The Involution Principle and $h$-positive Symmetric Functions

by

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Abstract

The criterion of $h$-positivity corresponds to the criterion that a polynomial representation of the general linear group of $V$ is a sum of tensor products of symmetric powers of $V$. Expanding the iterated exponential function as a power series yields coefficients whose positivity implies the $h$-positivity of the characteristic of the symmetric group character whose value on the permutation $w$ is the number of labeled forests with $c(w)$ vertices, where $c(w)$ is the number of cycles of $w$. Another example of an $h$-positive symmetric function is the characteristic of the top homology of the even-ranked subposet of the partition lattice. In this case, the positive coefficients of the characteristic refine the tangent number $E_{2n-1}$ into sums of powers of two.

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Chapter 1

Symmetric Functions and $h$-positivity

1.1 Symmetric Functions

The following questions are the same question posed in three different contexts.

1. When is an $s$-positive symmetric function $h$-positive?

2. When is a polynomial representation of $GL(V)$ isomorphic to a sum of tensor products of symmetric powers of $V$?

3. When is an $\mathfrak{S}_n$ character a sum of characters induced from trivial representations on Young subgroups?

In this section, the meaning of these questions and their equivalence is discussed. First, $h$-positivity is defined as the basic facts about symmetric functions are recalled without proof. A better exposition may be found in [4], which includes proofs of all of the facts compiled here with the exception of Proposition 2, which is proven in the next section. Symmetric functions are indexed by partitions of $n$.

A partition of $n$ is a sequence which is a composition of $n$, and whose terms are weakly decreasing.

**Definition 1.** (Partitions of a Positive Integer $n$) The set $\text{Par}(n)$ of partitions of the positive integer $n$ is the set of sequences of nonnegative integers

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$$

such that both $\sum_{i \geq 1} \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$.

If the partition $\lambda$ has $l$ nonzero parts, then the integer $l$ is called the length of the partition $\lambda$. In this case, the partition $\lambda$ will also be denoted by the sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$.

Let $(x_i)_{i \geq 1}$ be a set of indeterminates and let $n$ be a nonnegative integer.
Definition 2. (Monomial Symmetric Functions)
The monomial symmetric function of degree $n$ indexed by $\lambda \in \text{Par}(n)$ is the formal power series
\[ m_\lambda = \sum_{(a_1, a_2, a_3, \ldots)} x_1^{a_1} x_2^{a_2} x_3^{a_3} \ldots \]
where the sequence $(a_1, a_2, a_3, \ldots)$ ranges over all distinct permutations of $(\lambda_1, \lambda_2, \lambda_3, \ldots)$.

The module $\Lambda^n$ is the $\mathbb{Z}$-module $\mathbb{Z}[(m_\lambda)_{\lambda \in \text{Par}(n)}]$. Its elements are called the homogeneous symmetric functions of degree $n$. Evidently, if $f \in \Lambda^n$, then given any permutation $u$ of the positive integers, the formal power series $f$ is invariant under the action of $u$. That is,
\[ f(x_1, x_2, x_3, \ldots) = f(x_{u(1)}, x_{u(2)}, x_{u(3)}, \ldots). \]
Conversely, any formal power series $g$ in the indeterminates $(x_i)_{i \geq 1}$ whose terms are each of degree $n$, and which is invariant under the action of any given permutation $u$ of the positive integers is a linear combination of the monomial symmetric functions. Hence, the usual definition of the module $\Lambda^n$ as the set of such formal power series $g$ coincides with the definition of $\Lambda^n$ given here. Notice that the $\mathbb{Z}$-module, $\Lambda = \bigoplus_{n \geq 0} \Lambda^n$ inherits the structure of a graded ring as a subring of $\mathbb{Z}[x_1, x_2, x_3, \ldots]$. Thus, the ring $\Lambda$ is called the ring of symmetric functions.

Definition 3. (Complete Symmetric Functions)
The $n$th complete symmetric function is the formal power series
\[ h_n = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n}. \]
The complete symmetric function of degree $n$ indexed by $\lambda \in \text{Par}(n)$ is the formal power series
\[ h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_l}, \]
where $\lambda$ has $l$ parts.

Thus, the generating function $H(t)$ for the sequence $(h_n)_{n \geq 0}$ with $h_0 := 1$ is the formal power series,
\[ 1 + H(t) := \sum_{n \geq 0} h_n t^n = \prod_{i \geq 1} \frac{1}{1 - x_i t} \quad (1.1) \]
A symmetric function that has nonnegative, integer coefficients when written in the basis of complete symmetric functions is an $h$-positive symmetric function.
Definition 4. (h-positivity)
The symmetric function \( f \in \Lambda^n \) is defined to be h-positive if

\[
f = \sum_{\lambda \in \text{Par}(n)} a_\lambda h_\lambda,
\]

and the coefficient \( a_\lambda \) is a nonnegative integer.

In general, u-positivity may be defined for any basis \( \{(u_\lambda)_{\lambda \in \text{Par}(n)}\} \) of the homogeneous symmetric functions \( \Lambda^n \). Say that the symmetric function \( f \in \Lambda^n \) is u-positive if the formal power series \( f \) may be written as a linear combination of the basis functions \( u_\lambda \) with nonnegative coefficients.

Counting the multiplicity of the monomials which appear in \( h_\lambda \) yields the following result.

Proposition 1. (h-m Transition Matrix)

\[
h_\lambda = \sum_{\mu \in \text{Par}(n)} N_{\lambda \mu} m_\mu
\]

where \( N_{\lambda \mu} \) is the number of matrices with nonnegative, integer entries whose row sums are \( \lambda \) and column sums are \( \mu \).

Thus,

\[
\prod_{i \geq 1} \frac{1}{1 - x_i y_j t} = 1 + \sum_{n \geq 1} t^n \sum_{\lambda \in \text{Par}(n)} \sum_{\mu \in \text{Par}(n)} N_{\lambda \mu} m_\lambda m_\mu
\]

\[
= 1 + \sum_{n \geq 1} t^n \sum_{\lambda \in \text{Par}(n)} m_\lambda h_\lambda. \quad (1.2)
\]

A \( \mathbb{Q} \)-basis of \( \Lambda^n \otimes_{\mathbb{Z}} \mathbb{Q} \) is spanned by the power sum symmetric functions.

Definition 5. (Power Sum Symmetric Functions)
The \( n \)th power sum symmetric function is the formal power series

\[
p_n = \sum_{i \geq 1} x_i^n.
\]

The power sum symmetric function of degree \( n \) indexed by \( \lambda \in \text{Par}(n) \) is the formal power series

\[
p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_l},
\]

where \( \lambda \) has \( l \) parts.

A combinatorial definition of the numbers \( \phi^\lambda(\mu) \) is given in Definition 20. These numbers are the entries of the \( p-h \) transition matrix.

Proposition 2. (p-h Transition Matrix)

\[
p_\mu = \sum_{\lambda \in \text{Par}(n)} \phi^\lambda(\mu) h_\lambda.
\]
Proposition 2 is a basic fact proven, for example, in [9]. Its proof is given in the next section. The entries of the transition matrix $\phi^\lambda(\mu)$ determine the monomial character $\phi^\lambda$ of $S_n$ indexed by $\lambda$ whose central role in the study of $h$-positivity is discussed below.

A third $\mathbb{Z}$-basis of $\Lambda^n$ are the Schur functions, which will not appear here except in the introduction.

**Definition 6.** (Schur Symmetric Functions and $s$-$m$ Transition Matrix)
The Schur symmetric function of degree $n$ indexed by $\lambda \in \text{Par}(n)$ is the formal power series

$$s_\lambda = \sum_{\mu \in \text{Par}(n)} K_{\lambda \mu} m_\mu,$$

where $K_{\lambda \mu}$ is the number of semi-standard Young tableaux of shape $\lambda$ and content $\mu$.

Semi-standard Young tableaux of shape $\lambda$ and content $\mu$ are matrices with non-negative integer entries whose positive entries are $\mu$. Furthermore, these positive entries are top-left justified, strictly increasing in columns, weakly increasing in rows, and the number of positive entries in the $i$th row is $\lambda_i$.

**Proposition 3.** $h$-$s$ Transition Matrix

$$h_\mu = \sum_{\mu \in \text{Par}(n)} K_{\lambda \mu} s_\lambda.$$

In particular, every $h$-positive symmetric function is $s$-positive.

1.2 Representations of $GL(V)$

The direct sum of two representations $W$ and $X$ of $GL(V)$ is the representation $W \oplus X$ where for $g \in GL(V)$, $g$ acts on $W \oplus X$ by linearly extending the action $g(w \oplus x) = g(w) \oplus g(x)$. The tensor product of two representations of $GL(V)$ is the representation $W \otimes X$, where for $g \in GL(V)$, $g$ acts on $W \otimes X$ by linearly extending the action $g(w \otimes x) = g(w) \otimes g(x)$. Thus, the ring of polynomial $GL(V)$ representations $R(GL(V))$ is a ring whose addition operation is the direct sum $\oplus$ and whose multiplication operation is the tensor product $\otimes$.

Let $\Lambda_m$ denote the ring of symmetric functions in $m$ variables. Let $\Lambda_m^n$ denote the ring of homogeneous symmetric functions of degree $n$ in $m$ variables. Define the projection

$$\rho_m : \Lambda \to \Lambda_m$$

$$\rho_m f(x_1, x_2, x_3, \ldots) = f(x_1, x_2, \ldots, x_m, 0, 0, 0, \ldots)$$

Note that the restriction of $\rho_m$ to $\Lambda^n$ for $m \geq n$ gives an isomorphism between $\Lambda^n$ and $\Lambda_m^n$. 

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Theorem 1. Suppose $V$ is a vector space over $\mathbb{C}$ of dimension $m$. The map which sends a representation of $GL(V)$ to its character,

$$\text{ch} : R(GL(V)) \to \Lambda_m$$

is a ring isomorphism.

Hence $R(GL(V))$ is in fact a graded ring.

$$R(GL(V)) = \bigoplus_{n \geq 0} R_n(GL(V))$$

where $R_n(GL(V)) := \text{ch}^{-1}\Lambda_m^n$. Thus $R_n(GL(V))$, the homogeneous, polynomial representations of $GL(V)$ of degree $n$, consists of those polynomial representations of $GL(V)$ whose character is a homogeneous symmetric function in $m$ variables of degree $n$. Let $S^n(V)$ denote the $n$th symmetric power of $V$, and suppose $\dim_{\mathbb{C}}(V) = m \geq n$. Then, evidently,

$$\text{ch}(S^n(V)) = \rho_n h_n$$

Thus,

$$\text{ch}(S^\lambda(V) \otimes \cdots \otimes S^\lambda(V)) = \rho_m h_\lambda$$

where $\lambda \in \text{Par}(n)$ is a partition with $l$ parts. Hence, Theorem 1 has the following corollary.

Corollary 1. For a polynomial representation $W$ of $GL(V)$, $f = \text{ch}(W)$ is $h$-positive if and only if

$$W \cong \bigoplus_{\lambda \in \text{Par}(n)} (S^\lambda(V) \otimes \cdots \otimes S^\lambda(V))^{\otimes_{\mathbb{A}} \lambda}.$$

Above, $V^{\otimes k}$ denotes the $k$-fold direct sum of $V$.

The representation ring $R(GL(V))$ is endowed with another product which corresponds to the operation of composing two representations and which will appear in Chapter 4. Suppose that $\phi : GL(U) \to GL(V)$ is a representation of $GL(U)$ and that $\psi : GL(V) \to GL(W)$ is a representation of $GL(V)$. Suppose $\dim_{\mathbb{C}}(U) = m$ and $\dim_{\mathbb{C}}(V) = l$. The function $\text{ch}(\psi \circ \phi)$ is defined to be the plethystic product or simply the plethysm of $\text{ch}(\psi)$ with $\text{ch}(\phi)$. The symmetric function $\text{ch}(\psi \circ \phi)$ may be defined more explicitly in terms of $\text{ch}(\psi)$ and $\text{ch}(\phi)$. If $\text{ch}(\phi) = \tilde{g}(x_1, \ldots, x_m) = \sum_{i=1}^l x^{a_i}$ and $\text{ch}(\psi) = \tilde{f}(x_1, \ldots, x_l)$ then $\tilde{f}[\tilde{g}] := \text{ch}(\psi \circ \phi) = \tilde{f}(x^{a_1}, \ldots, x^{a_l})$. For symmetric functions $f, g \in \Lambda$ such that $\rho_m g = \tilde{g}$ and $\rho_l f = \tilde{f}$, plethysm is defined so that $\rho_m f[g] = \tilde{f}[\tilde{g}]$.

Definition 7. (Plethysm of Symmetric Functions) Suppose that the symmetric function $g$ is a sum of monomials with coefficient one, and $f$ is a symmetric function. Thus, in multi-index notation, $g = \sum_{i \geq 1} x^{a_i}$. Then, the plethysm of $f$ with $g$ is the symmetric function,

$$f[g] := f(x^{a_1}, x^{a_2}, x^{a_3}, \ldots).$$
1.3 \( \mathfrak{S}_n \) modules

The polynomial representations of \( GL(V) \) are closely related to representations of the symmetric groups \( \mathfrak{S}_n \). For some \( \mathfrak{S}_n \) module \( M \), a given homogeneous, degree \( n \), polynomial representation \( W \) of \( GL(V) \) may be written in the form,

\[
W \cong V^{\otimes n} \otimes_{\mathbb{C}[S_n]} M
\]  

(1.3)

where \( V^{\otimes n} \) denotes the \( n \)-fold tensor product of \( V \). \( V^{\otimes n} \) is an \( \mathfrak{S}_n \) module since linearly extending the map given by \( u(v_1 \otimes \ldots \otimes v_n) = (v_{u(1)} \otimes \ldots \otimes v_{u(n)}) \) to \( V^{\otimes n} \) gives an \( \mathfrak{S}_n \) action. Let \( R(\mathfrak{S}_n) \) denote the representation ring of \( \mathfrak{S}_n \). Then, there is a bijection \( \phi_n \) defined by,

\[
\phi_n : R(\mathfrak{S}_n) \to R_n(GL(V))
\]

\[
\phi_n(M) = V^{\otimes n} \otimes_{\mathbb{C}[S_n]} M.
\]  

(1.4)

Evidently, \( \phi_n \) is an isomorphism of \( R(\mathfrak{S}_n) \) and \( R_n(GL(V)) \) as abelian groups with addition \( \oplus \), and thus \( \phi_n \) is an isomorphism of vector spaces over \( \mathbb{C} \). Now, suppose \( N \) is an \( \mathfrak{S}_n \) module and \( L \) is an \( \mathfrak{S}_l \) module. Then,

\[
\phi_n(N) \otimes \phi_l(L) \cong (V^{\otimes n} \otimes_{\mathbb{C}[S_n]} N) \otimes (V^{\otimes l} \otimes_{\mathbb{C}[S_l]} L)
\]

\[
\cong V^{\otimes n+l} \otimes_{\mathbb{C}[S_{n+l}]} \text{ind}_{\mathfrak{S}_n \times \mathfrak{S}_l}^{\mathfrak{S}_{n+l}} N \otimes L.
\]

Hence define:

\[
N \times L := \text{ind}_{\mathfrak{S}_n \times \mathfrak{S}_l}^{\mathfrak{S}_{n+l}} N \otimes L
\]

Extend \( \times \) linearly to obtain a product on \( \bigoplus_{n \geq 1} R(\mathfrak{S}_n) \) such that \( \phi_{n+l}(N \otimes L) = \phi_n(N) \otimes \phi_l(L) \). Thus, we have the following proposition.

**Proposition 4.** Define a map,

\[
\phi : \bigoplus_{n \geq 1} R(\mathfrak{S}_n) \to R(GL(V))
\]

as the linear extension of the maps

\[
\phi_n(N) = V^{\otimes n} \otimes_{\mathbb{C}[S_n]} N
\]

for \( N \in R(\mathfrak{S}_n) \). Then, \( \phi \) is a ring homorphism.

A representation \( \phi : \mathfrak{S}_n \to GL(V) \) is determined by its character. The character \( \chi \) of \( \phi \) is the function,

\[
\chi : \mathfrak{S}_n \to \mathbb{C}
\]

\[
\chi(u) = \text{tr}(\phi(u)),
\]

where \( \text{tr}(\phi(u)) \) is the trace of the linear map \( \phi(u) \). Every character is a class function which is to say that it is constant on the conjugacy classes of \( \mathfrak{S}_n \).
Definition 8. For \( u \in \mathfrak{S}_n \) and \( \lambda \in \text{Par}(n) \), define,

\[
\text{type}(u) = \lambda,
\]

if the lengths of the cycles of \( u \) are the parts of \( \lambda \).

The permutations \( u, v \in \mathfrak{S}_n \) belong to the same conjugacy class if and only if \( \text{type}(u) = \text{type}(v) \). Hence, the conjugacy classes of \( \mathfrak{S}_n \) are indexed by partitions of \( n \).

The Frobenius characteristic associates a symmetric function to each class function.

Definition 9. (Frobenius Characteristic)

The Frobenius Characteristic of the \( \mathfrak{S}_n \) class function \( \chi \) is the symmetric function

\[
\Phi(\chi) := \frac{1}{n!} \sum_{u \in \mathfrak{S}_n} \chi(u)p_{\text{type}(u)}
\]

The Frobenius characteristic of an \( \mathfrak{S}_n \) module \( N \) is by definition the Frobenius characteristic of its character.

Theorem 2. Suppose \( N \) is an \( \mathfrak{S}_n \) module whose character is \( \chi_N \). Then, the linear extension of the map taking \( N \) to its Frobenius characteristic \( \Phi(\chi_N) \) is a ring isomorphism,

\[
\Phi : \bigoplus_{n \geq 1} R(\mathfrak{S}_n) \to \Lambda.
\]

The Schur-Weyl duality motivates the definition of \( \Phi \) and gives the relationship between the character of the \( \mathfrak{S}_n \) module \( N \) denoted \( \chi_N \) and the character of the \( GL(V) \) module \( \phi(N) = V^{\otimes n} \otimes_{\mathbb{C}[\mathfrak{S}_n]} N \).

Theorem 3. (Schur-Weyl Duality)

Suppose \( N \) is an \( \mathfrak{S}_n \) module whose character is \( \chi_N \) and \( \text{dim}_\mathbb{C}(V) = m \). Then,

\[
\text{ch}(V^{\otimes n} \otimes_{\mathbb{C}[\mathfrak{S}_n]} N) = \rho_m \Phi(\chi_N)
\]

Suppose \( \text{dim}(V) = m \). Let \( \Phi^{-1} \) denote the inverse of the Frobenius characteristic, and suppose \( \phi \) is defined as in Proposition 4. Then, the Schur-Weyl duality states that the composition of maps,

\[
\Lambda \xrightarrow{\Phi^{-1}} \bigoplus_{n \geq 1} R(\mathfrak{S}_n) \xrightarrow{\phi} R(GL(V)) \xrightarrow{\text{ch}} \Lambda_m
\]

satisfies

\[
\rho_m = \text{ch} \circ \phi \circ \Phi^{-1}.
\]

For example, denote the identity \( \mathfrak{S}_n \) module by \( 1_{\mathfrak{S}_n} \). Then,

\[
V^{\otimes n} \otimes_{\mathbb{C}[\mathfrak{S}_n]} 1_{\mathfrak{S}_n} \cong S^n(V).
\]
Hence,
\[ \Phi(1_{\mathfrak{S}_n}) = h_n. \]

Since \( \Phi \) is a ring isomorphism,
\[ \Phi(1_{\mathfrak{S}_{\lambda_1}} \times \cdots \times 1_{\mathfrak{S}_{\lambda_l}}) = h_{\lambda}, \]
where \( \lambda \) is a partition of \( n \) with \( l \) parts. Recall that \( 1_{\mathfrak{S}_{\lambda_1}} \times \cdots \times 1_{\mathfrak{S}_{\lambda_l}} = \text{ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_l}} 1. \)

A subgroup of the form \( \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_l} \) is called a Young subgroup or a parabolic subgroup. Thus, we have the following corollary to Theorem 2.

**Corollary 2.** Suppose \( N \) is an \( \mathfrak{S}_n \) module whose character is \( \chi_N \). \( f = \Phi(\chi_N) \) is \( h \)-positive if and only if
\[ N \cong \bigoplus_{\lambda \in \text{Par}(n)} (\text{ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_l}} 1)^{\mathfrak{S}_\lambda}. \]

Denote the irreducible representation of \( \mathfrak{S}_n \) indexed by \( \lambda \in \text{Par}(n) \) by \( S^\lambda \). Then \( S^\lambda(V) := V^{\otimes n} \otimes_{\mathfrak{S}[\mathfrak{S}_n]} \mathfrak{S}_\lambda \) is the irreducible representation of \( GL(V) \) indexed by \( \lambda \). Furthermore,
\[ \rho_m \text{ch}(S^\lambda(V)) = \Phi(S^\lambda) = s_\lambda. \]

Thus, the class function \( \chi \) is the character of an \( \mathfrak{S}_n \) module if and only if \( \Phi(\chi) \) is \( s \)-positive. Proposition 3 states that if \( \Phi(\chi) \) is \( h \)-positive then it is \( s \)-positive. In fact, if \( \Phi(\chi) \) is \( h \)-positive then Corollary 2 states that \( \chi \) is not only a character but also the character of an \( \mathfrak{S}_n \) module which is the sum of trivial modules induced from Young subgroups. Now, suppose that it is not known whether the \( \mathfrak{S}_n \) module \( N \) is the sum of trivial modules induced from Young subgroups, or equivalently whether \( \Phi(N) \) is \( h \)-positive. Suppose that \( \chi_N \) is the character of \( N \). Then, using the definition of \( \Phi \) and Proposition 2,
\[ \Phi(N) := \frac{1}{n!} \sum_{u \in \mathfrak{S}_n} \chi_N(u)p_{\text{type}}(u) = \frac{1}{n!} \sum_{\lambda \in \text{Par}(n)} \sum_{u \in \mathfrak{S}_n} \chi_N(u)\phi^\lambda(u)h_{\lambda}. \]

Hence, we have the following corollary of Proposition 2.

**Corollary 3.** \( \Phi(N) \) is \( h \)-positive if and only if for each \( \lambda \in \text{Par}(n) \)
\[ \sum_{u \in \mathfrak{S}_n} \chi_N(u)\phi^\lambda(u) \]
is a positive integer.

This corollary explains the central importance of the monomial character \( \phi^\lambda \) in determining the \( h \)-positivity of the Frobenius characteristic of an \( \mathfrak{S}_n \) module.
Chapter 2

The Combinatorics of the Monomial Character

2.1 The Compositional Formula

Definition 10. (Hypergraph)
A hypergraph $h = (V(h), E(h))$ consists of a vertex set $V(h)$ together with an edge set $E(h)$. The edge set $E(h) = \{A_1, A_2, \ldots, A_{h_1}\}$ is a collection of subsets of $V(h)$ such that each subset $A_i$ has cardinality at least two.

If each edge $A_i$ of the hypergraph has cardinality exactly two, then the hypergraph $h$ is a graph with vertex set $V(h)$. If the elements in each edge $A_i$ of a hypergraph $h$ are assigned an order then the hypergraph $h$ becomes a directed hypergraph.

Definition 11. (Directed Hypergraph)
A directed hypergraph $h = (V(h), E(h))$ consists of a vertex set $V(h)$ together with an edge set $E(h)$. The edge set $E(h) = \{A_1, A_2, \ldots, A_{h_1}\}$ is a collection of ordered subsets of $V(h)$ such that each ordered subset $A_i$ has cardinality at least two.

Let $D_S$ denote the set of all directed hypergraphs $h$ whose vertex set $V(h)$ is the set $S$. If each edge $A_i$ of the directed hypergraph has cardinality exactly two, then the hypergraph $h$ is a directed graph with vertex set $V(h)$.

Suppose that the function $w$ assigns an element of the ring $R$ to each element of the set $S$. For example, say that the ring $R$ is the integers $\mathbb{Z}$.

$$w : S \to \mathbb{Z}.$$ 

The function $w$ may be extended to the set $2^S$ of subsets of $S$, by setting

$$w(A) = \sum_{s \in A} w(s),$$

where $A \in 2^S$. Then,

$$w(A \sqcup B) = w(A) + w(B)$$

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where $A \coprod B$ denotes the disjoint union (coproduct) of $A$ and $B$ for $A, B \in 2^S$. Suppose $w_1, w_2$ are functions from $S_1, S_2$ respectively to the integers. Then define the function $w = w_1 \coprod w_2$ from $S_1 \coprod S_2$ to the integers by setting,

$$\begin{align*}
w(s) := \begin{cases} 
w_1(s), & s \in S_1 \\
w_2(s), & s \in S_2 \end{cases}.
\end{align*}$$

Let $S_1 \times S_2 = \{(s_1, s_2) | s_1 \in S_1, s_2 \in S_2\}$ denote the product of $S_1$ and $S_2$. Then define a function,

$$\tilde{w} : S_1 \times S_2 \to \mathbb{C}$$

by

$$\tilde{w}(s_1, s_2) = w(s_1)w(s_2).$$

Hence, if $A$ and $B$ are both subsets of $S$, then $w(A \times B) = w(A)w(B)$. Throughout this section, if the function $w$ has as its domain the disjoint union $S_1 \coprod S_2$, then the function $\tilde{w}$ whose domain is $S_1 \times S_2$ will also be called $w$. That is, it will be written that

$$w(s_1, s_2) = \tilde{w}(s_1, s_2) = w(s_1)w(s_2),$$

where $s_1 \in S_1$ and $s_2 \in S_2$.

In this section, the combinatorial interpretation of the composition of formal power series is recalled. A proof of the compositional formula along with many examples can be found in [6].

Suppose that the set $\mathcal{F}_{[n]}$ is a subset of the directed hypergraphs $\mathcal{D}_{[n]}$ with vertex set $[n]$. Let $F(x)$ be the weighted exponential generating function for the sequence of hypergraphs $\mathcal{F}_{[n]}$. That is,

$$F(x) = \sum_{n \geq 0} w(\mathcal{F}_{[n]}) \frac{x^n}{n!},$$

where $w(\mathcal{F}_{[0]}) := f_0$. Suppose, similarly, that

$$G(x) = \sum_{n \geq 0} w(\mathcal{G}_{[n]}) \frac{x^n}{n!},$$

where $w(\mathcal{G}_{[0]}) := g_0$.

Then we have the following result.

**Proposition 5.** (Multiplication Formula)

$$F(x)G(x) = \sum_{n \geq 0} w \left( \coprod_{B_1 \subseteq [n]} \mathcal{F}_{B_1} \times \mathcal{G}_{[n]-B_1} \right) \frac{x^n}{n!}$$
Proof.

\[ F(x)G(x) = \sum_{n \geq 0} \sum_{k=0}^{n} \binom{n}{k} w(\mathcal{F}_k)w(\mathcal{G}_{n-k}) \frac{x^n}{n!} \]

\[ = \sum_{n \geq 0} \sum_{B_1 \subseteq [n]} w(\mathcal{F}_{B_1})w(\mathcal{G}_{[n]-B_1}) \frac{x^n}{n!} \]

\[ = \sum_{n \geq 0} w \left( \prod_{B_1 \subseteq [n]} \mathcal{F}_{B_1} \times \mathcal{G}_{[n]-B_1} \right) \frac{x^n}{n!}. \]

\[ \square \]

Let \( \Pi^k_S \) denote the set of partitions of the set \( S \) into \( k \) blocks.

**Definition 12.** (Partitions of the set \( S \) into \( k \) blocks) A partition \( \Lambda = \{\Lambda_1, \ldots, \Lambda_k\} \) of the set \( S \) into \( k \) blocks is a collection of \( k \) disjoint subsets of the set \( S \) whose union is \( S \).

An element \( \Lambda_i \) of a partition \( \Lambda \) is called a block of the partition. Let \( \Pi_S \) denote the set of all partitions of the set \( S \).

\[ \Pi_S := \coprod_{k \geq 1} \Pi^k_S. \]

The set \( \Pi_S \) is also a lattice as will be explained in a later section.

Let \( \text{Comp}_k S \) be the set of partitions of \( S \) into \( k \) ordered blocks.

**Definition 13.** (Ordered Partitions of the Set \( S \)) An ordered partition \( B = (B_1, \ldots, B_k) \) of the set \( S \) into \( k \) blocks is an ordered \( k \)-tuple of disjoint subsets of set \( S \) whose union is \( S \).

Thus, if the tuple \( B = (B_1, \ldots, B_k) \) is an ordered partition of \( S \), then the unordered set \( \{B_1, \ldots, B_k\} \) is a partition of \( S \) into \( k \) blocks. The set of all ordered partitions of \( S \) will be denoted by \( \text{Comp} S \).

\[ \text{Comp} S = \coprod_{k \geq 1} \text{Comp}_k S \]

Throughout this section, the set \( \{1, \ldots, n\} \) will be denoted by \([n]\), and the set \( \Pi^k_{[n]} \) will be denoted by the standard notation \( \Pi^k_{[n]} \).

Notice that \( G(x) \) has a zero constant term in the following proposition which is proved by iterating Proposition 5 for such \( G(x) \).
Proposition 6. If

\[ G(x) = \sum_{n \geq 1} w(G_{[n]}) \frac{x^n}{n!}, \]

then

\[ G^k(x) = \sum_{n \geq 1} w \left( \prod_{B \in \text{Comp}_k [n]} G_{B_1} \times \cdots \times G_{B_k} \right) \frac{x^n}{n!}. \]

☐

Proposition 7. (Compositional Formula)

If, for \( f_0 \in R \)

\[ F(x) = f_0 + \sum_{n \geq 1} w(\mathcal{F}_{[n]}) \frac{x^n}{n!} \]

and

\[ G(x) = \sum_{n \geq 1} w(G_{[n]}) \frac{x^n}{n!} \]

then

\[ F(G(x)) = f_0 + \sum_{n \geq 1} w \left( \prod_{\Lambda \in \Pi_n} \mathcal{F}_{[\Lambda]} \times G_{\Lambda_1} \times \cdots \times G_{\Lambda_k} \right) \frac{x^n}{n!}. \]

Proof.

\[ F(G(x)) = f_0 + \sum_{k \geq 1} w(\mathcal{F}_{[k]}) \frac{(G(x))^k}{k!} \]

then using Proposition 6.

\[ F(G(x)) = f_0 + \sum_{k \geq 1} w(\mathcal{F}_{[k]}) \frac{1}{k!} \sum_{n \geq 1} w \left( \prod_{B \in \text{Comp}_k [n]} G_{B_1} \times \cdots \times G_{B_k} \right) \frac{x^n}{n!} \]

\[ = f_0 + \sum_{n \geq 1} \sum_{k \geq 1} w(\mathcal{F}_{[k]}) w \left( \prod_{\Lambda \in \Pi_n} G_{\Lambda_1} \times \cdots \times G_{\Lambda_k} \right) \frac{x^n}{n!} \]

\[ = f_0 + \sum_{n \geq 1} w \left( \prod_{\Lambda \in \Pi_n} \mathcal{F}_{[\Lambda]} \times G_{\Lambda_1} \times \cdots \times G_{\Lambda_k} \right) \frac{x^n}{n!} \]

☐

When for each positive integer \( n \) the value \( w(\mathcal{F}_{[n]}) \) is equal to one and \( f_0 = 1 \), then

\[ \exp(x) = F(x) = 1 + \sum_{n \geq 1} \frac{x^n}{n!}. \]

In this special case, the compositional formula is called the exponential formula.
Proposition 8. (Exponential Formula)

Suppose

\[ G(x) = \sum_{n \geq 1} w(G_{[n]}) \frac{x^n}{n!} \]

then

\[ \exp(G(x)) = 1 + \sum_{n \geq 1} w \left( \coprod_{\Lambda \in \Pi_n} G_{\Lambda_1} \times \cdots \times G_{\Lambda_A} \right) \frac{x^n}{n!} \]

For example, the exponential formula states that the exponential generating function for the number of graphs with vertex set \([n]\) is given by exponentiating the exponential generating functions for connected graphs with vertex set \([n]\).

2.2 Rooted Trees and Forests

Let \(T_{[n]}\) be the set of rooted trees with vertex set \([n]\). That is, \(T_{[n]}\) consists of trees labeled by \([n]\) with a distinguished vertex.

Set

\[ T(x) = \sum_{n \geq 1} \#(T_{[n]}) \frac{x^n}{n!}. \]

In fact,

\[ T(x) = \sum_{n \geq 1} n^{n-1} \frac{x^n}{n!}. \]

Cayley's formula, which implies that the number of rooted trees is \(n^{n-1}\) has been given numerous proofs [6]. Let \(F_{[n]}\) be the set of rooted forests with vertex set \([n]\). That is, \(F_{[n]}\) consists of graphs whose connected components are rooted trees. Set

\[ F(x) = 1 + \sum_{n \geq 1} \#(F_{[n]}) \frac{x^n}{n!}. \]

Proposition 9.

\[ T \left( \frac{x}{\exp(x)} \right) = x \]

or equivalently

\[ F \left( \frac{\log x}{x} \right) = x \]

Proof. By the exponential formula,

\[ F(x) = \exp T(x). \]

Thus, the two parts of the proposition are indeed equivalent. Now, a rooted tree consists of a distinguished vertex \(k\) which is connected to the roots of a rooted forest
with vertex set \([n] - k\). That is,

\[
\mathcal{T}_n \cong \prod_{k=1}^{n} \{k\} \times \mathcal{F}_{[n]-k}.
\]

Thus, by the multiplication formula, Proposition 5,

\[
T(x) = x F(x),
\]

\[
T(x) = x \exp T(x).
\]

Thus,

\[
\frac{T(x)}{\exp T(x)} = x,
\]

and

\[
T \left( \frac{x}{\exp x} \right) = x,
\]

which is to say that \(T(x)\) and \(x/\exp x\) are compositional inverses. \(\square\)

### 2.3 Trees with Increasing Leaves

Say that a leaf of a rooted tree labeled by \([n]\) is increasing if it is greater than its parent.

Let \(\mathcal{I}_{[n]}\) denote the set of rooted trees with vertex set \([n]\) such that all leaves of the tree are increasing. Let \(\mathcal{G}_{[n]}\) denote the set of rooted trees whose vertex set consists of the blocks of a partition \([n]\) such that no leaf vertex of the tree is a single-element block. Let \(\mathcal{H}_{[n]}\) denote the set of rooted forests whose vertex set consists of the blocks of a partition \([n]\) such that no leaf vertex of the forest is a single-element block.

Evidently, there is a bijection:

\[
\phi : \mathcal{I}_{n} \rightarrow \mathcal{G}_{n}
\]

If \(i \in \mathcal{I}_n\), form the tree \(\phi(i)\) by replacing each vertex \(v\) which is the parent of some leaves \(l_1, \ldots, l_k\) with the subset \(\{l_1, \ldots, l_k, v\} \subseteq [n]\) which consists of the vertex \(v\) and its children leaves, and removing the leaves \(l_1, \ldots, l_k\) from the tree. Conversely, suppose \(b \in \mathcal{G}_{[n]}\), form the tree \(\phi^{-1}(b)\) by replacing each node \(\{l_1, \ldots, l_k, v\}\) which is a block with more than one element with the least element in the block, say \(v\), and from the remaining elements of the block forming leaves \(l_1, \ldots, l_k\) each of which is a single element of \([n]\), and such that each leaf has the vertex \(v\) as its parent.

Now, \(b \in \mathcal{G}_{[n]}\) consists of a distinguished subset \(S\) together with a forest \(h \in \mathcal{H}_{[n]}\) none of whose components is a single element since such would form a leaf in the tree \(b\). That is,
\[ \mathcal{G}_{[n]} \cong \prod_{S \subset [n]} \left( S \times \prod_{\Lambda \in \Pi_{[n]-S} \Lambda_i \neq 1} \mathcal{G}_{\Lambda_{\Lambda_i}} \right). \]

Let \( G(x) \) be the weighted exponential generating function for the set \( \mathcal{G}_{[n]} \). Then by the multiplication formula, Proposition 5, and the compositional formula, Proposition 7:

\[ G(x) = (e^x - 1)e^{G(x) - x} = (1 - e^{-x})e^{G(x)}. \]

Thus,

\[ \log\left( \frac{e^{G(x)}}{e^{G(x)}} \right) = 1 - e^{-x}. \]

Thus,

\[ e^{G(x)} = F\left( \frac{\log\left( \frac{e^{G(x)}}{e^{G(x)}} \right)}{e^{G(x)}} \right) = F(1 - e^{-x}). \]

For \( n \geq 1 \) and \( \Lambda_i \subset [n] \), set \( \text{sgn}(\Lambda_i) = (-1)^{\#(\Lambda_i)-1} \). Then,

\[
\begin{align*}
\text{e}^{G(x)} &= F(1 - e^{-x}) \\
&= 1 + \sum_{n \geq 1} \text{sgn} \left( \prod_{\Lambda \in \Pi_n \Lambda_i \neq 1} (\mathcal{F}_{[\Lambda_i]} \times \Lambda_1 \ldots \times \Lambda_{i_\Lambda}) \right) \frac{x^n}{n!} \quad (2.1)
\end{align*}
\]

### 2.4 Cycles and Permutations

The set \( \mathcal{Z}_{[n]} \) consists of directed graphs which are \( n \)-cycles.

**Definition 14.** The set \( \mathcal{Z}_{[n]} \) denotes directed cycles with vertex set \([n]\).

If \( z \in \mathcal{Z}_{[n]} \), set

\[ \text{sgn}(z) = (-1)^{n-1}. \]

Then,

\[ \log(1 + x) = -\sum_{n \geq 1} (n - 1)! \frac{(-x)^n}{n!} = \sum_{n \geq 1} \text{sgn} \mathcal{Z}_{[n]} \frac{x^n}{n!}, \]

and

\[ -\log(1 - x) = \sum_{n \geq 1} (n - 1)! \frac{(x)^n}{n!} = \sum_{n \geq 1} \# \mathcal{Z}_{[n]} \frac{x^n}{n!}. \]

**Definition 15.** \( \mathcal{S}_{[n]} \) denotes directed graphs with vertex set \([n]\) whose connected components are directed cycles

A permutation may be regarded as a directed graph. Given a permutation \( u \in \mathcal{S}_n \) construct a graph with vertex set \([n]\) whose directed edges go from \( i \) to \( u(i) \) for each
\( i \in [n] \). For example, for \( z \in \mathcal{Z}_n \subset \mathfrak{S}_n \), the graph \( z \) may be regarded as a cyclic permutation of \( n \). Henceforth, the notation \( \mathfrak{S}_n \) will be used to denote the group of permutations of \( n \) elements as well as the set \( \mathcal{S}_n \) of directed graphs with vertex set \([n]\) whose connected components are directed cycles.

For \( u \in \mathfrak{S}_n \), define

\[
c(u) := \# \{ \text{connected components of } u \} = \# \{ \text{cycles of } u \} \quad (2.2)
\]

Set

\[
\text{sgn}(u) = (-1)^{n-c(u)}
\]

which coincides with the usual definition of the character \( \text{sgn} \) on the group \( \mathfrak{S}_n \). Evidently, the usual decomposition of a permutation into disjoint cycles gives a bijection

\[
\phi: \prod_{\Lambda \in \Pi_n} (\mathcal{Z}_{\Lambda_1} \times \cdots \times \mathcal{Z}_{\Lambda_l}) \to \mathfrak{S}_n. \quad (2.3)
\]

Furthermore, if \( \phi^{-1}(u) = (z_1, \ldots, z_{l_\Lambda}) \in \mathcal{Z}_{\Lambda_1} \times \cdots \times \mathcal{Z}_{\Lambda_l} \), then

\[
\text{sgn}(\phi^{-1}(u)) = \text{sgn}(z_1) \cdots \text{sgn}(z_{l_\Lambda}) = (-1)^{\#(\Lambda_1)-1} \cdots (-1)^{\#(\Lambda_l)-1} = (-1)^{n-c(u)} = \text{sgn}(u). \quad (2.4)
\]

Thus, the bijection \( \phi \) and the exponential formula, Proposition 8, imply

\[
1 + \sum_{n \geq 1} \text{sgn}(\mathfrak{S}_n) \frac{x^n}{n!} = \exp(\log(1 + x)) = 1 + x,
\]

and

\[
1 + \sum_{n \geq 1} \#(\mathfrak{S}_n) \frac{x^n}{n!} = \exp(-\log(1 - x)) = \frac{1}{1 - x}.
\]

Both of these formulæ are immediately verifiable.

\section{2.5 A Combinatorial Formula for the Monomial Character}

\( \mathcal{P}_n \) consists of directed graphs which are paths.

**Definition 16.** \( \mathcal{P}_n \) denotes directed paths with vertex set \([n]\).

Evidently, \( \mathcal{P}_n \cong \mathfrak{S}_n \) since permutations may be written in one-line notation as well as cycle notation. For \( P \in \mathcal{P}_n \), set

\[
w(P) = h_n
\]

where for \( i \geq 1 \), \( h_i \) are commuting indeterminates. Then, set
\[ H(x) = \sum_{n \geq 1} h_n x^n = \sum_{n \geq 1} w \left( P_{[n]} \right) \frac{x^n}{n!}. \]

By the compositional formula, Proposition 7,

\[ \log H = \sum_{n \geq 1} w \left( \prod_{\Lambda \in \Pi_n} Z_{[\Lambda]} \times P_{\Lambda_1} \times \cdots \times P_{\Lambda_{1_{\Lambda}}} \right) \frac{x^n}{n!}. \]

Set

\[ Z^P_{[n]} := \prod_{\Lambda \in \Pi_n} Z_{[\Lambda]} \times P_{\Lambda_1} \times \cdots \times P_{\Lambda_{1_{\Lambda}}}. \]

Then the set \( Z^P_{[n]} \) consists of tuples \((z, P_{\Lambda_1}, \ldots, P_{\Lambda_{1_{\Lambda}}})\) such that \( z \) is a cyclic permutation of paths, \( P_{\Lambda_1}, \ldots, P_{\Lambda_{1_{\Lambda}}} \), and such that \( \Lambda \in \Pi_n \). Now, a directed cycle of paths which partition \([n]\) may itself be regarded as a directed cycle with vertex set \([n]\). Indeed, there is a bijection

\[ \psi : Z^P_{[n]} \rightarrow Z^P_{[n]} \]

where \( Z^P_{[n]} \) is defined below.

**Definition 17.** \( Z^P_{[n]} \) is the set of tuples \((z, P_{\Lambda_1}, \ldots, P_{\Lambda_{1_{\Lambda}}})\) such that \( z \) is a directed cycle with vertex set \([n]\) which contains the paths, \( P_{\Lambda_1}, \ldots, P_{\Lambda_{1_{\Lambda}}} \), and such that \( \Lambda \in \Pi_n \).

For \((z, P_{\Lambda_1}, \ldots, P_{\Lambda_{1_{\Lambda}}}) \in Z^P_{[n]}\), set

\[ w(z, P_{\Lambda_1}, \ldots, P_{\Lambda_{1_{\Lambda}}}) = w \left( \psi^{-1}(z, P_{\Lambda_1}, \ldots, P_{\Lambda_{1_{\Lambda}}}) \right) = (-1)^{|\Lambda|} h_{\#(\Lambda_1)} \cdots h_{\#(\Lambda_{1_{\Lambda}})} = (-1)^{|\Lambda|-1} h_{\text{type}(\Lambda)} \]

Thus, we have the following proposition.

**Proposition 10.**

\[ \log H = \sum_{n \geq 1} w \left( Z^P_{[n]} \right) \frac{x^n}{n!}. \]

By Proposition 10,

\[ \sum_{i \geq 1} \log H(x_i t) = \sum_{n \geq 1} \frac{t^n}{n!} p_n(x) w \left( Z^P_{[n]} \right). \]

For \( z \in Z^P_{[n]} \), set

\[ \tilde{w}(z) := p_n(x) w(z) \]

so that

\[ \sum_{i \geq 1} \log H(x_i t) = \sum_{n \geq 1} \frac{t^n}{n!} \tilde{w} \left( Z^P_{[n]} \right). \]
Thus, by the exponential formula, Proposition 8,

\[
\exp \left( \sum_{i \geq 1} \log H(x_i t) \right) = 1 + \sum_{n \geq 1} \frac{t^n}{n!} \tilde{w} \left( \prod_{\Lambda \in \Pi_n} \mathcal{Z}_{\Lambda_1}^P \times \cdots \times \mathcal{Z}_{\Lambda_n}^P \right). \tag{2.5}
\]

Evidently, the usual decomposition of a permutation into cycles (2.3) induces a bijection

\[
\mathcal{G}_{[n]}^P := \prod_{\Lambda \in \Pi_n} \mathcal{Z}_{\Lambda_1}^P \times \cdots \times \mathcal{Z}_{\Lambda_n}^P \xrightarrow{\phi} \mathcal{G}_{[n]}^P
\]

where \( \mathcal{G}_{[n]}^P \) is defined as follows.

**Definition 18.** \( \mathcal{G}_{[n]}^P \) is the set of tuples \((u, P_{\Lambda_1}, \ldots, P_{\Lambda_k})\) such that \(u\) is a permutation of \(n\) and \(u\) contains paths \(P_{\Lambda_1}, \ldots, P_{\Lambda_k}\) whose vertex sets partition \([n]\).

For \((u, P_{\Lambda_1}, \ldots, P_{\Lambda_k}) \in \mathcal{G}_{[n]}^P\), set

\[
\tilde{w}(u, P_{\Lambda_1}, \ldots, P_{\Lambda_k}) = \tilde{w}(\phi^{-1}(u)) = (-1)^{\sigma(u) p_{\text{type}(u)}(x)} (-1)^{l_{\text{type}(\Lambda)}} = \text{sgn}(u) p_{\text{type}(u)}(x) (-1)^{n + l_{\text{type}(\Lambda)}}. \tag{2.6}
\]

Then, by (2.5)

\[
\exp \left( \sum_{i \geq 1} \log H(x_i t) \right) = 1 + \sum_{n \geq 1} \frac{t^n}{n!} \tilde{w} \left( \mathcal{G}_{[n]}^P \right). \tag{2.6}
\]

In order to further refine the above formula (2.6), one may define the set \( \mathcal{G}_{[n]}^P \) by defining subsets \( C_{u\Lambda} \).

\[
C_{u\Lambda} := \left\{ s \mid s = (u, P_{\Lambda_1}, \ldots, P_{\Lambda_k}) \in \mathcal{G}_{[n]}^P, \quad \text{type}(\Lambda) = \lambda \right\}
\]

Thus, we have the following equivalent definition deduced in [9].

**Definition 19.** \( C_{u\lambda} \) is the set of tuples \((u, P_{\Lambda_1}, \ldots, P_{\Lambda_k})\) such that \(u\) is a permutation of \(n\) and \(u\) contains paths \(P_{\Lambda_1}, \ldots, P_{\Lambda_k}\) of lengths \(\lambda_1, \lambda_2, \ldots, \lambda_{\lambda_k}\) such that the vertex sets of the paths partition \([n]\).

Evidently if \(\text{type}(u) = \text{type}(u')\), then \(\#C_{u\lambda} = \#C_{u'\lambda}\) so that the function \(\#C_{u\lambda}\) of the permutation \(u\) is a class function on \(\mathcal{G}_n\). The monomial character \(\phi^\lambda\) indexed by \(\lambda\) is defined as follows.

**Definition 20.** (Monomial Character)

\[
\phi^\lambda(u) := \text{sgn}(u)(-1)^{n - l_{\lambda}} \#(C_{u\lambda}).
\]
Thus, returning to equation (2.6)

\[
\exp \left( \sum_{i \geq 1} \log H(x_i t) \right) = 1 + \sum_{n \geq 1} \frac{t^n}{n!} \hat{w} \left( \mathcal{S}_{[n]}^P \right)
\]

\[
= 1 + \sum_{n \geq 1} \frac{t^n}{n!} \hat{w} \left( \prod_{u \in \mathcal{S}_n} \prod_{\lambda \in \text{Par}(n)} C_{u, \lambda} \right)
\]

\[
= 1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{u \in \mathcal{S}_n} \sum_{\lambda \in \text{Par}(n)} \#(C_{u, \lambda}) \text{sgn}(u)p_{\text{type}(u)}(-1)^{n+\lambda_\lambda} h_\lambda.
\]

Hence, supposing that \(1 + H(t) = 1/ \prod_{i \geq 1} (1 - y_i t) = 1 + \sum_{i \geq 1} h_i(y) t\), we have the following proposition.

**Proposition 11.**

\[
\exp \left( \sum_{i \geq 1} \log H(x_i t) \right) = 1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{u \in \mathcal{S}_n} \sum_{\lambda \in \text{Par}(n)} \phi^\lambda(u)p_{\text{type}(u)}(x) h_\lambda(y)
\]

The above proposition may be compared with the following proposition [6] [4].

**Proposition 12.**

\[
\exp \left( \sum_{i \geq 1} \log H(x_i t) \right) = 1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{u \in \mathcal{S}_n} p_{\text{type}(u)}(x)p_{\text{type}(u)}(y)
\]

**Proof.**

\[
\sum_{i \geq 1} \log H(x_i t) = \sum_{i \geq 1} \log \left( \prod_{j \geq 1} \frac{1}{1 - t x_i y_j} \right)
\]

\[
= \sum_{i \geq 1} \sum_{j \geq 1} \log \left( \frac{1}{1 - x_i y_j t} \right)
\]

\[
= \sum_{i \geq 1} \sum_{j \geq 1} \sum_{n \geq 1} \frac{(x_i y_j t)^n}{n}
\]

\[
= \sum_{n \geq 1} \frac{t^n}{n!} p_n(x)p_n(y)
\]

Then, \(\sum_{n \geq 1} \frac{t^n}{n!} (n - 1)! p_n(x)p_n(y)\) may be considered the exponential generating function for circular permutations of length \(n\) weighted by \(p_n(x)p_n(y)\).

Thus, applying the exponential formula, Proposition 8, proves the proposition. \(\square\)

Equating the coefficient of \(p_{\text{type}(u)}(x)\) in Proposition 11 and Proposition 12 proves Proposition 2 recorded in the Introduction and also proven in [9].

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Chapter 3

The Iterated Exponential

3.1 Coefficients of the Iterated Exponential

The iterated exponential

\[ y = x^{x^{x^{\ddots}}} \]

is the limit, when it exists, of the sequence

\[ x_1 = x, x_2 = x^x, x_3 = x^{(x^x)}, \ldots, x_k = x^{(x^{(x^{(x_{k-1})})})}, \ldots \]

Euler determined the interval of convergence of \( y \) for \( x \in \mathbb{R} \). When \( \frac{1}{e^e} \leq x \leq e^{\frac{1}{e}} \), \( y = x^{x^{x^{\ddots}}} \) converges to a limit. Whenever \( x > 1 \), the sequence \( (x_k)_{k \geq 1} \) increases monotonically

\[ x_1 < x_2 < x_3 < \cdots. \]

If additionally, \( x > e^{\frac{1}{e}} \), the sequence diverges as it increases without bound. Whenever \( x < 1 \), the sequence \( (x_k)_{k \geq 1} \) oscillates

\[ x_1 < x_2 > x_3 < x_4 > \cdots. \]

The odd subsequence increases monotonically to a limit

\[ x_1 < x_3 < x_5 < \cdots, \]

while the even subsequence decreases monotonically to a limit

\[ x_2 > x_4 > x_6 > \cdots. \]

If additionally \( x > \frac{1}{e^e} \), the limits of the odd and even subsequences are the same and \( y \) converges. Otherwise, the limits are distinct, and \( y \) does not converge. On its interval of convergence, \( y \) satisfies \( y = x^y \) so that \( x = y^{1/y} \) is the partial inverse of \( y \). For more information on the history of the iterated exponential, and a further discussion of its analytic properties see [3].

It is possible to expand \( y \) as a power series about \( x = 1 \) whose positive coefficients
have both a combinatorial and a representation-theoretic description. Richard Stanley first conjectured that the coefficients of this power series are positive and suggested to investigate their combinatorial significance. John Stembridge first gave the coefficients the representation-theoretic description (3.1). Before Stanley’s conjecture is proven in Theorem 4, the iterated exponential will be expanded as a power series about \( x = 1 \) whose coefficients are a sum of both positive and negative terms. The iterated exponential \( y \) satisfies the functional equation

\[
y = x^y
\]

Thus,

\[
\frac{\log y}{y} = \log x.
\]

Substituting the variable \( z = x - 1 \) in the above equation gives

\[
\frac{\log y}{y} = \log(1 + z).
\]

Thus by Proposition 5,

\[
y = F\left(\frac{\log y}{y}\right) = F(\log(1 + z)).
\]

Recall that for an \( n \)-cycle \( u \in \mathcal{Z}_{[n]} \), \( \text{sgn}(u) = (-1)^{n-1} \) so that

\[
\log(1 + z) = \sum_{n \geq 1} \text{sgn}(\mathcal{Z}_{[n]}) \frac{z^n}{n!}.
\]

Then, by the compositional formula, Proposition 7,

\[
y = F(\log(1 + z)) = \sum_{n \geq 1} \text{sgn} \left( \prod_{\Lambda \in \Pi_n} \mathcal{F}_{[\Lambda]} \times \mathcal{Z}_{B_1} \times \cdots \times \mathcal{Z}_{B_k} \right) \frac{z^n}{n!}.
\]

As in (2.3), the usual decomposition of a permutation into disjoint cycles, gives a bijection

\[
\prod_{\Lambda \in \text{Part}_{[n]}} \mathcal{F}_{[\Lambda]} \times \mathcal{Z}_{B_1} \times \cdots \times \mathcal{Z}_{B_k} \cong \prod_{u \in \mathcal{S}_n} \mathcal{F}_{[c(u)]}
\]

Then define the set \( \mathcal{Y}_{[n]} \) by

\[
\mathcal{Y}_{[n]} := \left\{ (F, u) \left| \begin{array}{c} u \in \mathcal{S}_n \\ F \in \mathcal{F}_{[c(u)]} \end{array} \right. \right\}
\]

\[
\cong \prod_{u \in \mathcal{S}_n} \mathcal{F}_{[c(u)]}
\]
Recall as defined in (2.2) that \( c(u) \) denotes the number of cycles of \( u \in \mathcal{S}_n \). Furthermore, as in (2.4), the weight function \( \text{sgn} \) on cycles \( \mathcal{Z}_n \) extends to the usual sign character on \( \mathcal{S}_n \). Thus, \( \text{sgn}(f, w) = \text{sgn}(w) \) so that

\[
y_n = \text{sgn}(\mathcal{Y}_n) = \sum_{u \in \mathcal{S}_n} (c(u) + 1)^{c(u) - 1} \text{sgn}(u) = n! \langle \chi, \text{sgn} \rangle,
\]

where \( \chi \) denotes the \( \mathcal{S}_n \) character given by \( \chi(u) = (c(u) + 1)^{c(u) - 1} \) so that \( \chi(u) \) is the number of forests with vertex set \( [c(u)] \), and \( \text{sgn} \) is the sign character for \( \mathcal{S}_n \).

**Theorem 4.** The coefficients \( y_n \) of the iterated exponential

\[
y = x^x = \sum_{n \geq 0} y_n \frac{(x - 1)^n}{n!}
\]

expanded about \( x = 1 \) satisfy

\[
y_n > 0.
\]

**Proof.** To prove this we will construct an involution

\[
\iota : \mathcal{Y}_n \rightarrow \mathcal{Y}_n
\]

such that \( \text{sgn}(\iota((F, u))) = -\text{sgn}((F, u)) \) for \( (F, u) \notin \text{Fix}(\iota) \), and such that if \( (F, u) \in \text{Fix}(\iota) \), then \( \text{sgn}((F, u)) = 1 \). Consequently,

\[
\text{sgn}(\mathcal{Y}_n) = \text{sgn}((\text{Fix}(\iota)) = \#(\text{Fix}(\iota)) \geq 0.
\]

In the proof, \( (F, u) \in \mathcal{Y}_n \) will be considered to be a forest, \( F \), with vertex set \( \{z_1, \ldots, z_{c(u)}\} \), the cycles of \( u \in \mathcal{S}_n \). In turn, each cycle \( z \) will be considered to be a directed graph, and \( z \) will be considered to be the set of its vertices, together with a cycle structure, so that \( z = (\min(z), u(\min(z)), u^2(\min(z)), \ldots) \).

Suppose that \( v \) is a vertex of \( F \), where \( (F, u) \in \mathcal{Y}_n \). Say that the vertex \( v \) is pre-active if one of the conditions 1-4 below holds.

1. The vertex \( v \) is an even cycle and the image \( u(\min v) \) of the minimal element \( \min v \) of the cycle \( v \) is less than all of the one-cycle leaf children of \( v \).

2. The vertex \( v \) is an odd cycle with at least one one-cycle, leaf child, and the minimum element \( \min(v) \) of the cycle \( v \) is less than all of the one-cycle, leaf children of \( v \).

3. The vertex \( v \) is an even cycle with a one-cycle leaf child \( l \) such that \( l \) is less than the image \( u(\min v) \) of the minimal element \( \min v \) of the cycle \( v \).
4. The vertex $v$ is an odd cycle with only one child $c$. The vertex $c$ is a one-cycle such that $c > \min(v)$. Furthermore, the vertex $c$ has a one-cycle, leaf child $l$ such that $c > l$.

Notice that in case the vertex $v$ is an even cycle then either condition 1 or condition 3 holds. Now, for $1 \leq i \leq 4$ define a subset $P_i$ of the vertices $V(F)$ of the forest $F$ by saying that the vertex $v$ is in the set $P_i$ if condition $i$ holds for the vertex $v$.

Consider all of those pre-active vertices with no pre-active descendants. Call the set of these vertices $P$. Among the vertices in the set $P$, find the vertex $z$ whose minimal element $\min z$ is the least. That is, the vertex $z$ is the vertex whose minimal element $\min z$ is less than any element in any other vertex $v$ in the set $P$. Say that the vertex $z$ is active. Now, the action of the involution $\iota$ may be defined according to which of the conditions 1-4 holds for the vertex $z$. In case $i$, condition $i$ holds for the vertex $z$ for $1 \leq i \leq 4$. In each case, as the action of the involution $\iota$ is defined below, the active vertex $z'$ of the forest $\iota F$ will be identified, and it will be determined which of the conditions 1-4 hold for the active vertex $z'$.

In case 1, condition 1 holds for the vertex $z$. The vertex set $V(F')$ of the forest $F' := \iota F$ is given by

$$V(F') := V(F) - z \cup z' \cup m$$

where $m$ is set equal to $u(\min z)$, and if $z' = (\min z, m = u(\min z), z_3, z_4, \ldots, z_{\#z})$, then the vertex $z'$ is set equal to $(\min z, z_3, z_4, \ldots, z_{\#z})$. Throughout the proof, given some vertex $w \in V(F)$, let $E_w$ denote the set of edges adjacent to the vertex $w$ in the forest $F$. That is,

$$E_w := \left\{ \{w, v\} \mid v \in V(F), \{w, v\} \in E(F) \right\}.$$

Throughout the proof, a second notation $E_w'$ will be used. Suppose that $w$ is a vertex of $F$ and $w'$ is a vertex of $F'$. Then, the subset $E_w'$ of edges of $F'$ is defined by

$$E_w' := \left\{ \{w', v\} \mid v \in V(F) \cap V(F'), \{w, v\} \in E(F) \right\}.$$

Notice that if $v \neq z$ then $v$ is in both $V(F)$ and $V(F')$ so that the pair $\{w', v\}$ may be an edge of $F'$. The definition of the forest $F'$ may now be completed by determining the edge set $E(F')$.

$$E(F') := E(F) - E_z \cup E_z' \cup \{z', m\}.$$

Since none of the descendents of the vertex $z \in V(F)$ are pre-active none of the descendents of the vertex $z' \in V(F')$ are pre-active except possibly the vertex $u(\min z)$. However, the vertex $u(\min z)$ is a one-cycle, not an even cycle, and since it is a leaf, it is not pre-active. Furthermore, $z'$ retains the element $\min(z)$, so the element $\min z'$ is minimal among the pre-active vertices, and the vertex $z'$ is active in the forest $F'$. Now, the vertex $z'$ is an odd cycle with a one-cycle leaf child $m$ in the forest $F'$. Furthermore, $\min z' = \min z < l$. If $l \neq m$ is a one cycle leaf child of $z'$ in $F'$, then $l$ is a child of $z$ in the forest $F$. Hence, since $z$ satisfies condition 1, $m < l$. 32
It follows that condition 2 holds for the vertex $z'$ in the forest $F'$.

In case II, condition 2 holds for the active vertex $z$. Since condition 2 holds, there exists a minimal one-cycle, leaf child $m$ of $z$ such that $m > \min z$. The vertex set $V(F')$ of the forest $F' := iF'$ is given by

$$V(F') := V(F) - z \cup z' - m$$

where if $z = (\min z, z_2, z_3, \ldots, z_{\#z})$, then the vertex $z'$ is set equal to $(\min z, m, z_2, z_3, \ldots, z_{\#z})$. The edge set of $F'$ is set equal to

$$E(F') := E(F) - E_z \cup E_z'^{z'}$$

In this case, notice that the set of edges $E_z'^{z'}$ does not include the edge $\{z', m\}$ since $m$ is not an edge of the forest $F'$.

None of the descendants of $z'$ may be pre-active since none of the descendants of $z$ are pre-active. Again, $z'$ retains the element $\min z = \min z' < m$, so $z'$ is active in $F'$. Now, the vertex $z'$ is an even cycle. Furthermore, if $l$ is a one-cycle, leaf child of $z'$ in the forest $F'$ then $l$ is a one-cycle leaf child of the vertex $z$ in the forest $F$. Thus, since condition 2 holds for the vertex $z$ in $V(F)$, it follows that $\min z' = \min z < m < l$. Since $z' = (\min z, m, z_2, z_3, \ldots, z_{\#z})$, the active vertex $z'$ in $V(F')$ satisfies condition 1.

In case III, condition 3 holds for the active vertex $z$. The vertex set $V(F')$ of the forest $F' := iF'$ is given by

$$V(F') = V(F) - z \cup z' \cup m$$

where, as in case I, $m$ is set equal to $u(\min z)$, and if $z = (\min z, m = u(\min z), z_3, z_4, \ldots, z_{\#z})$, then the vertex $z'$ is set equal to $(\min z, z_3, z_4, \ldots, z_{\#z})$. If the vertex $v$ is the child of some other vertex $p$ in $V(F)$, then the edge set of $F'$ is set equal to

$$E(F') := E(F) - E_z \cup E_z^m \cup \{m, p\} \cup \{z', p\} \cup \{z', m\}.$$  

If the vertex $v$ is not the child of any other vertex in $V(F)$, then the edge set of $F'$ is set equal to

$$E(F') := E(F) - E_z \cup E_z^m \cup \{z', m\}.$$  

Since condition 3 holds in this case, there exists a one-cycle, leaf child $l$ of the vertex $z$ such that $l < m$. Now, the only possible vertex in $F'$ which may be pre-active among the descendants of $z$ is $m$. In the forest $F'$, the vertex $l$ is a child of the vertex $m$. Condition 2 may not hold for the vertex $m \in V(F')$, since $\min m = m > l$. Furthermore, condition 2 may not hold for the vertex $m \in V(F')$, since although the vertex $l$ may be the only child of $m$ in the forest $F'$, it is a leaf of $F'$. Furthermore, the vertex $z'$ retains the element $\min z = \min z'$, so the vertex $z'$ is active in the forest $F'$. Now, the vertex $z'$ is an odd cycle that has an only child $m$ in the forest $F'$ such that $\min z' = \min z < m$. Since condition 3 holds for the vertex $z$ in $V(F)$, there exists a one-cycle leaf child $l$ of the vertex $z$ in the forest $F$ such that $m > l$. Then,
the vertex \( l \) is a one-cycle leaf child of the vertex \( m \) in the forest \( F' \). Hence, condition 4 holds for the vertex \( z' \).

In case IV, condition 4 holds for the active vertex \( z \). Since condition 4 holds, there exists a vertex \( m \) which is the only child of \( z \), and furthermore \( m > \min z \). The vertex set \( V(F') \) of the forest \( F' := i(F) \) is given by

\[
V(F') = V(F) - z \cup z' - m,
\]

where, as in case II, if \( z = (\min z, z_2, z_3, \ldots, z_{#z}) \), then the vertex \( z' \) is set equal to \( (\min z, m, z_2, z_3, \ldots, z_{#z}) \). If the vertex \( z \) is the child of some other vertex \( p \) in \( V(F) \), then the edge set of \( F' \) is set equal to

\[
E(F') := E(F) - E_m \cup E_m^z - \{z, p\} \cup \{z', p\}.
\]

Notice that the edge \( \{z', z\} \) is not in \( E_m^z \), since \( z \) is not in \( V(F') \). If the vertex \( z \) is not the child of any other vertex in \( V(F) \), then the edge set of \( F' \) is set equal to

\[
E(F') := E(F) - E_m \cup E_m^z.
\]

None of the descendants of the vertex \( z' \) in the forest \( F' \) may be pre-active since none of the descendants of the vertex \( m \) in the forest \( F \) are pre-active. Again, the vertex \( z' \) retains the element \( \min z = \min z' \), so the vertex \( z' \) is active in the forest \( F' \). Now, the vertex \( z' \) is an even cycle. Since condition 4 holds for the vertex \( z \) in \( V(F) \), there exists a one-cycle leaf child \( l \) of the vertex \( m \) such that \( m > l \). Then, the vertex \( l \) is a one-cycle leaf child of the vertex \( z \) in \( V(F') \). Since \( z' \) is equal to \( (\min z, m, z_2, z_3, \ldots, z_{#z}) \), condition 3 holds for the active vertex \( z' \).

It must be checked that \( i(i(F)) = F \)

In case I, the vertex set \( V(F') \) is equal to

\[
V(F') = V(F) - z \cup z' \cup m
\]

Since the active vertex \( z' \) in the forest \( F' \) satisfies condition 2 the vertex set \( V(F'') \) is equal to

\[
V(F'') = V(F') - z' \cup z'' - m'
\]

In fact, it is easy to verify that \( m' = m \) so that \( z'' = z \). Hence,

\[
V(F'') = V(F) - z \cup z' \cup m - z' \cup z - m = V(F).
\]

In this case, the edge set \( E(F') \) is equal to

\[
E(F') = E(F) - E_z \cup E_z^{z'} \cup \{z', m\}
\]

Since the active vertex \( z' \) in the forest \( F' \) satisfies condition 2 the vertex set \( E(F'') \) is equal to

\[
E(F'') = E(F') - E_{z'} \cup E_{z'}^{z''}
\]
Then since the vertex \( z'' \) is equal to the vertex \( z \),
\[
E(F'') = E(F) - E_\bar{z} \cup E'_z \cup \{z', m\} - E_{z'} \cup E''_z
\]
Since the edge \( \{z', m\} \) is in the set \( E_{z'} \), it follows that
\[
E(F'') = E(F).
\]

In case II, the vertex set \( V(F') \) is equal to
\[
V(F') = V(F) - z \cup z' - m
\]
Since the active vertex \( z' \) in the forest \( F' \) satisfies condition 1, the vertex set \( V(F'') \) is equal to
\[
V(F'') = V(F') - z' \cup z'' \cup m'
\]
Again, it is easy to verify that \( m' = m \) so that \( z'' = z \). Hence,
\[
V(F'') = V(F) - z \cup z' - m - z' \cup z \cup m = V(F).
\]
In this case, the edge set \( E(F') \) is equal to
\[
E(F') = E(F) - E_\bar{z} \cup E'_z
\]
Since the active vertex \( z' \) in the forest \( F' \) satisfies condition 2, the vertex set \( E(F'') \) is equal to
\[
E(F'') = E(F') - E_{z'} \cup E''_{z} \cup \{z'', m'\}
\]
where \( z'' = z \) and \( m' = m \). Thus,
\[
E(F'') = E(F) - E_{z'} \cup E'_z \cup E_{z'} - E_{z'} \cup E''_{z} \cup \{z, m\}.
\]
Since the edge \( \{z, m\} \) is in the set \( E_z \) but \( \{z', m\} \) is not an edge in the set \( E_{z'} \), it follows that
\[
E(F'') = E(F).
\]

In case III, checking that \( V(F'') = V(F) \) is entirely analogous to checking that \( V(F'') = V(F) \) in case I. In case III, the vertex set \( V(F') \) is equal to
\[
V(F') = V(F) - z \cup z' \cup m
\]
Since the active vertex \( z' \) in the forest \( F' \) satisfies condition 2, the vertex set \( V(F'') \) is equal to
\[
V(F'') = V(F') - z' \cup z'' - m'
\]
It is easy to verify that \( m' = m \) so that \( z'' = z \). Hence,
\[
V(F'') = V(F) - z \cup z' \cup m - z' \cup z - m = V(F).
\]
In this case, if the vertex \( v \) is the child of some other vertex \( p \) in \( V(F) \), then the edge set of \( F' \) is set equal to

\[
E(F') = E(F) - E_z \cup E_{z'}^m - \{m, p\} \cup \{z', p\} \cup \{z', m\}.
\]

Then the active vertex \( z' \) will be the child of the vertex \( p \) in the forest \( F' \). Hence, since condition 4 holds for the active vertex \( z' \), the edge set \( E(F'') \) is equal to

\[
E(F'') = E(F') - E_{m'} \cup E_{m'}^{z''} - \{z', p\} \cup \{z'', p\},
\]

where \( z'' = z \) and \( m' = m \). Thus,

\[
E(F'') = E(F) - E_z \cup E_{z}^m - \{m, p\} \cup \{z', p\} \cup \{z', m\} - E_{m} \cup E_{m}^z - \{z', p\} \cup \{z, p\}.
\]

The vertices \( m \) and \( z' \) do not appear in \( F'' \). Also, notice that the edge \( \{z, p\} \) is in \( E(F'') \). Hence,

\[
E(F'') = E(F).
\]

Now suppose that the vertex \( z \) is not the child of any other vertex in \( V(F) \). Then, the edge set of \( F' \) is set equal to

\[
E(F') := E(F) - E_z \cup E_{z}^m \cup \{z', m\}.
\]

The active vertex \( z' \) will not be the child of any other vertex in \( V(F') \). Hence, since condition 4 holds for the active vertex \( z' \), the edge set \( E(F'') \) is equal to

\[
E(F'') = E(F') - E_{m'} \cup E_{m'}^{z''},
\]

where \( z'' = z \) and \( m' = m \). Thus,

\[
E(F'') = E(F) - E_z \cup E_{z}^m \cup \{z', m\} - E_{m} \cup E_{m}^z
\]

Since the vertex \( \{z', m\} \) is in the set \( E_m \), it again follows that

\[
E(F'') = E(F).
\]

In case IV, checking that \( V(F'') = V(F) \) is entirely analogous to checking that \( V(F'') = V(F) \) in case II. In case IV, the vertex set \( V(F') \) is equal to

\[
V(F') = V(F) - z \cup z' - m
\]

Since the active vertex \( z' \) in the forest \( F' \) satisfies condition 3, the vertex set \( V(F'') \) is equal to

\[
V(F'') = V(F') - z' \cup z'' \cup m'
\]

Again, it is easy to verify that \( m' = m \) so that \( z'' = z \) Hence,

\[
V(F'') = V(F) - z \cup z' - m - z' \cup z \cup m = V(F).
\]
If the vertex $z$ is the child of some other vertex $p$ in $V(F)$, then the edge set of $F'$ is set equal to

$$E(F') = E(F) - E_m \cup E^z_m - \{z, p\} \cup \{z', p\}.$$ 

Then, the active vertex $z'$ will be the child of the vertex $p$ in the forest $F'$. Hence, since condition 3 holds for the active vertex $z'$, the edge set $E(F'')$ is equal to

$$E(F'') = E(F') - E_{z'} \cup E^m_{z'} - \{m', p\} \cup \{z'', p\} \cup \{z', m\},$$

where $z'' = z$ and $m' = m$. Thus,

$$E(F'') = E(F) - E_m \cup E^z_m - \{z, p\} \cup \{z', p\} - E_{z'} \cup E^m_{z'} - \{m, p\} \cup \{z, p\} \cup \{z, m\}.$$ 

The edge $\{z', p\}$ is in the set $E_{z'}$. Furthermore, in both $F$ and $F''$ the vertex $z$ is only adjacent to the edges $\{z, p\}$ and $\{z, m\}$, while $m$ is not adjacent to $p$ in $F$. Hence,

$$E(F'') = E(F).$$

If the vertex $z$ is not the child of any other vertex in $V(F)$, then the edge set of $F'$ is set equal to

$$E(F') = E(F) - E_m \cup E^z_m.$$ 

The active vertex $z'$ will not be the child of any other vertex in $V(F')$. Hence, since condition 3 holds for the active vertex $z'$, the edge set $E(F'')$ is equal to

$$E(F'') := E(F') - E_{z'} \cup E^m_{z'} \cup \{z'', m'\},$$

where $z'' = z$ and $m' = m$. Thus,

$$E(F'') = E(F) - E_m \cup E^z_m - E_{z'} \cup E^m_{z'} \cup \{z, m\}.$$ 

Then, the vertex $m$ is the only child of $z$ in both the forest $F$ and the forest $F''$. Hence, again

$$E(F'') = E(F).$$

Recall that if $(F, u) \in \mathcal{Y}_{[n]}$ then the permutation $u$ coincides with the vertex set $V(F)$ of the forest $F$. Hence, the number $c(u)$ of cycles of $u$ is the number $\#V(F)$ of vertices of the forest $F$. Thus,

$$\text{sgn}(F, u) = \text{sgn}(u) = (-1)^{n-c(u)} = (-1)^{n-\#V(F)}.$$ 

In cases I and III, $\#V(F') = \#V(F) + 1$, while in cases II and IV, $\#V(F') = \#V(F) - 1$. Hence, if the forest $F$ is not in the fixed set $\text{Fix}(\iota)$ of the involution $\iota$, then

$$\text{sgn}(\iota(F, u)) = -\text{sgn}(F, u).$$

The fixed set $\text{Fix}(\iota)$ of the involution $\iota$ consists of forests none of whose vertices are pre-active. In particular, all of the vertices of the forest $F$ must be odd cycles. Since the vertex set of $F$ is the set of cycles of the permutation $u$ where $(F, u) \in \mathcal{Y}_{[n]}$, $n$
all of the cycles of \( u \) must be odd. Thus, if the forest \( F \) is in the fixed set \( \text{Fix}(\iota) \), then
\[
\text{sgn}(F, u) = \text{sgn}(u) = 1.
\]

Thus,
\[
y_n = \text{sgn}(\mathcal{Y}_n) = \#\text{Fix}(\iota) > 0
\]

A forest \( F \) is in the fixed set \( \text{Fix}(\iota) \) if and only if all of its vertices are odd cycles and in addition properties 1-2 below hold.

1. If any vertex \( z \) in the forest \( F \) has a one-cycle, leaf child, then there must exist some one-cycle, leaf child \( l \) of the vertex \( z \) such that \( l < \min z \).

2. If any vertex \( z \) in the forest \( F \) has an only child \( m \), and this only child \( m \) has a one-cycle leaf child, then \( m < \min z \).

If property 1 above holds, then condition 2 cannot hold for the vertex \( z \). If property 2 above holds, then condition 4 cannot hold for the vertex \( z \). Thus, if both properties hold and \( z \) is an odd cycle, then \( z \) is not pre-active.

### 3.2 The Character \( \chi(u) = (c(u) + 1)^{c(u)-1} \)

The iterated exponential \( y = x^{x^{\cdot^{\cdot^x}}} \) may be generalized by setting \( H = \sum_{n \geq 1} h_n x^n \) for commuting, algebraically independent variables \( h_n \) and then considering the function \( Y = (1 + H)^{(1 + H)^{(1 + H)^{\cdot^{\cdot^x}}}} \). For instance, in this case, what is the coefficient of \( x^n h_\lambda \) in \( Y \)?

To answer this question, proceed as in the case of the iterated exponential. Since
\[
Y = (1 + H)^Y,
\]
\[
\log Y = Y \log(1 + H).
\]

Thus, by Proposition 9
\[
Y = F \left( \frac{\log Y}{Y} \right) = F(\log(1 + H)),
\]

where \( F(x) \) is the exponential generating function for rooted forests. Applying the compositional formula, Proposition 7, and using Proposition 10,
\[
F(\log(1 + H)) = 1 + \sum_{n \geq 1} \frac{x^n}{n!} \left( \prod_{\Lambda \in \Pi_n} \mathcal{F}_{\mathcal{H}_\Lambda} \times \mathcal{Z}_{\Lambda_1}^P \times \cdots \times \mathcal{Z}_{\Lambda_{\Lambda}}^P \right).
\]

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Thus, using Definition 19

\[ F(\log(1 + H)) = 1 + \sum_{n \geq 1} \frac{x^n}{n!} w \left( \prod_{u \in S_n} F_{[c(u)]} \times \prod_{\lambda \in \text{Par}(n)} C_{u, \lambda} \right) \]

\[ = 1 + \sum_{n \geq 1} \frac{x^n}{n!} \sum_{u \in S_n} \sum_{\lambda \in \text{Par}(n)} w(F_{[c(u)]}) \#(C_{u, \lambda}) \text{sgn}(u)(-1)^{n+\lambda} h_{\lambda}, \]

where \( w(F_{[c(u)]}) = (c(u) + 1)^{c(u)-1} = \chi(u) \) is the number of rooted forests with vertex set \([c(u)]\). Then the definition of the monomial character Definition 20 and the definition of \( \chi \) imply that

\[ F(\log(1 + H)) = 1 + \sum_{n \geq 1} \frac{x^n}{n!} \sum_{\lambda \in \text{Par}(n)} \sum_{u \in S_n} \phi^\lambda(u) \chi'(u) h_{\lambda} \]

\[ = 1 + \sum_{n \geq 1} \Phi(\chi_n) x^n. \]

In general, suppose that \( \chi_f(u) = \tilde{f}_{c(u)} \) where \( \tilde{f}_{c(u)} \) is an arbitrary function of \( c(u) \). Then, \( \chi_f(u) \) is a class function on \( S_n \). Furthermore, in this more general context, replacing \( F(x) \) with \( \tilde{F}(x) = \sum_{n \geq 0} \tilde{f}_n \frac{x^n}{n!} \) in the above argument proves the following proposition.

**Proposition 13.** For any class function that depends only on the number of cycles of a permutation, \( \chi_f(u) = \tilde{f}_{c(u)} \), the Frobenius characteristic of \( \chi_f \) in the basis of homogeneous symmetric functions is given by

\[ \tilde{F}(\log(1 + H)) = \tilde{f}_0 + \sum_{n \geq 1} \Phi(\chi_f(w)) = x^n. \]

Returning to the case of the iterated exponential,

\[ \Phi(\chi) = \sum_{\lambda \in \text{Par}(n)} \sum_{u \in S_n} \#(C_{u, \lambda}) \text{sgn}(u)(-1)^{n-\lambda} \chi(u) h_{\lambda}. \]

Thus, since \( \#(C_{u, \lambda}) = 1 \) for all \( u \in S_n \) the coefficient \( a_{1^l} \) of \( h_{1^l} \) in \( \Phi(\chi) \) is

\[ a_{1^l} = \sum_{u \in S_l} (-1)^{c(u)}(-1)^l(c(u) + 1)^{c(u)-1}. \]

Now, for \( l = l_{\lambda} \), compare \( a_{1^l} \) with the coefficient \( a_{\lambda} \) of \( h_{\lambda} \) in \( \Phi(\chi) \).

\[ a_{\lambda} = \sum_{u \in S_n} \#(C_{u, \lambda})(-1)^{c(u)}(-1)^l(c(u) + 1)^{c(u)-1}. \]
Recall that

\[ C_{u\lambda} := \left\{ (u, P_{\lambda_1}, \ldots, P_{\lambda_\Lambda}) \middle| \begin{array}{l}
    u \in \mathcal{S}_n \\
    \text{for } 1 \leq i \leq l_\Lambda, 
    P_{\lambda_i} \in \mathcal{P}_{\lambda_i} \\
    P_{\lambda_i} \subset u \\
    \Lambda \in \Pi_n \\
    \text{type}(\Lambda) = \lambda
  \end{array} \right\}. \]

Thus,

\[
\begin{align*}
  a_\lambda &= \sum_{u \in \mathcal{S}_n} \sum_{x \in C_{u\lambda}} (-1)^{c(u)}(-1)^i(c(u) + 1)^{c(u)-1} \\
  &= \sum_{\lambda \in \Pi_n} \sum_{\{P_{\lambda_1}, \ldots, P_{\lambda_\Lambda}\} \subset \mathcal{P}_{\lambda}} \sum_{u \in \mathcal{S}_n} (-1)^{c(u)}(-1)^i(c(u) + 1)^{c(u)-1}. 
\end{align*}
\]

Recall that a permutation \( u \in \mathcal{S}_n \) containing paths \( \{P_{\lambda_1}, \ldots, P_{\lambda_\Lambda}\} \) such that the labels of the paths partition \([n]\), defines a permutation \( v \in \mathcal{S}_l \) of the paths \( P_{\lambda_i} \) such that \( c(v) = c(u) \). Hence,

\[
\begin{align*}
  a_\lambda &= \sum_{\lambda \in \Pi_n} \sum_{\{P_{\lambda_1}, \ldots, P_{\lambda_\Lambda}\} \subset \mathcal{P}_{\lambda}} \sum_{u \in \mathcal{S}_l} (-1)^{c(v)}(-1)^i(c(v) + 1)^{c(v)-1} \\
  &= \sum_{\lambda \in \Pi_n} \sum_{\{P_{\lambda_1}, \ldots, P_{\lambda_\Lambda}\} \subset \mathcal{P}_{\lambda}} a_i^l \\
  &= \binom{n}{\lambda_1, \ldots, \lambda_l} \frac{1}{m_1!m_2!m_3!\cdots} a_i^l,
\end{align*}
\]

where \( m_i \) is the multiplicity of \( i \) in \( \lambda \). Thus, in particular \( a_\lambda > 0 \). The formula

\[
a_\lambda = \binom{n}{\lambda_1, \ldots, \lambda_l} \frac{1}{m_1!m_2!m_3!\cdots} a_i^l
\]

was first proven by John Stembridge.
3.3 A Related Character $\tilde{\chi}$

Suppose that $\text{type}(u) = \lambda$, and $c(u) = l$. Then define the character $\tilde{\chi} : \mathfrak{S}_n \rightarrow \mathbb{C}$ by

$$\tilde{\chi}(u) = n! \frac{1}{(\lambda_1 - 1)! (\lambda_2 - 1)! \cdots (\lambda_l - 1)!} (l + 1)^{l-1}.$$ 

Note that

$$\sum_{u \in \mathfrak{S}_n : \text{type}(u) = \lambda} \tilde{\chi}(u) = n! \left( \begin{array}{c} n \\ \lambda_1, \ldots, \lambda_l \end{array} \right) \frac{1}{m_1! m_2! \cdots (l + 1)^{l-1}}$$

$$= n! \sum_{\Lambda \in \Pi_n : \text{type}(\Lambda) = \lambda} (l + 1)^{l-1}.$$

For an arbitrary class function $f$, consider

$$S(f) = \sum_{\lambda \in \Pi(n)} \sum_{u \in \mathfrak{S}_n : \text{type}(u) = \lambda} \frac{f(u)}{(\lambda_1 - 1)! (\lambda_2 - 1)! \cdots (\lambda_l - 1)!}$$

Using the usual decomposition (2.3) of a permutation into disjoint cycles,

$$S(f) = \sum_{\Lambda \in \Pi_n} \sum_{x \in Z_{\Lambda_1} \times \cdots \times Z_{\Lambda_l}} \frac{f(\text{type}(\Lambda))}{(\lambda_1 - 1)! (\lambda_2 - 1)! \cdots (\lambda_l - 1)!}$$

$$= \sum_{\Lambda \in \Pi_n} f(\text{type}(\Lambda))$$

Consider the inner product $\langle \tilde{\chi}, \text{sgn} \rangle := \frac{1}{n!} \sum_{u \in \mathfrak{S}_n} \tilde{\chi}(u) \text{sgn}(u)$.

$$\langle \tilde{\chi}, \text{sgn} \rangle = \sum_{\lambda \in \Pi(n)} \sum_{u \in \mathfrak{S}_n : \text{type}(u) = \lambda} \frac{\text{sgn}(u)(l + 1)^{l-1}}{(\lambda_1 - 1)! (\lambda_2 - 1)! \cdots (\lambda_l - 1)!}$$

$$= \sum_{\Lambda \in \Pi_n} (-1)^{n+\Lambda}(l + 1)^{l-1}.$$

Comparing the above with equation (2.1) shows that $\langle \tilde{\chi}, \text{sgn} \rangle$ is the number of rooted forests with increasing leaves.
Chapter 4

Homology of the Even-Ranked Subposet of the Partition Lattice

4.1  Partition Lattice and Subposets

The elements of the set $\Pi_n$ may be ordered by refinement. If $\Lambda, M \in \Pi_n$, then

$$\Lambda \leq M$$

if each block $\Lambda_j \in \Lambda$ is contained in some block $M_k \in M$. That is,

$$\Lambda_j \subset M_k.$$

**Definition 21.** (The Partition Lattice $\Pi_n$)

$\Pi_n$ denotes the set $\Pi_n$ partially ordered by refinement.

A chain in a poset $P$ is a totally ordered subset of $P$. A maximal chain of $P$ is a chain not properly contained in another chain. A poset $P$ whose maximal chains are all of the same length is said to be ranked. The rank of an element $x \in P$ in the ranked poset $P$ is the maximum cardinality of those chains whose elements are all less than $x$. For example, $\Pi_n$ is a ranked poset. If $\Lambda \in \Pi_n$ has $l_{\Lambda}$ blocks, then $n - l_{\Lambda}$ is the rank of $\Lambda$. Let $\Lambda \wedge M$ denote the partition with blocks $\Lambda_j \cap M_k$ for each pair of blocks $(\Lambda_k, M_j) \in \Lambda \times M$. Then, $\Lambda \wedge M$ is the greatest lower bound of $\Lambda$ and $M$.

Since, additionally, $\Pi_n$, has a maximal element $\hat{1}$ consisting of the single block $[n]$, $\Pi_n$ is indeed a lattice. Define

$$\text{Par}_e[n] := \prod_{k=1}^{[n/2]} \text{Par}_{2k}[n].$$

Order $\text{Par}_e[n]$ by refinement and for any $\Lambda \in \text{Par}_e[n]$, set $\Lambda < \hat{1}$. This poset is denoted $\Pi_{2n}^e$. 

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Definition 22. (The Even-Rank-Selected Subposet of the Partition Lattice) The poset $\Pi_{2n}^e$ is the subposet of $\Pi_{2n}$ consisting of partitions with an even number of blocks together with an adjoined maximal element $\hat{1}$

In general, $\Pi_{2n}^e$ is not a lattice.

4.2 Poset Homology

Suppose $P$ is a poset with a maximal element $\hat{1}$ and a minimal element $\hat{0}$, and $K$ is a field. Then, let $C_i(P)$ be the set of $i$-chains in $P$.

$$C_i(P) := \{(\hat{0} < x_0 < x_2 < \cdots < x_i < \hat{1}) \mid \text{for } 1 \leq j \leq i, x_j \in P\}$$

Definition 23. (ith Chain Group of the poset $P$) $C_i(P)$ is the $K$-span of basis vectors indexed by $C_i(P)$

Suppose the longest chain of $P$ is of length $l$, then let $C(P) := \bigoplus_{i=0}^{l} C_i(P)$. Note that we have defined the $-1$th Chain Group to be the one-dimensional vector space over $K$ spanned by $(\hat{0} < \hat{1})$.

Definition 24. (Boundary Operator) The linear transformation

$$\partial_i : C_i(P) \to C_{i-1}(P)$$

is the linear extension of

$$\partial_i(\hat{0} < x_0 < x_2 < \cdots < x_i < \hat{1}) = \sum_{j=0}^{i} (-1)^j (\hat{0} < x_0 < x_2 < \cdots \hat{x}_j \cdots < x_i < \hat{1}) .$$

$$(\hat{0} < x_0 < x_2 < \cdots \hat{x}_j \cdots < x_i < \hat{1})$$ is the chain $(\hat{0} < x_3 < x_2 < \cdots < x_{j-1} < x_{j+1} < \cdots < x_i < \hat{1})$ with the element $x_j$ omitted. Since $\partial_i \circ \partial_{i+1} = 0$, we make the following definition.

Definition 25. (Reduced Homology of a Poset $P$)

$$\tilde{H}_i(P) = \frac{\ker \partial_i}{\text{im } \partial_{i+1}}$$

For a Cohen-Macaulay poset $P$ whose longest chain is of length $l$,

$$\tilde{H}(P) := \bigoplus_{i=0}^{l} \tilde{H}_i(P) = \tilde{H}_l(P) = \ker \partial_l . \quad (4.1)$$

In particular, $\tilde{H}_i(P) = 0$ for $i \neq l$. In turn, for a poset $P$ whose longest chain is of $l$ and whose reduced homology satisfies $\tilde{H}(P) = \tilde{H}_l(P)$,

$$\dim \tilde{H}(P) = |\mu_P(\hat{0}, \hat{1})| \quad (4.2)$$
The posets $\Pi_n$ and $\Pi_{2n}^e$ are Cohen-Macaulay posets whose reduced homology therefore satisfies (4.1) and (4.2). Evidently, the action of $\mathcal{S}_{2n}$ on $[n]$ induces an action on $\Pi_{2n}$ and $\Pi_{2n}^e$. Thus, $C(\Pi_{2n})$ and $C(\Pi_{2n}^e)$ are $\mathcal{S}_{2n}$ modules. Furthermore, for $u \in \mathcal{S}_{2n}$

$$\partial_i \circ u = u \circ \partial_i$$

Hence, $\tilde{H}(\Pi_{2n})$ and $\tilde{H}(\Pi_{2n}^e)$ are $\mathcal{S}_{2n}$ modules. For example, [8], as $\mathcal{S}_n$ modules,

$$\tilde{H}(\Pi_n) = \tilde{H}_n(\Pi_n) \cong \text{Lie}_n \otimes \text{sgn}$$

where

$$\text{Lie}_n := \text{ind}_{\mathcal{S}_n}^{\mathcal{S}_{2n}} e_{\mathcal{S}_n}^{2n i}.$$

### 4.3 Sundaram’s Conjecture

> From the point of view of [8], the reduced homology of rank-selected subposets of Cohen-Macaulay lattices encodes combinatorial invariants of the lattice. In this section, the work of Sundaram to understand the $\mathcal{S}_{2n}$ module $\tilde{H}(\Pi_{2n}^e) = \tilde{H}_{n+1}(\Pi_{2n}^e)$ is described. Recall that to determine the $\mathcal{S}_{2n}$ module $\tilde{H}(\Pi_{2n}^e)$, the symmetric function which is its Frobenius characteristic $\Phi \tilde{H}(\Pi_{2n}^e)$ may equivalently be determined. Also, recall Definition 7, the definition of the plethystic product of symmetric functions $f$ and $g$, denoted $f[g]$ which corresponds to the composition of representations of $GL(V)$. Sundaram has found a plethystic recurrence for $\Phi \tilde{H}(\Pi_{2n}^e)$ in [11].

**Theorem 5.** Sundaram [11]

$\Phi \tilde{H}(\Pi_{2n}^e)$ is the sum of terms of degree $2n$ in the symmetric function,

$$\left( \Phi \tilde{H}(\Pi_{2(n-1)}^e) - \Phi \tilde{H}(\Pi_{2(n-2)}^e) + \cdots + (-1)^{n-2}\Phi \tilde{H}(\Pi_2^e) + (-1)^{n-1}h_1 \right) \sum_{j \geq 1} h_j,$$

whose terms of even degree are all of degree $2n$.

Recall that $h_n$ denotes the $n$th homogeneous symmetric function. Sundaram is able to deduce from the plethystic recurrence in Theorem 5 a recurrence for the coefficients of $\Phi(\tilde{H}(\Pi_{2n}^e))$ in the homogeneous basis.

**Theorem 6.** Sundaram [11]

$$\Phi \tilde{H}(\Pi_{2n}^e) = \sum_{i=2}^n b_i(n) h_2^i h_1^{2(n-i)}$$

where

$$b_i(n) = \sum_{k \geq 0} \binom{2(n-i) + k}{k} \sum_{r \geq 1} (-1)^{r-1} \binom{i - k}{2r - k} b_{i-k}(n-r),$$

with the initial conditions $b_2(n) = 1$, for $n \geq 2$, and $b_i(n) = 0$, unless $2 \leq i \leq n$. 

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Sundaram's conjecture that $b_i(n) > 0$ is proven in Theorem 8. Hence, Corollary 4 states that $\Phi \tilde{H}(\Pi_{2n}^e)$ is $h$-positive. Equivalently, $\tilde{H}(\Pi_{2n}^e)$ is isomorphic to a sum of trivial modules induced from Young subgroups to $\mathfrak{S}_{2n}$. Furthermore, Sundaram has determined the dimension of $\tilde{H}(\Pi_{2n}^e)$ to be $(2n)!E_{2n-1}/2^{2n-1}$, where $E_{2n-1}$ is the tangent number.

**Theorem 7.** Sundaram [11]

$$\dim(\tilde{H}(\Pi_{2n}^e)) = \frac{(2n)!}{2^{2n-1}} E_{2n-1}$$

Thus, by (6),

$$\frac{(2n)!}{2^{2n-1}} E_{2n-1} = \sum_{i=2}^{n} b_i(n) \frac{(2n)!}{2^i}.$$ 

Thus,

$$E_{2n-1} = \sum_{i=2}^{n} b_i(n)2^{2n-i-1}.$$  \hspace{1cm} (4.3)

Hence, Corollary 5 states that (4.3) gives a refinement of the tangent number $E_{2n-1}$ into sums of powers of two. This Corollary and its relation to André permutations studied by [1] is discussed in the next section.

**Theorem 8.** For each $2 \leq i \leq n$,

$$b_i(n) > 0,$$

where $b_i(n)$ is defined recursively by,

$$b_i(n) = \sum_{k \geq 0} \binom{2(n-i)+k}{k} \sum_{r \geq 1} (-1)^{r-1} \binom{i-k}{2r-k} b_{i-k}(n-r),$$

with the initial conditions $b_2(n) = 1$, for $n \geq 2$, and $b_i(n) = 0$, unless $2 \leq i \leq n$.

**Proof.** First, interpret $b_i(n)$ combinatorially by defining a set $\mathcal{B}_{i,n}$ together with a function $\text{sgn} : \mathcal{B}_{i,n} \to \{1, -1\}$ such that:

$$b_i(n) = \sum_{B \in \mathcal{B}_{i,n}} \text{sgn}(B)$$
Figure 4-1: $r_1$ and $k_1$ for the first term $B_1$ of a sequence $B$.

Let $B_{i,n}$ be the set of sequences $(B_j)_{j=0}^{J_B}$ of the following form,

$$B_j = \left\{ \begin{array}{l}
  r_j, k_j \\
  y_{j,1}, \ldots, y_{j,2R_j-K_j} \\
  (x_{j,1}, z_{j,1}), \ldots, (x_{j,K_j}, z_{j,K_j})
\end{array} \right| \begin{array}{l}
  r_j, k_j, y_{j,t}, x_{j,t} \in \mathbb{N} \\
  \text{conditions 1-12 below are satisfied}
\end{array}$$

1. $k_0 = i$, $r_0 = n$,
2. $K_0 = 0$, $K_j = k_{j-1} - k_j$ for $1 \leq j \leq J_B$
3. $R_0 = 0$, $R_j = r_{j-1} - r_j$ for $1 \leq j \leq J_B$

Hence, $\{(x_{0,1}, z_{0,1}), \ldots, (x_{0,K_0}, z_{0,K_0})\} = \emptyset = \{y_{0,1} < \cdots < y_{0,2R_0-K_0}\}$
4. With the lexicographic order on 2-tuples $(x_{j,1}, z_{j,1}) < \cdots < (x_{j,K_j}, z_{j,K_j})$
5. $y_{j,1} < \cdots < y_{j,2R_j-K_j}$
6. $2 \leq r_j < r_{j-1}$ for $0 \leq j \leq J_B$
7. $2 \leq k_j \leq k_{j-1}$ for $0 \leq j < J_B$, and $k_{J_B} = 2$
8. $K_j \leq 2R_j$
9. $2R_j \leq k_j - 1$
10. $k_j \leq x_{j,t} < r_{j-1}$. If $X_j = 1$, then $k_j < r_{j-1}$ 11. $z_{j,t} \in \{0, 1\}$, and if $x_{j,t} < k_{j-1}$ then $z_{j,t} = 0$.
12. $0 \leq y_{j,t} < k_j$

Notice that these properties imply that,
13. $k_j \leq r_j$

Otherwise, in case $j \neq J_B$, property 10 implies

$$k_{j+1} < r_j < k_j.$$  

If $j = J_B$, then by property 6, and 7

$$k_j = 2 \leq r_j.$$

A typical sequence, $(B_j)_{j=0}^{J_B} \in B_{i,n}$, is plotted in Figures 1-7.

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Figure 4-2: \((x_{1,1}, z_{1,1})\) and \((x_{1,2}, z_{1,2})\) for the first term \(B_1\) of a sequence \(B\) with \(K_1 = 2\). The distinct boxes \((x_{1,1}, z_{1,1})\) and \((x_{1,2}, z_{1,2})\) may lie anywhere within the shaded region.

Figure 4-3: The first term \(B_1\) of a sequence \(B\). A choice of \(y_{1,1}\) and \(y_{1,2}\) for \(B_1\) with \(2R_1 - K_1 = 2\) completes the description of \(B_1\). The distinct boxes \(y_{1,1}\) and \(y_{1,2}\) may lie anywhere within the shaded region.

Figure 4-4: \(r_2\) and \(k_2\) for the first term \(B_2\) of a sequence \(B\) plotted together with the first term \(B_1\).
Figure 4-5: \((x_{2,1}, z_{2,1})\) for the second term \(B_2\) of a sequence \(B\) with \(K_2 = 1\) plotted together with the first term \(B_1\). The box \((x_{2,1}, z_{2,1})\) may lie anywhere within the shaded region.

Figure 4-6: The second term \(B_2\) of a sequence \(B\) plotted together with the first term \(B_1\). A choice of \(y_{2,1}\) for \(B_2\) with \(2R_1 - K_1 = 1\) completes the description of \(B_2\). The box \(y_{2,1}\) may lie anywhere within the shaded region.

Figure 4-7: The sequence \(B = (B_j)_{j=0}^4\) with five terms.
The following notation will be used to describe the data contained in a term $B_j$.

\[ X_j = \{ (x_{j,1}, z_{j,1}), \ldots, (x_{j,K_j}, z_{j,K_j}) \} \]
\[ Y_j = \{ y_{j,1}, \ldots, y_{j,2R_j-K_j} \} \]

Furthermore, it will be assumed that $\{k_j, r_j, X_j, Y_j\}$ indicates data that satisfy 1-13 for some choice of $r_{j-1}$ and $k_{j-1}$, which may be specified. In particular, these data will be denoted by $C_{k_{j-1}, r_{j-1}}$.

\[ C_{k_{j-1}, r_{j-1}} = \{ B_j = \{k_j, r_j, X_j, Y_j\} | B_j \text{ satisfies } 1-12, \text{ given } r_{j-1}, k_{j-1} \} \]

For example,

\[ B_{i,n} = \left\{ \begin{array}{c}
B_{j = 0} \\
B_j = \{k_j, r_j, X_j, Y_j\} \in C_{k_{j-1}, r_{j-1}} \text{ for } 1 \leq j \leq J_B \\
B_0 = \{ r_0 = n, k_0 = i \}
\end{array} \right\} \]

The sign of a term of a sequence is defined as $\text{sgn}(B_j) = (-1)^{R_j-1}$. The sign of a sequence is defined as the product of the signs of its terms. Hence, $\text{sgn}(\{B_j\}_{j=1}^{J_B}) = \prod_{j=1}^{J_B} \text{sgn}(B_j)$.

The sum of the signed sequences in $B_{i,n}$ satisfies the recurrence,

\[ \sum_{B \in B_{i,n}} \text{sgn}(B) = \sum_{(B_j)_{j=1}^{J_B} \in B_{i,n}} \text{sgn}(B_1) \prod_{j=2}^{J_B} \text{sgn}(B_j) \]

Assume that $B_j = \{k_j, r_j, X_j, Y_j\}$. Then,

\[ \sum_{B \in B_{i,n}} \text{sgn}(B) = \sum_{B_1 \in C_{i,n}} \text{sgn}(B_1) \sum_{\{B_j\}_{j=1}^{J_B} \in C_{k_{j-1}, r_{j-1}}} \prod_{j=2}^{J_B} \text{sgn}(B_j) \]

\[ = \sum_{B_1 \in C_{i,n}} \text{sgn}(B_1) \sum_{B' \in B_{k_1, r_1}} \text{sgn}(B') \quad (4.4) \]

Recall that for $B_1 = \{r_1, k_1, X_1, Y_1\} \in C_{i,n}$, $k_1 = i - K_1$ and $r_1 = n - R_1$. Furthermore, by properties 1, 6 and 7, $2 \leq k_1 \leq i$ and $2 \leq r_1 < n$. Thus,

\[ 0 \leq K_1 \leq i - 2, \]

and

\[ 1 \leq R_1 \leq n - 2. \]

Now, define

\[ C_{i,n}^{K,R} := \{ B_1 \in C_{i,n} | K_1 = K, R_1 = R \} \]

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Then, the recursion (4.4) becomes

\[
\sum_{B \in \mathcal{B}_{i,n}} \text{sgn}(B) = \sum_{K=0}^{i-2} \sum_{R=1}^{n-2} \sum_{B_1 \in \mathcal{C}^{K,R}_{i,n}} \text{sgn}(B_1) \sum_{(B'_i)_{i=0}^{i-K} \in \mathcal{B}_{i-K,n-R}} \text{sgn}(B').
\]

Note that for \(B_1 \in \mathcal{C}_{i,n}\) such that \(K = K_1\) and \(R = R_1\),

\[
\text{sgn}(B_1) = (-1)^{R_1-1} = (-1)^{R_1-1}
\]

Thus,

\[
\sum_{B \in \mathcal{B}_{i,n}} \text{sgn}(B) = \sum_{K=0}^{i-2} \sum_{R=1}^{n-2} \#(\mathcal{C}^{K,R}_{i,n})(-1)^{R_1-1} \sum_{(B'_i)_{i=0}^{i-K} \in \mathcal{B}_{i-K,n-R}} \text{sgn}(B'). \tag{4.5}
\]

Now, compare (4.5) with Sundaram's recursion. Sundaram's recursion is

\[
b_{i,n} = \sum_{K=0}^{i-2} \binom{2(n-i)+K}{K} \sum_{R=1}^{n-2} (-1)^{R_1-1} \binom{i-K}{i-2R} b_{i-K,n-R}
\]

Evidently the recursion (4.4) for \(\sum_{B \in \mathcal{B}_{i,n}} \text{sgn}(B)\) is the same recursion as Sundaram's recursion for the numbers \(b_{i,n}\) provided that

\[
\#(\mathcal{C}^{K,R}_{i,n}) = \binom{2(n-i)+K}{K} \binom{i-K}{i-2R} \tag{4.6}
\]

The equality (4.6), and hence the equivalence of the recursion (4.5) and Sundaram's recursion, may be verified by enumerating the set \(\mathcal{C}^{K,R}_{i,n}\). If \(B_1 = \{r_1, k_1, x_1, y_1\} \in \mathcal{C}^{K,R}_{i,n}\), then, as noted above, \(k_1 = i - K\), and \(r_1 = n - R\). The data in \(B_1\) must satisfy properties 1-12. Already, properties 1-3, 6, and 7 evidently hold regardless of the additional data \(X_j\) and \(Y_j\). Properties 8 and 9 also must hold if \(\binom{i-K}{i-2R}\) is non-zero. In particular, if \(\binom{i-K}{i-2R} > 0\), then \(i - 2R \geq 0\) which verifies property 9, and \(2R - K \geq 0\) which verifies property 8. Finally, the data \(X_1\) and \(Y_1\) must be chosen such that properties 4, 5, 10, 11, and 12 hold.

If \(k_1 \leq x_{1,t} < i\) then \(z_{1,t} = 0\) by property 10. If \(i \leq x_{1,t} < n\), then \(z_{1,t} \in \{0, 1\}\) by property 11. Hence, there are \(\binom{2(n-i)+K}{K}\) possibilities for the data \(X_1 = \{(x_{1,1}, z_{1,1}), \ldots < (x_{1,K}, z_{1,K})\}\) so that \(X_1\) satisfies properties 4, 10, and 11.

By property 12, \(y_{1,t}\) satisfies \(1 \leq y_{1,t} \leq k_1 = i - K_1\). Hence there are \(\binom{i-K}{i-2R-K}\) choices for \(Y_1 = \{y_{1,1}, \ldots < y_{1,2R-K}\}\) so that \(Y_1\) satisfies properties 5, and 12.

Thus, it is verified that \(\mathcal{C}^{K,R}_{i,n}\) has the required cardinality so that the recursion (4.4) for \(\sum_{B \in \mathcal{B}_{i,n}} \text{sgn}(B)\) is the same recursion as Sundaram's recursion for the numbers \(b_{i,n}\).
Now, we construct an involution

$$\iota : B_{k,n} \rightarrow B_{k,n}.$$  

The image of the involution \( \iota \) will be denoted by

$$\left( B'_q \right)_{q=1}^{Q_B} = \iota \left( B_q \right)_{q=0}^{Q_B}.$$  

The involution \( \iota \) is described in six cases. In the \( i \)th case \( j \in S_i \) where

$$j = \min_{q \in \cup_{i=1}^6 S_i} q,$$

and where \( S_i \) is defined below. Let \( B \) denote the sequence \( (B_j)_{j=1}^{J_B} \) with \( J_B \) terms.

\[
S_1(B) = \left\{ q \left| R_q > 1 \right. \quad K_q \in \{0,1\} \right\}
\]

\[
S_2(B) = \left\{ q \left| R_q > 1 \right. \quad K_q \geq 2 \text{ if } q \neq J_B \quad K_q > 2 \text{ if } q = J_B \right\}
\]

\[
S_3(B) = \left\{ J_B \left| B_{J_B}, R_{J_B} > 1 \right. \quad K_{J_B} = 2 \right\}
\]

\[
S_4(B) = \left\{ q \left| B_q, R_q = 1 \right. \quad K_q \in \{0,1\}, \quad K_{q+1} = 0 \right\}
\]

\[
S_5(B) = \left\{ q \left| B_q, R_q = 1 \right. \quad K_q = 2 \right\}
\]

\[
S_6(B) = \left\{ J_B \left| K_{J_B} = 2 \right. \quad R_{J_B} = 1 \right\}
\]

In Cases I and II, \( \iota \) acts to "add a term at \( j \)" to the sequence \( (B_q)_{q=0}^{Q_B} \). Set \( B'_q = B_q \) for \( q < j \), and set \( B'_q = B_{q-1} \) for \( q > j + 1 \). Then, \( Q_{B'} = Q_B + 1 \). Set \( r'_j = r_{j-1} - 1 \). Set \( r'_{j+1} = r_j \). Set \( k'_{j+1} = k_j \). In Case I, set \( k'_j = k_j \). In Case II, set \( k'_j = k_{j-1} - 2 \). Set \( (x'_{j,t}, z'_{j,t}) = (x_{j,t}, z_{j,t} \cup \{K'_j, K'_{j+1}\}) \) for \( 1 \leq t \leq K'_j \), and \( (x'_{j,t+1}, z'_{j,t+1}) = (x_{j,t}, z_{j,t}) \) for \( 1 \leq t \leq K'_{j+1} \). Finally, set \( y'_{j+1,t} = y_{j,t+1} + 2R_{j+1} - K'_{j+1} \) for \( 1 \leq t \leq 2R'_j - K'_j \), and set \( y'_{J_B+1,t} = y_{J_B+1,t} \) for \( 1 \leq t \leq 2R'_{J_B+1} - K'_{J_B+1} \).

In Case III, \( \iota \) acts to "modify the final term, \( B_{J_B} \)". Set \( B'_q = B_q \) for \( q < j \). Set \( k'_j = k_j \), set \( (x'_{j,t}, z'_{j,t}) = (x_{j,t}, z_{j,t}) \) for \( 1 \leq t \leq K'_j \), and set \( R'_j = 1 \).

In Cases IV and V, \( \iota \) acts to "delete a term at \( j \)" in the sequence \( (B_q)_{q=0}^{Q_B} \). Set \( B'_q = B_q \) for \( q < j \), and \( B'_q = B_{q+1} \) for \( q > j \). In these cases, set \( r'_j = r_{j+1} \) and set \( k'_j = k_{j+1} \). Set \( (x'_{j,t}, z'_{j,t}) = (x_{j+1,t}, z_{j+1,t}) \) for \( 1 \leq t \leq K_{j+1} \), and set \( (x'_{j,t+K_{j+1}}, z'_{j,t+K_{j+1}}) = (x_{j,t+K_{j+1}}, z_{j,t+K_{j+1}}) \)
Figure 4-8: A sequence \((B_j)_{j=0}^2\). Since, \(R_1 = 2\) and \(K_1 = 1\), this sequence is in case I and \(\iota\) acts to add a term at 1. Thus, the image of \(\iota\) is the sequence plotted in Figure 9.

\[
\begin{array}{|c|c|c|c|c|}
\hline
k_2 & k_1 & k_0 & r_2 & r_1 & r_0 \\
\hline
2 & 5 & 6 & 9 & 11 & 13 \\
\hline
\end{array}
\]

Figure 4-9: A sequence \((B_j)_{j=0}^3\). Notice that \(R_1 = 1\), \(K_1 = 1\), \(K_2 = 0\), and \(y_{2,2} < y_{1,1}\). Hence, this sequence is in case III and \(\iota\) acts to delete a term at 1. The image of \(\iota\) is the sequence plotted in Figure 8.

\[
\begin{array}{|c|c|c|c|c|}
\hline
k_3 & k_2 & k_1 & k_0 & r_3 & r_2 & r_1 & r_0 \\
\hline
2 & 5 & 6 & 9 & 11 & 12 & 13 \\
\hline
\end{array}
\]

\((x_{j,t}, z_{j,t})\) for \(1 \leq t \leq K_j\). Set \(y_{j,t} = y_{j+1,t}\) for \(1 \leq t \leq 2R_{j+1} - K_{j+1}\), and set \(y_{j,t+2R_{j+1} - K_{j+1}} = y_{j,t}\) for \(1 \leq t \leq 2R_j - K_j\).

In Figure 8, a sequence is plotted. Its image under \(\iota\) is plotted in Figure 9.

In Figure 10, a sequence is plotted. Its image under \(\iota\) is plotted in Figure 11.

In Case VI, \(\iota\) acts to “modify the final term, \(B_{jB}\)”. Set \(B'_{q} = B_q\) for \(q < j\), and set \(k'_{j} = k_j = 2\) \((x'_{j,t}, z'_{j,t}) = (x_{j,t}, z_{j,t})\) for \(1 \leq t \leq K_{jB} = 2\). Set \(r'_{jB} = r_{jB-1} - 2\), set \(y_{jB,1} = 0\), and set \(y_{jB,2} = 1\).

In Figure 12, a sequence is plotted. Its image under \(\iota\) is plotted in Figure 13.

Now, it must be proven that \((B'_{q})_{q=0}^{Q'}\) satisfies properties 1-12 in cases I-VI. In all cases, it is evident that \((B'_{q})_{q=0}^{Q'}\) has properties 1-6. For the remaining properties, it must be verified that \((B'_{q})_{q=0}^{Q'}\) has these given properties by considering each of the six cases.

In cases II-VI, it is evident that \((B'_{q})_{q=0}^{Q'}\) has property 7. In case I, it must be verified that \(k'_{j} > 2\) since \(B'_{j}\) is not the final term of the sequence \((B'_{q})_{q=0}^{Q'}\). In case I, if \(K_j = 0\) then

\[k'_{j} = k_j = k_{j-1} > 2.\]
Figure 4-10: A sequence \((B_j)_{j=0}^2\). Note that \(R_1 = 1\), \(K_1 = 1\), but \(K_2 \neq 0\), so \(\iota\) does not act at \(j = 1\). Since \(R_2 = 2\) and \(K_2 = 4\), this sequence is in case II and \(\iota\) acts to add a term at 2. Thus, the image of \(\iota\) is the sequence plotted in Figure 11.

Figure 4-11: A sequence \((B_j)_{j=0}^3\). Since \(R_1 = 1\), \(K_1 = 1\), but \(K_2 \neq 0\), so \(\iota\) does not act at \(j = 1\). Note that \(R_2 = 1\), \(K_2 = 2\), \(K_3 = 0\) \((x_{3,2}, z_{3,2}) < (x_{2,1}, z_{2,1})\), \(x_{3,2} \geq k_1 = 6\), and \(z_{3,1} = 0\). Hence this sequence is in case IV and \(\iota\) acts to delete a term at 2. Thus, the image of \(\iota\) is the sequence plotted in Figure 10.

Figure 4-12: A sequence \((B_j)_{j=0}^3\). Note that \(R_1 = 1\), \(K_1 = 1\), but \(K_2 \neq 0\) so \(\iota\) does not act at 1. Note that \(R_2 = 1\), \(K_2 = 2\), but \((x_{3,2}, z_{3,2}) > (x_{2,1}, z_{2,1})\), so \(\iota\) does not act a 2 either. Since \(J_B = 3\) and \(K_3 = 2\) and \(R_3 = 2\), this sequence is in case III. Thus, the image of \(\iota\) is the sequence plotted in Figure 13.
Figure 4-13: A sequence $(B_j)^3_{j=0}$. Note that $R_1 = 1$, $K_1 = 1$, but $K_2 \neq 0$ so $i$ does not act at 1. Note that $R_2 = 1$, $K_2 = 2$, but $(x_{3,2}, z_{3,2}) > (x_{2,1}, z_{2,1})$, so $i$ does not act a 2 either. Since $J_B = 3$ and $K_3 = 2$ and $R_3 = 1$, this sequence is in case VI. Thus, the image of $i$ is the sequence plotted in Figure 12.

In case I if $K_j = 1$, then $4 \leq 2R_j \leq k_{j-1}$, thus

$$k_j' = k_{j-1} - 1 \geq 3.$$  

In all cases, $(B')^Q_{q=0}$ has property 8. In case I, $K_j \in \{0, 1\}$. Thus,

$$K_j', K_{j+1}' \in \{0, 1\} \leq 2R_j.$$  

In case II, $K_j' = 2 \leq 2R_j$. Furthermore,

$$K_{j+1}' + 2 = K_j' + K_{j+1}' = K_j \leq 2R_j = 2R_j' + 2R_{j+1}' = 2 + 2R_{j+1}'.$$  

In case III, $K_j' = 2 \leq 2R_j$. In cases IV and V,

$$K_j' = K_j + K_{j+1} \leq 2R_j + 2R_{j+1} = 2R_j'.$$

In case VI, $K_{j_B}' = 2 \leq 2R_{j_B}'$.

In all cases, $(B')^Q_{q=0}$ has property 9. In cases I, II, and III, $2R_j' = 2 \leq k_{j-1}'$. In cases I, $2 > k_{j-1} - k_j = k_{j-1} - k_j'$. In case II, $2 = k_{j-1} - k_j'$. Hence, in cases I and II,

$$2R_{j+1}' + 2 = 2R_j \leq k_{j-1} \leq k_j' + 2.$$  

In case IV, $y_{j,1}' < y_{j,2}' < \cdots < y_{j,2R_j'-K_j'} = y_{j,2R_j-K_j}$. Since $y_{j,2R_j-K_j} < k_{j+1}$, it follows that

$$2R_j' - K_j' < k_{j+1} = k_j'.$$

In case V, it is trivial to verify that $(B')^Q_{q=0}$ has property 9. In Case VI,

$$R_{j_B}' \leq R_{J_B} \leq k_{j-1} = k_{j-1}.'$$
In all cases, \((B')^{Q_{B'}}_{q=0}\) has property 10. In case I, if \(K'_j = 0\), then
\[
k'_j = k_j < r_{j-1} = r'_{j-1}.
\]
Otherwise, in case I, if \(K'_j = 1\), then
\[
k'_j = k_j \leq x'_{j,1} = x_{j,1} < r_{j-1} = r'_{j-1}.
\]
In case I, by property 13,
\[
k'_{j+1} = k_j \leq r_j < r'_{j}.
\]
In case II, if \(x_{j,K_j-2} \geq k_{j-1} - 2\), then
\[
x'_{j,t} = x_{j,K_j-2+t} \geq x_{j,K_j-2} \geq k_{j-1} - 2 = k'_j.
\]
Otherwise, in case II, if \(x_{j,K_j-2} < k_{j-1} - 2\), then for \(1 \leq t \leq K_j - 2\), \(z_{j,t} = 0\). Furthermore,
\[
(x_{j,1}, z_{j,1}) < (x_{j,2}, z_{j,2}) < \cdots < (x_{j,K_j}, z_{j,K_j}).
\]
Hence,
\[
k_j \leq x_{j,1} < x_{j,2} < \cdots < x_{j,K_j-2}.
\]
Hence,
\[
k_{j-1} - 2 = k_j + K_j - 2 \leq x_{j,K_j-2}
\]
which is a contradiction.

In case II,
\[
x'_{j,t} = x_{j,K'_j+1+t} < r_{j-1} = r'_{j-1}.
\]
In case II,
\[
k'_{j+1} = k_j \leq x_{j,t} = x'_{j+1,t}.
\]
Furthermore, in case II, since \(x_{j,K_j} < r_{j-1}\), it follows that \(x_{j,K_j-2} < r_{j-1} - 1\). Otherwise, since \(x_{j,K_j} < r_{j-1}\) by property 10, it follows that
\[
x_{j,K_j-2} = x_{j,K_j-1} = x_{j,K_j} = r_{j-1} - 1.
\]
Hence,
\[
x'_{j+1,K'_j+1} = x_{j,K'_j+1} = x_{j,K_j-2} < r_{j-1} - 1 = r'_j.
\]
In case III, \((B')^{Q_{B'}}_{q=0}\) evidently has property 10. In cases IV and V,
\[
k'_j = k_{j+1} \leq x_{j+1,1} = x'_{j,1} \leq x'_{j,1}.
\]
Furthermore, in cases IV and V,
\[
x'_{j,t} \leq x'_{j,2K_j-K'_j} = x_{j,2K_j-K_j} < r_{j-1} = r'_{j-1}.
\]
In case VI, \((B')^{Q_{B'}}_{q=0}\) evidently has property 10.

In all cases, \((B')^{Q_{B'}}_{q=0}\) has property 10. In cases I and II, if \(x'_{j,t} = x_{j,K'_j+1+t} < k_{j-1} = \)
\(k'_{j-1}\), then
\[z'_{j,t} = z_{j,k'_{j+1}+t} = 0.\]
If \(x'_{j+1,t} < k'_j\) then \(x_{j,t} = x'_{j+1,t} < k'_j \leq k'_{j-1} = k_{j-1}\). Hence,
\[z'_{j+1,t} = z_{j,t} = 0.\]

In case III, \((B')^{Q_B'}_{q=0}\) evidently has property 11. In cases IV and V, if \(1 \leq t \leq K_{j+1}\)
and if \(x'_{j,t} = x_{j+1,t} < k_{j-1} = k'_{j-1}\), then since \(k_{j-1} \leq k_j\), it follows that
\[z'_{j,t} = z_{j+1,t} = 0.\]
If \(1 \leq t \leq K_j\) and if \(x'_{j,t+K_{j+1}} = x_{j,t} < k_{j-1} = k'_{j-1}\) then
\[z'_{j,t+K_{j+1}} = z_{j,t} = 0.\]

In case VI, \((B')^{Q_B'}_{q=0}\) evidently has property 11.

12. In cases I and II,
\[y'_{j,t} = y_{j,t+2R_{j+1} - K_{j+1}} < k_j \leq k'_j.\]
Similarly,
\[y'_{j+1,t} = y_{j,t} < k_j = k'_{j+1}.\]
In case III, \((B')^{Q_B'}_{q=0}\) trivially has property 11. In case IV, if \(1 \leq t \leq 2R_{j+1} - K_{j+1}\)
then
\[y'_{j,t} = y_{j+1,t} < k_{j+1} = k'_j.\]
If \(1 \leq t \leq 2R_j - K_j\), then
\[y'_{j,t+2R_{j+1} - K_{j+1}} = y_{j,t} < k_j = k_{j+1} = k'_j.\]
In case V, \((B')^{Q_B'}_{q=0}\) trivially has property 11. In case VI, \((B')^{Q_B'}_{q=0}\) evidently has property
11.

13. In case I, it must be verified that \(k'_j > 2\).
If \(K_j = 0\) then \(k'_j = k_j = k_{j-1} > 2\). If \(K_j = 1\), then \(4 \leq 2R_j \leq k_{j-1}\), thus
\(k'_j = k_{j-1} - 1 \geq 3\) Cases II,III,IV,V,VI are evident.

In the following, set \(B' := \iota(B)\) as above. Set \(B'' := \iota(B')\). The verification \(B'' = B\) proceeds in cases. As above, set
\[j = \min_{q \in \cup_{k=1}^n S_k(B)} q,\]
and set
\[j' = \min_{q \in \cup_{k=1}^n S_k(B')} q.\]
In case I, \( j \in S_1(B) \). Thus, \( r_j' = r_{j-1} - 1 \) and \( k_j' = k_j = k_{j+1} \). Hence,

\[
R_j' = 1,
\]

and

\[
K_{j+1}' = 0.
\]

Furthermore,

\[
K_j' = K_j \in \{0, 1\},
\]

and

\[
y_j+1,2R_{j+1}'-K_{j+1}' = y_j,2R_{j+1}'-K_{j+1}' < y_j,2R_{j+1}'-K_{j+1}' + 1 = y_j'.
\]

Hence, if \( j \in S_1(B) \), then \( j' \in S_4(B') \).

In case II, \( j \in S_2(B) \). Thus, \( r_j' = r_{j-1} - 1 \), and \( k_j' = k_{j-1} - 2 \). Hence,

\[
R_j' = 1,
\]

and

\[
K_j' = 2.
\]

Furthermore,

\[
(x_j',K_j'+1, z_j'+1, K_j'+1) = (x_j,K_j'+1, z_j,K_j'+1) < (x_j,K_j'+1, z_j,K_j'+1) = (x_j', z_j'),
\]

and if \( x_j'+1,t = x_j,t < k_{j-1} \), then

\[
z_j'+1,t = z_j,t = 0.
\]

Hence, if \( j \in S_2(B) \), then \( j' \in S_5(B') \).

For the cases I and II, the data in \( B, B' \), and \( B'' \) are compared below. In both cases I and II, for \( q < j \),

\[
B''_q = B'_q = B_q,
\]

while for \( q > j \),

\[
B''_q = B'_{q+1} = B_q.
\]

In both of these cases, for \( q = j \),

\[
r_j'' = r_j' = r_j,
\]

\[
k_j'' = k_j' = k_j,
\]

\[
(x''_j,t, z''_j,t) = (x_j'+1,t, z_j'+1,t) = (x_j,t, z_j,t) \text{ for } 1 \leq t \leq K_j'+1,
\]

\[
(x''_{j,t+K_j'+1}, z''_{j,t+K_j'+1}) = (x_j',z_j',t) = (x_j,K_j'+1+t, z_j,K_j'+1+t) \text{ for } 1 \leq t \leq K_j',
\]

\[
y_j'' = y_j'+1,t = y_j,t \text{ for } 1 \leq t \leq 2R_{j+1}' - K_j',
\]

\[
y_j,2R_{j+1}'-K_{j+1}' = y_j'+1,t = y_j,t+2R_{j+1}'-K_{j+1}' \text{ for } 1 \leq t \leq 2R_j' - K_j'.
\]

Hence, in cases I and II, \( B'' = B \).
In case III, \( j \in S_3(B) \). Thus, \( K'_{J_B} = K_{J_B} = 2 \), and while \( R'_{J_B} = 2 \),
\[
R''_{J_B} = 1.
\]

Hence, \( j' \in S_6(B') \).

In case III, note that \( 2 = 2R_{J_B} - K_{J_B} = 2R''_{J_B} - K''_{J_B} \) so that,
\[
y_{J_B,1} < y_{J_B,2} = y_{J_B,2R_{J_B}-K_{J_B}} \leq k_{J_B} = 2.
\]

Thus,
\[
y_{J_B,1} = 1 = y''_{J_B,1},
\]

and
\[
y_{J_B,2} = 2 = y''_{J_B,2}.
\]

The only other data changed by \( i \) in case III and VI are \( R_{J_B} \), and \( R'_{J_B} \). Since, \( R'_{J_B} = 1 \),
\[
R''_{J_B} = 2 = R_{J_B}.
\]

In case VI, \( j \in S_6(B) \). Thus, \( K'_{J_B} = K_{J_B} = 2 \) and while \( R'_{J_B} = 1 \),
\[
R''_{J_B} = 2 = R_{J_B}.
\]

Hence, \( j' \in S_3(B') \). Now, \( 0 = 2R_{J_B} - K_{J_B} = 2R''_{J_B} - K''_{J_B} \) so that,
\[
\{y_{J_B,1}, \ldots, y_{J_B,2R_{J_B}-K_{J_B}}\} = \emptyset = \{y''_{J_B,1}, \ldots, y''_{J_B,2R_{J_B}-K_{J_B}}\}.
\]

Again, the only other data changed by \( i \) are \( R_{J_B} \), and \( R'_{J_B} \). Since, \( R'_{J_B} = 2 \),
\[
R''_{J_B} = 1 = R_{J_B}.
\]

In case IV, \( j \in S_4(B) \). Thus, \( r'_j = r_{j+1} \), and, since \( K_{j+1} = 0 \), \( k'_j = k_{j+1} = k_j \).

Hence,
\[
R'_j = R_{j+1} + R_j \geq 2,
\]

and
\[
K'_j = K_j \in \{0, 1\}.
\]

Hence, \( j' \in S_1(B') \)

In case V, \( j \in S_5(B) \). Thus, \( r'_j = r_{j+1} \), and \( k'_j = k_{j+1} \). Hence,
\[
R'_j = R_{j+1} + R_j \geq 2,
\]

and
\[
K'_j = K_{j+1} + K_j \geq 2,
\]

since \( K_j \geq 2 \). If \( k'_j = k_{j+1} = 2 \) and \( K'_j = 2 \) so that
\[
0 = K_{j+1} = k_{j+1} - k_{j},
\]

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which is impossible since then $k_{j+1} = k_j = 2$, but $k_q = 2$ only when $q = J_B$
Hence, $j' \in S_2(B')$

For the cases IV and V, the data in $B$, $B'$, and $B''$ are compared below. In both cases I and II, for $q < j$,
\[ B''_q = B'_q = B_q, \]
while for $q > j + 1$,
\[ B''_q = B'_{q-1} = B_q. \]
For the $j$th term, in both case IV and V,
\[ r''_j = r'_{j-1} - 1 = r_{j-1} - 1 = r_j, \]
since $R_j = 1$. For the $j + 1$st term, in both of these cases,
\[ r''_{j+1} = r'_{j} = r_{j+1}. \]
In case IV,
\[ k''_j = k'_j = k_{j+1} = k_j, \]
while in case V,
\[ k''_j = k'_{j-1} - 2 = k_{j-1} - 2 = k_j, \]
since $K_j = 2$. In both cases,
\[ k''_{j+1} = k'_j = k_{j+1}. \]
Thus, $K''_j = K_j$ and $K''_{j+1} = K_{j+1}$. In both cases IV and V, for $1 \leq t \leq K''_j = K_j$,
\[ (x''_{j,t}, z''_{j,t}) = (x'_{j, K''_{j+1}+t}, z'_{j, K''_{j+1}+t}) = (x_{j,t}, z_{j,t}), \]
and for $1 \leq t \leq K''_{j+1} = K_{j+1}$,
\[ (x''_{j+1,t}, z''_{j+1,t}) = (x'_{j+1, t}, z'_{j+1, t}) = (x_{j+1,t}, z_{j+1,t}). \]
In both cases for $1 \leq t \leq 2R''_j - K''_j = 2R_j - K_j$,
\[ y''_{j,t} = y'_{j, t + 2R''_{j+1} - K''_{j+1}} = y_{j,t}, \]
and for $1 \leq t \leq 2R''_{j+1} - K''_{j+1} = 2R_{j+1} - K_{j+1}$,
\[ y''_{j+1,t} = y'_{j+1, t} = y_{j+1,t}. \]

Hence, in all cases, $B'' = B$, and $\iota$ is an involution.

It must be verified that $\text{sgn}(B) = -\text{sgn}(\iota(B))$.
Recall that,
\[ \text{sgn}(B) = \prod_{j=1}^{J_B} (-1)^{R_j - 1} = (-1) \sum_{j=1}^{J_B} R_j (-1)^{J_B} = (-1)^{r_0 - r_B} (-1)^{J_B}. \]
Again set $B' = \iota(B)$, and $j = \min \{ q \in \cup_{i=1}^{3} S_i(B) \} q$, and assume that in the $i$th case, $j \in S_i(B)$. In case I and II,

$$J_{B'} = J_B + 1$$

while

$$r_{J_{B'}}^j = r_{J_B+1} = r_J,$$

since $J_B \geq j$. In cases III and VI,

$$J_{B'} = J_B.$$

In case III,

$$r_{J_{B'}}^j = r_J - 1,$$

while in case VI,

$$r_{J_{B'}}^j = r_J + 1.$$

In case IV and V,

$$J_{B'} = J_B - 1,$$

while

$$r_{J_{B'}}^j = r_{J_B-1} = r_J,$$

since $J_B \geq j$. Thus, in all cases, $\text{sgn}(B) = -\text{sgn}(\iota(B))$. Finally, note that if $(B_j)_{j=1}^{J_B}$ has a term $B_j$ such that $R_j > 1$, then

$$j \in \cup_{i=1}^{3} S_i(B).$$

Hence if $B_j^{J_B} \in \text{Fix}(\iota)$, then $R_j = 1$ for $1 \leq j \leq J_B$. Thus, $r_0 - r_J = J_B$, so that

$$\text{sgn}(B) = (-1)^{r_0-r_J} (-1)^{J_B} = (-1)^{2J_B} = 1.$$

Now, the fixed set may be characterized as those sequences $B$ such that $\cup_{i=1}^{3} S_i(B) = \emptyset$. Already, it has been shown that for such a sequence, $R_j = 1$ for $1 \leq j \leq J_B$. Thus, since $K_j < 2R_j = 1$, for $1 \leq j \leq J_B$,

$$K_j \in \{0, 1, 2\}.$$

Also, if $B \in \text{Fix}(\iota)$,

$$K_{J_B} \neq 2.$$

Otherwise $\{J_B\} = \{S_3 \cup S_4\}$. Next, suppose $K_j \in \{0, 1\}$, then if $K_{j+1} = 0$,

$$y_{j+1, 2R_{j+1} - K_{j+1}} \geq y_{j,1}.$$

Otherwise, $j \in S_4$. Now, suppose $K_j = 2$, then either

$$(x_{j+1, K_{j+1}}, z_{j+1, K_{j+1}}) \geq (x_{j,1}, z_{j,1}).$$
or both $\exists x_{j+1,t}$ such that

$$x_{j+1,t} \leq k_{j-1},$$

and

$$z_{j+1,t} = 0.$$

Otherwise, $j \in S_5$.

Note, for example, that when $K_j = 2$, $K_{j+1} \neq 0$.

This completes both the characterization of the fixed set $\text{Fix}(\iota)$ of the involution $\iota$, and the proof that

$$b_{n,i} = \#(\text{Fix}(\iota))$$

□

**Corollary 4.** The Frobenius characteristic of the reduced top homology of $\Pi_{2n}^e$ is $h$-positive.

$$\Phi \tilde{H}(\Pi_{2n}^e) = \sum_{i=2}^{n} b_{i}(n) h_2 h_1^{2(n-i)}$$

Equivalently, $\tilde{H}(\Pi_{2n}^e)$ is a sum of trivial modules induced from Young subgroups,

$$\tilde{H}(\Pi_{2n}^e) \cong \bigoplus_{i=2}^{n} (\text{ind}_{\langle \mathfrak{S}_2 \rangle \times \langle \mathfrak{S}_1 \rangle \times \langle \mathfrak{S}_1 \rangle \times 2(n-i)}^{\mathfrak{S}_{2n}} 1_{\langle \mathfrak{S}_2 \rangle \times \langle \mathfrak{S}_1 \rangle \times \langle \mathfrak{S}_1 \rangle \times 2(n-i)})^{\otimes b_i(n)}$$
4.4 The Tangent Number and André Permutations

Let \((u_1, u_2, \ldots, u_n)\) denote the permutation \(u \in \mathcal{S}_n\) written in one-line notation. An down-up or alternating permutation in \(u \in \mathcal{S}_n\) is a permutation that satisfies \(u_1 > u_2 < u_3 > u_4 < \cdots u_n\).

**Definition 26.** (Euler Number \(E_n\))

\(E_n\) is the number of down-up permutations \(u \in \mathcal{S}_n\).

Evidently, for \(n \geq 1\), the Euler numbers satisfy the recursion,

\[
2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}
\]

Since the exponential generating function for the Euler numbers is

\[
\sum_{n \geq 0} E_n \frac{x^n}{n!} = \tan(x) + \sec(x)
\]

**Corollary 5.**

\[E_{2n-1} = \sum_{i=2}^{n} b_i(n) 2^{2n-i-1}\]

\(E_{2n-1}\) is called a tangent number and \(E_{2n}\) is called a secant number.

The tangent number has been shown by Foata and Schützenberger in [1] to be the number of André permutations with \(k\) peaks. Let \(\mathcal{A}_n\) denote the subset of \(\mathcal{S}_n\) which are André permutations. Let \((u_1, u_2, \ldots, u_n)\) denote the permutation \(u \in \mathcal{S}_n\) written in one-line notation. Let \(IL_j = (u_i, \ldots, u_j)\) be the largest interval ending with \(u_j\) such that for \(i \in IL_j\), \(i \geq u_j\). Similarly let \(IR_j = (u_j, \ldots, u_r)\) be the largest interval beginning with \(u_j\) such that for \(i \in IR_j\), \(i \geq u_j\). Set \(L_j = IL_j - u_j\) and \(R_j = IR_j - u_j\). Note that \(j\) is a trough or a valley if and only if both \(L_j\) and \(R_j\) are nonempty.

**Definition 27.** (André Permutations) \(\mathcal{A}_n\) is the set of permutations \((u_1, u_2, \ldots, u_n)\) in \(\mathcal{S}_n\) such that

1. \(u\) has no two consecutive descents
2. for each trough \(j\), \(\max(L_j) < \max(R_j)\)

**Proposition 14.**

\[E_{n-1} = \#(\mathcal{A}_n)\]

Foata and Schützenberger study the André permutations with \(k\) peaks \(\mathcal{A}_{n,k}\). The position \(j\) is a peak if both \(L_j\) and \(R_j\) are nonempty. If \(j \neq 1\) and \(j \neq n\) then this coincides with the usual criterion \(u_{j-1} < u_j > u_{j+1}\). Any refinement of the set of
André permutations, or the set of down-up permutations, gives a refinement of the
tangent numbers.

\[ E_{n-1} = \sum_{k=1}^{\lfloor n/2 \rfloor} \#(A_{n,k}) \]

Furthermore,

**Proposition 15.**

\[ E_{2n-1} = 2^{n-1} \#(A_{2n-1,n}) \]

Comparing with Corollary 5 shows that the numbers \( b_i(n) \) in fact refine \( \#(A_{2n-1,n}) \)
into sums of powers of two. Thus, we have the following corollary of Theorem 8.

**Corollary 6.**

\[ \#(A_{2n-1,n}) = \sum_{i=2}^{n} b_i(n) 2^{n-i}. \]
Bibliography


