PRICE EXPECTATIONS AND ADJUSTMENT TIME

IN

NEOCLASSICAL MONETARY GROWTH MODELS

by

Stephen D. Lewis*

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I. Introduction

This paper investigates three simple one-sector models of monetary growth based upon the work of Tobin (1965) and Sidrauski (1967). The differences between models depends upon the specification of price expectations and the corresponding implications for stability. The comparative statics for each model are identical. It is only the dynamic behaviour which differs. For stable models, dynamic adjustment implied by comparative static derived policies (non-discretionary policies) is only asymptotic, and there is reason to believe that adjustment times may be long. For unstable models, non-discretionary policies lack meaning (except possibly for the one unique path of a saddle point equilibrium) since the policy target will never be reached. In what follows, alternative policies to those obtained from comparative static considerations are derived with the aid of optimal control theory. Attention is focused on the possibility of implementing a discretionary monetary policy that will result in a specified target being reached in a finite period of time.

The next section briefly summarizes the comparative statics of each model. In section III, a discretionary policy for Model I is derived with the aid of optimal control theory. Based upon the results obtained for Model I, a less rigorous approach is adopted in deriving discretionary policies for Models II and III in section IV. Conclusions are found in section V.
II. Model Specification and Comparative Static Analysis

Notation and important elements of function specification are given in Table 1. The three models are summarized in Table 2 where it is seen that they are based upon the formulation of disposable income by Tobin (1965) and money market equilibrium brought about by instantaneous price changes. In Model I, the expected rate of price change is "static" which insures stability. In Model II is identical to that of Sidrauski (1967) in which the expected rate of price change is formulated according to the "adaptive expectations model." Model III follows most closely the original specification of Tobin (1965) with expectations based upon "perfect myopic foresight" in which the equilibrium is a saddle point.

The comparative statics of all three models are identical. Setting $k = 0$ and where appropriate $p/p = \pi$ and $m = 0$, the models reduce to

$$sf(k) - nk - (1 - s)nL(k, \pi) = 0$$  \hspace{1cm} (1)

$$m = L(k, \pi)$$  \hspace{1cm} (2)

$$\pi = \theta - n$$  \hspace{1cm} (3)

Substituting (2) and (3) into (1), equations

$$m = (sf(k) - nk)/(1 - s)n$$  \hspace{1cm} (4)

$$m = L(k, \theta - n)$$  \hspace{1cm} (5)

are easily obtained and based upon the assumptions in Table 1 are graphed in Figure 1 where $\theta_0$ is associated with an initial steady-state of $(k_0, m_0)$.
TABLE 1
Notation and Assumptions

<table>
<thead>
<tr>
<th>1. Variables:</th>
<th></th>
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<tbody>
<tr>
<td>k</td>
<td>capital-labour ratio with time derivative $k^\prime$</td>
</tr>
<tr>
<td>p</td>
<td>price level with time derivative expressed in relative terms as $p/p$</td>
</tr>
<tr>
<td>m</td>
<td>per capita real money balances with time derivative $m$</td>
</tr>
<tr>
<td>$\pi$</td>
<td>expected rate of change in prices with time derivative $\pi$</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>2. Parameters:</th>
<th></th>
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<tbody>
<tr>
<td>s</td>
<td>saving ratio with $0 &lt; s &lt; 1$</td>
</tr>
<tr>
<td>$\theta$</td>
<td>rate of increase in nominal money balances, a monetary policy instrument</td>
</tr>
<tr>
<td>n</td>
<td>rate of growth of labour or &quot;natural&quot; rate of growth</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>expectations coefficient in adaptive expectations model</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>3. Functions:</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(k)$</td>
<td>&quot;well-behaved&quot; aggregate per capita production function</td>
</tr>
<tr>
<td>$L(k, \pi)$</td>
<td>per capita real money demand with $L &gt; 0$, partial derivatives $L_k &gt; 0$, $L_\pi &lt; 0$ and elasticity $(dL/dk)k/L &gt; 1$</td>
</tr>
<tr>
<td>$\phi(k, m)$</td>
<td>rate of price change from money market equilibrium condition. Assuming $L_\pi &lt; 0$, then $\phi_k &gt; 0$ and $\phi_m &lt; 0$.</td>
</tr>
</tbody>
</table>
TABLE 2

Summary of Models

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td></td>
<td>k = sf(k) - nk - (1 - s)(θ - μ)m</td>
<td></td>
</tr>
<tr>
<td>I</td>
<td>m = L(k, π)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>π = θ - μ</td>
<td></td>
</tr>
<tr>
<td></td>
<td>k = sf(k) - nk - (1 - s)(θ - μ)m</td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>m = L(k, π)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>p/p = θ - μ - (L_k + L_π)/L</td>
<td></td>
</tr>
<tr>
<td></td>
<td>π = γ(p/p - μ)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>k = sf(k) - nk - (1 - s)(θ - μ)m</td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>m = L(k, p/p) → p/p = φ(k, m)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>m = m(θ - p/p - μ)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>π = p/p</td>
<td></td>
</tr>
</tbody>
</table>
\[ m = L(k, \theta_0 - n) \]

\[ m = L(k, \theta_e - n) \]

\[ m = \frac{sf(k) - nk}{1 - s}n \]

\( m_0 \)

\( m_e \)

\( k_0 \)

\( k_e \)

FIGURE 1
and $\theta_e$ with a final steady-state $(k_e, m_e)$. An increase in $\theta$ from $\theta_0$ to $\theta_e$ as represented in this figure produces the easily derived comparative static results $d\theta / d\theta < 0$ and $dk / d\theta > 0$. For sufficiently small values of $k$ (i.e., when $sf_k - n > 0$), it is possible that $d\theta / d\theta > 0$.

Attention is focused on the movement from points A to C and the time required to accomplish this result. From Model I, Figure 1 can be used instead of a phase diagram and the asymptotic dynamic behaviour for a non-discretionary monetary policy can be represented along the segment BC. Initially $\pi$ is increased through $\pi = \theta + n$, while instantaneous price adjustment as represented by a movement from A to B assures money market equilibrium. The equation for $k$ explains the movement from B towards C. While Figure 1 also represents the comparative statics for Models II and III, phase diagrams are required to adequately depict the dynamic behaviour of these models.
III. Discretionary Policy for Model I

The dynamic behaviour of Model I is described by one non-linear differential equation, and in terms of optimal control theory, a discretionary monetary policy can be formulated as a "minimum-time" or optimal-time problem. Accordingly, Model I is first expressed in standard state-space notation. The Hamiltonian (H) is formed, and using Pontryagin's Minimum Principle the control to minimize H is obtained. The nature of the switching function is investigated from the behaviour of the costate variable, and finally a control law is derived.

After substitution, the basic differential equation for Model I is expressed as

$$\dot{k} = sf(k) - nk - (1 - s)(\theta - \pi)L(k, \pi)$$

(6)

The optimal-time problem is one in which the initial time is $t_0 = 0$ while the final time $t_f$ is to be determined. Letting $x = k - k_e$ be the state variable representing deviations of $k$ from $k_e$ with $x_0 = k_0 - k_e$ the initial deviation, then $\dot{x} = \dot{k}$. Finally $x(t_f) = 0$ implies $k(t_f) = k_e$ indicating that the final steady-state has been reached. With $k_e$ representing the steady-state capital-labour ratio, equation (4) and (5) can be solved for $\theta_e$, the rate of growth of money corresponding to $\dot{k} = 0$. Assuming some constant $C > 0$, then $\theta(t) = \theta_e + Cu(t)$ with $|u(t)| \leq 1$ specifies the policy instrument as a function of a control variable $u(t)$. For $C > \theta_e$, it is possible for the policy instrument to assume negative values.
The minimum time problem becomes:

\[ \minimize_{t_f} \int_0^{t_f} dt \text{ with respect to } u \text{ subject to} \]

\[ x = sf(x + k_e) - n(x + k_e) - (1 - s)(\theta_e + \pi_e)L(x + k_e, \pi_e) \]
\[ x_0 = k_0 - k_e \]
\[ |u| \leq 1 \]

The corresponding Hamiltonian for this problem is

\[ H = 1 + \lambda[ sf(x + k_e) - n(x + k_e) - (1 - s)(\theta_e + \pi_e)L(x + k_e, \pi_e)] \]
\[ + \lambda[-(1 - s)L(x + k_e, \pi_e)]u \]

where \( \lambda \) is the costate variable. Pontryagin's Minimum Principle is expressed as

\[ \lambda^*[-(1 - s)L(x^* + k_e, \pi_e)]u^* \leq \lambda^*[-(1 - s)L(x^* + k_e, \pi_e)]u \]

where an asterisk denotes optimum values. Since by assumption

\( (1 - s)L(x^* + k_e, \pi_e)C > 0 \), the control which minimizes \( H \) is given by

\[ u = \text{sgn} \{ \lambda \} \]

where \( \text{sgn} \) denotes the signum function.

The costate variable must satisfy

\[ \lambda = -\beta H/\beta x = -\lambda[ sf_k - n - (1 - s)(\theta_e + Cu - \pi_e)k_e ] \] (7)
which yields the solution

\[ \lambda(t) = \lambda_0 \exp \left\{ -\int_0^t \ldots \right\} \, dt \]

(8)

with \( \ldots \) the bracketed term of equation (7). From equation (8), \( \exp \{ \ldots \} > 0 \) which implies \( \text{sgn} \{ \lambda(t) \} = \text{sgn} \{ \lambda_0 \} \). Furthermore since \( \lambda_0 \neq 0 \), \( \lambda(t) \) cannot be zero for finite \( t \) and therefore no switching will occur. Consequently the control which minimizes the Hamiltonian is given by

\[ u(t) = +1 \quad \text{for all } t > 0 \]

or

\[ u(t) = -1 \quad \text{for all } t > 0. \]

Finally, it can be proved that the time-optimal control law which moves any initial state of Model I to zero in minimum time is given by

\[ u^* = \text{sgn} \{ x \} \]

(9)

provided that a time-optimal control exists.

The use of this control law and the dynamic behaviour of Model I is represented in the phase diagram of Figure 2. The phase line associated with a steady-state \((k_0, m_0)\) is labelled \((k)_0\) and similarly \((k)_e\) corresponds to \((k_e, m_e)\). Asymptotic adjustment associated with the non-discretionary policy target setting, \( \theta_e \), is given along segment BC of \((k)_e\) analogous to that of Figure 1. Since in Figure (2) \( k_0 - k_e < 0 \), control law (9) yields \( u = -1 \). Denoting the discretionary setting of monetary policy as
Potential $k$ curves

FIGURE 2
\( \theta = \theta_0 - C \) and its corresponding phase line as \((k)\), consider

\[
(k)_e = sf(k) - nk - (1 - s)(\theta - \pi_e)L(k, \pi_e)
\]

and

\[
(k)_e = sf(k) - nk - (1 - s)(\theta - \pi_e)L(k, \pi_e)
\]

(10)

In general, depending upon the magnitude and sign of \((\theta - \pi_e)\), the slope of \((k)_e\) can be positive or negative and may alter sign over the interval of concern \((k_0, k_e)\). Thus, \((k)_e\) is represented by several curves in Figure 2. Since however \((\theta_0 - \pi_e) > (\theta - \pi_e), (k)_e > (k)_e\) and both are positive when evaluated at \(k = k_0\). As \(k\) approaches \(k_e, \lim_{k \to k_e} (k)_e = 0\) represents the asymptotic behavior discussed earlier while for equation (10) with \(k = k_e, (k)_e > 0\). This last inequality is the basis upon which \((k_e, \lambda_e)\) can be reached in a finite period of time.

The usual shift in a phase line such as between \((k)_0\) and \((k)_e\) is given by \((\partial k/\partial \theta)k = k_0 = -(1 - s)nL > 0\) where use has been made of \(d\lambda/d\theta = 1\) obtained from \(\pi = \theta - n\). The control law for Model I is explained from another viewpoint by considering a second phase line shift between \((k)_e\) and \((k)_e\) of the form \((\partial k/\partial \theta)k = k_e, \pi = \pi_e = -(1 - s)L(k, \pi_e) < 0\). Thus while \(\pi\) remains at its steady-state value \(\pi_e, \theta_e\) will differ from \(\theta_e\) during the adjustment process in which the control law is employed. This second shift indicates that the fastest way to obtain \(k_e\) when \((k - k_e) < 0\) \((k - k_e > 0)\) is to manipulate the policy instrument such that \(k\) is as great (small) as possible until the new steady-state is reached.

It should be emphasized that while comparative static based policies (i.e., \(dk/d\theta > 0\)) require that \(k\) and \(\theta\) move in the same direction, policies based upon a "minimum-time" problem indicate an inverse association between \(k\) and \(\theta\). The difference of course is between maintaining a larger capital-
labour ratio once obtained which requires an increase in the rate of monetary growth and the movement from an initial to final equilibrium capital-labour ratio which, if expectations are "static" and adjusted to the latter, is most quickly obtained by a decrease and not an increase in the monetary policy instrument. Once the final equilibrium capital-labour ratio is reached, however, the rate of monetary growth must be set at the rate corresponding to the comparative static equations if further movement is to be avoided.
IV. Discretionary Policies for Model II and III

The difference between Model I and Models II and III is found in the formulation of expectations. Allowing for "adaptive expectations", a necessary and sufficient condition for a stable equilibrium is $\gamma < -L/L_{II}$ while with "perfect myopic foresight" the equilibrium is a saddle point. Model II can be reduced to two differential equations expressed as

$$\dot{k} = sf(k) - nk - (1 - s)(\theta - \pi)L(k, \hat{\pi})$$  \hspace{1cm} (11)

$$\dot{\pi} = (\theta - n - \pi - kL_k/L)/(1 + \gamma L_\pi/L)$$  \hspace{1cm} (12)

which, except for minor changes of no importance here, are identical to equations (17) and (18) of Sidrauski (1967).

Ideally it would be nice to duplicate the development of Model I and construct a "minimum-time" problem for Model II. Model I consisted of one state and one control variable and though non-linear it was possible to solve the control problem and obtain a control law analytically. Model II, however, consists of two state variables and one control variable which along with the non-linearities inherent in the model prevent an analytic solution from being obtained.\textsuperscript{10} The system represented by equations (11) and (12) can however be put into the form of a vector differential equation as

$$\dot{X}(t) = A[X(t)] + B[X(t)]u(t)$$  \hspace{1cm} (13)

with column vectors $X(t) = [\pi(t) - \pi_e, k(t) - k_e]^T$, $\dot{X}(t) = [\dot{\pi}(t), \dot{k}(t), \hat{\pi}]^T$ and $u(t)$ formulated as in Model I. Equation (13) is recognized as a system of non-linear differential equations, and it is well-known that the
number of theorems in control theory which apply to this situation are few. One theorem which carries over to equation (13) from linear systems in the context of a "minimum-time" problem is the well-known "Bang-Bang Principle" which states that if a normal optimal control exists, its components will be piece-wise constant functions of time given by the constraints on \( u(t) \).

In Figure 3, initial and final equilibrium points are given by \((\pi_0, k_0)\) and \((\pi_e, k_e)\) respectively. Assuming stability, either point is represented by a node, but phase lines have been omitted for clarity. Instead, phase lines when \( u(t) = +1 \) and \( u(t) = -1 \) with corresponding equilibrium values \((\pi_+, k_+)\) and \((\pi_-, k_-)\) have been inserted. If the control follows the "Bang-Bang Principle", it is the dynamic behavior associated with these curves which is relevant. Typical first-order qualitative dynamic behavior for \((\pi_+, k_+)\) is indicated by arrows. It is assumed that the intersection of the region associated with \((\pi_-, k_-)\) for which this type of qualitative behavior remains valid and the corresponding region associated with \((\pi_+, k_+)\) includes the points \((\pi_0, k_0)\) and \((\pi_+, k_+)\). This assumption insures that (1) an optimal control exists, (2) that qualitative dynamic behavior is dominated by the linear components of equation (13) as represented by the arrows in Figure 3, and (3) that the stability condition \( \gamma < -L/L_\pi \) does not reverse sign during the adjustment process.

The problem then is to find a combination of time paths associated with \( u(t) = \pm 1 \) which will move the system from \((\pi_0, k_0)\) to \((\pi_e, k_e)\) in finite time. Examination of Figure 3 indicates that initially \( \theta \) should be at its upper limit. At some point the time path \((\pi(t), k(t))\) associated
FIGURE 3
with \( u(t) = +1 \) will cross a time path associated with \( u(t) = -1 \) (such as point D) which in addition passes through \((\pi_e, k_e)\). This latter path is a policy switching curve which allows \((\pi_e, k_e)\) to be reached in a finite period of time. If Model II is examined under conditions of instability the "bang-bang" control is the reverse of the stable case just considered. The unstable equilibrium is a saddle point and the corresponding time paths are depicted in Figure 4. Following a procedure similar to that of the stable case \((\pi_e, k_e)\) can be reached by a control sequence of -1, +1. This reversal of policy settings is easily explained. In the stable (unstable case), only a policy setting of \( u(t) = +1 \) (\( u(t) = -1 \)) causes \( \pi(t) \) and \( k(t) \) to move along a time path away from \( (\pi_0, k_0) \) and generally towards \( (\pi_+ , k_+) \). At some point after the trajectory has overshot \( (\pi_e, k_e) \), a change to \( u(t) = -1 \) (\( u(t) = +1 \)) will cause the trajectory to alter course so that the equilibrium is finally reached.

As the final example, the dynamic equations for Model III are obtained by solving the money market equilibrium condition for \( \dot{p}/p \) and substituting into the equations for \( \dot{k} \) and \( \dot{m} \) with the result

\[
\dot{k} = sf(k) - nk - (1 - s)(\theta - \phi(k, m)(L(k, \phi(k,m)))
\]

\[
\dot{m} = m(\theta - \phi(k, m) - n)
\]

Two non-linear differential equations are encountered which can also be expressed in the form of equation (13). As in Model II, if an optimal control exists for the "minimum-time" problem of Model III, it will be of the "bang-bang" type. Analogous to the development of Model II, the appropriate phase lines and associated dynamic behavior (arrows) for
FIGURE 4
\((k_e, m_e)\) are given in Figure 5. To avoid confusion, representation of dynamic behaviour for points \((k_+, m_+)\) and \((k_-, m_-)\) have been deleted. For Model III, the "bang-bang" policy which attains equilibrium in finite time is given by \(u(t) = -1\) along curve AD and \(u(t) = -1\) along curve DC. These results are comparable to those for the unstable version of the previous model as should be expected since as the expectations coefficient approaches infinity the limiting form for Model II is Model III.
V. Conclusion

The focus of this paper has been to examine alternative discretionary policies to the usual non-discretionary ones. For monetary growth models employing "static", "adaptive", and "myopic" price expectations, it was possible to obtain discretionary "bang-bang" type policies by utilizing the underlying dynamic behaviour of the various models. Unlike comparative static based policies which only allow asymptotic adjustment to a steady-state, the discretionary monetary policies reported here resulted in a policy target being reached in a finite period of time. With "static" price expectations, it was possible to obtain an optimal control law without policy switching which required changes in policy instruments which were just the opposite of what would be predicted by comparative static analysis. In the more complicated "adaptive" and "myopic" price expectation models, an explicit control law could not be derived, but by analysis of the appropriate phase diagrams "bang-bang" type policies with switching patterned after a "minimum time" problem were obtained. Policies for the stable version of the "adaptive" expectations model compared to unstable versions of the "adaptive" and "myopic" expectations models were found to be mirror images of each other, reflecting differences in their underlying dynamic structure.

Non-discretionary policies have been shown to be inferior in the context of a "minimum-time" problem. This result weakens the arguments for constancy in the rate of monetary growth associated with among others the
name of Friedman (1968). In addition, support is provided for the results of Lerner-Petersen (1971) who obtained a "bang-bang" monetary policy as a method of reducing the cyclical movement of income inherent in Friedman's conception of monetary influences. Cooper-Fischer (1972) in an autoregressive model with various lags in monetary policy have shown that the maintenance of growth in the money supply at a constant rate is never optimal and that feedback controls are found to be stabilizing. While their optimal control problem is different from the one considered above, both their results and those reported here suggest policy alternatives to simple non-discretionary rules.

Finally, no estimates have been given for the period of time required for adjustment to take place or conversely what can be expected for the magnitude of adjustment time given realistic restrictions on monetary policy. Analytical techniques are incapable of answering these questions, but numerical techniques are available. Thus the speed of adjustment can be determined but such an undertaking is beyond the scope of this paper. It has only been established here that complete adjustment can be accomplished in a finite time interval as opposed to the incomplete asymptotic behavior usually encountered in models of economic growth. In addition, a weakness of "bang-bang" type policies is that they may cause a cyclical movement in the capital-labour ratio which through the aggregate production function implies a similar movement in per capita output. Alternate formulations to handle this objectionable feature of the "minimum-time" problem are currently under investigation.
Footnotes

1. Simulations of neoclassical monetary growth models of the type to be considered here for a wide variety of parameter values and initial conditions indicates that adjustment times are quite long and not much different in magnitude from those of non-monetary growth models which are known to converge on a steady-state slowly. See Lewis (1974).

2. See for example Stein (1971).

3. This property of Tobin's model was pointed out by Nagatani (1970).


5. Generally time subscripts are omitted unless needed for clarity.

6. Discussion of costate variable behaviour is patterned after Athans-Falb (1966) especially Section 7-10.

7. See Athans-Falb (1966) for a proof of the control law for a first-order non-linear system.

8. To avoid unnecessary duplication only the case for $k_0 - k_e < 0$ is considered.

9. An extensive statement of stability conditions for Model II as well as other monetary growth models is found in Hadjimichalakis (1971).

10. Numerical solutions (e.g., Bryson-Ho (1969)) are possible, but the objective of this study is to show the possibility of complete adjustment in a finite interval of time not to calculate actual values.


12. For the general specification of Models II and III given here, it is impossible to determine whether or not the time-optimal control problem is normal. Examination of the necessary conditions for the occurrence of a singular control were ambiguous (see Athans-Falb (1966) or Bryson-Ho (1969)).
on the singular problem). The singular problem can only be resolved when the functions and parameters in the model are specified exactly. Since such an exercise is beyond the objectives of this study, singular controls are not considered further, and the analysis is carried out as if the time-optimal control problem is normal. If there is a singular control that is optimal over a finite time interval, then the "bang-bang" controls to be derived can only be considered to produce complete adjustment in a "finite" period of time as opposed to "minimum" time.

13. A complete derivation of phase diagrams is contained in Sidrauski (1967).

14. It has been assumed that the lower limit on \( \theta \) does not violate the restriction that the opportunity cost of holding real money balances is positive. Otherwise, the incentives for holding both \( k \) and \( m \) would vanish. More simply, \( \theta \) is constrained to prevent the complications of a liquidity trap from occurring.

15. Actually this final requirement would only be of concern if numerical solutions of this model were being studied. Under these circumstances, it is possible (for \( L_\pi/L \) initially close to \( \gamma \)) that during the adjustment process \( (1 + \gamma L_\pi/L) \) reverses sign implying a switch in the mode of dynamic behaviour. Such a situation would alter the control sequences to be set forth for Model II, and it is not difficult to devise situations for which an optimal solution does not exist. All of this is of no consequence here, however, since the qualitative first-order behaviour upon which controls are derived assume that \( (1 + \gamma L_\pi/L) \) is constant.

16. Assuming \( \dot{k} \) and \( \dot{\pi} \) are defined in the portion of the \((\pi, k)\) plane of concern, there exists an unique solution to \((\pi, k)\). Furthermore,
Bibliography

if a characteristic of $(\pi_-, k_-)$ passes through $(\pi_e, k_e)$, it is unique and therefore the trajectory from some point $D$ to $(\pi_e, k_e)$ can be found.