to his net worth, but he loses the flow $\tilde{\phi} p_t k_t s_t dt$ because he owns the fraction $\tilde{\phi}$ of his own equity. He will then choose not to steal if and only if

$$p_t k_t \left( \phi - \tilde{\phi} \right) s_t \leq 0 \iff \tilde{\phi} \geq \phi$$

So he will chose $s = 0$. The extended problem then boils down to the constrained portfolio problem in Definition 3, and the expert can be allowed to chose his $(\hat{e}, g, \hat{k}, \hat{\theta}, \tilde{\phi})$ on his own, subject only

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47 I'm already using the fact it will be markov in the aggregate state variables $X = (\nu, x)$. In general it would be $H(\omega, t, n)$. 

51
to the equity constraint. This requires observing his choice of \( \phi \), i.e. how much of his own equity he keeps.

1.8 Appendix B: More proofs

1.8.1 Proof of Proposition 6

Proof. Without any financial frictions, idiosyncratic risk can be perfectly shared and has zero price in equilibrium. Capital then must be priced by arbitrage

\[
g_{t, t} + \mu_{p,t} + \sigma_{\sigma_{p,t}} + \frac{a - t (g_{t,t})}{p_t} = \pi_t (\sigma + \sigma_{p,t})
\]

(1.25)

and experts face the same portfolio problem as consumers, with the exception of the choice of the growth rate \( g \), pinned down by the static FOC

\[
\ell' (g_t) = p_t
\]

We have, in effect, a standard representative agent model with a stationary growth path with risk-free interest rate:

\[
r_t = \rho + \psi g_t - \frac{1}{2} (1 + \psi) \gamma \sigma^2
\]

and price of aggregate risk

\[
\pi_t = \gamma \sigma
\]

In a stationary equilibrium the price of capital is constant so we have \( \mu_{p,t} = \sigma_{p,t} = 0 \), and replacing all of this in (1.25) gives (1.8). For the agents' problem to be well defined we need \( \rho - (1 - \psi) g^* + (1 - \psi) \frac{1}{2} \sigma^2 > 0 \), since otherwise they could achieve infinite utility.

1.8.2 Proof of Proposition 9

Proof. From (1.16) we see that if \( \sigma_{\nu} = 0 \) then \( \sigma_{x,t} = 0 \). Furthermore, the idiosyncratic volatility of capital, \( \nu_t \) is then deterministic because it is the solution to an ODE (1.2). We can replace \( \nu_t \) with \( t \) in the Markov equilibrium (and obtain a time-dependent equilibrium). The only possibly stochastic state variable is \( x_t \), but we have seen that it can only have a stochastic drift. However, since all equilibrium objects are functions of \( x \) and time \( t \), then by (1.10) we see that \( x_t \) is the deterministic solution to a time-dependent ODE.

1.8.3 Proof of Proposition 10

Proof. We solve the experts' problem with the added constraint that \( \theta_t = \bar{\theta}_t B \). The FOC for \( \bar{\theta}_t \) yields:

\[
B \sigma_{\alpha,t} = B \pi_t \frac{\gamma}{\gamma - 1} + \frac{\gamma - 1}{\gamma} B \sigma_{\xi,t}
\]

52
with an analogous condition for consumers. Let's build the equilibrium with no balance sheet channel, where \( \sigma_{\xi,t} = \sigma_{\zeta,t} = \sigma_{p,t} = 0 \), it follows that \( B\sigma'_{x,t} = B\sigma'_{w,t} \), and from market clearing we obtain

\[
B\sigma'_{x,t} = 0
\]

In addition, from the formula for \( \sigma_{n,t} \)

\[
\sigma_{n,t} = \phi \frac{1}{x_t} \sigma + \bar{\theta}_t B
\]

Because \( \sigma \) is in the row space of \( B \), we can write \( \sigma = \kappa B \) for some \( \kappa \in \mathbb{R}^{d'} \), so we get \( \sigma_{n,t} = h_t B \), and \( \sigma_{x,t} = x_t (\sigma_{n,t} - \sigma) = m_t B \). Then we obtain

\[
BB'm' = 0
\]

from which it follows that \( m_t = 0 \), and \( \sigma_{x,t} = 0 \). By the same argument as in Proposition 9, \( x_t \) is deterministic.

1.9 Appendix C: solving for the equilibrium

The strategy to solve for the equilibrium when uncertainty shocks hit the economy is to first use optimality and market clearing conditions to obtain expressions for equilibrium objects in terms of the stochastic processes for \( p, \xi, \zeta \), and then use Ito's lemma to map the problem into a system of partial differential equations. In order to obtain a non-degenerate stationary long-run distribution for \( x \), I also introduce turnover among experts: they retire with independent Poisson arrival rate \( \tau > 0 \). When they retire they don't consume their wealth right away, they simply become consumers. Without turnover, experts want to postpone consumption and approach \( x_t \to 1 \) as \( t \to \infty \). Turnover modifies experts' HJB slightly:

\[
\rho \frac{\psi - 1}{1 - \psi} = \max_{\hat{c}, \hat{\xi}, \hat{\zeta}} \left[ \frac{\hat{c}^{1-\psi}}{1-\psi} \rho \hat{\xi}^{\psi-1} + \frac{\tau}{1 - \gamma} \left( \left( \frac{\zeta}{\xi} \right)^{1-\gamma} - 1 \right) + \mu_n \hat{c} + \mu_\xi - \frac{\gamma}{2} \left( \sigma_n^2 + \sigma_\xi^2 - 2(1-\gamma)\sigma_n \sigma_\xi + \sigma_\zeta^2 \right) \right]
\]

(1.26)

With Poisson intensity \( \tau \) the expert retires and becomes a consumer, losing the continuation utility of an expert, but gaining that of a consumer. For this reason, consumers' wealth multiplier \( \zeta \) appears in experts' HJB equation. Consumers have the same HJB equation as before. The FOC for consumption for experts and consumers are:

\[
\hat{c} = \rho \frac{1}{\psi} \hat{\xi}^{\psi-1} \psi
\]

\[
\hat{c} = \rho \frac{1}{\psi} \hat{\zeta}^{\psi-1} \psi
\]
So market clearing in the consumption goods market requires:

\[ \rho^{\frac{1}{\psi}} \left( \xi^{\frac{1}{\psi}} x + \zeta^{\frac{1}{\psi}} (1 - x) \right) = \frac{a - \lambda}{p} \quad (1.27) \]

Equation (1.16) provides a formula for \( \sigma_x \)

\[ \sigma_x = \frac{(1 - x)^{\frac{1}{\gamma}} \left( \frac{\zeta_x - \xi_x}{\zeta - \xi} \right) \sigma_{\nu} \sqrt{\nu_t}}{1 - (1 - x)^{\frac{1}{\gamma}} \left( \frac{\zeta_x - \xi_x}{\zeta - \xi} \right) \sigma_{\nu} \sqrt{\nu_t}} \]

We can use Ito’s lemma to obtain expressions for

\[ \sigma_p = \frac{p_{\nu}}{p} \sigma_{\nu} \sqrt{\nu_t} + \frac{p_x}{p} \sigma_x, \quad \sigma_\xi = \frac{\zeta_x}{\zeta} \sigma_{\nu} \sqrt{\nu_t} + \frac{\xi_x}{\zeta} \sigma_x, \quad \sigma_\zeta = \frac{\zeta_\nu}{\zeta} \sigma_{\nu} \sqrt{\nu_t} + \frac{\xi_x}{\zeta} \sigma_x \]

and the definition of \( \sigma_x \) from (1.10) to obtain an expression for

\[ \sigma_n = \sigma + \sigma_p + \frac{\sigma_x}{x} \]

Then we use experts FOC for aggregate risk sharing (1.12) to obtain an expression for the price of aggregate risk

\[ \pi = \gamma \sigma_n + (\gamma - 1) \sigma_\xi \]

Consumers’ exposure to aggregate risk is taken from (1.13):

\[ \sigma_w = \frac{\pi}{\gamma} - \frac{1}{\gamma} \sigma_\zeta \]

Experts’ exposure to idiosyncratic risk is given by \( \sigma_n = \frac{2}{x} \nu \). We can now use consumers’ budget constraint to obtain the drift of their wealth (before consumption)

\[ \mu_w = r + \pi \sigma_w \]

and plugging into their HJB equation we obtain an expression for the risk-free interest rate

\[ r = \rho \frac{1}{1 - \psi} + \frac{\psi}{1 - \psi} \rho^{\frac{1}{\psi}} \left( \frac{\psi - 1}{\psi} \right) \sigma_{\nu} - \mu_\zeta - \frac{\gamma}{2} \left( \sigma_w^2 + \sigma_\zeta^2 - 2(1 - \gamma) \sigma_w \sigma_\zeta \right) \]

where the only term which hasn’t been solved for yet is \( \mu_\zeta \). We use the FOC for capital (1.11) and the expression for the risk-free interest rate and plug into the formula for \( \mu_n \) from equation (1.6) to get

\[ \mu_n = r + \frac{1}{x^2} (\phi \nu)^2 + \pi \sigma_n \]

\( \underbrace{\alpha_x pk(\phi \nu)} \)}
In equilibrium experts receive the risk-free interest on their net worth, plus a premium for the idiosyncratic risk they carry through capital, $\gamma \frac{1}{x^2} (\phi \nu)^2$, and a risk premium for the aggregate risk they carry, $\pi \sigma _n$. This allows us to compute the drift of the endogenous state variable $x$ in terms of known objects, from (1.10) (appropriately modified for turnover) and (1.11)

$$
\mu _x = \mu _n - \dot{\theta} - \tau + \frac{\alpha - \zeta}{p} - r - \pi (\sigma + \sigma _p) - \frac{\gamma}{x} (\phi \nu)^2 + (\sigma + \sigma _p)^2 - \sigma _n (\sigma + \sigma _p)
$$

Turnover works to reduce the fraction of aggregate wealth that belongs to experts through the term $-\tau$. Using Ito’s lemma we get expressions for the drift of $p$, $\xi$, and $\zeta$:

$$
\mu _p = \frac{\nu _p}{p} \mu _\nu + \frac{\nu _x}{\xi} \mu _\mu + \frac{1}{2} \left( \frac{p \nu _p}{p} \sigma _p^2 \nu + 2 \frac{p \nu _x}{p} \sigma _\nu \sqrt{\nu _t} \sigma _x + \frac{p \nu _x}{p} \sigma _x^2 \right)
$$

$$
\mu _\xi = \frac{\xi _p}{\xi} \mu _\nu + \frac{\xi _x}{\xi} \mu _\mu + \frac{1}{2} \left( \frac{\xi _p}{\xi} \sigma _\nu ^2 \nu + 2 \xi _p \sigma _\nu \sqrt{\nu _t} \sigma _x + \xi _p \sigma _x ^2 \right)
$$

$$
\mu _\zeta = \frac{\zeta _p}{\zeta} \mu _\nu + \frac{\zeta _x}{\zeta} \mu _\mu + \frac{1}{2} \left( \frac{\zeta _p}{\zeta} \sigma _\nu ^2 \nu + 2 \frac{\zeta _p}{\zeta} \sigma _\nu \sqrt{\nu _t} \sigma _x + \frac{\zeta _p}{\zeta} \xi _x ^2 \right)
$$

Finally, experts’ HJB (1.26) and their FOC for capital (1.11) provide two second order partial differential equations in $p$, $\xi$, and $\zeta$. Together with (1.27) they characterize the markov equilibrium. The idiosyncratic volatility $\nu _t$ moves exogenously in $(0, \infty)$, and we conjectured that $x \in (0, 1)$. The system never reaches any of its boundaries, so they play no role in the solution. I verify this numerically after obtaining the equilibrium solution. We must only make sure that $p$, $\xi$, and $\zeta$ are strictly positive and the transversality conditions are satisfied.

### 1.10 Appendix D: Stochastic moral hazard

I first provide a proof of the equivalence between shocks to $\nu$ and shocks to $\phi$.

#### 1.10.1 Proof of Proposition 12

**Proof.** By inspection of the equations that characterize the equilibrium, we see $\phi$ and $\nu$ enter only in the idiosyncratic risk experts take $\sigma _{n,t} = \frac{1}{x} \phi \nu _t$ and the pricing equation for capital

$$
\gamma _t + \mu _{p,t} + \sigma _{p,t} \phi _t + \frac{\alpha - \zeta _t}{\gamma} - r_t = (\sigma + \sigma _{p,t}) \pi _t + \frac{\gamma}{x_t} (\phi \nu _t)^2
$$

Then we can take $\phi \nu$ as the state variable that characterizes the Markov equilibrium, and it doesn’t matter whether $\phi$ or $\nu$ is being shocked, except for the implementation $\theta _t = \sigma _{n,t} - \frac{\phi}{x} (\sigma + \sigma _{p,t})$. \(\square\)

In order to illustrate the solution technique when the boundary $x = 1$ is reached, I solve the model for intertemporally linear preferences, $\psi = 0$. Otherwise, the solution technique is similar to the one for the baseline model, with different boundary conditions. With infinite EIS, experts
postpone consumption until \( x = 1 \). Consumers must then consume all the economy’s output \((a - u_t)K_t\). From the FOC for consumption, we know that \( \zeta = \rho \) and \( \xi > \rho \) while \( x < 1 \). This already implies that \( \sigma_\zeta = \mu_\zeta = 0 \), so consumers don’t have a hedging motive for aggregate risk sharing. The general equilibrium interest rate and price of risk adjust endogenously to keep them indifferent between consuming and saving. The equation for \( \sigma_x \) then simplifies to:

\[
\sigma_x = \frac{(1 - x)x^{\frac{1-\gamma}{\gamma}\xi^{-\frac{\xi}{\gamma}}}}{1 - (1 - x)x^{\frac{1-\gamma}{\gamma}\xi^{-\frac{\xi}{\gamma}}}} \sigma_\phi
\]

and the HJB for consumers yields a simple formula for the risk-free interest rate:

\[
r = \rho - \frac{1}{2} \frac{\pi^2}{\gamma}
\]

The equation for market clearing can now be dropped, so we have two second-order partial differential equations for \( p(\phi, x) \) and \( \xi(\phi, x) \), given by experts’ HJB and the pricing equation for capital. The boundary behavior is different, however. Not only do we have reflective boundaries at \( \phi = 0 \) and \( \phi = 1 \), but we also have an absorbing state at \( x = 1 \). When the system reaches \( x = 1 \), consumers go bankrupt and drop out of the economy. With \( \tau = 0 \), this is an absorbing state, and the economy only has experts from then on. Since consumers are out of the picture, experts must now consume so \( \xi = \rho \) is required at the \( x = 1 \) boundary. Since consumers cannot lend to experts, nor share aggregate risk with them, \( \sigma_x = \mu_x = 0 \) (which is consistent with \( x = 1 \) being an absorbing state). The risk-free interest rate is now priced by experts’ HJB equation \( r^{BC} = \rho - \frac{1}{2} \frac{(p^{BC})^2}{\gamma} - \frac{1}{2} \gamma (\phi \nu)^2 \).

Experts’ FOC for capital is a second-order ODE for \( p^{BC} (\phi) \), with boundary conditions given by the reflective boundaries, \( p^{BC}_\phi (0) = p^{BC}_\phi (1) = 0 \).
Chapter 2

Optimal Financial Contracts with Recontracting

2.1 Introduction

Consider the problem of a risk-averse agent who wants to raise funds and share risk with a risk-neutral principal, but faces a moral hazard problem: he can secretly steal/divert funds to a private account. Since the private benefit is purely pecuniary, if the agent could commit to a long-term contract that controls his consumption the first best with full insurance could be implemented. In order to obtain a tractable but binding moral hazard problem, I introduce recontracting: at any point the agent can offer a new contract to the principal and rollover his obligations. This allows him to use the previously stolen funds to consume and invest and creates a moral hazard problem. This setting is quite natural in macro and financial applications. It is in fact a discrete-time analog\(^1\) of the setting I use in Di Tella (2013). In order to obtain a tractable solution, I allow the agent to steal after privately observing the current noise. This imposes a linear structure on the model that greatly simplifies the analysis. The main result in this paper is that the optimal contract can be characterized as the solution to a standard portfolio problem with a linear “skin in the game” constraint. And because the model requires very few restrictions on preferences and shocks, it can be easily incorporated into general equilibrium macro and financial models.

I consider an investment game where the agent trades and uses capital to produce consumption goods. In order to share risk, he sells a cash flow to a principal, who values it with a state price density. In order to make the problem easy to use in macro and financial applications, I distinguish between non-observable idiosyncratic shocks (which will create the moral hazard problem by “hiding” the agent’s stealing) and observable aggregate shocks, which not only affect the return of the project but also general equilibrium conditions. An important assumption is that the principal does not price idiosyncratic risk, so in the first best there is full insurance for idiosyncratic risk. I also impose

\(^1\)In that paper the volatility of the idiosyncratic shock is itself an aggregate (observable) stochastic variable.
few constraints on preferences and on the stochastic process for shocks, and take general equilibrium prices as arbitrary functions of the history of aggregate shocks.

A moral hazard problem arises because the agent can secretly divert capital to a private account and sell it. He only gets \( \phi \in (0,1) \) units of capital per unit diverted, so stealing is inefficient. I first focus on contracts that implement no stealing at all time, and then show this is without loss of generality. I use a timing convention as in Edmans and Gabaix (2011) or DeMarzo and Fishman (2007), which greatly simplifies the analysis. Whereas in standard moral hazard problems the private action is taken before the realization of shocks, here I assume that the agent first observes the shocks and then decides how much to steal. Incentive compatibility needs to be satisfied state-by-state (for every shock realization), rather than in expectation, constraining the set of admissible contracts and imposing a linear structure.

Private fund diversion is, however, not enough to create a moral hazard problem on its own. A salient feature of the setting is that the private benefit from stealing is purely pecuniary. The agent doesn’t get utility from stealing per se, only from consumption, so if he is committed to a long-term contract that controls his consumption, the first best with full idiosyncratic insurance can be achieved. To create a moral hazard problem I introduce a form of limited commitment. At the beginning of any period the agent can offer a new contract that gives the principal at least the same continuation value as under the original contract. This allows him to use his stolen funds to consume and invest. By stealing the agent increases his hidden savings, but reduces the observed return of the project. In order to induce no stealing the contract gives the agent some “skin in the game”, so that if the observed return is low his legitimate net worth must fall to undo the private benefit of stealing. I first assume that whenever the agent recontracts (or writes the contract at \( t = 0 \)), his full hidden savings are observable. I then relax this assumption and show that the agent willingly reveals his full hidden savings as soon as possible. The intuition for this result is that since incentive compatibility constraints don’t interact with the agent’s wealth (legitimate or hidden), it is always optimal to put his hidden savings into the contract.

I then consider the case of state-contingent private action. The timing convention which simplifies the contract when we want to implement no stealing at all times also introduces the possibility of implementing a “stealing function”. It might be the case that by allowing the agent to steal after some realization of the shocks, the contract might be able to relax risk-sharing sufficiently to overcome the cost of stealing. However, this is not the case, and it is optimal to implement no stealing at all times.

This paper is most closely connected to the literature on financial contracting with pecuniary benefits, such as DeMarzo and Fishman (2007) and DeMarzo and Sannikov (2006). In both models the agent can inefficiently divert funds to a private account. He can then use those funds to secretly consume, but he is committed to a long-term contract. Because preferences are linear in those papers, the contract can be solved assuming all stolen funds must be immediately consumed, and this turns out to be optimal for the agent even if he could secretly save. The drawback of
this approach is that with risk-averse preferences the contract becomes more difficult to solve, making it less appealing for macro and financial applications which typically require risk-aversion. Edmans and Gabaix (2011) and Edmans, Gabaix, Sadzik, and Sannikov (2011) also use the timing convention in this paper to derive tractable incentive contracts in a setting with full commitment (In DeMarzo and Fishman (2007) stealing also occurs after the agent observes the cash flow). In a macro setting, Brunnermeier and Sannikov (2012) use a similar setting with cash flow diversion and lack of commitment (no lock-ups) in a continuous-time setting where shocks and the private action occur simultaneously. In their setting, however, there are no idiosyncratic shocks, but aggregate shocks are taken to be unobservable. By explicitly allowing for both observable aggregate shocks and non-observable idiosyncratic shocks I can better distinguish the type of risk-sharing that is constrained by the moral hazard problem. He and Krishnamurthy (2011) use a short-term contracts approach, with the same observability restrictions as in Brunnermeier and Sannikov (2012). In Di Tella (2013) I use a continuous-time analog of the setting in this paper, with the addition that the volatility of the idiosyncratic shock is an aggregate (observable) stochastic variable.

The rest of the paper is organized as follows. In Section 2 I introduce the model and solve for the first best optimal contract without moral hazard. I also show that with full commitment to long-term contracts that control the agent’s consumption the first best can be achieved. In Section 3 I introduce recontracting and characterize optimal contracts that implement no stealing at all times, under the assumption that all hidden savings are fully revealed upon recontracting. I then relax this assumption and show it is without loss of generality. Finally, I consider state-contingent private actions, and show it is optimal to implement no stealing at all times. Section 4 concludes.

2.2 The model

2.2.1 Setup

Time is discrete with finite horizon \( t \in \mathcal{T} = \{0, 1, \ldots, T\} \). A risk-averse agent can trade and use capital to produce, and wants to share risk with a principal who is risk-neutral with respect to idiosyncratic risk.

At the beginning of period \( t \), the agent buys \( k_t \) units of capital at price \( p_t \), he produces output \( ak_t \) in consumption goods (the numeraire), and obtains at the end of the period

\[
\tilde{k}_{t+1} = k_t(1 + e_t + u_t - s_t)
\]

units of capital. Here \( e_t \sim iid \) is an observable aggregate shock with interval support \([\underline{e}, \overline{e}]\), \( u_t \sim iid \) a private idiosyncratic shock with interval support \([\underline{u}, \overline{u}]\), both with \( \mathbb{E}[e_t] = \mathbb{E}[u_t] = 0 \). After observing \( e_t \) and \( u_t \), the agent can steal capital \( s_t \). His stealing must be consistent with a bad realization of the idiosyncratic shock (otherwise the agent can be punished), so he is restricted to

---

2 Otherwise there wouldn't be a moral hazard problem in their setting.
choosing \( s_t \in [0, u_t - u] \). To make sure \( \bar{k}_t \geq 0 \), I assume \( \xi + u \geq -1 \). With some abuse of notation, I will refer to \( R_t = e_t + u_t - s_t \) as the “return” of capital, which is observable. For each unit of capital stolen the agent can keep only a fraction \( \phi \in (0,1) \), which he must immediately sell at price \( p_{t+1} \), so that his hidden savings evolve:\(^3\)

\[
S_{t+1} = S_t(1 + r_t) + \phi p_{t+1} k_t s_t
\]

The parameter \( \phi \) captures the severity of the moral hazard problem. With \( \phi = 0 \), for example, there is no moral hazard. Because \( \phi < 1 \), stealing is inefficient: in the first best where \( s_t \) is observable there is no stealing (\( s_t = 0 \) always). In fact, no stealing will always be optimal. The approach I take here is to first solve for the optimal contract that implements no stealing, and then show that this is actually optimal.

Capital is traded in a competitive market with price \( p_t \), and there is a complete financial market with state price density \( \eta_t \) (with an implied risk-free interest rate \( r_t \)). Both the price of capital and the state price density are functions of the history of aggregate shocks \( e^t = (e_0, e_1, ... e_{t-1}) \), with \( e^0 = \emptyset \). Notice the aggregate shocks that affect the return to capital may also have general equilibrium effects on prices. I don’t model the general equilibrium here. The objective is to derive the optimal contract for any general equilibrium prices. In particular, the financial market does not price idiosyncratic risk \( u_t \). I will typically suppress the dependence of \( p_t \) and \( \eta_t \) on \( e^t \) to simplify notation.

The agent has an observable bank balance starting with \( b_0 > 0 \) funds, which evolves:

\[
b_{t+1} = b_t(1 + r_t) + ak_t + p_{t+1} k_t (1 + R_t) - p_t k_t (1 + r_t) - c_t (1 + r_t) - F_{t+1}
\]

(2.1)

where \( c_t \) is the agent’s consumption at the beginning of period \( t \), and \( F_{t+1} \) a cash payment from the agent to the principal at the end of period \( t \).

Timing. To summarize, each period \( t \) the following happens:

1. The agent chooses capital \( k_t \) and consumes \( c_t \)
2. shocks \((e_t, u_t)\) are observed by the agent, who then chooses \( s_t \)
3. the return \( R_t = e_t + u_t - s_t \) and the aggregate shock \( e_t \) are observed by everyone.
4. the agent pays \( F_{t+1} \) to the principal
5. the agent secretly sells\(^4\) his stolen capital \( \phi k_t s_t \), at price \( p_{t+1} \) and adds it to his hidden savings \( S_{t+1} \)

\(^3\)Here I assume his hidden savings yield the risk-free interest rate \( r_t \) which will be explained below.

\(^4\)We could also imagine this happening at the beginning of period \( t + 1 \).
Histories. We already introduced the history of aggregate shocks \( e^t = (e_0, e_1, ..., e_{t-1}) \). The history of capital returns \( R^t = (R_0, R_1, ..., R_{t-1}) \) is also observable, so the observable history\(^5\) is \( h^t = (e^t, R^t) \). The agent also observes the history of his private idiosyncratic shock \( u^t = (u_0, u_1, ..., u_{t-1}) \), so his privately observed history \( x^t = (e^t, u^t) \). The stealing activity of the agent is a contingent plan \( s = \{ s_t(x^t, e_t, u_t) \} \). Therefore, the history of returns \( R^t \) is a function of the history of shocks and the contingent stealing plan of the agent. With some abuse of notation I will sometimes write \( s = 0 \) to mean the stealing strategy \( s \) with \( s_t(x^{t+1}) = 0 \) for all \( t \).

Contracts. A contract \( C = (c, k, F, b) \) specifies consumption \( c = \{ c_t(h^t) \geq 0; t \in T \} \), capital \( k = \{ k_t(h^t) \geq 0; t \in T \} \), cash flows \( F = \{ F_{t+1}(h^t, e_t, R_t); t \in T \} \), and the bank account of the agent \( b = \{ b_t(h^t); t \in T \} \). Given a contract \( C \) the agent chooses his stealing plan \( s = \{ s_t(x^t, e_t, u_t); t \in T \} \) to solve

\[
\max_s \mathbb{E}^s \left[ \sum_{t=0}^{T} \beta^t u(c_t(h^t)) \right]
\]

where the expectation is taken over the shocks \( e \) and \( u \), and under the stealing plan \( s \). Call \( s(C) \) the optimal stealing plan for the agent given contract \( C \). The principal values the continuation contract \( C \) after history \( h^t \) as the conditional present value of the promised cash flow (he just sells the equilibrium payoff on the financial market) given by: \(^7\)

\[
J_t(h^t) = \mathbb{E}^{s(C)} \left[ \sum_{u=t}^{T} \eta_{u+1} F_{u+1}(h^{u+1}) | h^t \right]
\]

The principal’s participation constraint is \( J_0(h^0) \geq 0 \), and in order to make the problem well defined I add a solvency constraint: \( b_t(h^t) - J_t(h^t) \geq 0 \) for all \( h^t \). The agent must always have enough funds to cover the present value of cash flow he promised to the principal. We say a contract is feasible if \( b \) satisfies the budget constraint (2.1) with \( b(h^0) = b_0 \) given, the solvency constraint \( b_t(h^t) - J_t(h^t) \geq 0 \) for all \( h^t \) and the participation constraint of the principal \( J_0 \geq 0 \). A feasible contract is incentive compatible if \( s(C) = 0 \). An incentive compatible contract is optimal if it maximizes the agent’s expected utility.

\[
\mathbb{E}^{x=0} \left[ \sum_{t=0}^{T} \beta^t u(c_t(h^t)) \right]
\]

It will be useful to define the legitimate net worth of the agent as \( n_t(h^t) = b_t(h^t) - J_t(h^t) \) for all \( h^t \). We can characterize feasible contracts with the following result:

**Proposition 16.** If \( C = (c, k, F, b) \) is a feasible contract, then the agent’s legitimate net worth

---

\(^5\) All histories at time \( t = 0 \) are \( \emptyset \).

\(^6\) Since he controls his stealing activity, \( R^t = e_t + u_t - s_t \) is superfluous for the agent.

\(^7\) The expectation is also conditional on the initial bank balance of the agent, \( b_0 \) which is known. When I introduce recontracting later, the principal will also condition on the recontracting history.
\( n_t(h^t) = b_t(h^t) - J_t(h^t) \) satisfies:

\[
\begin{align*}
    n_{t+1}(h^t, e_t, R_t) &= n_t(h^t)(1 + r_t) + a_k(h^t) + n_{t+1}(h^t)(1 + R_t) \\
    &- p_k(h^t)(1 + r_t) - c_t(h^t)(1 + r_t) - m_{t+1}(h^t, e_t, R_t)
\end{align*}
\]

(2.2)

for some process \( m = \{m_{t+1}(h^t, e_t, R_t); t \in \mathcal{T}\} \) such that

\[
\mathbb{E}^s(C) \left[ \frac{\eta_{t+1}}{\eta_t} m_{t+1}(h^t, e_t, R_t) | h^t \right] = 0
\]

(2.3)

with \( n_t(h^t) \geq 0 \) for all \( h^t \), and \( n(h^0) \leq b_0 \).

Conversely, given processes \( c, k, n \text{ and } m \) satisfying equations (2.2) and (2.3), and with \( n_t(h^t) \geq 0 \) for all \( h^t \) and \( n(h^0) \leq b_0 \), there is a cash flow \( F \) and bank balance \( b \) such that \( (c, k, F, b) \) is a feasible contract.

**Proof.** In the first direction, take a feasible contract \( C = (c, k, F, b) \). From the definition of \( n_t \) we have (suppressing the history dependence)

\[
    n_{t+1} = b_{t+1} - J_{t+1}
\]

\[
    n_{t+1} = b_t(1 + r_t) + a_k + p_{t+1}k_t(1 + R_t) - p_k(k_t(1 + r_t) - c_t(1 + r_t) - F_{t+1} - J_{t+1}
\]

and we can write (where the notation is straightforward)

\[
    J_t = \mathbb{E}^s(C) \left[ \sum_{u=t}^{T} \frac{\eta_{u+1}}{\eta_t} F_{u+1} \right] = \mathbb{E}^s(C) \left[ \frac{\eta_{t+1}}{\eta_t} (F_{t+1} + J_{t+1}) \right]
\]

So if we define \( m_{t+1}(h^t, e_t, R_t) = F_{t+1}(h^t, e_t, R_t) + J_{t+1}(h^t, e_t, R_t) - J_t(h^t)(1 + r_t) \) we have

\[
\mathbb{E}^s(C) \left[ \frac{\eta_{t+1}}{\eta_t} m_{t+1}(h^{t+1}) \right] = 0
\]

and we obtain equation (2.2) and (2.3) as desired. In addition, because a feasible contract has \( b_t(h^t) - J_t(h^t) \geq 0 \) for all \( h^t \) and \( J_0 \geq 0 \), we get \( n_t(h^t) \geq 0 \) for all \( h^t \) and \( n(h^0) = b_0 - J_0 \leq b_0 \).

In the other direction, first let’s say that \( n(h^0) = b_0 \) and we will set \( J_t(h^t) = 0 \) for all \( h^t \) so that \( b_t(h^t) = n_t(h^t) \) for all \( h^t \) satisfies the budget constraint. To do this, let \( F_{t+1}(h^{t+1}) = m_{t+1}(h^{t+1}) \) and we can verify that in fact

\[
    J_t = \mathbb{E}^s(C) \left[ \sum_{u=t}^{T} \frac{\eta_{u+1}}{\eta_t} F_{u+1} \right] = \mathbb{E}^s(C) \left[ \sum_{u=t}^{T} \frac{\eta_{u+1}}{\eta_t} m_{u+1} \right] = \mathbb{E}^s(C) \left[ \sum_{u=t}^{T} \frac{\eta_{u+1}}{\eta_t} \mathbb{E}^s(C) \left[ \frac{\eta_{u+1}}{\eta_u} m_{u+1} \right] \right] = 0
\]

Because \( n_t(h^t) \geq 0 \) we also have \( b_t(h^t) - J_t(h^t) \geq 0 \). If instead \( n_0(h^0) < b_0 \), let \( J_0(h^0) = b_0 - n_0(h^0) \) and \( F_1(h^1) = m_1(h^1) + J_0(1 + r_0) \). All the other \( F_t \) for \( t = 2...T + 1 \) are as before. So \( J_t(h^t) = 0 \) for
all \( h^t \) with \( t \geq 1 \), and \( J_0(h^0) = b_0 - n_0(h^0) \) as desired. Now \( b_0(h^0) = b_0 \) as required by feasibility and \( b_t(h^t) = n_t(h^t) \) for all \( h^t \) with \( t \geq 1 \) satisfied the budget constraint (we use money from the bank balance to cancel the outstanding debt to the principal in period \( t = 1 \)). The resulting contract is therefore feasible as desired.

In light of Proposition 16, we can work with \((c, k, n, m)\) instead of \((c, k, F, b)\). Equations (2.2) and (2.3) take the form of a dynamic budget constraint. The agent can use \( m \) to share risk with the financial market via de principal.

2.2.2 First Best without moral hazard

As a benchmark, let us first solve for the optimal contract without moral hazard: \( s_t \) is now contractible and can be set to 0 after all histories \( x^{t+1} \).

**Proposition 17** (First best). The optimal contract without moral hazard \( C = (c, k, n, m) \) solves the following problem

\[
\max_{(c, k, n, m)} \mathbb{E}^{s=0} \left[ \sum_{t=0}^{T} \beta^t u(c_t(h^t)) \right]
\]

\[
n_{t+1}(h^{t+1}) = n_t(h^t)(1 + r_t) + ak_t(h^t) + p_{t+1}k_t(h^t)(1 + R_t)
\]

\[
- p_{t}k_t(h^t)(1 + r_t) - c_t(h^t)(1 + r_t) - m_t(h^{t+1})
\]

\[
\mathbb{E}^{s=0} \left[ \frac{\eta_{t+1}}{\eta_t} m_{t+1}(h^{t+1})|h^t \right] = 0
\]

\[
n_0(h^0) \leq b_0 \quad n_t(h^t) \geq 0 \quad \forall h^t
\]

Notice the expectation in the budget constraint is now taken under no stealing. The first best contract is the solution to a standard portfolio problem. An important property of this portfolio problem is that it has full insurance for idiosyncratic risk \( u_t \), since the principal is effectively risk-neutral with respect to idiosyncratic risk while the principal is risk-averse.

**Proposition 18.** The first best contract without moral hazard has full insurance against idiosyncratic risk. For two histories \( h^t = (e^t, R^t) \), and \( h^t = (\tilde{e}^t, \tilde{R}^t) \) which differ only in the idiosyncratic shock, i.e. \( e^t = \tilde{e}^t \), we have \( c_t(h^t) = c_t(h^t) \).

**Proof.** This is a standard result. \(\square\)

---

\(\text{We can also add an initial payment in the contract to do this, but it is not necessary to obtain the result.}\)
2.2.3 Moral hazard with full commitment

Now we can add moral hazard back in. In order to implement no stealing, we need to add the incentive compatibility constraint to the problem.

\[
0 \in \arg \max_{s \in \{s_t(x^{t+1})\}} E^s \left[ \sum_{t=0}^{T} \beta^t u(c_t(h^t)) \right]
\]

Because the first best contract has full insurance with respect to idiosyncratic risk, as per Proposition 18, the stealing activity of the agent has no effect on his consumption and hence his utility under this contract. Because the private benefit of stealing is purely pecuniary, and the contract controls his consumption, stealing does not benefit the agent, and so the first best contract is, in fact, incentive compatible.

**Proposition 19.** Under full commitment, the first best contract is also incentive compatible and hence the optimal contract.

One approach to make the moral hazard problem binding is to make the agent’s consumption unobservable, so that he can use his hidden savings to consume. If the agent must immediately consume what he steals, we have the setting in DeMarzo and Sannikov (2006), for example. Here instead, I will introduce recontracting.

2.3 Recontracting

In this section I introduce recontracting. At the beginning of period \( t \) the agent has contract \( C = (c, k, F, b) \). He can offer a new contract \( \tilde{C} = (\tilde{c}, \tilde{k}, \tilde{F}, \tilde{b}) \) to the principal. The new contract specifies consumption \( \tilde{c} \), capital \( \tilde{k} \), cash payments \( \tilde{F} \) and the observable bank account of the agent \( \tilde{b} \) for the continuation histories of \( h^t \). Assume for now that, just as the agent’s initial bank balance \( b_0 \) was observable, when the agent recontracts he reveals his hidden savings \( S \) and adds them to his observable bank balance \( b_t \), so \( \tilde{b}_t(h^t) = b_t(h^t) + S \). I will later show how this is without loss of generality. If the continuation value for the principal with the new contract is higher than with the old one, then the principal accepts the new contract. Otherwise, they keep the old contract.

Let \( V_t(h^t, C, S) \) be the value function of an agent that after history \( h^t \) has contract \( C \) and \( S \geq 0 \) in hidden savings (the result of previous stealing activity). Faced with contract \( C \) the agent choses a stealing strategy \( s = \{s_t(x^t, e_t, u_t); t \in T\} \) and a stopping time\(^{10} \) \( \tau \), to maximize

\[
E^s \left[ \sum_{t=0}^{\tau} \beta^t u(c_t(h^t)) + \beta^\tau V_\tau(h^\tau, C, S, x^\tau) \right].
\]

We say a feasible contract is *incentive compatible* if the optimal stealing strategy \( s(C) = 0 \). As before, an incentive compatible contract is *optimal* if it maximizes the agent’s utility.

---

\(^9\)With hidden savings and unobservable consumption the problem becomes considerably more difficult unless the agent is risk-neutral. This is in fact the case considered in DeMarzo and Sannikov (2006).

\(^{10}\)A stopping time is a random variable \( \tau(x^T) \in T \) adapted to the filtration generated by the private history \( x^t \).
Proposition 20. The value function for the agent at any $t$, after any history $h^t$ with incentive compatible contract $C$ and hidden savings $S$ takes the form

$$V_t(h^t, C, S) = V_t(e^t, b_t(h^t) - J_t(h^t) + S)$$

and is strictly increasing and concave in the second argument. A feasible contract $C = (c, k, n, m)$ has

$$m_{t+1}(h^t, e_t, R_t) = \overline{m}_{t+1}(h^t, e_t) + p_{t+1}k_t(h^t)(1 - \phi)(R_t - e_t - 1) \quad \forall h^t, e_t, R_t, t < T$$

Proof: The proof is by backwards induction, as follows.

Last period $t = T$. At the beginning of period $t = T$, after history $h^t$ the agent has incentive compatible contract $C = (c, k, F, \bar{b})$ and hidden savings from previous stealing activity $S$. Let $V_T(h^T, C, S)$ be his value function. He can keep the old contract $C$, in which case he gets

$$V_T^{NR}(h^T, C, S) = u(c_T(h^T))$$

Since this is the last period, he can’t do anything with his hidden savings $S_{T+1}$, so whatever stealing he does has no effect on his utility and he might as well not steal. There is in fact no moral hazard problem. The continuation value of $C$ for the principal is therefore

$$J_T(h^T) = \mathbb{E}^s_{T=0} \left[ \frac{\eta_{T+1}}{\eta_T} F_{T+1}(h^{T+1})|h^T \right]$$

Note that the agent’s hidden savings $S$ are irrelevant for this computation. The agent can instead offer a new continuation contract $\tilde{C} = (\tilde{c}, \tilde{k}, \tilde{F}, \tilde{b})$. For this contract there is no moral hazard as well, so the continuation value for the principal if he accepts the new contract is

$$\tilde{J}_T = \mathbb{E}^s_{T=0} \left[ \frac{\eta_{T+1}}{\eta_T} \tilde{F}_{T+1}(h^{T+1})|h^T \right]$$

He will accept if and only if $\tilde{J}_T \geq J_T(h^T)$. This condition is in fact the principal’s participation constraint for the new contract. As before we can work with $(\tilde{c}, \tilde{k}, \tilde{n}, \tilde{m})$, so the agent can design the new contract solving the following problem.

$$V_T^R(h^T, C, S) = \max_{c, k, n'(e_T, R_T), m(e_T, R_T)} u(c)$$

st:

$$n'(e_T, R_T) = n_T(1 + r_T) + ak_T + p_{T+1}k_T(1 + R_T) - cT(1 + r_T) - c_T(1 + r_T) - m(e_T, R_T)$$

$$\mathbb{E}^s_{T=0} \left[ \frac{\eta_{T+1}}{\eta_T} m(e_T, R_T)|h^T \right] = 0$$

$$n_T \leq b_T(h^T) - J_T(h^T) + S \quad n'(e_T, R_T) \geq 0 \forall e_T, R_T$$
The old contract $C$ of course satisfies the constraints (because $S \geq 0$), so $V_T^R(h^T, C, S) \geq V_T^{NR}(h^T, C, S)$ for any $C$, any $h^T$ and any $S \geq 0$. It is always optimal for the agent to recontract at time $t = T$.

$$V_T(h^T, C, S) = \max \{ V_T^R(h^T, C, S); V_T^{NR}(h^T, C, S) \} = V_T^R(h^T, C, S)$$

Notice furthermore that $S$ and $C$ enter the problem only through $b_T(h^T) - J_T(h^T) + S$. So we can write $V_T^R(h^T, C, S) = V_T^R(e^T, b(h^T) - J(h^T) + S)$. Any contract $C$ that leaves the agent with the same total funds (legitimate net worth $b - J$ plus hidden funds $S$) will yield the same continuation utility (and continuation contract). In addition, a higher $b - J + S$ relaxes the constraints, so $V_T(e^T, b - J + S)$ is increasing in the second argument. And because the constraints are linear in $b - J + S$ and the objective function concave, we see that $V_T(e^T, b - J + S)$ is concave in the second argument. The history of aggregate shocks $e^T$ still matters because it affects prices ($p$ and $\eta$). We conclude that at $t = T$ it is always optimal for the agent to recontract, and his continuation utility at the beginning of period $t = T$ is $V_T(e^T, b - J + S)$ which is strictly increasing and concave in the second argument.

**Periods before the last $t < T$.** At the beginning of period $t < T$, after history $h^t$, the agent has feasible contract $C$, and hidden savings $S$. Assume the value function for next period, for any incentive compatible contract $\tilde{C}$, takes the form $V_{t+1}(h^{t+1}, \tilde{C}, S') = V_{t+1}(e^{t+1}, \tilde{b} - \tilde{J} + S')$, increasing and concave in the second argument, and where I denote $\tilde{C}$ the contract the agent has at the beginning of period $t + 1$ (it could be different from $C$) and $\tilde{b}$, $\tilde{J}$ and $S'$ are also the values of these variables at $t + 1$ under contract $\tilde{C}$. Under contract $C$, the agent can choose how much to steal contingent on the shocks $e_t$ and $u_t$ he observes:

$$\max_{s_t(e_t, u_t)} \mathbb{E}^s \left[ \beta V_{t+1}(e^{t+1}, b_{t+1}(h^t, e_t, R_t) - J_{t+1}(h^t, e_t, R_t) + S'(h^t, e_t, u_t)) | h^t \right]$$

$$st: \quad s_t(e_t, u_t) \in [0, u_t - u] \quad \forall e_t, u_t$$

where $S'(h^t, e_t, u_t) = S(1 + r_t) + \phi p_{t+1}k_t(h^t)s_t(e_t, u_t)$. Since we want to implement no stealing for all $e_t$ and $u_t$, we can get rid of the expectation, and since $V_{t+1}(e^{t+1}, b' - J' + S')$ is increasing in the second argument, we only need to optimize $b - J + S = n + S$. Using Proposition 16 we obtain the condition that for each $(e_t, u_t)$

$$s_t(e_t, u_t) \in \arg \max_{s \in [0, u_t - u]} \left\{ \begin{array}{l}
 n_t(1 + r_t) + ak_t + p_{t+1}k_t(1 + e_t + u_t - s) - p_tk_t(1 + r_t) \\
 -c_t(1 + r_t) - m_{t+1}(h^t, e_t, 1 + e_t + u_t - s) + S(1 + r_t) + \phi p_{t+1}k_t s
\end{array} \right\}$$

$\text{11 Also remember that the price of capital } p_{t+1} \text{ and the interest rate } r_t \text{ are functions of the history of aggregate shocks, but as before I suppress the notation to avoid clutter.}$

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where I suppress the dependence on the history to simplify notation (e.g. \( k_t(h^t) \to k_t \), or \( m_{t+1}(h^t, e_t, R_t) \to m_{t+1}(e_t, R_t) \)). This in turn simplifies to

\[
s_t(e_t, u_t) \in \arg \max_{s \in [0, u_t - y]} \{ p_{t+1} k_t(1 + e_t + u_t - s) - m_{t+1}(e_t, 1 + e_t + u_t - s) + \phi p_{t+1} k_t s \}
\]

Notice that the agent's hidden savings \( S \) and his legitimate net worth \( n_t = b_t - J_t \) are not relevant for the decision of how much to steal. Not stealing \( (s = 0) \) is optimal for all \( e_t \) and \( u_t \) if and only if \( \forall (e_t, u_t, s \in [0, u_t - y]) \)

\[
p_{t+1} k_t(1 + e_t + u_t - s) - m_{t+1}(e_t, 1 + e_t + u_t - s) + \phi p_{t+1} k_t s \leq p_{t+1} k_t(1 + e_t + u_t) - m_{t+1}(e_t, 1 + e_t + u_t)
\]

\[
\iff m_{t+1}(e_t, 1 + e_t + u_t) - m_{t+1}(e_t, 1 + e_t + u_t - s) - p_{t+1} k_t(1 - \phi)s \leq 0
\]

This in turn implies that the function \( g(e_t, y) = m_{t+1}(e_t, y) - p_{t+1} k_t(1 - \phi)y \) is decreasing in \( y \) for all \( y \in [u, \bar{u}] \) and for all \( e_t \). The function \( g(e_t, y) \) must therefore be almost everywhere differentiable with respect to \( y \), for any \( e_t \), with at most a countable number of downward jump discontinuities. The function \( m_{t+1}(e_t, R_t) \) inherits some of these properties: it must also be almost everywhere differentiable with respect to \( R \) with at most a countable number of jump discontinuities, with derivative (where it exists):

\[
\frac{\partial m_{t+1}(e_t, R_t)}{\partial R_t} \leq p_{t+1} k_t(1 - \phi) \quad \forall e_t, R_t
\]

Because the agent is risk-averse (recall the value function for next period is concave) and the principal risk-neutral with respect to idiosyncratic risk, and because to achieve full insurance we would like to set \( m_{t+1}(e_t, R) = \bar{m}_{t+1}(e_t) + p_{t+1} k_t u_t \) we can without loss of generality restrict our attention to contracts where

\[
m_{t+1}(e_t, R_t) = \bar{m}_{t+1}(e_t) + p_{t+1} k_t(1 - \phi)(R_t - e_t - 1)
\]

The agent's legitimate net worth is then

\[
n_{t+1} = n_t(1 + r_t) + a k_t + p_{t+1} k_t(1 + R_t) - p_t k_t(1 + r_t)
\]

\[
- c_t(1 + r_t) - \bar{m}_{t+1}(e_t) - p_{t+1} k_t(1 - \phi)(R_t - e_t - 1)
\]

And as a function of \( e_t \) and \( u_t \) (which he observes), his total funds net period \( n_{t+1} + S_{t+1} \) are

\[
n_{t+1} + S_{t+1} = (n_t + S_t)(1 + r_t) + a k_t + p_{t+1} k_t(1 + e_t) - p_t k_t(1 + r_t) - c_t(1 + r_t)
\]

\[
- \bar{m}_{t+1}(e_t) + p_{t+1} k_t \phi(u_t - s) + p_{t+1} k_t s
\]
Stealing does not affect his future total funds and so we verify the contract is incentive compatible. As a result of the incentive compatibility constraint, the agent ends up exposed to his idiosyncratic risk through $\phi p_{t+1} k_t u_t$. But at least he is able to get rid of fraction $1 - \phi$ of this risk. On the other hand, moral hazard does not limit his ability to share aggregate risk as he sees fit. He must only satisfy the present value condition:

$$
\mathbb{E}_{\pi} = 0 \left[ \frac{\eta_{t+1}}{\eta_t} m_{t+1}(e_t, R_t) | h^t \right] = \mathbb{E}_{\pi} = 0 \left[ \frac{\eta_{t+1}}{\eta_t} m_{t+1}(e_t) + \frac{\eta_{t+1}}{\eta_t} p_{t+1} k_t (1 - \phi)(u_t - s) | h^t \right] = 0
$$

$$
\iff \mathbb{E}_{\pi} = 0 \left[ \frac{\eta_{t+1}}{\eta_t} m_{t+1}(e_t) | h^t \right] = 0
$$

If the agent starts period $t$ with contract $C$ which is incentive compatible, then if he does not recontract, he obtains utility

$$
V_t^{NR}(h^t, C, S) = u(c_t(h^t)) + \beta \mathbb{E}_{\pi} = 0 [V_{t+1}(e_t^{t+1}, n_{t+1}(h^{t+1}) + S(1 + r_t)) | x^t]
$$

The principal, on the other hand, knows that if he stays with contract $C$ which is incentive compatible he obtains continuation value $J_t$ (and recall he doesn’t need to know $S_t$ to make sure the contract is incentive compatible). The agent can offer a new contract $\tilde{C}$, revealing his hidden savings and adding them to his bank balance, and the new contract must provide the principal with at least the same level of continuation value: $\tilde{J}_t \geq J_t(h^t)$. So an agent with incentive compatible contract $C$, after history $h^t$, and with hidden savings $S$, can offer the principal a new continuation contract and obtain utility

$$
V_t^R(h^t, C, S) = \max_{(c, k, n'(e_t, R_t), \tilde{m}(e_t))} u(c) + \beta \mathbb{E}_{\pi} = 0 [V_{t+1}(e_t^{t+1}, n'(e_t, R_t))]
$$

$$
st : \quad n'(e_t, R_t) = n_t(1 + r_t) + ak + p_{t+1} k(1 + e_t) - p_t k(1 + r_t) - c(1 + r_t) - \tilde{m}(e_t) + p_{t+1} k_t \phi (R_t - e_t - 1)
$$

$$
\mathbb{E}_{\pi} = 0 \left[ \frac{\eta_{t+1}}{\eta_t} \tilde{m}(e_t) | h^t \right] = 0
$$

$$
n_t \leq b_t(h^t) - J_t(h^t) + S \quad n'(e_t, R_t) \geq 0 \quad \forall e_t, R_t
$$

As before, the old contract $C$ and the agent’s hidden savings $S$ enter only through $b_t(h^t) - J_t(h^t) + S$, so $V_t^R(h^t, C, S) = V_t^R(e^t, b - J + S)$, and because a larger $b_t(h^t) - J_t(h^t) + S$ relaxes the constraint, $V_t^R$ is increasing in the second variable. The constraints are linear and the objective function concave in $b - J + S$, so the value function must be concave as well. In addition, the old

\[12\]the expectation is conditional on his information set, but this is equivalent to using $h^t$ since the agent’s knowledge of his past stealing behavior is not relevant as long as we know $S_t$. 68
contract \( C \) also satisfies the constraints, so we have \( V_t^R(h^t, C, S) \geq V_t^{NR}(h^t, C, S) \) and therefore

\[
V_t(h^T, C, S) = V_t^R(e^t, b(h^t)) - J(h^t) + S
\]

is the value function for the agent. The agent always chooses to recontract (and reveal his hidden savings to the principal). By induction (backwards) we obtain the desired result. 

**Optimal contract under recontracting.** In light of Proposition 20, we can characterize the optimal contract under recontracting as the solution to the following portfolio problem. 

**Proposition 21.** The optimal contract with recontracting \( C = (c, k, n, m) \) solves the following problem:

\[
\max_{c(h^t), k(h^t), n(h^t, e_t)} \mathbb{E} \left[ \sum_{t=0}^{T} \beta^t u(c_t(h^t)) \right]
\]

\[
st : \quad n_{t+1}(h^{t+1}) = n_t(h^t)(1 + r_t) + ak_t(h^t) + pt+1k_t(h^t)(1 + e_t) - ptk_t(h^t)(1 + r_t) - c_t(h^t)(1 + r_t) - m_t+1(h^t, e_t, R_t) \quad \forall h^t, e_t, R_t
\]

\[
m_{t+1}(h^t, e_t, R_t) = \bar{m}(h^t, e_t) + pt+1k_t(h^t)\phi(R_t - e_t - 1) \quad \forall h^t, e_t, R_t, t < T
\]

\[
\mathbb{E} \left[ \frac{\eta^{h+1}}{\bar{m}_{t+1}(h^t, e_t)} | h^t \right] = 0 \quad \forall h^t
\]

\[
n_0(h^0) \leq b_0 \quad n_t(h^t) \geq 0 \quad \forall h^t
\]

Proposition 21 differs from a standard portfolio problem only in the incentive compatibility constraint, which forces the agent to keep a fraction \( \phi \) of his idiosyncratic risk \( pt+1k_tu_t \). On the other hand, the agent is free to use \( m_{t+1}(h^t, e_t) \) to share aggregate risk. Moral hazard does not limit the agent’s ability to share aggregate risk, since it does not interact with his stealing activity.

**Partial revelation of hidden savings.** I have assumed up to this point that if the agent retracts, his hidden savings are fully revealed and added to his legitimate bank balance. This is without loss of generality, because the incentive compatibility constraints don’t depend on his wealth (legitimate or otherwise). If the agent has \( S \) hidden savings, he can chose to reveal \( \bar{S} \geq S \) and add them to his legitimate net worth, keeping \( S - \bar{S} \) hidden. Revealing hidden savings will not affect his incentive compatibility constraints, but it will relax his budget constraint

\[
n_t(e_t, R_t) = n_t(1 + r_t) + ak + pt+1k(1 + e_t) - ptk(1 + r_t) - c(1 + r_t)
\]

\[
-\bar{m}(e_t) + pt+1k_t\phi(R_t - e_t - 1)
\]

\[
n_t \leq b_t(h^t) - J_t(h^t) + \bar{S} \quad n_t(e_t, R_t) \geq 0 \quad \forall e_t, R_t
\]
and his continuation utility next period is $V_{t+1}(e^{t+1}, R_t) + (S_t - \bar{S})(1 + r_t)$ so that his total funds next period $n'(e_t, R_t) + S_{t+1}$ are not affected by this decision. But since the solvency constraint $n'(e_t, R_t) \geq 0 \; \forall e_t, R_t$ applies only to the legitimate net worth, by fully revealing his hidden savings he relaxes the problem and improves his continuation utility. This means that even though we ignored truthful revelation conditions (because we assumed that hidden savings became observable), the agent will willingly reveal all of his hidden savings, and the contract remains incentive compatible even when hidden savings don’t become observable upon recontracting.

2.3.1 State-contingent private action.

Up to this point I have focused on implementing no stealing after any history of shocks. The timing convention that stealing happens after the agent observes the shocks $e_t$ and $u_t$ make the problem tractable, but it also opens up the possibility of making stealing contingent on these shocks. It could be optimal to allow the agent to steal after some shocks in order to improve the sharing of idiosyncratic risk between the agent and the principal. I will show here, however, that this is not the case. Implementing no stealing always is in fact optimal. The intuition is as follows: if the agent is stealing in equilibrium we can always offer him a contract that just gives him whatever he was stealing in exchange for not stealing anymore. Since stealing involved a loss of capital (the agent keeps only a fraction $\phi \in (0, 1)$ of what he steals) the new arrangement dominates the old one. It remains to be seen whether just giving him what he was previously stealing will in fact induce him to stop stealing. The following result says that this is indeed the case.

**Proposition 22.** It is optimal to implement no stealing always: $s_t(x^t, e_t, u_t) = 0 \forall (x^t, e_t, u_t)$.

**Proof.** We know there is no moral hazard problem at $t = T$ and implementing no stealing is therefore optimal at that point. We also obtained the value function $V_T(h^T, C, S) = V_T(e^T, b_T(h^T) - J_T(h^T) + S)$, increasing in the second argument. Now, towards induction, take any other $t < T$, assuming the value function for next period takes the same form. After observing $(e_t, u_t)$ the agent chooses $s$ to solve (as before, dropping the dependence on $h^t$ to simplify notation)

$$\max_s p_{t+1}k_t(1 + e_t + u_t - s) - m_t(e_t, 1 + e_t + u_t - s) + \phi p_{t+1}k_t s$$

If the contract implements some $s_t(e_t, u_t)$ then a necessary and sufficient condition is $\forall (e_t, u_t, s' \in [0, u_t - u])$

$$p_{t+1}k_t(1 + e_t + u_t - s') - m_t(e_t, 1 + e_t + u_t - s') + \phi p_{t+1}k_t s' \leq$$

$$p_{t+1}k_t(1 + e_t + u_t - s_t(e_t, u_t)) - m_t(e_t, 1 + e_t + u_t - s_t(e_t, u_t)) + \phi p_{t+1}k_t s_t(e_t, u_t)$$

or equivalently $\forall (e_t, u_t, s' \in [0, u_t - u])$

$$m_t(e_t, 1 + e_t + u_t - s_t(e_t, u_t)) - m_t(e_t, 1 + e_t + u_t - s') - p_{t+1}k_t(1 - \phi)(s' - s_t(e_t, u_t)) \leq 0 \quad (2.4)$$
I will show how we can modify the contract to implement no stealing and leave both the agent and the principal better off. Define the new $m$ like this:

$$
\hat{m}(e_t, R_t) = m(e_t, R_t - s_t(e_t, R_t - e_t - 1)) + (1 - \phi)p_{t+1}k_ts_t(e_t, R_t - e_t - 1) - A(e_t)
$$

for some $A(e_t)$ which will allow us to shift surplus between agent and principal without affecting incentives. Under $\hat{m}$ he chooses $s$ to solve:

$$
\max_{s \in [0, u_t - y]} p_{t+1}k_t(1 + e_t + u_t - s) - \hat{m}(e_t, 1 + e_t + u_t - s) + \phi p_{t+1}k_t s
$$

and plugging in the new $\hat{m}$ we get:

$$
\max_{s \in [0, u_t - y]} p_{t+1}k_t(1 + e_t + u_t - s) - m(e_t, 1 + e_t + u_t - s) - (1 - \phi)p_{t+1}k_ts_t(e_t, u_t - s) + \phi p_{t+1}k_t s + A(e_t)
$$

In particular, if he chooses $s = 0$ he gets $p_{t+1}k_t(1 + e_t + u_t) - m(e_t, 1 + e_t + u_t - s_t(e_t, u_t)) - (1 - \phi)p_{t+1}k_ts_t(e_t, u_t) + A(e_t)$, which is what he would have gotten under the old contract by stealing $s_t(e_t, u_t)$, plus $A(e_t)$. Not stealing is optimal for every $e_t$ and $u_t$ if and only if $\forall(e_t, u_t, s' \in [0, u_t - y])$

$$
p_{t+1}k_t(1 + e_t + u_t - s') - m(e_t, 1 + e_t + u_t - s' - s_t(e_t, u_t - s)) - (1 - \phi)p_{t+1}k_ts_t(e_t, u_t - s') + \phi p_{t+1}k_t s' \leq
$$

$$
p_{t+1}k_t(1 + e_t + u_t) - m(e_t, 1 + e_t + u_t - s_t(e_t, u_t)) - (1 - \phi)p_{t+1}k_ts_t(e_t, u_t)
$$

which after some algebra becomes:

$$
m(e_t, 1 + e_t + u_t - s_t(e_t, u_t)) - m(e_t, 1 + e_t + u_t - s' - s_t(e_t, u_t - s'))
$$

$$
-(1 - \phi)p_{t+1}k_ts_t(e_t, u_t - s') - (1 - \phi)p_{t+1}k_t(s' - s_t(e_t, u_t)) \leq 0
$$

Define $\Delta = s' + s(e_t, u_t - s')$. We know $\Delta \in [0, u_t - y]$ because $s_t(e_t, u_t - s') \leq u_t - s'$ and both terms are non-negative. Then we rewrite the condition above $\forall(e_t, u_t, \Delta \in [0, u_t - y])$

$$
m(e_t, 1 + e_t + u_t - s_t(e_t, u_t)) - m(e_t, 1 + e_t + u_t - \Delta) - (1 - \phi)p_{t+1}k_t(\Delta - s_t(e_t, u_t)) \leq 0
$$

which is true because of (2.4), that is, because $s_t(e_t, u_t)$ was optimal under $m$. Under the new contract, the principal gets

$$
\mathbb{E}
\left[
\eta_{t+1} \left( m(e_t, u_t - s_t(e_t, u_t)) + (1 - \phi)p_{t+1}k_ts_t(e_t, u_t) - A(e_t) \right) | h^t \right]
$$

$$
= \mathbb{E}_t \left[ \eta_{t+1} m(e_t, u_t - s_t(e_t, u_t)) | h^t \right] + \mathbb{E}_t \left[ \eta_{t+1} \left( (1 - \phi)p_{t+1}k_ts_t(e_t, u_t) - A(e_t) \right) | h^t \right]
$$
the first term is what he was getting before, and the second term is zero if we set

\[ A(e_t) = \mathbb{E} \left[ \frac{\eta_{t+1}}{\eta_t} \left( (1 - \phi) p_{t+1}^l k_t s_t(e_t, u_t) \right) | h^t \right] \geq 0 \]

Then the principal is equally well off, but the agent is better off because he pays less (recall if he doesn’t steal he gets what he was getting under the old contract plus \( A(e_t) \geq 0 \)). By the same induction argument as before, this is true for every period, after any history.

\[ \Box \]

### 2.4 Conclusion

This paper provides a tractable model of dynamic moral hazard which is readily applicable to macro and financial general equilibrium models. The two main features are (i) recontracting: the agent can offer a new contract to the principal at the beginning of any period, and (ii) the agent takes his private stealing action after he observes the shocks. The timing convention (ii) imposes a linear structure on the incentive scheme where incentive compatibility conditions must hold state-by-state rather than in expectation. This linearity in turn helps us deal with the hidden savings problem: the agent always wants to immediately recontract and fully reveal his hidden savings. The optimal contract can therefore be characterized as the solution to a standard portfolio problem which takes general equilibrium prices as given and imposes a “skin in the game” constraint: the agent must keep some exposure to the return of his project so he doesn’t steal. This limits his ability to share idiosyncratic risk, but not his ability to share aggregate risk.
Chapter 3

Moral Hazard and the Balance Sheet Channel

3.1 Introduction

The financial system can play a crucial role in business cycles. If more productive agents have a very large exposure to aggregate risk, their balance sheets will be hit disproportionally after bad aggregate shocks. This will reduce their demand for assets, driving asset prices and growth down, and amplifying and propagating the initial shock. And because it takes time for the balance sheets to be rebuilt, even transitory shocks can lead to protracted balance sheet recessions. However, we don’t have a good explanation for why these agents are so exposed to aggregate risk in the first place. Most papers in the literature impose ad-hoc constraints on agents’ ability to share aggregate risk\(^1\).

In Di Tella (2013) I show that in standard models of balance sheet recessions driven by Brownian TFP shocks, the balance sheet channel completely vanishes if agents are allowed to write contracts on the aggregate state of the economy. I then show how uncertainty shocks, in contrast to TFP shocks, will endogenously create incentives for more productive agents to take on a disproportionate fraction of aggregate risk. In this paper I take a different approach: I use a standard growth model driven by a Brownian TFP shock, and study under what conditions the presence of moral hazard can distort aggregate risk-sharing and create a balance sheet channel. The main result is that this will only happen if the agent’s private action exposes him to aggregate risk. I then build a model economy where a balance sheet channel arises, even though agents are able to write contracts on the aggregate state of the economy.

I use a standard continuous-time growth model with two types of agents: experts and consumers. Both have the same preferences and can trade and use capital to produce consumption goods, but experts have higher productivity. They would like to raise funds and share risk in a financial market,

\(^1\)See for example Bernanke and Gertler (1989) and Kiyotaki and Moore (1997), or more recently Brunnermeier and Sannikov (2012), He and Krishnamurthy (2011), or Kiyotaki, Gertler, and Queralto (2011).
but face a moral hazard problem. As in standard models of moral hazard, such as DeMarzo and Sannikov (2006) or Brunnermeier and Sannikov (2012), if the agent shirks he reduces the observable return of the project but receives a private benefit. Where the setting here differs from the standard setup is that the private benefit carries exposure to both aggregate and idiosyncratic risk. This can be interpreted as the reduced form of a richer game where the agent must invest some of his own funds in a risky activity in order to take advantage of his shirking, or where shirking increases return volatility which he must mask using his own funds to prevent detection. The main theoretical result is that the optimal contract will create a distortion in aggregate risk-sharing that will create a balance sheet channel only if shirking increases the agent’s exposure to aggregate risk. The intuition is as follows: since agents are risk averse, the contract can over-expose them to aggregate risk to deter them from taking a private action (shirking) that further increases their exposure to aggregate risk. By doing this it can reduce the agent’s exposure to his own observable return, and hence to his own idiosyncratic risk, so there is a tradeoff between aggregate and idiosyncratic risk-sharing.

To see how this will create a balance sheet channel, recall that the only difference between experts and consumers is that experts are more productive. Experts will therefore have a higher demand for capital proportionally to their net worth, and will lever up more. Other things equal, they will have a proportionally larger exposure to their idiosyncratic risk and will therefore have a bigger incentive to offload their equity stake in their project onto the market; that is, to reduce their “skin in the game”. In order to credibly promise to not shirk, they will instead take on more aggregate risk. In other words, because experts are more productive and therefore will manage more capital, they have larger incentives to take advantage of the tradeoff between aggregate and idiosyncratic risk-sharing. In equilibrium experts will take a disproportionate fraction of aggregate risk. When a negative shock hits the economy, their balance sheets will be hit harder and they will sell capital to consumers, driving asset prices down and reducing growth and output. This is the balance sheet channel.

In contrast, if shirking does not expose the agent to aggregate risk, the contract can completely separate incentives from aggregate risk sharing and the balance sheet channel will vanish. It is perfectly possible for experts to be very highly leveraged and yet carry a small fraction of aggregate risk, proportional to their net worth. When a negative shock hits the economy both experts and consumers will loose net worth, but balance sheets will play no role in amplifying or propagating the aggregate shock.

An important assumption is that financial contracts can be made contingent on the (observable) aggregate shocks hitting the economy. This can be achieved, for example, by letting agents freely trade a market index. This is the main difference between this paper and most models in the literature on balance sheet recessions, such as Kiyotaki and Moore (1997) and Bernanke, Gertler, and Gilchrist (1999), or more recently Brunnermeier and Sannikov (2012) and He and Krishnamurthy (2011). In those papers contracts are constrained and cannot be written on the aggregate state of the economy. This creates a mechanic relationship between leverage and exposure to aggregate risk,
and since experts are more productive they will be more leveraged than consumers, and hence more exposed to aggregate risk. Thus incomplete contracts underlie the balance sheet channel in those models. In contrast, in Di Tella (2013) I allow agents to write optimal contracts on all observable variables. I keep the standard moral hazard setup and explore the role of the type of structural shock hitting the economy. I show that more productive agents have incentives to be very highly exposed to uncertainty shocks in order to take advantage of endogenously stochastic investment possibility sets. Here instead, I focus on the benchmark Brownian TFP shocks, but generalize the moral hazard setup to allow the private action to affect the agent’s exposure to aggregate risk. Both papers help understand different aspects of the aggregate risk-sharing puzzle behind balance sheet recessions and financial crises.

Krishnamurthy (2003) also shows how complete markets sever the feedback from asset prices back to experts’ balance sheets in a discrete-time model based on Kiyotaki and Moore (1997). In his model, however, experts may still be overexposed to aggregate risk because they might face a tradeoff between insurance and obtaining funds up front. A similar tradeoff is explored in Rampini, Sufi, and Viswanathan (2012). Limited commitment on the insurers side can also prevent full aggregate risk-sharing. Krishnamurthy (2003) explores precisely this mechanism, similarly to Holmstrom and Tirole (1996). Geanakoplos (2009) instead emphasizes the role of heterogeneous beliefs. More optimistic agents place a higher value on risky assets such as capital, and are naturally more exposed to aggregate risk. The balance sheet channel in my model, in contrast, does not rely on heterogeneous beliefs. A completely different approach is taken by Myerson (2012), who builds a model of credit cycles where the interaction of different generations of bankers can generate endogenous credit cycles, even without aggregate shocks.

Empirically, several papers make the case for balance sheet recessions. Sraer, Chaney, and Thesmar (2011), for example, use local variations in real estate prices to identify the impact of firm collateral on investment. They find each extra dollar of collateral increases investment by $0.06. Gabaix, Krishnamurthy, and Vigneron (2007) provide evidence for balance sheet effects in asset pricing. They show that the marginal investor in mortgage-backed securities is a specialized intermediary, instead of a diversified representative agent. Adrian, Etula, and Muir (2011) use shocks to the leverage of securities broker-dealers to construct an “intermediary SDF” and use it to explain asset returns.

**Layout.** The rest of the paper is organized as follows. Section 2 introduces the setup of the model and the contracting problem. In Section 3 I characterize the equilibrium and obtain the main results. In Section 4 I build an economy with a balance sheet channel and solve it numerically. Section 5 concludes.
3.2 The Model

3.2.1 Setup

Preferences and technology. Consider an economy populated by two types of agents: experts and consumers. They have the same CRRA preferences over consumption

\[
E \left[ \int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right]
\]

with discount factor \( \rho \) and relative risk aversion \( \gamma \). There are two goods: consumption goods and capital goods, and the only difference between experts and consumers is that experts use capital more productively. Over a short period of time, if an agent of type \( i \in \{c, e\} \) holds \( k_{i,t} \) unit of capital, he produces an output flow in consumption goods

\[
y_{i,t} = (a_i - \iota(g_{i,t}))k_{i,t}
\]

with \( a_i \in \{a_c, a_e\} \) and \( a_c < a_e \), \( \iota(g_{i,t}) \) is the investment in consumption goods required to achieve the expected growth rate of his capital \( g_{i,t} \). Capital then evolves over that short period of time:

\[
\frac{dk_{i,t}}{k_{i,t}} = g_{i,t} dt + \sigma dZ_t + \nu dW_{i,t}
\]

where \( Z = \{Z_t \in \mathbb{R}; \mathcal{F}_t, t \geq 0\} \) is an aggregate brownian motion, and \( W_i = \{W_{i,t}; \mathcal{F}_t, t \geq 0\} \) an idiosyncratic brownian motion for expert \( i \), in a probability space \((\Omega, P, \mathcal{F})\) equipped with a filtration \( \{\mathcal{F}_t\} \) with the usual conditions.

Markets. Experts can trade capital continuously at a competitive price \( p_t > 0 \), which we conjecture follows an Ito process:

\[
\frac{dp_t}{p_t} = \mu_{p,t} dt + \sigma_{p,t} dZ_t
\]

The price of capital depends on the aggregate shock \( Z \) but not on the idiosyncratic shocks \( \{W_i\}_{i \in [0,1]} \), and it's determined endogenously in equilibrium. The total value of the aggregate capital stock is \( p_t k_t \) and it constitutes the total wealth of the economy, since this is the only real asset.

---

\(^2\) This formulation where capital is exposed to aggregate risk is equivalent to a standard growth model where TFP \( a_t \) follows a Brownian Motion. Then if \( k_t \) is physical capital, \( k_t = a_t k_t \) is "effective capital". To preserve scale invariance we must also have investment costs proportional to effective capital \( k_t \), which makes sense if we think investment requires diverting capital from consumption to investment (or in a richer model with labor).

\(^3\) The aggregate shock can be multidimensional, \( Z_t \in \mathbb{R}^d \) with \( d \geq 2 \), so the economy could be hit by many aggregate shocks. There is no loss in intuition from taking \( d = 1 \) and focusing on a single aggregate shock.
There is also a complete financial market with SDF $\eta_t$:

$$\frac{d\eta_t}{\eta_t} = -r_t dt - \pi_t dZ_t$$

Here $r_t$ is the risk-free interest rate and $\pi_t$ the price of aggregate risk $Z$. Both are determined endogenously in equilibrium. I am already using the fact that idiosyncratic risks $\{W_i\}_{i \in [0,1]}$ have price zero in equilibrium because they can be aggregated away. Even though there is a complete financial market, agents' access to it will be limited by a moral hazard problem similar to Brunnermeier and Sannikov (2012) and He and Krishnamurthy (2011).

**Moral hazard and contracts.** Agents can take a private action $s_{i,t}$ ("shirking") that reduces the observable return of capital by that amount, so the observable return from investing a dollar in capital is

$$dP_{i,t}^k = \left( g_{i,t} + \mu_{p,t} + \sigma_{p,t}^2 + \frac{a_i t - t (g_{i,t})}{p_t} - s_{i,t} \right) dt + \sigma_{p,t} dZ_t + \nu dW_{i,t}$$

Shirking creates a pecuniary private benefit for the agent per dollar invested in capital $p_t k_{i,t}$

$$dh_{i,t} = \alpha s_{i,t} dt + \beta s_{i,t} dZ_t + \lambda s_{i,t} dW_{i,t}$$

where $\alpha \in (0, 1)$ and $\beta, \lambda \geq 0$, so that shirking is inefficient. With $\beta = \lambda = 0$ we would have the standard setting in moral hazard problems, interpreted as diverting capital to a private account and obtaining a fraction $\alpha$ per unit diverted, as in DeMarzo and Sannikov (2006). Here I generalize the moral hazard to study under which conditions it can create a balance sheet channel. Shirking $s$ may also expose the agent to aggregate and idiosyncratic risk through the terms $\beta s_{i,t} dZ_t$ and $\lambda s_{i,t} dW_{i,t}$. This can have several interpretations. Under one interpretation, the agent cannot simply divert funds to a private account, but can hire an inefficient supplier for the firm. This is costly for the project and it only benefits him if he secretly acquires some equity in the supplier, exposing himself to the risk contained therein. Under an alternative interpretation, the fund diversion also increases the volatility of the observable capital return, but the agent must mask this increase in volatility with his own funds to avoid being detected. Instead of committing to a moral hazard story, I focus on the reduced form representation which will be relevant for the balance sheet channel.

At every $t$, a short-term contract is signed specifying the agent's consumption $c_{i,t}$, capital $k_{i,t}$, growth $g_{i,t}$, and a payment he will make at time $t+dt$, contingent on both the aggregate shock and market indices.
hitting the economy $dZ_t$ (which is observable) and his own return $dR_{i,t}$. After this the relationship is broken up and another contract is signed, and so on. The agent deals with a principal who simply sells the equilibrium cash flow he receives in the financial market, so the agent can be thought of as contracting with the market directly. I restrict attention to affine contracts as in He and Krishnamurthy (2011), which specify a fixed fee $f_{i,t}dt$ and a linear exposure to aggregate risk $\theta_{i,t}$ and the observable return $\phi_{i,t}$. I will focus on the case where it is optimal to implement no shirking always $s_{i,t} = 0$. The agent's budget constraint is

$$dn_{i,t} = n_{i,t}r_{t}dt + pt_k_i,t \left( dR_{i,t}^k - r_{t}dt \right) - f_{i,t} - \theta_{i,t}dZ_t - \phi_{i,t}p_t_k_i,t dR_{i,t}^k - c_{i,t}dt + p_t k_{i,t}dh_t$$

with a standard solvency constraint $n_t \geq 0$ to make the problem well defined. In order to be taken by the market, the contract must satisfy a no-arbitrage condition

$$f_{i,t} + \phi_{i,t}p_t k_i,t (E_t^{\pi=0} \left[ dR_{i,t}^k \right]) = \pi_{t} (\theta_{i,t} + \phi_{i,t}p_t k_i,t (\sigma + \sigma_{p,t}))$$

where the expectation is taken under no shirking (because it is optimal to implement no shirking). The expected return for the market must cover the cost of the aggregate risk, both from the direct exposure $\theta_{i,t}$ and from the aggregate risk contained in the observable return $dR_{i,t}$. So in terms of the structural shocks $Z$ and $W_i$, we obtain the following budget constraint

$$\frac{dn_{i,t}}{n_{i,t}} = (\mu_{n_{i,t}} - \hat{\mu}_t)dt + \sigma_{n_{i,t}}dZ_{t} + \hat{\sigma}_{n_{i,t}}dW_{i,t}$$

(3.1)

$$\mu_{n_{i,t}} = r_{t} + p_t k_i,t \left( E_t^{\pi=0} \left[ dR_{i,t}^k \right] - r_t \right) - \pi_t \phi_{i,t}p_t k_i,t (\sigma + \sigma_{p,t}) - \pi_t \theta_{i,t} + (\alpha - 1 + \phi_{i,t})p_t k_i,t s_{i,t}$$

$$\sigma_{n_{i,t}} = (1 - \phi_{i,t})p_t k_i,t (\sigma + \sigma_{p,t}) - \theta_{i,t} + \beta p_t k_i,t s_{i,t}$$

$$\hat{\sigma}_{n_{i,t}} = (1 - \phi_{i,t})p_t k_i,t \hat{\nu} + \lambda p_t k_i,t s_{i,t}$$

where the hat denotes the variable is normalized by net worth, e.g. $k_{i,t} = \frac{k_{i,t}}{n_{i,t}}$. Of course in an incentive compatible contract the agent will chose $s_{i,t} = 0$. Assume the value function for an agent with net worth $n > 0$ is $V^\epsilon(\xi_{t}, n)$ and $V^c(\zeta_{t}, n)$, for experts and consumers respectively, for some processes $\xi = \{\xi_{i,t}; t \geq 0\}$ and $\zeta = \{\zeta_{i,t}; t \geq 0\}$ which capture the general equilibrium investment possibility sets each type of agent faces and follow laws of motion $\frac{d\xi_{i,t}}{\xi_{i,t}} = \mu_{\xi_{i,t}}dt + \sigma_{\xi_{i,t}}dZ_{i}$ and $\frac{d\zeta_{i,t}}{\zeta_{i,t}} = \mu_{\zeta_{i,t}}dt + \sigma_{\zeta_{i,t}}dZ_{i}$. Furthermore, assume $V^\epsilon$ and $V^c$ are strictly increasing and concave in $n$. 

---

6The contract could also be contingent on the return of other agents $R_{j,t}^k$ with $j \neq i$, but there is no loss in generality from dropping this term, since it will never be optimal to expose the agent to that risk since it doesn’t help with incentives and is costly for him due to risk-aversion.

7When the price of aggregate risk is positive $\pi_t > 0$, the standard argument applies: for a contract with equilibrium shirking $s_{i,t} > 0$, we could consider instead an alternative contract that just gives the agent the cash flow he gets by shirking and induces him to not shirk $s_{i,t}' = 0$. Because shirking is inefficient the new contract leaves the principal better off.
and twice continuously differentiable in both arguments. We will later verify this is indeed the case in equilibrium. For experts, \( V^e(\xi_t, n) \) must solve the following HJB equation:

\[
\rho V^e(\xi_t, n) = \max_{\alpha, \xi_t, \sigma, \phi} \frac{(\phi n)^{1-\gamma}}{1-\gamma} + \mathbb{E}_t[e^{\xi_t}] [dV^e(\xi_t, n)]
\]

where \( \mathbb{E}_t[e^{\xi_t}] [dV^e(\xi_t, n)] = V_n^e n \mu_n + V^e \xi_t \xi_t + \frac{1}{2} V_n^e \xi_t \xi_t + \frac{1}{2} V_n^e \sigma_n \sigma_n + \frac{1}{2} V_n^e \sigma_n \sigma_n + \frac{1}{2} V_n^e \xi_t \sigma_n \sigma_n + \frac{1}{2} V_n^e n \sigma_n \sigma_n \), subject to the budget constraint (3.1) evaluated at \( s = 0 \), feasibility constraints \( \hat{c}, \hat{k} \geq 0 \), and an IC constraint:

\[
0 \in \arg \max_{s \geq 0} \frac{(\phi n)^{1-\gamma}}{1-\gamma} - \rho V(x, n) + \mathbb{E}_t[e^{\xi_t}] [dV^e(\xi_t, n)]
\]

and a transversality condition

\[
\lim_{t \to \infty} e^{-\rho t} \mathbb{E}[V^e(\xi_t, n_t)] = 0
\]

The IC constraint says that, having promised to not shirk (\( s = 0 \)) it must be optimal for the agent to in fact not shirk, taking into account that he will act optimally from \( t + dt \) on. It boils down to

\[
V_n^e n (\alpha - 1 + \phi) + (V_n^e n \sigma_n + V_n^e \sigma_n \xi_n) \beta + V_n^e n \sigma_n \lambda \leq 0
\]  

(3.2)

Notice that if \( \beta = \lambda = 0 \) as in the standard setup, this IC constraint would boil down to \( 1 - \phi_{i,t} \geq \alpha \): the agent must keep some "skin in the game" to prevent him from diverting funds. This is precisely the condition in Brunnermeier and Sannikov (2012), except that in that paper contracts cannot be written on the aggregate state of the economy, so \( \hat{\theta}_t = 0 \) is an additional constraint\(^8\). When the agent's private action also exposes him to aggregate and idiosyncratic risk, the IC constraint generalizes to (3.2). Consumers have an analogous problem, with \( \zeta_t \) taking the place of \( \xi_t \): the only difference is their productivity is lower \( a_e > a_c \).

**Equilibrium.** Denote the set of experts \( \mathcal{E} = \{0, 1\} \) and the set of consumers \( \mathcal{C} = \{1, 2\} \). We denote the aggregate capital stock \( k_t \) and take the initial capital stock \( k_0 \) and its distribution among agents \( \{k_{i,0}\}, \{k_{i,0}\} \) as given, with \( \int_{\mathcal{E} \cup \mathcal{C}} k_{i,0} \eta_t = k_0 \). Let \( k_{i,0} > 0 \) and \( k_{j,0} > 0 \) so that all agents start with strictly positive net worth.

**Definition 23.** An equilibrium is a set of aggregate stochastic processes adapted to the filtration generated by \( Z \): the price of capital \( \{p_t\} \), the state price density \( \{\eta_t\} \), and the aggregate capital stock \( \{k_t\} \), and a set of stochastic processes for each expert \( i \in \mathcal{E} \) and each consumer \( i \in \mathcal{C} \) (each adapted to their information\(^9\)): net worth and wealth \( \{n_{i,t}\} \), consumption \( \{c_{i,t}\} \), capital holdings \( \{k_{i,t}\} \), growth \( \{g_{i,t}\} \), and contracts \( \{\theta_{i,t}, \phi_{i,t}\} \), such that:

1. Initial net worth satisfies \( n_0^i = p_0^i k_0^i \) for all \( i \in \mathcal{E} \cup \mathcal{C} \).

---

\(^8\)This is also the case in He and Krishnamurthy (2011)

\(^9\) The filtration generated by \( Z \) and \( W_t \)
2. \( \{n_{i,t}, c_{i,t}, k_{i,t}, g_{i,t}, \theta_{i,t}, \phi_{i,t}\} \) solve agent i’s problem, with \( n_{0,t} = p_{0}k_{i,0} \) for all experts and consumers.

3. **Market Clearing:**

\[
\int_{\mathcal{EUC}} c_{i,t} di = \int_{\mathcal{EUC}} (a_{i} - g_{i,t}) k_{i,t} di \\
\int_{\mathcal{EUC}} k_{i,t} di = k_{t} \\
\int_{\mathcal{EUC}} \left((1 - \phi_{i,t}) p_{t} k_{i,t} (\sigma + \sigma_{p,t}) - \theta_{i,t}\right) di = p_{t} k_{t} (\sigma + \sigma_{p,t})
\]

4. **Law of motion of aggregate capital:**

\[
dk_{t} = \left(\int_{\mathcal{EUC}} g_{i,t} k_{i,t} di\right) dt + k_{t} \sigma dZ_{t}
\]

The market clearing conditions for the consumer goods and capital market are standard. The condition for market clearing in the financial market is derived as follows. Each agent sells a cash flow with loading \( \phi_{i,t} p_{t} k_{i,t} (\sigma + \sigma_{p,t}) + \theta_{i,t} \) on the aggregate shock \( dZ_{t} \) and loading \( \phi_{i,t} p_{t} k_{i,t} \nu \) on his own idiosyncratic shock \( dW_{i,t} \). We can aggregate these securities into a market index eliminating the idiosyncratic risk, and if we normalize it so that it has an identity loading on aggregate risk, there is a total supply \( \int_{\mathcal{EUC}} \phi_{i,t} p_{t} k_{i,t} (\sigma + \sigma_{p,t}) + \theta_{i,t} di \) of this market index, which is in zero net demand. We obtain \( \int_{\mathcal{EUC}} \sigma n_{i,t} di = \int_{\mathcal{EUC}} p_{t} k_{i,t} (\sigma + \sigma_{p,t}) di = p_{t} k_{t} (\sigma + \sigma_{p,t}) \) as desired. By Walras’ law, the market for risk-free debt clears automatically.

### 3.3 Solving the model

#### 3.3.1 First best with no moral hazard.

If the private action is actually observable and contractible, we can set \( s_{i,t} = 0 \) and forget about the IC constraint. We obtain a standard AK growth model where the distribution of wealth doesn’t play any role\(^{10}\) and therefore the is no balance sheet channel. Experts hold all the capital in the economy and there is perfect risk-sharing. Idiosyncratic risk is aggregated away, and aggregate risk is shared proportionally to net worth.

**Proposition 24 (First Best).** If \( p - (1 - \gamma)g^* + (1 - \gamma)\frac{1}{2} \sigma^2 > 0 \) and without any financial frictions, there is a stationary growth equilibrium, where the price of capital is \( p^* \) and the growth rate \( g^* \), given by:

\[
\nu'(g^*) = p^* \\
p^* = \frac{a_{e} - \nu(g^*)}{p - (1 - \gamma)g^* + (1 - \gamma)\frac{1}{2} \sigma^2}
\]

\(^{10}\)Except to determine consumption, of course.
Proof. Without any financial frictions, idiosyncratic risk can be perfectly shared and has zero price in equilibrium. Capital will be held by experts who are more productive and must be priced by arbitrage

\[ g_{t,t} + \mu_{p,t} + \sigma_{p,t} + \frac{a - t(g_{t,t})}{p_t} - r_t = \pi_t (\sigma + \sigma_{p,t}) \quad (3.5) \]

The growth rate \( g \) is pinned down by the static FOC

\[ t'(g_{t,t}) = p_t \]

and experts face the same portfolio problem as consumers. We have, in effect, a standard representative agent model with a stationary growth path with risk-free interest rate:

\[ r_t = \rho + \gamma g_t - \frac{1}{2} (1 + \gamma) \gamma \sigma^2 \]

and price of aggregate risk

\[ \pi_t = \gamma \sigma \]

In a stationary equilibrium the price of capital is constant so we have \( \mu_{p,t} = \sigma_{p,t} = 0 \), and replacing all of this in (3.5) gives (3.4). For the agents' problem to be well defined we need \( \rho - (1 - \gamma) g^* + (1 - \gamma) \frac{\gamma}{2} \sigma^2 > 0 \), since otherwise they could achieve infinite utility. □

This result relies on the assumption of complete financial markets. This allows experts to hold all the capital in the economy (they are more productive) while holding no idiosyncratic risk and a fraction of aggregate risk just proportional to their net worth. We can imagine them financing their operation using all equity. Their equity gets aggregated into a market index, which is then used by experts and consumers to share aggregate risk. In order to create a balance sheet channel, aggregate risk-sharing must be prevented or distorted. I now introduce moral hazard to study when it can help create a balance sheet channel.

### 3.3.2 Moral hazard

When the private action is not contractible, the IC constraint creates a financial friction. Conjecture that the value function of experts takes the form

\[ V^e (\xi_t, n) = \frac{\xi_t^n}{1 - \gamma} \]

where the process \( \xi \) follows

\[ \frac{d\xi_t}{\xi_t} = \mu_{\xi,t} dt + \sigma_{\xi,t} dZ_t \]

and captures the stochastic investment possibility sets the expert faces. A high \( \xi_t \) means investment possibility sets are favorable and a lot of utility out of his net worth.

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The expert's HJB then takes the following form (dropping the $i$ subscript for simplicity)

$$
\frac{1}{1-\gamma} \max_{\hat{c}, \hat{k}, \hat{g}, \hat{\theta}, \phi} \left( \frac{\hat{c} n}{1-\gamma} + (\xi_t n)^{1-\gamma} \left( \mu_n + \mu_{\xi, t} - \frac{\gamma}{2} \sigma_n^2 - \frac{\gamma}{2} \sigma_{\xi, t}^2 + (1-\gamma)\sigma_n \sigma_{\xi, t} - \frac{\gamma}{2} \sigma_{\xi, t}^2 \right) \right)
$$

subject to $\hat{c}, \hat{k} \geq 0$ and

$$
\frac{dn}{n} = (\mu_n - \hat{c}_t)dt + \sigma_n dZ_t + \tilde{\sigma}_n dW_t
$$

$$
\mu_n = r_t + p_t \hat{k} \left( E_t^{\hat{k}} \int_{t}^{\infty} \left[ dR^k \right] - r_t \right) - \pi_t \phi p_t \hat{k} (\sigma + \sigma_{p, t}) - \pi_t \hat{\theta}
$$

$$
\sigma_n = (1-\phi) p_t \hat{k} (\sigma + \sigma_{p, t}) - \hat{\theta}
$$

$$
\tilde{\sigma}_n = (1-\phi) p_t \hat{k} \nu
$$

and IC constraint

$$
(\alpha - 1 + \phi) + ((1-\gamma)\sigma_{\xi, t} - \gamma \sigma_n) \beta - \gamma \tilde{\sigma}_n \lambda \leq 0
$$

with the transversality condition $\lim_{t \to \infty} e^{-\gamma t} E \left[ \frac{1-\gamma n_{t-\gamma}}{1-\gamma} \right] = 0$. Notice the net worth $n$ drops from the HJB, so the optimal policy $\{\hat{c}, \hat{k}, g, \hat{\theta}, \phi\}$ does not depend on the agent's net worth. An expert with twice the net worth will get twice the consumption and capital, and his contract will also be scaled up proportionally (so he will have the same $\phi$ and $\theta$). The growth rate of his capital $g$ will be the same however, given by the static first order condition

$$
l'(g) = p_t
$$

This is a standard Tobin's $q$ condition: the marginal cost of creating an extra unit of capital is equal to its market price. Consumers have an analogous HJB and the same linearity property and the same FOC for growth.

**Lemma 25.** [Linearity] Growth is determined by a static FOC

$$
l'(g_t) = p_t
$$

the same for all agents. In addition, all experts chose the same $\{\hat{c}_e, \hat{k}_e, \hat{\theta}_e, \phi_e\}$ and all consumers the same $\{\hat{c}_c, \hat{k}_c, \hat{\theta}_c, \phi_c\}$.

This result allows us to simplify the state space. We don't need to keep track of the whole distribution of net worth across experts and consumers. We can instead focus on a single ratio of aggregate expert to consumers wealth:

$$
x_t = \frac{N_t}{p_t k_t} \in (0, 1)
$$
We then look for a Markov equilibrium in \( x_t \)

\[
p_t = p(x_t), \quad \xi_t = \xi(x_t), \quad \zeta_t = \zeta(x_t), \quad r_t = r(x_t), \quad \pi_t = \pi(x_t), \quad \psi_t = \psi(x_t)
\]

where \( \psi(x) \in [0, 1] \) is the fraction of aggregate capital managed by experts. We have a recursive definition for equilibrium:

**Definition 26.** A Markov Equilibrium is a set of aggregate functions \((p, \xi, \zeta, r, \pi, \psi, g)(x)\) and policy functions \((\hat{e}_e, \hat{k}_e, \hat{\theta}_e, \phi_e)(x)\) for experts and \((\hat{e}_c, \hat{k}_c, \hat{\theta}_c, \phi_c)(x)\) for consumers, and a law of motion for the aggregate state variable \( dx_t = \mu(x)dt + \sigma(x)dZ_t \) such that:

1) \( \xi \) and \( \zeta \) solve the experts' and consumers' HJB equations, and \((\hat{e}_e, \hat{k}_e, \hat{\theta}_e, \phi_e)\) and \((\hat{e}_c, \hat{k}_c, \hat{\theta}_c, \phi_c)\) are the corresponding policy functions, taking \( \{p, \xi, \zeta, r, \pi, \psi\} \) and the law of motion of \( x \) as given.

2) Market Clearing:

\[
\hat{e}_e x p + \hat{e}_c (1 - x) p = \psi a_e + (1 - \psi) a_c - \psi(g)
\]

\[
\hat{k}_e = \frac{\psi}{px}, \quad \hat{k}_c = \frac{1 - \psi}{p(1 - x)}
\]

\[
(\phi_e \psi + \phi_c (1 - \psi)) (\sigma + \sigma_p) + \hat{\theta}_e x + \hat{\theta}_c (1 - x) = 0
\]

3) The law of motion for \( x \) is derived from experts' and consumers' policy functions

\[
\frac{\mu_x}{x} = \mu_{n,e} - \hat{e}_e - \hat{k}_e - \mu_p - \sigma_p \sigma + (\sigma_p + \sigma)^2 - \sigma_{n,e}(\sigma + \sigma_p)
\]

\[
\frac{\sigma_x}{x} = \sigma_{n,e} - \sigma - \sigma_p
\]

The recursive definition of equilibrium abstracts from the aggregate effective capital stock. Given the recursive equilibrium we can compute the law of motion of \( k_t \) and recover all the elements of the non-recursive definition of equilibrium. The state variable \( x \) captures the health of balance sheets. In the first best without moral hazard, experts should hold all the capital in the economy. However, when \( x \) is low, experts' aggregate net worth is small relative to the value of the capital stock of the economy, so they offload capital onto consumers who are less productive. This drives the price of capital and growth down and reduces output. In this sense, balance sheets have real effects. However, in order to have a balance sheet channel, we need the endogenous state variable \( x \) which captures the health of balance sheets to be exposed to aggregate risk, so that when a negative shock hits the economy it is amplified by weakened balance sheets. Since \( x \) is the only state variable for this economy, this requires \( \sigma_x(x) > 0 \).

**Definition 27.** We say balance sheets have real effects if equilibrium objects depend on \( x \). We say there is a balance sheet channel if balance sheets have real effects and, in addition, the
endogenous state variable \( x \) is exposed to aggregate shocks, i.e.: \( \sigma_x > 0 \).

**Aggregate risk sharing.** The FOC for \( \hat{\theta} \) are

\[
\pi = \gamma \sigma_{n,e} - (1 - \gamma) \sigma_x + \chi_e \gamma \beta 
\]

for the expert, where \( \chi_e \) is the Lagrange multiplier on the IC constraint. The expert can trade aggregate risk at price \( \pi \). Increasing his exposure to aggregate risk is costly because he is risk averse, captured by the term \( \gamma \sigma_{e,n} \), and because the value he gets out of a dollar \( \xi \) might be correlated with the aggregate state of the economy (intertemporal hedging motive). In addition, by taking aggregate risk he can relax his IC constraint: intuitively, if he is very highly exposed to aggregate risk through \( \theta \) he won't want to shirk, which further exposes him to aggregate risk through \( pk\beta_s \). For consumers, we have an analogous equation:

\[
\pi = \gamma \sigma_{n,c} - (1 - \gamma) \sigma_z + \chi_c \gamma \beta 
\]

Putting the two together and using the market clearing condition in the financial market we get the following expression:

\[
\sigma_{e,n} = \sigma + \sigma_p + (1 - x) \left( \frac{1 - \gamma}{\gamma} (\sigma_x - \sigma_z) + \beta (\chi_c - \chi_e) \right) 
\]

(3.9)

This equation says that experts will take, proportionally to their net worth, an exposure to aggregate risk equal to the total volatility of aggregate wealth \( \sigma + \sigma_p \), plus a relative hedging motive term \( (1 - x) \frac{1 - \gamma}{\gamma} (\sigma_x - \sigma_z) \) plus an incentive provision term \( (1 - x) \beta (\chi_c - \chi_e) \). The first term just reflects the fact that both experts and consumers are equally risk averse, so they will split total volatility proportionally to their net worth. However, since they face different problems (experts are more productive) the value of a dollar for experts is bigger than for consumers, and the gap between them is not constant: it changes in response to aggregate shocks. This creates a relative hedging motive: if \( \gamma < 1 \) experts want to have more net worth when the gap is bigger, to take advantage of better investment possibility sets (low price of capital, etc.). This is sometimes called “keeping dry-powder” effect. On the other hand, if agents are risk averse, they want to stabilize their utility across states. If \( \gamma > 1 \) the wealth effect dominates and they want to have more net worth when the gap between experts and consumers is small and they need more net worth to achieve any given level of utility (relative to consumers).

Finally, I call \( \Omega = \beta (\chi_c - \chi_e) \) the aggregate risk-sharing wedge created by moral hazard. Both experts and consumers want to take on aggregate risk in order to relax their IC constraints. By distorting their aggregate risk-sharing they are able to improve their idiosyncratic risk-sharing. This tradeoff between aggregate and idiosyncratic risk-sharing is at the core of the mechanism that can
create a balance sheet channel.

Using Ito’s lemma to compute \( \sigma_\xi = \frac{\xi'}{\xi} \sigma_x \) and \( \sigma_\zeta = \frac{\zeta'}{\zeta} \sigma_x \) and using the definition of \( \sigma_x = x(\sigma_{nx} - \sigma - \sigma_p) \) we get the following expression for the exposure to aggregate risk of the endogenous state variable \( x \).

**Lemma 28.** The volatility of the endogenous state variable \( x \), is given by

\[
\sigma_x = \frac{\Omega}{1 - (1 - x)^{\frac{1-\gamma}{\gamma}} (\frac{\xi'}{\xi} - \frac{\zeta'}{\zeta}) (1 - x)^{\frac{1-\gamma}{\gamma}}}
\]

This result makes clear that what is requires for a balance sheet channel is a positive wedge \( \Omega > 0 \).

**Proposition 29.** A balance sheet channel with \( \sigma_x \neq 0 \) can only appear if shirking increases agents’ private exposure to aggregate risk, i.e.: \( \beta > 0 \).

**Corollary.** In the standard moral hazard setting with \( \beta = \lambda = 0 \), there is no balance sheet channel.

Recall a balance sheet channel requires \( \sigma_x \neq 0 \). This can only happen if aggregate risk sharing is distorted by the moral hazard wedge \( \Omega \). A non-zero \( \sigma_x \) requires that the first best risk-sharing where each agent takes a proportional fraction of aggregate risk be distorted. In principle, this distortion can come from an endogenous relative hedging motive \( \frac{1-\gamma}{\gamma} (\sigma_\xi - \sigma_\zeta) \), but the relative hedging motive is endogenous: without a balance sheet channel there wouldn’t be a hedging motive to distort aggregate risk-sharing, and therefore no balance sheet channel. The moral hazard wedge, instead, may appear even without a balance sheet channel, therefore creating a distortion in aggregate risk-sharing that induces a balance sheet channel. This balance sheet channel will in turn create a relative hedging motive which may amplify it. But the initial impetus must come from the moral hazard wedge. In the standard moral hazard setup, where shirking is interpreted as stealing and does not affect the agent’s exposure to aggregate risk, there can’t be a balance sheet channel. Furthermore, notice it’s the impact of the private action on the exposure to aggregate risk \( \beta \) what matters, not idiosyncratic risk \( \lambda \).

Having \( \beta > 0 \) is not sufficient, however. We also need the IC constraint to be binding differently for experts and consumers (as captured by the difference in the Lagrange multipliers). This makes sense, they can’t both increase their exposure to aggregate risk to relax their IC constraints, so what matters is who has the strongest incentives to do so. A particular case where a balance sheet will not appear, even with \( \beta > 0 \) is if contracts on the observable return are not allowed at all (no equity issuance). Then \( \phi_e = \phi_c = 0 \) and the IC constraint is not binding for either type (they are residual claimants), so the wedge \( \Omega = \beta(\chi_e - \chi_c) = 0 \).

**Proposition 30.** If contracts cannot be written on the agent’s return (\( \phi_e = \phi_c = 0 \)) then there won’t be a balance sheet channel: \( \sigma_x = 0 \).
3.4 Building an economy with a balance sheet channel

I make the following assumptions to simplify the model. First, log preferences will eliminate the relative hedging motive and avoid the need to actually solve for $\xi(x)$ and $\zeta(x)$. Since the impact of shirking on idiosyncratic risk doesn't play an important role, set $\lambda = 0$ to get an IC constraint:

$$\alpha - 1 + \phi - \sigma_n \beta \leq 0 \quad (3.11)$$

In the first best the agent would want to set $\sigma_n = \sigma + \sigma_p = \sigma$ and $\phi = 1$, so for a binding moral hazard problem we will need $\alpha - \sigma \beta > 0$. Set $g = \iota = 0$, abstracting from growth. Finally, to obtain a stationary distribution, introduce retirement: with Poisson intensity $\tau$ agents die. New agents are born to keep the population constant and they inherit the wealth of dying agents. By distributing this wealth unequally between new experts and consumers we can induce a drift in the direction of some $x \in (0,1)$. The drift of $x$ gets a new term $\tau (\bar{x} - x)$. If we interpret $\rho = \bar{\rho} + \tau$ as the sum of the real discount factor $\bar{\rho} > 0$ and the arrival rate of death $\tau > 0$ we don't need to change the HJB equations.

**First Best with no moral hazard.** The first best without moral hazard has all capital held by experts $\psi(x) = 1$, and full idiosyncratic risk sharing $\phi_e(x) = 1$. Balance sheets don't have real effects, and furthermore aggregate risk is shared proportionally to net worth, $\sigma_n e = \sigma_n c = \sigma$. The price of capital is constant $p^* = \frac{2\alpha}{\rho}$, and the price of aggregate risk is given by $\pi^* = \sigma$.

**Moral hazard.** The economy with moral hazard can be understood using three main equations. First, using the FOC for $\phi$ and $\dot{\theta}$, and the IC constraint itself we get

$$1 - \phi_i = \frac{-\pi \beta + \alpha}{1 + (p_k i \beta \nu)^2} \quad (3.12)$$

The fraction of their equity the agent keeps $1 - \phi_i$ depends on the equilibrium price of risk $\pi$, and on the fraction of his net worth he invests in capital $p_k i$. The more he invests in capital, the larger the idiosyncratic risk he will face other things equal, so he has more incentives to offload his return on the market, even at the cost of increasing his exposure to aggregate risk to satisfy the IC constraint. The numerator should be positive if the IC constraint is binding, and less than 1 if implementing $s = 0$ is optimal\(^\dagger\) so $1 - \phi \in (0,1)$.

Second, experts are more productive than consumers, so they will want to hold more capital, proportionally to their net worth. Using the FOC for $\dot{k}$ and $\dot{\theta}$ we obtain the following condition

$$\frac{a_k}{p} + \mu_p + \sigma \sigma_p - r \leq \pi (\sigma + \sigma_p) + \gamma p_k i ((1 - \phi_i) \nu)^2 \quad (3.13)$$

\(^\dagger\)The net benefit for the contract of a positive $s$ is $(\alpha - 1 - \pi \beta) s$ and we want it to be $\leq 0$ for $s = 0$ to be optimal.
with equality if \( \dot{k} > 0 \). Experts find capital more attractive, so in equilibrium they will always demand some capital\(^{12} \), \( k_{e,t} > k_{c,t} \geq 0 \). So we get the following pricing equation for capital using the market clearing conditions in the capital market and the above condition for experts (who must always hold some capital, otherwise consumers wouldn’t hold capital either)

\[
\frac{a_c}{p} \mu_p + \sigma_p - r = \pi(\sigma + \sigma_p) + \gamma \frac{\psi}{x} ((1 - \phi)\nu)^2
\]

As a consequence of expert’s higher productivity with capital, \( a_e > a_c \), they will hold more capital and will therefore face, other things equal, more idiosyncratic risk. They will then sell more equity than consumers, as per (3.12), and in order to satisfy their IC constraint will take on more aggregate risk. This leads to the third equation, for the wedge \( \Omega = \beta(x_e - x_c) \). Using the FOC for \( \phi \) and the \( \theta \) we get an expression for \( x_i \), plug it into the expression for \( \Omega \) and use (3.12) to obtain:

\[
\Omega = \frac{\phi_e - \phi_c}{\beta} > 0 \tag{3.15}
\]

and the volatility of the endogenous state variable \( \sigma_z \) is then

\[
\sigma_z = \Omega x(1 - x) > 0 \tag{3.16}
\]

Putting these three relationships together we get the following picture: experts, being more productive than consumers, want to hold more capital proportionally to their net worth. They consequently face more idiosyncratic risk so they want to offload a larger fraction of their return on the market. To continue credibly promising \( s = 0 \) they must instead distort their aggregate risk-sharing more than consumers, so a positive aggregate risk-sharing wedge arises \( \Omega > 0 \), giving \( x \) a positive loading on the aggregate shock: \( \sigma_z > 0 \). When the economy gets hit by a negative shock, experts will lose net worth more than proportionally and the fraction of total wealth belonging to them \( x \) will fall. This is what will drive the dynamics of the model.

Because experts are more productive than consumers, when they are very wealthy they will hold all the capital in the economy, \( \psi(x) = 1 \) for \( x \geq \hat{x} \). When \( x \) falls below this value, they will start offloading capital onto consumers, and \( \psi(x) \) will fall below 1. From the FOC and market clearing condition for consumption, we get

\[
\psi = \frac{\rho - a_c}{a_e - a_c}
\]

So when \( x \geq \hat{x} \) the price of capital is at its first best \( p(x) = p^* = \frac{a_e}{\rho} \) for \( x \geq \hat{x} \), and as \( \psi(x) \) falls with \( x \), the price of capital falls as well. When \( x \rightarrow 0 \), \( \psi(x) \rightarrow 0 \) and the price of capital approaches its lower bound \( p(x) \rightarrow p = \frac{a_c}{\rho} \), the value if consumers had to hold it forever. Using the IC constraint

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\(^{12}\)The equation has the right hand side increasing in \( \dot{k} \) after we substitute \( 1 - \phi_4 \).
and market clearing in the financial market we get, along with \( \sigma_{n,c} = \sigma_{n,e} + \Omega \) we get
\[
\sigma + \sigma_p + \Omega(1 - x) = \frac{\alpha - 1}{\beta} + \frac{\phi_e}{\beta} \tag{3.17}
\]

Use Ito's lemma to obtain \( \sigma_p = \frac{\nu_x}{p} \sigma_x \) and plug into (3.17) the formula for \( \Omega, \phi_e = 1 - \frac{-\pi \beta + \alpha}{1 + (\frac{\nu}{\beta})^2} \), and \( \phi_e = 1 - \frac{-\pi \beta + \alpha}{1 + (\frac{\nu}{\beta})^2} \) to obtain an first order ODE for the price of capital \( p(x) \). Since in the region \( x \geq \hat{x} \), the price of capital \( p = \frac{a_c}{p} \), we also know \( p'(x) = 0 \) and (3.17) becomes an equation for \( \pi(x) = \hat{\pi}(x) \) for \( x \geq \hat{x} \). We can then find \( \hat{x} \) as the point where consumers start to buy capital by using their condition for demand for capital (3.13) with an equality. Re-arranging we get:
\[
-\frac{1}{\hat{x}} \frac{(-\pi \beta + \alpha)^2}{1 + (\frac{\nu}{\beta})^2} \nu^2 - \frac{a_c - a_e}{p^*} = 0
\]

This pins down \( \hat{x} \in [0, 1] \). If the equation does not have a solution in \([0, 1]\) it means that either \( \hat{x} = 0 \) (experts always hold all capital) or \( \hat{x} = 1 \) (experts never hold all the capital stock). Focus on the case with an interior solution. Now use the pricing equation for capital (3.14), together with \( \psi = \frac{\sigma_p a_c}{a_e - a_c} \) to find the price of aggregate risk \( \pi \) as a function of \( x \) and \( p \), and hence \( \Omega, \sigma_x \) and \( \phi_e \) as functions of \( x \) and \( p \). We can then solve (3.17) with boundary condition \( p(\hat{x}) = \frac{a_c}{p} \). To check that this is, in fact, an equilibrium we need to make sure that for all \( x \), the IC constraint is binding \( \alpha - \pi \beta \geq 0 \) and that implementing \( s = 0 \) is optimal \( \alpha - \pi \beta - 1 \leq 0 \).

**Numerical solution.** I use the following parameter values for the numerical solution. Expert productivity is \( a_e = 1 \), while consumers have \( a_c = 0.5 \). The discount factor is \( \rho = 0.05 \) and the volatility of capital is \( \sigma = 0.02 \) and \( \nu = 0.3 \). The moral hazard comes from \( \alpha = 0.5 \) and \( \beta = 0.3 \). With these values, the first best value of capital is \( p^* = \frac{a_c}{p} = 20 \). The lowest possible value for capital, corresponding to consumers holding it forever is \( q = \frac{a_c}{p} = 10 \).

Figure 3-1 shows the price of capital for different values of \( x \) and the fraction of capital managed...
by experts. Both increase with \( x \), until we reach \( \hat{x} \) when all capital is held by experts and the price reaches its first best level. Also, in this case \( p(0) = \frac{\sigma_x}{\rho} = 10 \). This figure by itself is not surprising: with moral hazard or other contractual frictions, a positive relationship between \( x \) and the price of capital will emerge even if there is no balance sheet channel. We need to look at the dynamics of \( x \).

Figure 3-2 shows the drift and volatility of \( x \), as well as the aggregate risk-sharing wedge \( \Omega \) which underlies them. Notice how the volatility \( \sigma_x \) vanishes at both borders even though the wedge \( \Omega \) is bounded away from zero. The drift \( \mu_x \) is positive at \( x = 0 \) and negative at \( x = 1 \), pushing the system towards the interior of the domain (this is because agents retire). The upper and lower boundaries \( x = 0 \) and \( x = 1 \) are Feller entry boundaries: the system could start there and would immediately move to the interior, but they can never be reached if \( x_0 \) starts in the interior.

Figure 3-3 shows the price of aggregate risk \( \pi \), the total aggregate risk in the economy \( \sigma_p + \sigma \) and the risk-free interest rate \( r \).

Somewhat surprisingly, the price of aggregate risk initially drops as we move away from \( x = 1 \). This is because the incentives to distort aggregate risk-sharing increase faster than the total amount of aggregate risk initially. Both agents want to be “over-exposed” to aggregate risk, driving the price down. The slope of \( \pi \) eventually turns around as \( x \) falls further and the total aggregate volatility in the economy increases. Another surprising feature of the equilibrium is that the risk-free interest rate rises as \( x \to 0 \), in contrast to other models of balance sheet recessions where the interest rate drops as agents demand more safe assets. The reason is that risky assets are demanded when \( x \) is low to provide incentives and improve idiosyncratic risk sharing. Total market volatility, \( \sigma + \sigma_p \) meanwhile, increases as \( x \) falls, but eventually comes back down as \( \sigma_x \) approaches 0.
3.5 Conclusions

In this paper I have shown how moral hazard can help explain balance sheet recessions. I first show that it standard models of balance sheet recessions, even if financial frictions are derived from a moral hazard problem, optimal contracts allow agents to share aggregate risk and this eliminates the balance sheet channel. The reason for this is that in order to have incentives to not shirk, agents need to have some “skin in the game” and be exposed to their own observable return. This exposes them to both aggregate and idiosyncratic risk, but since there private action does not interact with aggregate risk, there is no need to distort aggregate risk-sharing to provide incentives. In equilibrium, this eliminates the balance sheet channel: experts and consumers share aggregate risk proportionally to their net worth, so that when a negative shock hits the economy, everyone loses net worth proportionally, and there is no amplification through weaker balance sheets. In consequence, investment possibility sets are not affected by the aggregate shock and therefore agents don’t have any intertemporal hedging motives which could distort proportional aggregate risk-sharing.

I then generalized the moral hazard setup to allow the agent’s private action (shirking) to also expose him to both aggregate and idiosyncratic risk. This can be interpreted as a reduced form representation of a richer game. I show that a balance sheet channel may appear only if shirking exposes the agent to aggregate risk through his private benefit. By “over-exposing” the agent to aggregate risk the contract can deter him from taking a private action that further exposes him to aggregate risk. A tradeoff between aggregate and idiosyncratic risk-sharing arises: the contact can reduce the agent’s equity holdings which expose him to idiosyncratic risk by increasing his exposure to aggregate risk. Since experts are more productive than consumers, they find capital more attractive and will want to leverage. They will therefore have a larger exposure to idiosyncratic risk than consumers, other things equal, and will have larger incentives to reduce their equity stakes, even at the price of taking on more aggregate risk. A wedge is aggregate risk-sharing will therefore arise where more productive experts will take on a disproportionate fraction of aggregate risk. When a negative shock hits the economy, their balance sheets will be hit harder. This will amplify and propagate the shock, forcing them to offload assets onto less productive consumers and reducing asset prices and output. This is the balance sheet channel.
Bibliography


