Goodwillie calculus and algebras over a spectral operad

by

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Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of
Ph.D. in Mathematics
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Abstract

The overall goal of this thesis is to apply the theory of Goodwillie calculus to the
category $Alg_\mathcal{O}$ of algebras over a spectral operad.

Its first part generalizes many of the original results of Goodwillie in [14] so that
they apply to a larger class of model categories and hence be applicable to $Alg_\mathcal{O}$.

The second part then applies that generalized theory to the $Alg_\mathcal{O}$ categories. The
main results here are: an understanding of finitary homogeneous between such cate-
gories; identifying the Taylor tower of the identity in those categories; showing that
finitary $n$-excisive functors can not distinguish between $Alg_\mathcal{O}$ and $Alg_{\mathcal{O}_{\leq n}}$, the cate-
gory of algebras over the truncated operad $\mathcal{O}_{\leq n}$; and a weak form of the chain rule
between the algebra categories, analogous to the one found in [1].

Thesis Supervisor: Mark J. Behrens
Title: Associate Professor
Acknowledgments

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Other than that, I thank my advisor Mark Behrens for putting up with me for these past few years, and my annoying friends and family for being so damn annoying in a sometimes mildly endearing way. As they know who they are, and as I'd rather not risk incurring anyone's wrath by forgetting a name, I'll leave it at that.
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Chapter 1

Overview

Our primary goal in this paper is to extend the notion of Goodwillie calculus so that it can be applied to the categories $Alg_{\mathcal{O}}$ of algebras over a spectral operad, and to then establish some of the basic results about such calculus.

There are hence two main parts to the paper, the first part dealing with general results about Goodwillie Calculus in reasonably general model categories, and a second part dealing with results which apply specifically to $Alg_{\mathcal{O}}$.

1.1 Part I

The main goal of this part is to establish a fairly general setup in which to do Goodwillie calculus, extending the treatment of Goodwillie in [14].

There are two main results. The first is the existence of universal $n$-excisive approximations, which is Theorem 4.12. The second is the classification of $n$-homogeneous functors $F: \mathcal{C} \to \mathcal{D}$ (at least under some finitary conditions) as having the form $F = \Omega^{\infty}((\bar{F} \circ (\Sigma^\infty)^{\times n})_{h\Sigma_n})$, where $\bar{F}: Stab(\mathcal{C})^{\times n} \to Stab(\mathcal{D})$ is a symmetric multilinear functor between the stabilizations. In the current write-up, this result corresponds to piecing together Proposition 6.7, which shows such functors factor through $Stab(\mathcal{D})$ (provided they are determined by their values on a small subcategory $C' \subset C$), Proposition 5.29, showing that, provided the target category is stable, a homogeneous functor is recovered from its cross effect, and Theorem 7.2, which
proves directly that multilinear symmetric functors factor through stabilizations on both source and target.

The following is a more detailed overview of the paper.

Chapter 2 establishes some of the general assumptions on model categories and functors between them used throughout the paper. Particularly important is Proposition 2.9, stating informally that, in any cofibrant model category, "linear/direct" colimits over a large enough ordinal commute with finite holims.

Chapter 3 establishes the basic definitions of excisive functors (via hocartesian/hococartesian cubes), and many of their basic properties, which are generally straightforward generalizations of the analogous properties found in [14]. Of some technical interest seems to be Proposition 3.15, which is currently necessary for the proof of Theorem 4.12 (check Remark 4.15 for more on this).

Chapter 4 then establishes the first main result, Theorem 4.12, for the existence of universal $n$-excisive approximations $P_nF$. These are constructed via Goodwillie's $T_nF$, except those are iterated a transfinite number of times. Excisiveness of this construction then follows by Goodwillie's usual proof combined with Proposition 2.9 and holds generally under the assumption that $\mathcal{C}, \mathcal{D}$ are cofibrantly generated model categories. The proof of universality seems currently more delicate, however, relying on Proposition 3.15, which currently requires the source category $\mathcal{C}$ to be either a pointed simplicial model category or a left Bousfield localization of one.

Chapter 5 establishes that, when the target category $\mathcal{D}$ is stable, homogeneous functors and multilinear functors determine each other (Propositions 5.26 and 5.29), and is a fairly straightforward generalization of the treatment in [14]. This requires no further assumptions on the categories.

Chapter 6 establishes that homogeneous functors can be delooped. This does require additional conditions on the categories. Namely, it is rather unclear how to generalize Goodwillie's proof of this when the $T_n$ are to be iterated a transfinite number of times, so that one is forced to assume $P_n$ can be constructed via countable iterations, and this, in turn, essentially amounts to assuming that, in the target category $\mathcal{D}$, countable directed hocolims commute with finite holims. It also establishes
the infinitely iterated result, Proposition 6.7.

Finally, Chapter 7, deals with showing Theorem 7.2. This too imposes more conditions on the model categories, most obviously requiring that the stabilizations (as defined in [20]) exist, and the further technical condition that in those stabilizations one can detect weak equivalences by looking at the $\Omega^{\infty-t}$ functors (see [20] for a discussion of this condition).

1.2 Part II

In Part II we then apply the theory developed in Part I to the specific case of functors between categories of the form $\text{Alg}_\mathcal{O}$.

Chapter 8 introduces the basic necessary definitions and notations about symmetric spectra $Sp^X$, model structures in that category, operads and algebras over them, and model structures in the categories of algebras.

Chapter 9 then begins the task of understanding (finitary) homogeneous functors between algebra categories by understanding their stabilization. The results here are Theorem 9.3 and its Corollary 9.8 showing that the stabilization of $\text{Alg}_\mathcal{O}$ is the module category $\text{Mod}_\mathcal{O}(1)$.

Chapter 10 completes the task of classifying (finitary) homogeneous functors between algebra categories by classifying homogeneous functors between their stabilizations, the module categories. The result here is Theorem 10.2, which generalizes the well known characterization in the case of spectra.

Finally, chapter 11 concludes the paper by establishing our main results about Goodwillie calculus in the $\text{Alg}_\mathcal{O}$ categories. In section 11.1 we finally establish Theorem 11.3, saying that the Goodwillie tower for $\text{Id}_{\text{Alg}_\mathcal{O}}$ is indeed the homotopy completion tower studied in [17]. Section 11.2 establishes Theorem 11.8, which is probably more surprising. It roughly says that, as far as (finitary) $n$-excisive functors are concerned, Goodwillie calculus can not distinguish the category $\text{Alg}_\mathcal{O}$ from $\text{Alg}_{\mathcal{O}_{\leq n}}$. Lastly, section 11.3 establishes Theorem 11.12, which shows the $\text{Alg}_\mathcal{O}$ categories satisfy at least a weak analogue of the chain rule from [1].
Additionally, there is also Appendix A, which is dedicated to proving Proposition A.8, a basic result about the positive flat stable model structure on $Sp^\Sigma$, and which is used in the proof of Theorem 9.3. Queerly, though this result can be viewed as one of the main reasons for why algebraic structures in $Sp^\Sigma$ admit projective model structures, the treatments in the literature that the author is aware of (such as [30], [15]) all rely instead on (sometimes diverse) immediate consequences of Proposition A.8 rather than the fuller result, which is more convenient in our context.
Part I

General Goodwillie calculus
Chapter 2

Setup for homotopy calculus

This chapter lists assumptions that appear repeatedly in the paper, and deduces some of their basic consequences.

Pervasive throughout the paper is the assumption that the model categories used are cofibrantly generated. We recall the definition (and set notation) in 2.1, along with some results about the functoriality of hocolimits and holimits in such model categories.

2.2 deals with the general assumptions made on functors between model categories.

2.3 deals with the further assumptions made when the model categories are also assumed simplicial.

2.4 deals with detailing the notions of spectra in a general simplicial model category that are used in chapter 7.

Finally, 2.5 deals with showing that our definitions are nicely compatible with replacing one of the model categories involved by a Quillen equivalent one.

2.1 Cofibrant generation

Definition 2.1. A model category $\mathcal{C}$ is said to be cofibrantly generated if there exist sets of maps $I$ and $J$, and regular cardinal $\kappa$ such that

1. The domains of $I$ are small relative to $I$ with respect to $\kappa$. 
2. The trivial fibrations are precisely the maps with the RLP$^1$ with respect to $I$.

3. The domains of $J$ are small relative to $J$ with respect to $\kappa$.

4. The trivial fibrations are precisely the maps with the RLP with respect to $J$.

5. Both the domains and targets of $I$ and $J$ are small relative to $I$ with respect to $\kappa$.

Remark 2.2. We refer to [18], section 10, for the precise definitions of “smallness relative to”, but we give a sketch definition here:

$X$ is said to be $\kappa$-small relative to $I$ if for any $\lambda \geq \kappa$ regular cardinal$^2$ and $\lambda$-sequence$^3$

$$Y_0 \to Y_1 \to Y_2 \to \cdots \to Y_\beta \to Y_{\beta+1} \to \cdots (\beta < \lambda)$$

where all maps $Y_\beta \to Y_{\beta+1}$ are relative $I$-cell complexes$^4$, the natural map

$$\text{colim}_{\beta<\lambda} \text{Hom}(X, Y_\beta) \xrightarrow{\sim} \text{Hom}(X, \text{colim}_{\beta<\lambda} Y_\beta)$$

is an equivalence.

Remark 2.3. Though the first condition in our definition is a particular case of the last, we choose to include it so as to draw attention to the fact that that condition is not present in the definition of cofibrantly generated model category in [18].

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Our goal is to implement Goodwillie’s construction of the polynomial approximations in this general context. Cofibrant generation allows us to do so, thanks to the following results:

$^1$Right Lifting Property.

$^2$Here, as in [18], cardinals are to be viewed as ordinals by selecting the initial ordinal with that cardinality.

$^3$We recall that a functor $Y_\lambda : \lambda \to C$ is called a $\lambda$-sequence if it is “well behaved” on limit ordinals, i.e. $Y_\alpha = \text{colim}_{\beta<\alpha} Y_\beta$ for $\alpha < \gamma$ a limit ordinal.

$^4$The more “simplicially minded” reader can likely ignore this, as in those contexts objects are often small with respect to all maps.
Proposition 2.5. Suppose $I$ is a set of maps whose domains are small relative to $I$ with respect to a regular cardinal $\kappa$.

Then, letting $\mathcal{C}^I$ denote the arrow category, there exists a functorial factorization

$$\mathcal{C}^I \rightarrow \mathcal{C}^I \times \mathcal{C}^I$$

factoring any map into a relative $I$-cell complex followed by an $I$-injective.

Proof. This is just Quillen's well known small object argument. A reference for this is section 2.1.2 of [19], or most other basic treatments of model categories. \(\square\)

Note that, according to the previous result, in any general cofibrantly generated model category $\mathcal{C}$, one can construct functorial cofibrant and fibrant replacement functors. We will denote fixed such functors by $Q$ and $R$ in the remainder of this paper.

We will also need to make special note of particularly nice types of diagram categories that feature prominently in Goodwillie calculus (see [2] for the following definitions):

Definition 2.6. A small category $A$ is called a direct category if there is an identity reflecting functor $A \rightarrow \lambda$, where $\lambda$ is an ordinal category.

A small category $A$ is called an inverse category if $A^{op}$ is a direct category.

It is well known that when $A$ is direct a projective model structure exists on any $\mathcal{C}^A$, for $\mathcal{C}$ any model category (check [2] for the characterization of this model structure). Dually, when $A$ is an inverse category, $\mathcal{C}^A$ always has an injective model structure. We will use the following basic properties about such model structures:

Proposition 2.7. Let $A$ be a direct category and $A'$ an initial subcategory (i.e., such that if $x \in A'$ then all maps $x' \rightarrow x$ are in $A'$). Note that $A'$ is then itself a direct category. Then if $F: A \rightarrow C$ is projective cofibrant, then so is the restriction $F|_{A'}$. Furthermore, the analogous result holds for cofibrations $F \rightarrow F'$.
Let $A, B$ be direct categories. Then $A \times B$ is a direct category and, if $F: A \times B \to C$ is projective cofibrant, then so are the restrictions $F(a, \bullet), F(\bullet, b)$ for any $a \in A, b \in B$. Furthermore, the analogous result holds for cofibrations $F \to F'$.

Proof. The first part is straightforward from the definition of the model structure, as the conditions required for $F|_{A'} \to F'|_{A'}$ to be a cofibration are a subset of the conditions for $F \to F'$ to be one.

As for the second part, it follows from noticing that in these projective model structures cofibrations are in particular pointwise cofibrations, and by further viewing the projective model structure on $C^{A \times B}$ as the projective model structure in $(C^A)^B$ over the projective model structure in $C^A$.

Proposition 2.8. Let $A$ be a direct category, and $C$ a model category which admits functorial factorizations as those described in Proposition 2.5. Then the projective model structure on $C^A$ also admits such functorial factorizations, which can be constructed using those in $C$.

Furthermore, if $C$ is assumed simplicial, and with simplicial factorization functors, then $C^A$ is itself a simplicial model category and the constructed factorizations are again simplicial functors.

Proof. This statement is a variation of Proposition 6.3 of [26], and the proof is essentially the same.

Proposition 2.9. Let $C$ be a cofibrantly generated model category with respect to some regular cardinal $\kappa$, which is also assumed chosen larger than the cardinalities of $I, J$, and the set of arrows between objects in them. Then $\kappa$-filtered homotopy colimits (taken over the $\kappa$ ordinal category) commute with $\kappa$-small homotopy limits (taken over inverse categories).

Before giving the proof, we recall some definitions. For more details, see [27]. Given a functor

$$C \xrightarrow{F} D$$
between model categories \( C \) and \( D \), we define the derived functors \( \mathbb{L}F, \mathbb{R}F : Ho(C) \to Ho(D) \) by

\[
\mathbb{L}F = Ran_{\kappa C} \gamma_D \circ F, \quad \mathbb{R}F = Lan_{\kappa C} \gamma_D \circ F
\]

where \( \gamma_C : C \to Ho(C) \), \( \gamma_D : D \to Ho(D) \) are the standard maps.

It is then well known that, when \( F \) is a left Quillen (resp. right Quillen) functor, \( \mathbb{L}F \) (resp. \( \mathbb{R}F \)) is the functor induced by \( F \circ Q \) (resp. \( F \circ R \)).

The statement of 2.9 is then that the following diagram (where \( A \) is assumed the ordinal category \( \kappa \) and \( B \) assumed inverse and \( \kappa \)-small) commutes up to a natural isomorphism.

\[
\begin{array}{c}
Ho(C^{A \times B}) \xrightarrow{\mathbb{R} \lim_{A}} Ho(C^A) \\
\downarrow \mathbb{L} \lim_{A} \downarrow \mathbb{L} \lim_{A}
\end{array} \quad (2.10)
\]

\[
Ho(C^B) \xrightarrow{\mathbb{R} \lim_{B}} Ho(C)
\]

Note, however, that we do not claim the natural isomorphism supplied in the proof can be chosen canonically (this should be of contrasted with the canonical map \( \lim_{A} \lim_{B} \to \lim_{B} \lim_{A} \)), as it depends at this level on the choice of bifibrant replacements used.

**Proof of 2.9.** The proof will be an adaptation of Schwede’s argument in [29], Lemma 1.3.2.

First notice that \( \mathbb{L} \lim_{A} \) can be computed directly by applying \( \lim_{A} \) whenever when one is dealing with a diagram \( X_{a,b} \) such that \( X_{a,b} \) is projective cofibrant for each \( b \). Dually \( \mathbb{R} \lim_{B} \) is computed directly by applying \( \lim_{B} \) when \( X_{a,*} \) is injective fibrant for all \( a \). But now notice that any general \( X_{a,b} \) can be functorially replaced modulo weak equivalences by a \( \tilde{X}_{a,b} \) satisfying both of those properties\(^5\). One hence needs only deal with representatives \( X_{a,b} \) for the elements of \( Ho(C^{A \times B}) \) that simultaneously satisfy those properties, and we always assume this in the remainder of the proof.

The assumption in the previous paragraph hence implies that in (2.10) the upper

\(^5\)To see this, consider any of the two natural model structures on \( C^{A \times B} \), either “projective after injective” or “injective after projective”. Now notice that functorial bifibrant replacements in any of those model structures satisfy the desired properties.
horizontal and left vertical maps can be computed directly. We next claim that so can the bottom horizontal map. More explicitly, the claim is that the assumption implies that \( \lim_{a} X_{a,b} \) is still injective fibrant. Since \( B \) be an inverse category, this amounts to showing that the maps \( \lim_{a} X_{a,b} \rightarrow \lim_{b \rightarrow b', f \neq id} \lim_{a} X_{a,b'} \) are fibrations, and this in turn amounts to checking the RLP of these maps with respect to the generating trivial cofibrations \( J \). Consider hence a map \( j \rightarrow j' \) in \( J \) together with some commutative square into the previous map, which corresponds to maps \( j \rightarrow \lim_{a} X_{a,b} \) and (for each \( b \downarrow b' \)) \( j' \rightarrow \lim_{a} X_{a,b'} \). Compactness then implies the existence of factorizations \( j \rightarrow X_{a,b}, j' \rightarrow X_{a,b',e} \). Here a priori \( a \) and the \( a_{b \rightarrow b'} \) have different values, but since the ordinal category \( \kappa \) is \( \kappa \)-filtered\(^6\) we can assume a fixed value \( a_{0} \) is chosen. We would now like to say that these factorizations amount to a factorization of the initial square by a square from \( j \rightarrow j' \) to \( X_{a_{0},b} \rightarrow \lim_{b \rightarrow b', f \neq id} X_{a_{0},b'} \). Again this a priori needs not be true, but using again the fact that \( \kappa \) is \( \kappa \)-filtered this can be guaranteed by choosing\(^7\) an appropriate \( a_{1} \geq a_{0} \). It is now obvious that the lifting property follows.

We now know that (under our assumption), all maps in (2.10) are computed directly except for the right vertical map. To compute this last map one needs hence to choose a (functorial) projective cofibrant replacement \( X \) of \( \lim_{b} X_{a,b} \).

Notice that one then has a natural composite

\[
\lim_{a} X \rightarrow \lim_{a} \lim_{b} X_{a,b} \rightarrow \lim_{b} \lim_{a} X_{a,b},
\]

and that we will be done if we prove that this map is a weak equivalence. In fact, we prove it is actually a trivial fibration, and this again amounts to verifying a RLP against the generating cofibrations \( I \). Hence consider a commutative square from \( i \rightarrow i' \) to the map above. Repeating the argument of the previous paragraph one sees that that square factors through \( \lim_{a_{1}} X_{a_{1},b} \rightarrow \lim_{b} X_{a_{1},b} \) for some \( a_{1} \), and since by construction this map is a trivial fibration, the proof is concluded.

\(^{6}\)That is to say, any \( \kappa \)-small subcategory has a cone over it, or, put more simply in this case, any subset of cardinality less than \( \kappa \) has a majorant.

\(^{7}\)Informally speaking the argument here is that the cardinality of the commutativity conditions to be imposed is less then \( \kappa \).
Remark 2.11. Notice that the previous result also follows (with no alterations to the proof) for other $\kappa$-filtered shapes of the diagram category $A$ provided one knows the domains and codomains of $I$ and $J$ satisfy compactness with respect to such shapes.

2.2 Functors

We now indicate our assumptions on functors. Throughout we will denote

$$F: C \to D$$

a functor between a simplicial model category $C$ and a model category $D$ (both still assumed cofibrantly generated), which is typically assumed left homotopical, i.e., such that $F \circ Q$ is homotopical\(^8\). The reason for this slight generalization is that, even for straight up homotopical functors, some of our constructions require precomposition with $Q$ anyway.

We do however notice that, when we allow $F$ to be merely left homotopical, our constructions will be presenting the Goodwillie tower of $F \circ Q$, not of $F$.

2.3 Simplicial categories

In the situation in which we are dealing with simplicial model categories $C, D$, we will sometimes (but not always) require the functor $F: C \to D$ between them to be a simplicial functor. When that is the case we would also like to know that our constructions of the Goodwillie tower still yield simplicial functors. Since cofibrant replacements are necessary when making those constructions, we need the following result, which is proposition 6.3 in [26]:

**Proposition 2.12.** Suppose $C$ a cofibrantly generated simplicial model category. Then

\(^8\)Recall that a functor is called homotopical if it preserves w.e.s.
there exist simplicially functorial factorizations

\[ \mathcal{C}^1 \to \mathcal{C}^1 \times \mathcal{C}^1 \]

factoring any map into a trivial cofibration followed by fibration, or as a cofibration followed by trivial fibration.

In particular, \( \mathcal{C} \) has simplicial cofibrant and fibrant replacement functors.

When dealing with simplicial model categories we will hence always further assume that the chosen replacement functors \( Q, R \) are taken to be simplicial.

**Remark 2.13.** Note that the definition of “cofibrantly generated” given in this paper is slightly more demanding than that used in [26].

### 2.4 Spectra on model categories

We now explain the notions of spectra that will be used in Chapter 7.

Assume \( \mathcal{C} \) a pointed simplicial model category, so that it is also in particular tensored over \( SSet_* \). Denote this tensoring by \( \wedge \).

Following Hovey in [20] one can then define Bousfield-Friedlander type spectra on \( \mathcal{C} \) as sequences \( (X_i)_{i \in \mathbb{N}} \), plus structure maps

\[ S^1 \wedge X_n \to X_{n+1}, \]

where \( S^1 \) is the standard simplicial circle \( \Delta^1/\partial \Delta^1 \).

We denote the category of such spectra by \( Sp(\mathcal{C}, S^1) \), or just \( Sp(\mathcal{C}) \) when no confusion should arise.

Then when \( \mathcal{C} \) is cofibrantly generated one has that \( Sp(\mathcal{C}) \) always has a cofibrantly generated projective model structure (i.e. a model structure where weak equivalences and fibrations are defined levelwise on the underlying sequences of the spectra), with
generating cofibrations and generating trivial cofibrations given respectively as

\[ I_{proj} = \bigcup_n F_nI, \quad J_{proj} = \bigcup_n F_nJ, \]

where \( I, J \) denote respectively the generating cofibrations and generating trivial cofibrations of \( C \), and \( F_n \) is the level \( n \) suspension spectra functor\(^9\).

To get the correct notion of stable w.e.'s on spectra one then needs to left Bousfield localize with respect to the maps

\[ S = \{ F_{n+1}(S^1 \wedge A_i) \to F_nA_i) \} \]

where \( A_i \) ranges over the (cofibrant replacements of, when necessary) domains and codomains of the maps in \( I \).

**Definition 2.14.** Let \( C \) be a cofibrantly generated pointed simplicial model category.

Then we define the stable model structure on \( Sp(C) \) as the (cofibrantly generated) left Bousfield localization with respect to \( S \), should it exist.

**Remark 2.15.** For a detailed treatment of left Bousfield localizations, check [18], chapters 3 and 4.

Notice that we crucially do not want to just assume \( C \) is left proper cellular, as in [20], where it is shown that that is sufficient for the model structure above to exist.

This is because left properness generally fails in the main examples we are interested in, those of algebras over an operad in symmetric spectra. This turns out to be ok, however, both because one can show directly in those cases that the description above produces a model category, and because even though Hovey requires left properness in many of his statements in [20], in the ones we shall need he is only really requiring the existence of the model structure just described.

We now adapt the treatment in [18] in order to show that one has a nice localization functor in \( Sp(C) \).

\(^9\) \( F_n \) can also be described as the left adjoint to the \( (X_i) \mapsto X_n \) functor. Explicitly \( (F_nX)_m = (S^1)^{m-n} \wedge X \), where one sets \( (S^1)^{(m-n)} = * \) when \( m - n < 0 \).
First notice that the projective model structure on $Sp(C)$ is simplicial (just apply the simplicial constructions levelwise).

Then set

$$\tilde{S} = \{ F_{n+1}(S^1 \wedge A_i) \to \tilde{F}_nA_i \}$$

the set of maps obtained by taking the cofibration factor of those in $S$ (i.e., these are the cofibrations appearing in the canonically chosen factorizations of those maps as a cofibration followed by a trivial fibration).

Further set $\tilde{S} = J_{proj} \cup \{ F_{n+1}(S^1 \wedge A_i) \otimes \Delta_m \coprod \{ F_{n+1}(S^1 \wedge A_i) \otimes \delta \Delta_m \to \tilde{F}_nA_i \otimes \Delta_m \} \}

Proposition 2.16. Let $X \in Sp(C)$. Then the stable localization of $X$ can be obtained by performing the Quillen small object argument on the map $X \to *$ with respect to the set of maps $\tilde{S}$.

Proof. This is just Proposition 4.2.4 from [18]. It should be noted that though the statement asks for the category to be left proper cellular none of those properties is used, and indeed all one needs is cofibrant generation (in order to run the small object argument).

We will generally denote by

$$\Sigma^\infty: C \rightleftarrows Sp(C): \Omega^\infty$$

the standard adjoint functors.

We will also require an additional niceness property about the model structure on $Sp(C)$, namely, we shall require that the stable equivalences be detectable via the $\tilde{\Omega}^{\infty-n}$ functors (these are the derived functors giving the $n$-th space of the spectrum after being made into an omega spectrum).

In [20] Hovey determines conditions under which this follows (and, importantly, those proofs do not require left properness):
Theorem 2.17 (Hovey). Suppose \( C \) is a pointed simplicial almost finitely generated model category. Suppose further that in \( C \) sequential colimits preserve finite products, and that \( \text{Map}(S^1, -) \) preserves sequential colimits.

Then stable equivalences in \( \text{Sp}(C) \) are detected by the \( \Omega^{\infty-n} \) functors.

Proof. This is just a particular case of Theorem 4.9 in [20].

Remark 2.18. We shall also need to make use of slightly more general kinds of spectra. Namely, we shall denote by \( \text{Sp}(C, A) \) the spectra in \( C \) constructed with respect to the endofunctor \( A \wedge \bullet \) (in the language of Hovey). In the particular case that \( A = (S^1)^n \) these can be thought of as spectra that only include the \( ni \)-th spaces.

Notice that the obvious analog of 2.17 still holds.

2.5 Composing functors with Quillen equivalences

Suppose we are given a Quillen equivalence

\[
F : C \rightleftarrows D : G.
\]

In this section we show that, up to a zig zag of equivalences of functors, studying homotopy functors into \( C \) is the same as studying homotopy functors into \( D \) (the analogous result for functors from \( C, D \) also holds, but we won’t be using it). This is the content of the following results.

Proposition 2.19. Suppose

\[
F : C \rightleftarrows D : G
\]

a Quillen equivalence (between cofibrantly generated model categories). Then \( F \circ Q_C \) and \( G \circ R_D \) are homotopy inverses, i.e., there are zig zags of w.e.s \( G \circ R_D \circ F \circ Q_C \sim id_C \), \( F \circ Q_C \circ G \circ R_D \sim id_D \).

Proof. By reasons of symmetry we need only to construct the zig zag \( G \circ R_D \circ F \circ Q_C \sim \)
\[ id_C \rightarrow \mathcal{Q}_C \rightarrow G \circ F \circ \mathcal{Q}_C \rightarrow G \circ R_D \circ F \circ \mathcal{Q}_C \]

where the natural transformation \( \mathcal{Q}_C \rightarrow id_C \) is of course given by w.e.s, while the composite natural transformation \( \mathcal{Q}_C \rightarrow G \circ R_D \circ F \circ \mathcal{Q}_C \) is given by w.e.s since it is adjoint to the obvious maps\(^{10}\) \( F \circ \mathcal{Q}_C \rightarrow R_D \circ F \circ \mathcal{Q}_C \).

\[ \square \]

**Corollary 2.20.** Suppose given a Quillen equivalence

\[ F: \mathcal{C} \rightleftharpoons \mathcal{D}: G \]

as before and \((E, W_E)\) a category with w.e.s. Then, when defined, the homotopy categories of homotopy functors \( Ho^h(E, \mathcal{C}) \) and \( Ho^h(E, \mathcal{D}) \) are equivalent.

\(^{10}\)Recall that a Quillen adjunction is a Quillen equivalence iff a map (with \( X \) cofibrant and \( Y \) fibrant) \( F(X) \rightarrow Y \) is a w.e. iff the adjoint map \( X \rightarrow R(Y) \) is.
Chapter 3

Goodwillie calculus: polynomial functors

In this chapter we define the generalized concepts of polynomial functor and construct the universal polynomial approximations $P_n F$ to a homotopy functor $F$. All definition and results are reasonably straightforward generalizations of those of Goodwillie in [14].

3.1 Cartesian cubes and total homotopy fibers.

We first set some notation.

For $S$ a finite set, denote by $\mathcal{P}(S)$ its poset of subsets, and additionally define $\mathcal{P}_i(S) = \{ A \in \mathcal{P}(A) : |A| \geq i \}$. Of particular importance will be $\mathcal{P}_1(S) = \mathcal{P}(S) - \{ \emptyset \}$.\footnote{This does NOT match Goodwillie's notation.} Additionally we shall also use the notation $\mathcal{P}_{\leq i}(S) = \{ A \in \mathcal{P}(A) : |A| \leq i \}$ for the "dual" subposets.

Notice that all of these categories are have finite nerves, and hence are both direct and inverse categories (Definition 2.6).

Definition 3.1. An $n$-cube in a category $C$ is a functor $\lambda_n : \mathcal{P}(S) \to C$, where $|S| = n$.\footnote{This does NOT match Goodwillie's notation.}
$\mathcal{X}$ is said to be **hocartesian** if the canonical map

$$\text{hocmp}(\mathcal{X}): \mathcal{X}_0 \to \text{holim}_{T \in \mathcal{P}_1(S)} \mathcal{X}_T$$

is a w.e.\(^2\). We call this map the **homotopy comparison map**, and denote it by $\text{hocmp}(\mathcal{X})$, or just $\text{hocmp}$ when this would not cause confusion.

Furthermore, $\mathcal{X}$ is said to be **strongly hocartesian** if the restrictions $\mathcal{X}|_{\mathcal{P}_1(T)}$ are hocartesian for any subset $T \subset S$ with $|T| \geq 2$.

There are also obvious dual definitions of **hococartesian** and **strongly hococartesian** $n$-cubes.

**Remark 3.2.** Note that hocartesianness is a homotopical property of the cube $\mathcal{X}$, as so is the map $\mathcal{X}_0 \to \text{holim}_{T \in \mathcal{P}_1(S)} \mathcal{X}_T$ (which we regard as a map in $\text{Ho}(\mathcal{C})$). Furthermore, that map can be constructed in the following way: take $\tilde{\mathcal{X}}$ an injective fibrant replacement of $\mathcal{X}$. Then $\tilde{\mathcal{X}}|_{\mathcal{P}_1(S)}$ is itself injective fibrant, since $\mathcal{P}_1(S)$ is a terminal subset of $\mathcal{P}(S)$ (see Lemma 2.7), so that $\text{lim}_{\mathcal{P}_1(S)} \tilde{\mathcal{X}}$ is indeed a holim, and the intended map is then given by $\tilde{\mathcal{X}}_0 \to \text{lim}_{T \in \mathcal{P}_1(S)} \tilde{\mathcal{X}}_T$.

**Remark 3.3.** Notice that any strongly hocartesian cube $\mathcal{X}$ is determined by its restriction to $\mathcal{P}_{n-1}(S)$. Indeed, consider $\tilde{\mathcal{X}}$ its injective fibrant replacement. Then its restriction $\tilde{\mathcal{X}}|_{\mathcal{P}_{n-1}(S)}$ is itself injective fibrant, and it trivially follows that its reextension (i.e., the right Kan extension) $\tilde{\mathcal{X}}'$ to $\mathcal{P}(S)$ is itself injective fibrant and the unit map $\tilde{\mathcal{X}} \to \tilde{\mathcal{X}}'$ is a pointwise equivalence.

We now turn to the issue of defining the total fiber of a cube, and proving its iterative properties. Suppose $\mathcal{C}$ is now a pointed model category with zero object $\ast$.

**Definition 3.4.** Consider the Quillen adjunction

$$\mathcal{C} \rightleftarrows \mathcal{C}^\perp$$

\(^2\)We now use the traditional notations $\text{holim}$ and $\text{hocolim}$ for the derived functors $R\text{lim}$ and $L\text{lim}$ introduced in 2.1.
(where \( C^4 \) has the injective model structure) with left adjoint \( X \mapsto (X \to *) \) and right adjoint \( (X \to Y) \mapsto \text{fiber}(X \to Y) \).

We then denote by hofiber the derived functor \( R \text{fiber} : \text{Ho}(C^4) \to \text{Ho}(C) \).

More generally, consider the Quillen adjunctions:

\[
C \leftrightarrows C^{P(S)}
\]

(where \( C^{P(S)} \) has the injective model structure) with left adjoint \( X \mapsto (\emptyset \mapsto X; \text{else} \mapsto *) \) and right adjoint \( X \mapsto \text{fiber}(X_\emptyset \to \lim_{T \in P_1(S)} X_T) \).

We now denote by tothofiber the derived functor \( \text{Ho}(C^{P(S)}) \to \text{Ho}(C) \).

**Remark 3.5.** Though not immediately obvious, our definition of hofiber does match the obvious alternate candidate definition: \( \text{holim}(X \to Y \leftarrow *) \). This follows, for instance, from Proposition A.2.4.4 in [23]. That the analogous result also holds for the higher dimensional variants can be proven by combining the following result with an induction argument.

We now show that the tothofiber can be computed iteratively.

**Proposition 3.6. Iterative definition of the total hofiber**

The following diagram commutes up to a natural isomorphism:

\[
\begin{array}{ccc}
\text{Ho}(C^{P(S \cup S' \cup W)}) & \xrightarrow{\text{hofiber}_{S'}} & \text{Ho}(C^{P(S' \cup W)}) \\
\downarrow & & \downarrow \\
\text{Ho}(C^{P(W)}) & \xrightarrow{\text{hofiber}_{S'}} & \text{Ho}(C^{P(W)})
\end{array}
\] \hspace{1cm} (3.7)

**Proof.** Notice first that we have obvious Quillen adjunctions (for the injective model structures)

\[
C^{P(W)} \leftrightarrows C^{P(S \cup W)}
\]

with the right adjoint given by \( \text{totfiber}_S \), the total fiber in the \( S \) direction (that this is a Quillen adjunction is clear from looking at the left adjoints).

It then follows that if we compute the paths in (3.7) by first performing an injective fibrant replacement, then no further replacements are needed, and one is left
with proving that \( \text{fibers}_{\text{SUS}'} = \text{fibers}_S \circ \text{fibers}_S \). But this is obvious: mapping into \( \text{fibers}_{\text{SUS}'}(\mathcal{X}) \) is the same as giving a map to \( \mathcal{X}_0 \) which restricts to the (uniquely determined) zero maps on the other \( \mathcal{X}_T \), and \( \text{fibers}_S \circ \text{fibers}_S(\mathcal{X}) \) clearly satisfies that same universal property.

\[ \square \]

The previous result can also be viewed as a consequence/particular case of the following:

**Proposition 3.8. Iterative definition of the homotopy comparison map**

The following diagram commutes up to a natural isomorphism:

\[
\begin{array}{ccc}
\text{Ho}(\mathcal{CP}(\text{SUS}'\cup W)) & \xrightarrow{\text{emp}_S} & \text{Ho}(\mathcal{CP}(\{\ast\} \cup \text{SUS}'\cup W)) \\
\downarrow \text{emp}_{\text{SUS}'} & & \downarrow \text{emp}_{\{\ast\} \cup \text{SUS}'} \\
\text{Ho}(\mathcal{CP}(\{\ast\} \cup \text{SUS}'\cup W)) & & \\
\end{array}
\]

\[ (3.9) \]

**Proof.** As in the previous proposition we have obvious Quillen adjunctions (for the injective model structures)

\[
\mathcal{CP}(\{\ast\} \cup W) \simeq \mathcal{CP}(\text{SUW})
\]

with the right adjoint given by \( \text{emp}_S \), so again it suffices to compare the results obtained by first performing an injective fibrant replacement and then applying the levelwise constructions. Again it is easy to check that \( \text{emp}_{\text{SUS}'} = \text{emp}_{\{\ast\} \cup \text{SUS}'} \circ \text{emp}_S \) by appealing to the universal property (this is probably clearer by looking at the left adjoints, which are manifestly equal).

\[ \square \]

**Definition 3.10.** Let \( X \) be any object in \( \mathcal{C} \) a pointed simplicial cofibrantly generated model category. Recall that \( \mathcal{C} \) is then a \( \text{SSets}_* \)-model category.

Define \( \mathcal{C}(X) = X \wedge (\Delta^1, \{0\}) \).

There is a standard map \( X \xrightarrow{\epsilon_X} \mathcal{C}(X) \) induced by \( (\{0, 1\}, \{0\}) \to (\Delta^1, \{0\}) \).

**Remark 3.11.** Note that, when \( X \) is cofibrant, then \( \mathcal{C}(X) \sim \ast \), and the standard map \( X \to \mathcal{C}(X) \) is a cofibration.
Definition 3.12. Fix $S$ a finite set.

Denote by $\mathcal{X}_X$ the left Kan extension of the diagram (where in the diagram we have $|S|$ copies of $C(X)$)

$$
\begin{array}{c}
X \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
C(X) & C(X) & \ldots & C(X) & C(X)
\end{array}
$$

(3.13)

along the inclusion $P_{\leq 1}(S) \to P(S)$.

Remark 3.14. Notice that, when $X$ is cofibrant, then (3.13) is a cofibrant $P_{\leq 1}(S)$ diagram, so that $\mathcal{X}_X$ is a strongly cocartesian cube.

More generally, when $X$ not cofibrant, we will also think of $\mathcal{X}_{Q(X)}$ as the strongly cartesian cube associated to $X$.

Proposition 3.15. Suppose that $C$ is a pointed simplicial cofibrantly generated model category.

The functors $P(S) \times P(S) \to C$ given by $(U, V) \mapsto \mathcal{X}_{X_U}(V)$ and $(U, V) \mapsto \mathcal{X}_{X_V}(U)$ are naturally isomorphic. Furthermore, this natural isomorphism is also natural in $X$.

Proof. Since $C(X)$ commutes with colimits, and $\mathcal{X}_* \text{ is constructed via a left Kan extension of its values on } P_{\leq 1}$, the result will follow if we prove it for the restriction of the functors to $P_{\leq 1}(S) \times P_{\leq 1}(S)$.

This then amounts to producing an automorphism of $C(C(X))$ which swaps $c_{C(X)}$ and $C(c_X)$. But $C(C(X))$ is just $(X \wedge (\Delta^1, \{0\})) \wedge (\Delta^1, \{0\})$, and the required maps induced by the inclusions $(\{0, 1\}, \{0\}) \to (\Delta^1, \{0\})$ into each of the product terms, so the required automorphism just follows from the symmetry isomorphisms for the monoidal structure in $SSet_*$. 

\[ \Box \]
3.2 Excisive functors

We can now finally define polynomial/excisive functors.

**Definition 3.16.** A homotopical functor $F: C \to D$ is said to be $d$-excisive if for any $X$ a strongly hococartesian $(d+1)$-cube in $C$ we have that $F(X)$ is a hocartesian cube in $D$.

**Remark 3.17.** Notice that, since hocartesianness and strong hococartesianness of cubes are homotopical properties, the excisiveness of an homotopy functor $F$ is entirely determined by looking at the functors

$$Ho(C^P(S)) \xrightarrow{Ho(F^P(S))} Ho(D^P(S))$$

which we denote by $hF_S$, for short.

We now prove some basic properties of excisive functors.

**Proposition 3.18.** If $F$ is $d$-excisive, then it is also $d'$-excisive for $d' \geq d$.

**Proof.** It suffices to prove that $d$-excisiveness implies $d+1$-excisiveness. Write a set $\bar{S} = S \amalg \{\ast\}$ for a set of cardinality $d + 1$, so that an $\bar{S}$-cube $X$ can be viewed as a map of $S$-cubes $X^b \to X^t$. By definition $X^b$ and $X^t$ are strongly hocartesian, so that by hypothesis $F(X^b)$ and $F(X^t)$ are cartesian. But then so must be $X$ by applying Proposition 3.8. $\square$

**Definition 3.19.** Suppose $C$ is pointed. A sequence of functors $F \to G \to H$ is a hofiber sequence if the composite is the $\ast$ functor, and $F$ is pointwise the homotopy fiber of $G \to H$. Or in other words, if the squares

$$\begin{align*}
F(c) &\xrightarrow{} \ast \\
\downarrow & & \downarrow \\
G(c) &\xrightarrow{} H(c)
\end{align*}
$$

are hocartesian.
**Remark 3.21.** Given a hofiber sequence $F \to G \to H$, consider

$$
\begin{array}{ccc}
F & \rightarrow & G \\
\downarrow & & \downarrow \\
\bar{F} & \rightarrow & \bar{G} \\
\end{array}
$$

(3.22)

where $\bar{G} \to \bar{H}$ is a fibrant replacement (pointwise in the arrow category) of $G \to H$ and $\bar{F} = \text{fiber}(\bar{G} \to \bar{H})$. It then follows it must be $F \tilde{\twoheadrightarrow} \bar{F}$, so that one concludes that the hofiber of a map of functors is well defined up to a zig zag of natural w.e.s.

**Proposition 3.23.** If $F \to G \to H$ is a hofiber sequence of homotopy functors with $G$ and $H$ both $d$-excisive, then so is $F$.

**Proof.** Given a $S$-cube $\mathcal{X}$, form the $S \amalg \{\ast\}$-cube $G(\mathcal{X}) \to H(\mathcal{X})$.

Now consider the following diagram:

$$
\begin{array}{ccc}
Ho(D^{P(S\amalg \{\ast\})}) & \xrightarrow{\text{cmp}_{S}} & Ho(D^{P(\{\ast, \ast\})}) \\
\downarrow_{\text{hofiber}_{\ast}} & & \downarrow_{\text{hofiber}_{\ast}} \\
Ho(D^{P(S)}) & \xrightarrow{\text{cmp}_{S}} & Ho(D^{P(\{\ast\})})
\end{array}
$$

(3.24)

By repeating the techniques of the proofs of Propositions 3.7 and 3.9 this diagram commutes up to natural isomorphism: namely, it suffices to perform the constructions by first choosing an injective fibrant replacement and then checking that $\text{cmp}_{S} \circ \text{fiber}_{\ast} = \text{fiber}_{\ast} \circ \text{cmp}_{S}$.

What we want to prove is then that $\text{cmp}_{S} F(\mathcal{X})$ is a w.e.. Since

$$
F(\mathcal{X}) = \text{hofiber}_{\ast}(G(\mathcal{X}) \to H(\mathcal{X})),
$$

this follows if we show $\text{hofiber}_{\ast}(\text{cmp}_{S}(G(\mathcal{X})) \to H(\mathcal{X}))$ is a w.e.. But this is clear since, by the $d$-excisiveness hypothesis, $\text{cmp}_{S}(G(\mathcal{X}))$ is a square for which two opposing arrows are w.e.s. □

**Lemma 3.25.** Suppose $F$ is $d$-excisive, and that $\mathcal{X}_{\ast}$ is a strongly hococartesian $e$-cube,
with \( e \geq d + 1 \). Then the map

\[
F(\mathcal{X}_0) \to \text{holim}_{T \in \mathcal{P}_{e-d}(S)} F(\mathcal{X}_T)
\]

is a w.e..

**Proof.** This being an homotopical property, we are free to replace \( F(\mathcal{X}) \) by an injective fibrant replacement \( \mathcal{Y} \), whose sub-cubes of dimension \( \geq d \) are all hocartesian (by the previous result).

Now let \( \bar{\mathcal{Y}} \) be the right Kan extension of the restriction \( \mathcal{Y}|_{\mathcal{P}_{e-d}(S)} \). \( \bar{\mathcal{Y}} \) is still fibrant since both adjoints preserve fibrant objects (by Proposition 2.7). The unit map \( \mathcal{Y} \to \bar{\mathcal{Y}} \) is tautologically a w.e. on the points in \( \mathcal{P}_{e-d}(S) \), but by induction it is also so on all other points, and the remaining maps can all be identified with \( \text{cmp}_S \) for some \( S \) with \( |S| \geq d \). But the map we want to check is a w.e. is precisely \( \mathcal{Y}_0 \to \bar{\mathcal{Y}}_0 \), hence we are done. \( \square \)

**Lemma 3.26.** Let \( A \) be a category covered by subcategories \( \{A_s\}_{s \in S} \) (i.e., any arrow in \( A \) lies in some \( A_s \)). Note the \( A_s \) are not assumed luff\(^3\)), and suppose given a functor \( F: A \to \mathcal{D} \).

Define an \( S \)-cube \( \mathcal{X} \) by \( T \mapsto \text{lim}(F|_{s \in T(A_s)}) \) (the empty intersection is taken to be all of \( A \)). Then \( \mathcal{X} \) is cartesian.

**Proof.** This is simply a matter of diagram chasing to check that the universal properties of the terms in \( \text{cmp}(\mathcal{X}) \) match. \( \square \)

**Proposition 3.27.** Let \( L: C^d \to \mathcal{D} \) be a homotopy functor which is 1-excisive (separately) in each variable, and denote by \( \Delta: \mathcal{C} \to C^d \) the diagonal functor. Then \( L \circ \Delta: \mathcal{C} \to \mathcal{D} \) is \( d \)-excisive.

**Proof.** Consider \( \mathcal{X} \) any strongly hococartesian \( S \)-cube, with \( |S| = d + 1 \).

Denote by \( L(\mathcal{X})^d \) the composite \( \mathcal{P}(S)^d \xrightarrow{\mathcal{X}^d} C^d \xrightarrow{L} \mathcal{D} \). We wish to show that \( L(\mathcal{X})^d \circ \Delta \) is a hocartesian cube, but by homotopy invariance this can equivalently

\(^3\)A subcategory of \( \mathcal{C} \) is called luff if it has the same set of objects.
be done after replacing $L(\mathcal{X})^d$ by an injective fibrant multicube $\mathcal{Y}$. By retracing the steps in the proof of Lemma 3.25 (and applying them inductively), one concludes that one might as well further replace $\mathcal{Y}$ by the right Kan extension $\mathcal{Y}$ of its restriction to $\mathcal{P}_d(S)^d$.

We know apply Lemma 3.26 to $\mathcal{P}_d(S)^d$: take $A_s = \{(T_1, \cdots, T_d) \in \mathcal{P}_d(S)^d : \forall i \in T_i\}$, or equivalently, $A_s$ is the undercategory $\langle \{s\}, \cdots, \{s\}\rangle \downarrow \mathcal{P}_d(S)^d$. It is easily seen that intersections $\bigcap_{s \in T}(A_s)$ with $T$ non empty are the undercategories $\langle T, \cdots, T\rangle \downarrow \mathcal{P}_d(S)^d$, and that, by a pigeonhole argument $\bigcup_{s \in S} A_s = \mathcal{P}_d(S)^d$.

Hence the cube constructed by Lemma 3.26 is precisely $\mathcal{Y} \circ \Delta$, and we will be done provided this cube is also fibrant (so that the limits are indeed holimits). This follows by checking that the adjunction $\mathcal{D}\mathcal{P}(S) \dashv \mathcal{D}\mathcal{P}(S)^d$ is Quillen, and this is in turn clear at the level of left adjoints since $\Delta : \mathcal{P}(S) \to \mathcal{P}(S)^d$ is inclusion of sublattices, so that $\text{Lan}_{\mathcal{H}}(y) = \mathcal{H}(\max(x \in \mathcal{P}(S) : x \leq y))$. \qed

**Remark 3.28.** Essentially the same proof applies to the case where $L$ is known to be $n_i \geq 1$ excisive in each variable, the result being that $L \circ \Delta$ is $n = n_1 + \cdots + n_d$ excisive. In that version of the proof $\mathcal{P}_d(S)^d$ is replaced by $\mathcal{P}_{n+1-n_1}(S) \times \cdots \times \mathcal{P}_{n+1-n_d}(S)$, everything else following similarly.
Chapter 4

The Taylor tower

We now turn to the task of defining the universal $n$-excisive approximations $P_n F$ to a homotopy functor $F$. Rather than receive a straight up functor $F \to P_n F$, we will have a zig zag $F \rightsquigarrow P_n F$ (with backward arrows being pointwise w.e.s). With this in mind, and to simplify notation, we assume from now on that $F$ takes values in cofibrant objects, that is to say, we assume $F$ is replaced by $Q_D \circ F$ if necessary.

**Definition 4.1.** Assume that $C$ is a pointed simplicial cofibrantly generated model category, and that $D$ is a cofibrantly generated model category. Let $X_{\cdot}$ denote the cubes of Definition 3.12.

Define $\tilde{t}_n F: F \to \tilde{T}_n F = \operatorname{holim}_{T \in \mathcal{P}_1(S)} F_{\cdot}(T)$ as the standard map to the holim, and $\tilde{t}_n F: F \rightsquigarrow \tilde{T}_n F$ as $Q(\tilde{t}_n F)$ the **canonical cofibrant factor** of $\tilde{t}_n F$.

We then define $t_n F: F \rightsquigarrow T_n F$ as the zig zag $F \rightsquigarrow F \circ Q \overset{\tilde{t}_n F \circ Q}{\rightsquigarrow} \tilde{T}_n F \circ Q = T_n F$.

**Remark 4.2.** Notice that the $\tilde{T}_n F$ defined above is not a homotopy functor, since $X_{\cdot}$ is not itself an homotopical construction. It is, however, a "left Quillen" functor, in the sense that $\tilde{T}_n F \circ Q$ is now a homotopy functor, since so is $X_{Q(\cdot)}$.

The need for the $\tilde{T}_n F$ construction is the following: ideally, one would like to just define $P_n F = \operatorname{hocolim}_\alpha T_n^\alpha F$. However, obtaining functorial cartesian cubes from an object $X$ seems to require making a cofibrant replacement first. Hence, so as not to make infinitely many cofibrant replacements when constructing the intended $P_n F$, we merely do it once and then iterate the intermediate $\tilde{T}_n F$ construction instead.
Definition 4.3. Let $\lambda$ be an ordinal. Define, by transfinite induction,

$$\tilde{T}_n^\lambda F = \tilde{T}_n(\tilde{T}_n^\lambda F),$$

if $\lambda = \tilde{\lambda} + 1$, a successor ordinal

$$= \text{colim}_{\beta < \lambda} \tilde{T}_n^\beta F,$$

if $\lambda$ a limit ordinal. \hfill (4.4)

Definition 4.5. Let $\kappa$ be the chosen regular cardinal with respect to which the target category $\mathcal{D}$ is cofibrantly generated.

Set $\tilde{P}_n F = T_n^\kappa F$, and denote by $\tilde{p}_n F: F \to \tilde{P}_n F$ the standard map.

We then define $p_n F: F \rightsquigarrow P_n F$ as the zig zag $F \overset{\sim}{\leftarrow} F \circ Q \overset{\tilde{p}_n F \circ Q}{\to} \tilde{P}_n F \circ Q = P_n F$.

Remark 4.6. Again notice that, just like in the case of $\tilde{T}_n F$, the functor $\tilde{P}_n F$ is not in general a homotopy functor. And, likewise, note that $P_n F$ is a homotopy functor, since so is $T_n F$ and the colimits used in the construction are actually hocolims (this is the reason to demand the map $F \to \tilde{T}_n F$ be a cofibration and that $F$ be pointwise cofibrant).

We now turn to the task of proving that $F \overset{p_n F}{\rightsquigarrow} P_n F$ is the universal zig zag from $F$ to an $n$-excisive functor. First we check that indeed $P_n F$ is $n$-excisive.

Lemma 4.7. Suppose $\mathcal{W}$ a cofibrant strongly hococartesian cube that is the left Kan extension of its restriction to $\mathcal{P}_{\leq 1}(S)$.

Then the map of cubes $F(\mathcal{W}) \overset{i_{\mathcal{W}} \circ \mathcal{W}}{\to} \tilde{T}_n F(\mathcal{W})$ factors through a hocartesian cube.

Proof. First notice that one needs only prove the result for $F \circ \mathcal{W} \overset{i_{\mathcal{F} \circ \mathcal{W}}}{\to} \tilde{T}_n F \circ \mathcal{W}$, as then the result for $\tilde{T}_n F(\mathcal{W})$ follows by applying $Q$, the cofibrant factor functor.

Now consider the following $\mathcal{P}(S) \times \mathcal{P}_{\leq 1}(S)$ diagrams in $\mathcal{C}$:

$$\mathcal{Y}_{U,V} = \mathcal{W}_U \text{ if } V = \emptyset$$

$$\text{colim}(\mathcal{W}_U \otimes \Delta^1) \leftarrow \mathcal{W}_U \otimes \{0\} \to \mathcal{W}_{U \cup V}) \text{ otherwise}$$

$$\mathcal{Z}_{U,V} = \mathcal{W}_U \text{ if } V = \emptyset$$

$$\mathcal{C}(\mathcal{W}_U) \text{ otherwise}$$

\hfill (4.8)
where maps in the $U$ direction are obvious, and those in the $V$ direction are induced by $\{1\} \to \Delta^1$. Furthermore, unwinding the definition of $C(\mathcal{W}_U) = \mathcal{W}_U \wedge (\Delta^1, \{0\}) = \text{colim}(\mathcal{W}_U \otimes \Delta^1 \leftarrow \mathcal{W}_U \otimes \{0\} \rightarrow \ast)$, we see there is an obvious functor $\mathcal{Y} \to \mathcal{Z}$. Notice also that, since the values of $\mathcal{W}_\ast$ are all cofibrant, the diagrams $\mathcal{Y}_{U_\ast}$ are $\mathcal{P}_{\leq 1}(S)$ cofibrant, and that there is an obvious map $\mathcal{Y}_{U_\ast} \to \mathcal{W}_{U_U \ast}$, where $\mathcal{W}_{U_U \ast}$ is itself $\mathcal{P}_{\leq 1}(S)$ cofibrant.

Now let $\mathcal{Y}, \mathcal{Z}$ be the $\mathcal{P}(S) \times \mathcal{P}(S)$ obtained by doing the left Kan extensions along the second variable. Notice that $\mathcal{Z}_{U,V} = \mathcal{X}_{\mathcal{W}_U}(V)$. By the above remarks about cofibrancy, we have a map $\mathcal{Y}_{U,V} \to \mathcal{W}_{U_U V}$ which is a pointwise equivalence (that the left Kan extension of the restriction $\mathcal{W}_{U_U V} \mid_{\mathcal{P}(S) \times \mathcal{P}_{\leq 1}(S)}$ is again $\mathcal{W}_{U_U V}$ is where we use that $\mathcal{W}$ was itself given by a left Kan extension).

The map we were trying to factor is precisely the full composite:

$$F(\mathcal{W}_\ast) \to \text{holim}_{V \in \mathcal{P}_1(S)} F(\mathcal{Y}_\ast_{\ast,V}) \to \text{holim}_{V \in \mathcal{P}_1(S)} F(\mathcal{Z}_\ast_{V})$$

so we will be done if we prove $\text{holim}_{V \in \mathcal{P}_1(S)} F(\mathcal{Y}_\ast_{V})$ a cartesian cube.

But comparison maps commute with $\text{holim}_{V \in \mathcal{P}_1(S)}$ (as holims are readily seen to commute with each other in the presence of injective model structures), so that it is enough to show that the $F(\mathcal{Y}_\ast_{V})$ are hocartesian cubes for each $V \neq \emptyset$, which by homotopy invariance is the same as showing that the $F(\mathcal{W}_\ast_{U,V})$ are hocartesian, which is now clear since in those cubes the maps in the $V$ directions are all w.e.s (and in fact isomorphisms).

\[\square\]

**Corollary 4.9.** For $F$ any homotopy functor, $P_n F$ is $n$-excisive.

**Proof.** First notice that, since $P_n F$ is a homotopy functor, we need only check that the cubes $\mathcal{W}$ described in Lemma 4.7 are sent to hocartesian cubes. Furthermore, since those cubes are pointwise cofibrant, we might as well just check hocartesianness for $\tilde{P}_n F \circ \mathcal{W}$.

But the result is now obvious from Lemma 4.7 and Proposition 2.9, as the latter shows that comparison maps commute with the hocolims in the $\tilde{P}_n$ construction, and,
by cofinality, those hocolims can be taken over the factoring strongly hocartesian cubes.

In order to check universality we'll also need the following:

Proposition 4.10.

- Let \( a \mapsto F_a \), where \( a \in A \) a finite category, be a compatible family of homotopy functors, and let \( F \) be another homotopy functor equipped with compatible natural transformations \( F \to F_a \) displaying \( F = \text{holim}_A F_a \). Then the natural transformations \( P_n F \to P_n F_a \) also display \( P_n F = \text{holim}_A P_n F_a \).

- Similarly, let \( \beta \mapsto F_\beta \), for \( \beta < \kappa \), a transfinite sequence of homotopy functors (\( \kappa \) assumed as in Proposition 2.9), and let \( F \) be another homotopy functor equipped with compatible natural transformations \( F_\beta \to F \) displaying \( F = \text{hocolim}_\beta F_\beta \). Then \( P_n F_\beta \to P_n F \) display \( P_n F = \text{hocolim}_\beta P_n F_\beta \).

Proof. For the first part, commutativity of holims with each other immediately gives that \( T_n F = \text{holim}_A T_n F_a \), so that the result then follows by applying transfinite induction and Proposition 2.9.

For the second part, first apply Proposition 2.9 to conclude the result for \( T_n F = \text{holim}_\beta T_n F_\beta \), and then transfinite induction plus the fact that hocolims commute with each other.

The first part of the previous result has the following consequence.

Corollary 4.11. Let \( F_T, T \in \mathcal{P}(S) \), be a cube of functors. Then \( P_n(\text{cmp}(F_*)) \) is w.e. to \( \text{cmp}(P_n F_*) \)

Theorem 4.12. Assume that the categories \( C, D \) are cofibrantly generated. Assume further that \( C \) is pointed simplicial, or a left Bousfield localization of a pointed simplicial model category.

Then \( F \overset{\eta}{\sim} P_n F \) exhibits \( P_n F \) as the universal \( n \)-excisive approximation to \( F \).
Proof. We first prove existence of factorization. Given a zig zag \( f : F \rightsquigarrow R \), with \( R \) assumed \( n \)-excisive, first replace it by \( f \circ Q : F \circ Q \rightsquigarrow R \circ Q \).

Given a weak map \( F(Q) \rightarrow R \), where \( R \) is assumed \( n \)-excisive, just consider the commutative diagram of weak maps:

\[
\begin{array}{ccc}
F \circ Q & \xrightarrow{f \circ Q} & R \circ Q \\
\downarrow \phi_n F \circ Q & & \downarrow \phi_n R \circ Q \\
P_n F & \xrightarrow{P_n f} & P_n R
\end{array}
\] (4.13)

Since \( \phi_n R \circ Q \) is clearly an equivalence when \( R \) is \( n \)-excisive (the \( \tilde{t}_n \) where constructed to ensure this, and \( \phi_n \) is merely their transfinite composition), we have that the zig zag \( P_n F \rightsquigarrow P_n R \rightsquigarrow R \circ Q \) provides a factorization \( \tilde{f} \) of \( f \circ Q \) through \( \phi_n F \circ Q \).

We now prove uniqueness of the factorization in the homotopy sense, i.e., that any two factorizations can be related by w.e.s.

Consider the diagram

\[
\begin{array}{ccc}
F & \xrightarrow{p_n F} & P_n F \\
\downarrow & & \downarrow \sim \\
P_n F & \xrightarrow{P_n (p_n F)} & P_n R
\end{array}
\] (4.14)

The factorization of \( \tilde{f} \circ p_n F \) constructed in the first part of the proof is given by the "lower followed by right" zig zag in the diagram. Which, given two of the vertical maps are known to be w.e.s, will follow if \( P_n (p_n F) \) is a zig zag of w.e.s.

This amounts to showing that \( P_n (p_n F \circ Q) \) is a w.e., and this, by the second part of Proposition 4.10, will follow if we show \( P_n (\tilde{t}_n F \circ Q) \) a w.e., which by Corollary 4.11 amounts to showing that \( P_n (F \circ X_Q(\bullet)) \) is a hocartesian cube of functors. And this is in turn is equivalent to showing that \( \tilde{P}_n (F \circ X_{\bullet}) \) gives a cartesian cube when applied to cofibrant objects.

But now notice that the cube \( \tilde{P}_n (F \circ X_{\bullet}) \) is isomorphic to the cube \( \tilde{P}_n F \circ X_{\bullet} \): indeed, that this is true at the \( \tilde{T}_n \) level follows from Proposition 3.15 (which actually establishes such isomorphisms for the \( \tilde{t}_n \) functor itself), and the claim for \( \tilde{P}_n \) follows by induction (with the additional remark that precomposing with \( X_{\bullet} \) obviously commutes
with limits of functors).

But now one finally concludes hocartesianness of \( \tilde{P}_n F \circ \mathcal{X}_* \) as an instance of \( P_n F \) being \( n \)-excisive, and the proof is complete.

\[
\square
\]

**Remark 4.15.** Much of the above follows just fine using alternate functorial definitions of the "cones" \( X \to C(X) \). However, such cones typically only satisfy a weak version of Proposition 3.15 (indeed, in the previous Theorem the extra conditions on \( C \) are merely to guarantee that proposition holds) and, as such, it seems technically hard to produce the corresponding "uniqueness" part of the previous proof.
Chapter 5

Goodwillie calculus and stable categories

Our ultimate goal in this chapter will be to adapt the proof of the following result of Goodwillie's: let $F: C \to D$ be a $d$-homogeneous homotopy functor, whose target category $D$ is stable, and let $cr_d F: C^d \to D$ be its cross effect. Then $F(X) = cr_d F(X, \cdots, X)_{h\Sigma_d}$.

5.1 Hofibers/hocofibers detect weak equivalences

We start by proving some basic results on stable categories. We recall the definition:

**Definition 5.1.** Let $D$ be a pointed simplicial (cofibrantly generated) model category. Then $D$ is said to be stable if the Quillen adjunction

$$(S^1, *) \wedge \bullet = \Sigma: D \rightleftarrows D: \Omega = Map((S^1, *), \bullet)$$

is actually a Quillen equivalence.

We will denote by $\tilde{\Sigma} = \Sigma \circ Q$, $\tilde{\Omega} = \Omega \circ R$ chosen derived functors.

**Remark 5.2.** Note that this definition can also be made non simplicially, though we will not be using it in such generality.
Proposition 5.3. Let $\mathcal{D}$ be a stable model category. Then every hococartesian square is also a hocartesian square.

Proof. Following the definition of the Goodwillie tower in the last chapter we have that $T_2 \text{Id} \sim \tilde{\Omega} \tilde{\Sigma}$ with the map $\text{Id} \sim T_2 \text{Id}$ that induced by the Quillen equivalence. Hence $P_2 \text{Id} \sim \text{Id}$, and the result follows.

Proposition 5.4. Let $\mathcal{D}$ be stable model category, and consider a map $X \xrightarrow{f} Y$. Then $F$ is a w.e. iff the hocofiber $\text{hocof}(f)$ is contractible.

Proof. Without loss of generality assume $f$ a cofibrant diagram. Then the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
\ast & \longrightarrow & \text{hocof}(f)
\end{array}
$$

(5.5)

is a homotopy pushout and hence, by the previous proposition, also a homotopy pullback, and the result is now obvious.

Proposition 5.6. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence of maps with $g \circ f = \ast$. Then, if the sequence is a hofiber sequence we have $\text{hof}(f) \sim \tilde{\Omega} \tilde{C}$. Conversely, if the sequence is a hocofiber sequence, we have $\text{hocof}(g) \sim \tilde{\Sigma} A$.

Proof. This is clear from the simplicial models for hocofibers/ hofibers.

Corollary 5.7. Let $\mathcal{D}$ be a stable category, and $X \xrightarrow{f} Y$ a map. Then $\text{hofiber}(f) \sim \tilde{\Omega} \text{hocof}(f)$.

Proof. This is clear from the proof of Proposition 5.4 and Proposition 5.6.

Corollary 5.8. Let $\mathcal{D}$ be stable model category, and consider a map $X \xrightarrow{f} Y$. Then $F$ is a w.e. iff the hocofiber $\text{hocof}(f)$ is contractible.

Proof. Obvious from Corollary 5.7 and Proposition 5.4.

Corollary 5.9. Let $\mathcal{D}$ be stable model category.

Let $\mathcal{X} : \mathcal{P}(S) \rightarrow \mathcal{D}$ be a d-cube. Then $\text{hototfiber} \mathcal{X} \sim \tilde{\Omega}^d \text{hototcofiber} \mathcal{X}$. 

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Proof. By using Proposition 3.6 and its dual (concerning cofibers), one sees that the result can be proven inductively. But then Proposition 5.7 and its "relative case" immediately provide both the base case and the induction step.

\[ \text{Proposition 5.10. Let } \mathcal{D} \text{ be a pointed model category, and } A \xrightarrow{f} B \xrightarrow{g} C \text{ a sequence of maps. Then } \text{hofiber}(f) = \text{hofiber}(\text{hofiber}(g \circ f) \to \text{hofiber}(g)). \]

Proof. Clear from first choosing fibrant diagrams.

\[ \text{Proposition 5.11. Let } \mathcal{D} \text{ be a stable model category, and } A \xrightarrow{f} B \xrightarrow{g} A \text{ a sequence of maps with } g \circ f = \text{id}_A. \text{ Then } \text{hocofiber}(f) \sim \text{hofiber}(g) \text{ and, furthermore, this equivalence is obtained from the natural sequence of maps } \text{hofiber}(g) \to B \to \text{hocofiber}(f). \]

Proof. Consider the obvious maps \( A \vee \text{hofiber}(g) \to B \to A \). Proposition 5.10 shows the first map to be an equivalence, so that \( B \sim A \vee \text{hofiber}(g) \) with the maps appearing in the proposition being the standard inclusions/projections, so that the result is now obvious.

Remark 5.12. Both of the previous results can be proven in a functorial sense, i.e., with the diagrams depending functorially on some other category. This allows one to prove the result that follows.

\[ \text{Corollary 5.13. Let } \mathcal{D} \text{ be a stable model category, and let } Z : (0 \to 1 \to 2)^{\times d} \to \mathcal{D} \text{ be a "double cube" which is a retract in each direction. Then} \]

\[ \text{hototcofiber}(Z|_{(0\to1)^{\times d}}) \sim \text{hototfiber}(Z|_{(1\to2)^{\times d}}) \]  

(5.14)

Furthermore, this equivalence is exhibited by the natural composition

\[ \text{hototfiber}(Z|_{(1\to2)^{\times d}}) \to Z(1, \cdots, 1) \to \text{hototcofiber}(Z|_{(0\to1)^{\times d}}) \]

Proof. This follows inductively by combining Proposition 5.11 and Proposition 3.6.
5.2 Cross-effects

Let $\mathcal{C}$ be a pointed simplicial cofibrantly generated model category.

**Definition 5.15.** Given a $d$-tuple of objects $X_1, \cdots, X_d$, define a cube

$$X^c_{X_1,\cdots,X_d}: \mathcal{P}(S) \to \mathcal{C}$$

by

$$X^c_{X_1,\cdots,X_d}(T) = \bigvee_{i \in S-T} X_i \vee \bigvee_{j \in T} C(X_j) \tag{5.16}$$

**Remark 5.17.** Note that when all of the $X_i$ are cofibrant the cube obtained is $\mathcal{P}(S)$-projective cofibrant, and that it is, in fact, the left Kan extension of its restriction to $\mathcal{P}_{\leq 1}(S)$, so that it is actually a strongly hoco-cartesian cube.

**Definition 5.18.** Let $F: \mathcal{C} \to \mathcal{D}$ a homotopy functor.

We set

$$cr_d F(X_1, \cdots, X_d) = \text{hototfibcr}(F \circ X^c_{X_1,\cdots,X_d}) \tag{5.19}$$

and define the cross effect $cr_d F = cr_d F \circ Q^X d$.

**Lemma 5.20.** Suppose $\mathcal{D}$ a stable model category, and let $F: \mathcal{C} \to \mathcal{D}$ be a $d$-excisive functor. Then $F$ is also $d-1$-excisive iff $cr_d F \sim \ast$.

**Proof.** The “if” direction is obvious from Remark 5.17.

For the “only if” direction, we must show that, if $cr_d F \sim \ast$, then $F$ sends any strongly hoco-cartesian cube $\mathcal{X}$ (which can as always be assumed the left Kan extension of a cofibrant restriction to $\mathcal{P}_{\leq 1}(S)$) to a hocartesian cube $F(\mathcal{X})$. Constructing the cube $\mathcal{X}'(T) = \text{colim}(C(\mathcal{X}(\emptyset)) \leftarrow \mathcal{X}(\emptyset) \to \mathcal{X}(T))$, we immediately obtain a strongly hocartesian $d+1$-cube $\mathcal{X} \to \mathcal{X}'$. Hence, by 3.6 and the assumption that $F$ is $d$-excisive, it will suffice to show that $\mathcal{X}'$ itself is sent to a hocartesian cube, and we have hence reduced to the case of cubes with $\mathcal{X}(\emptyset) \sim \ast$ (which can be replaced by cubes were it is actually $\mathcal{X}(\emptyset) = \ast$).

Such a cube $\mathcal{X}$ (again assumed left Kan extension of cofibrant restriction) is totally determined by specifying $d$ cofibrant objects $X_1, \cdots, X_d$. Given those, consider the
“double cube” $\mathcal{Z}: (0 \to 1 \to 2)^{\times d} \to \mathcal{D}$ given by “wedging” the one dimensional “double cubes” $\star \to X_i \to C(X_i)$.

We now apply Proposition 5.13 to $\mathcal{Z}$; $\mathcal{Z}|_{(1\to 2)^{\times d}}$ is the cube appearing in the definition of $cr_n F$; while $\mathcal{Z}|_{(0\to 1)^{\times d}}$ is the cube $\mathcal{X}$. Hence $hototcofiber(\mathcal{X}) \sim \star$, which by 5.9 also implies $hototfiber(\mathcal{X}) \sim \star$, as desired. □

**Definition 5.21.** Now let $L: \mathcal{C}^n \to \mathcal{D}$ be a symmetric multi-homotopy functor (see Section 7 for the definition). Set $\Delta_n L: \mathcal{C} \to \mathcal{D}$ by $\Delta_n L(X) = L(X, \cdots, X)_{h\Sigma_n}$.

(Here, as usual, we use functorially chosen cofibrant replacements to construct the hoorbits)

We thus now have constructions $cr_n$ transforming a homotopy functor into a symmetric $n$-homotopy functor, and $\Delta_n$ doing the reverse.

The following results (all of which are adapted from section 3 of [14]) show that, when the target category $\mathcal{D}$ is stable, these are “inverse constructions” when restricted to $n$-homogeneous functors and $n$-symmetric multilinear functors, respectively.

Indeed, that $\Delta_n$ takes multilinear functors to $n$-excisive functors is Proposition 3.27 (The usage of hoorbits is inconsequential since the target category $\mathcal{D}$ is assumed stable). That $L \circ \Delta_n$ is then also $n$-reduced (and hence homogeneous), i.e., $P_{n-1} F \sim \star$, follows from the following two propositions:

**Proposition 5.22.** If $\mathcal{C}^n \xrightarrow{L} \mathcal{D}$ is a $(1, \cdots, 1)$-reduced homotopy functor, then for any $X$ cofibrant, the map

$$((L \circ \Delta)(X) \xrightarrow{i_{n-1}(L \circ \Delta)} \overline{T}_{n-1}(L \circ \Delta)(X))$$

factors through a contractible object.

**Proof.** Consider the the maps

$$L(X, \cdots, X) \to \text{holim}_{\mathcal{P}_1(\mathcal{S}) \times n} L(\mathcal{X}(U_1), \cdots, \mathcal{X}(U_n))$$

$$\to \text{holim}_E L(\mathcal{X}(U_1), \cdots, \mathcal{X}(U_n))$$

$$\to \text{holim}_{\mathcal{P}_1(\mathcal{S})} L(\mathcal{X}(U_1), \cdots, \mathcal{X}(U_n))$$

(5.23)
where $E$ is the subset of $\mathcal{P}_1(S)^n$ such that for some $s$ one has $s \in U_s$, and $\mathcal{P}_1(S)$ denotes the “diagonal” subset of $\mathcal{P}_1(S)^n$.

Notice that $\mathcal{P}_1(S)^n \supset E \supset \mathcal{P}_1(S)$ so that the maps are just obtained by restriction.

We will now be done if we can show that $\text{holim}_E L(\mathcal{X}_X(U_1), \cdots, \mathcal{X}_X(U_n))$ is contractible. To see this, we prove that this holim is equivalent to the holim over $E^*$, the subset of $\mathcal{P}_1(S)^n$ such that for some $s$ one has $U_s = \{s\}$.

We show homotopy initiality of $E^*$ in $E$. In other words (using Theorem 8.5.5 from [27]), we need to show that for every $(U_1, \cdots, U_s) \in E$, $E^* \downarrow (U_1, \cdots, U_s)$ is contractible after realization. But, setting $H = \{s: s \in U_s\}$, this is a union $\bigcup_{h \in H} \prod_{s \notin h} \mathcal{P}_1(U_s)$ of contractible subspaces with contractible intersections, since

$$\bigcap_{h \in H'} \prod_{s \notin h} \mathcal{P}_1(U_s) \cong \prod_{s \notin H'} \mathcal{P}_1(U_s),$$

and hence itself contractible (by appealing, for instance, to the fact that the union is then a holim).

Finally one needs to show that $\text{holim}_{E^*} L(\mathcal{X}_X(U_1), \cdots, \mathcal{X}_X(U_n))$ is contractible, but this is obvious since it is always $L(\mathcal{X}_X(U_1), \cdots, \mathcal{X}_X(U_n)) \sim *$ on points of $E^*$. \hfill $\square$

**Proposition 5.24.** If $L: C^n \to D$, $D$ assumed stable, is a $(1, \cdots, 1)$-reduced homotopy functor, then $L \circ \Delta$ is $n$-reduced. If, further, $L$ is symmetric, then so is $\Delta_n L$.

**Proof.** We need to prove that $P_{n-1}(L \circ \Delta) = \tilde{T}_{n-1}^n L \circ Q$ is the trivial functor. But, this is immediate from the previous result, since we can then “interpolate” an infinite direct colim by contractible objects (in fact, it follows that even $\tilde{T}_{n-1}^n L \circ Q$ is itself the zero functor, where $\aleph_0$ denotes the smallest infinite ordinal). Hence $L \circ \Delta$ is $n$-reduced.

But the case of $\Delta_n L = (L \circ \Delta)_{h \in \Xi}$ then follows immediately, since, when the target is stable, the homotopy orbits commute with the construction of $P_{n-1}$.

\hfill $\square$

The converse, i.e., that $\sigma_n$ sends $n$-homogeneous functors to multilinear ones, is a particular case of the following:
Proposition 5.25. If $F$ is $n$-excisive then for $0 \leq m \leq n$ the functor $cr_{m+1}F$ is $(n-m)$ excisive in each variable.

Proof. Induction on $m$. The base case $m = 0$ is obvious.

For the induction step, notice that, for $F_{\vee A}(X) = \text{hofiber}(F(X \vee A) \to F(A))$ we have

$$
\partial r_{m+1}F(X_1, \ldots, X_m, A) \sim \partial r_m F_{\vee A}(X_1, \ldots, X_m)
$$

whenever the $X_i, A$ are cofibrant (this is just an application of 3.6), and hence $cr_{m+1}F(X_1, \ldots, X_m, A) \sim cr_m F_{\vee A}(X_1, \ldots, X_m)$ in general.

But now the result follows since $n$-excisiveness of $F$ implies $(n-1)$-excisiveness of $F_{\vee A}$. \qed

We now turn to the task of proving that the $\Delta_n$ and $cr_n$ are indeed inverse constructions when restricted appropriately. We first tackle the case of multilinear functors.

Proposition 5.26. Let $L$ be any symmetric multilinear functor. Then $cr_n(\Delta_n L) \sim L$.

Proof. As should be expected by now, it will suffice to produce a zig zag

$$
L \rightsquigarrow \partial r_n(\Delta_n L)
$$

that is an equivalence on cofibrant objects.

First notice that

$$
\partial r_n(\Delta_n L)(X_1, \ldots, X_n) = \text{hototfiber}(\Delta_n L(\mathcal{X}_{X_1, \ldots, X_n}^{\vee}))
$$

$$
= \text{hototfiber}(L(\mathcal{X}_{X_1, \ldots, X_n}^{\vee}, \ldots, \mathcal{X}_{X_1, \ldots, X_n}^{\vee})_{h \Sigma_n})
$$

$$
\sim \text{hototfiber}(L(\mathcal{X}_{X_1, \ldots, X_n}^{\vee}, \ldots, \mathcal{X}_{X_1, \ldots, X_n}^{\vee})_{h \Sigma_n})_{h \Sigma_n}
$$

(5.27)

where in the last step we use that the target category $D$ is stable.

Recalling that $\mathcal{X}_{X_1, \ldots, X_n}(T) = V_{i \in S-T} X_i \vee \bigvee_{j \in T} C(X_j)$ one gets, for any map $\pi: \mathcal{X} \to S-T$ a map $L(\mathcal{X}_{X_1, \ldots, X_n}^{\vee}, \ldots, \mathcal{X}_{X_1, \ldots, X_n}^{\vee})(T) \to L(X_{\pi(1)}, \ldots, X_{\pi(n)})$. In fact,
putting these all together and appealing to multilinearity of \( L \), one has that the map

\[
L(X_{i_1}, \ldots, X_n, \ldots, X_{i_1}, \ldots, X_n)(T) \xrightarrow{\sim} \prod_{\pi: \Sigma \to S-T} L(X_{\pi(1)}, \ldots, X_{\pi(n)})
\]

is a w.c. (when the \( X_i \) are cofibrant).

Since these equivalences are natural in \( T \), they can be reinterpreted as a map of cubes

\[
L(X^\sigma, \ldots, X^\sigma) \to \prod_{\pi: \Sigma \to \Sigma} Y_{\pi}
\]

where \( Y_{\pi} \) is the cube with \( Y_{\pi}(T) = L(X_{\pi(1)}, \ldots, X_{\pi(n)}) \) if \( \pi(n) \subset T \), and \( Y_{\pi}(T) = * \) otherwise.

But now one notices that when \( \pi \) is not a permutation, and hence not surjective, the cube \( Y_{\pi} \) is cartesian (it is constant on any direction \( s \notin \pi(n) \)), so we can ignore those factors. On the other hand, when \( \pi \) is a permutation, the cubes are \( * \) on all entries except the initial one \( L(X_{\pi(1)}, \ldots, X_{\pi(n)}) \), which is hence the totofiber. Furthermore, these factors are freely permuted by the \( \Sigma \) action, so one finally gets

\[
\text{hototfiber}(L(X^\sigma_{i_1}, \ldots, X_n, \ldots, X^\sigma_{i_1}, \ldots, X_n))_{h\Sigma_n} \sim L(X_1, \ldots, X_n)
\]

as intended.

\[\square\]

**Remark 5.28.** Following the steps of the previous proof one obtains a more "explicit" description of the zig zag \( L \sim \phi \tilde{c}_n(\Delta_n L) \). Indeed, consider the natural map

\[
L(X_1, \ldots, X_n) \xrightarrow{\iota=(\iota_1, \ldots, \iota_n)} L(Z, \ldots, Z) \]

where \( Z = \vee X_i \) and the \( \iota_i \) are the inclusions \( X_i \to Z \).

The previous proof then shows that the composite

\[
L(X_1, \ldots, X_n) \xrightarrow{\iota} L(Z, \ldots, Z) \to (L(Z, \ldots, Z))_{h\Sigma_n}
\]

hits precisely \( \tilde{c}_n(\Delta_n L) \), which by Corollary 5.13 is a summand of \( L(Z, \ldots, Z)_{h\Sigma_n} \).
Hence we have a commutative diagram (in the homotopy category of functors¹)

\[
\begin{array}{ccc}
L(X_1, \ldots, X_n) & \xrightarrow{\epsilon} & L(Z, \ldots, Z) \\
\downarrow{\theta} & & \downarrow \\
\operatorname{gr} \Delta_n L(X_1, \ldots, X_n) & \xrightarrow{\epsilon} & L(Z, \ldots, Z)_{n\Sigma_n}
\end{array}
\]

We are now in a position to prove the converse result:

**Proposition 5.29.** Let \( F \) be any \( n \)-homogeneous functor. Then \( \Delta_n(\operatorname{cr} F) \sim F \).

**Proof.** As before, it will suffice to produce a zig zag \( \Delta_n(\operatorname{cr} F) \xrightarrow{\gamma} F \) that is a w.e. on cofibrant objects.

This will be the map (best viewed as a map in the homotopy category of functors)

\[
\Delta_n(\operatorname{cr} F)(Z) = ((\operatorname{cr} F)(Z, \ldots, Z))_{n\Sigma_n} \xrightarrow{\gamma} F(Z)_{n\Sigma_n} = F(Z)
\]

induced by the map (best viewed as a map in the homotopy category of functors \( C \to D^{\Sigma_n} \)).

\[
(\operatorname{cr} F)(Z, \ldots, Z) \xrightarrow{\gamma} F(\coprod Z) \xrightarrow{F(f)} F(Z)
\]

where \( f : \coprod Z \to Z \) denotes the fold map.

To prove \( \gamma \) a w.c. it suffices, by 5.20, to show that \( \operatorname{cr} \gamma \) is.

In turn, from the previous result it will suffice to prove that the composite

\[
\operatorname{cr} F \xrightarrow{\theta} \operatorname{cr} \Delta_n \operatorname{cr} F \xrightarrow{\operatorname{cr} \gamma} \operatorname{cr} F
\]

is a w.c. And this in turn will follow if the composite \( \operatorname{cr} F(X_1, \ldots, X_n) \xrightarrow{\operatorname{cr} \gamma \circ \theta} \operatorname{cr} F(X_1, \ldots, X_n) \xrightarrow{\epsilon} F(Z) \) is homotopic to \( \epsilon \) (this is because, according to 5.13, \( \epsilon \) is a homotopy injection).

¹Technically, as written here, the reverse directions needed for the zigzags might only be w.e. on cofibrant objects.
This now follows from considering the following commutative diagrams

\[
\begin{align*}
& (\tilde{\tau}_n F)(X_1, \ldots, X_n) \xrightarrow{i} (\tilde{\tau}_n F)(Z, \ldots, Z) \\
& \downarrow \gamma \\
& (\tilde{\tau}_n \Delta_n \tilde{\tau}_n F)(X_1, \ldots, X_n) \xrightarrow{\epsilon} (\tilde{\tau}_n F)(Z, \ldots, Z)_{h \Sigma_n} \\
& \downarrow \tilde{\tau}_n(\gamma) \\
& (\tilde{\tau}_n F)(X_1, \ldots, X_n) \xrightarrow{\epsilon} F(Z)
\end{align*}
\]

(Notice that the right vertical composite is just the map that induced \( \gamma \))

\[
\begin{align*}
& (\tilde{\tau}_n F)(X_1, \ldots, X_n) \xrightarrow{i} F(Z) \\
& \downarrow F(\Pi_{i_0}) \\
& (\tilde{\tau}_n F)(Z, \ldots, Z) \xrightarrow{\epsilon} F(\Pi Z) \xrightarrow{F(f)} F(Z)
\end{align*}
\]

The result now follows from noticing that the composite \( F(Z) \xrightarrow{\Pi_{i_0}} F(\Pi_{i_0}) \xrightarrow{F(f)} F(Z) \) is homotopic to the identity.

\( \square \)
Chapter 6

Delooping homogeneous functors

The goal of this chapter is to obtain an adequate analogue of Goodwillie's Lemma 2.2 in [14], showing that homogeneous functors can always be delooped.

Unfortunately this will require us to make an additional hypothesis on the target category $D$:

**Hypothesis 6.1.** Assume that in $D$ one has that $P_1(S)$ shaped holims commute with countable directed hocolims.

**Remark 6.2.** Notice that then $P_n F$ can be computed as $T_n^{\aleph_0} F$, as in [14].

**Lemma 6.3.** If $F$ is any reduced functor then, up to natural equivalence, there is a fibration sequence

$$P_n F \to P_{n-1} F \to R_n F$$

in which the functor $R_n F$ is $n$-homogeneous.

**Proof.** This is just a repeat of the proof of Lemma 2.2 in [14], which we do not repeat here as it is fairly long. One merely does the obvious changes on replacing $X \ast T$ with $X(T)$ and precomposing everywhere with $Q$. \hfill \Box

**Remark 6.4.** The reason for introducing Hypothesis 6.1 is that there seems to be no clear way of generalizing Goodwillie’s proof to the transfinite case, as it is rather unclear what should take the place of the $P_0(n)^i$ appearing in the proof. The obvious guess, $P_0(n)\alpha$, for $\alpha$ an ordinal, does not seem to be easily workable.
Proposition 6.5. Suppose $\mathcal{D}$ a pointed simplicial model category, and suppose given a diagram of pointwise fibrant functors $\mathcal{C} \to \mathcal{D}$

\[
\begin{array}{ccc}
A & \longrightarrow & S \\
\downarrow & & \downarrow \\
K & \longrightarrow & C
\end{array}
\]

such that $S, K \sim \ast$.

Then there exists a w.e. square of functors

\[
\begin{array}{ccc}
\bar{A} & \longrightarrow & Map_*((I, \ast), C) \\
\downarrow & & \downarrow \\
\ast & \longrightarrow & C
\end{array}
\]  \hspace{1cm} (6.6)

yielding the associated map $A' \to \Omega C$ (where $\Omega$ denotes simplicial loops).

Proof. First notice that one can freely assume the vertical maps in 6.6 are levelwise fibrations by performing functorial factorizations.

Next, to replace $K$ by $\ast$ one simply does the pullback

\[
\begin{array}{ccc}
A' & \longrightarrow & A & \longrightarrow & S \\
\downarrow & & \downarrow & & \downarrow \\
\ast & \longrightarrow & K & \longrightarrow & C
\end{array}
\]

Next one uses mapping path objects:

\[
\begin{array}{ccc}
A' & \longrightarrow & S \\
\downarrow & & \downarrow \\
R & \longrightarrow & Map(I, C) \times_C S \\
\downarrow & & \downarrow \\
\ast & \longrightarrow & C
\end{array}
\]

where $Z$ is defined as the pushout of the lower corner (that the map $A \to Z$ is a w.e. follows from both squares being homotopy pullback squares).

Finally, one uses the obvious w.e. map $\ast \to S$ to obtain (notice $Map_*((I, \times$
\[ \ast, C) = Map(I, C) \times_C \ast \]

\[
\begin{array}{ccc}
\bar{A} & \longrightarrow & \text{Map}_\ast((I, \ast), C) \\
\downarrow & & \downarrow \\
R & \longrightarrow & \text{Map}(I, C) \times_C S \\
\downarrow & & \downarrow \\
\ast & \longrightarrow & C
\end{array}
\]

\[ \square \]

**Proposition 6.7.** Let \( C' \subset C \) any small subcategory, and \( F: C \to D \) a homogeneous functor, where \( D \) is further assumed to be simplicial. Then \( F|_{C'} \) factors (up to a ziz zag by w.e.) by \( Sp(C, \Sigma) \).

**Proof.** Iterative application of Lemma 6.3 and Proposition 6.5 yields two sequences of pointwise fibrant functors \( A_i \) and \( A'_i \) together with w.e.'s \( A_i \overset{\sim}{\rightarrow} \Omega A_{i+1} \).

Since the projective model structure on \( DC' \) exists (Theorem 13.3.2 in [27]), we get to that we can form \( A''_i \) together with maps \( A_i \overset{\sim}{\leftarrow} A''_i \overset{\sim}{\rightarrow} A'_i \).

Hence one may just as well replace \( A_i \) by \( A''_i \). Now inductively define \( A_i^{(n)} \) by setting \( A_i^{(2)} = A''_i \) and forming the pullbacks

\[
\begin{array}{ccc}
A_i^{(n+1)} & \overset{\sim}{\longrightarrow} & \Omega A_i^{(n)} \\
\downarrow & & \downarrow \\
A_i^{(n)} & \overset{\sim}{\longrightarrow} & \Omega A_i^{(n-1)}
\end{array}
\]

Finally one defines \( A_i^{(\infty)} = \lim_n A_i^{(n)} \), which come equipped with the desired spectrum maps \( A_i^{(\infty)} \overset{\sim}{\rightarrow} \Omega A_{i+1}^{(\infty)} \).

\[ \square \]
Chapter 7

Factoring multilinear symmetric functors through spectra

In this chapter we use the notion of spectra described in 2.4, along with the assumptions described there. Namely, we always assume that the spectra categories $Sp(C)$ are cofibrantly generated, that they have a localization functor as in Proposition 2.16, and that stable w.e.s are detected by the $\Omega^\infty - n$ functors.

**Definition 7.1.** Consider a (multi-)functor $F: C^n \to D$. There is an obvious left action$^1$ of $\Sigma_n$ on $C^n$, and we call the functor $F$ symmetric if there are natural transformations

$$F \xrightarrow{\mu_\sigma} F \circ \sigma$$

satisfying $\mu_{\sigma\sigma'} = \mu_{\sigma} \circ \mu_{\sigma'}$.

Furthermore, given symmetric functors $F, G$, we define a natural transformation between them to be a natural transformation $\eta: F \to G$ compatible with $\mu_{\sigma,F}, \mu_{\sigma,G}$ in the obvious way.

Our goal in this chapter is to prove a theorem of the following kind.

**Theorem 7.2.** Suppose given a symmetric functor $F: C^n \to D$ between nice enough model categories which is linear in each coordinate. Then $F$ is of the form$^2$ $\Omega^\infty \circ F \circ$

---

$^1$Which by abuse of notation we just denote by the corresponding elements $\sigma \in \Sigma_n$

$^2$That is to say, there is a zig zag of w.e.s between the two.
where $\mathcal{F} : \text{Sp}(\mathcal{C})^\times_n \to \text{Sp}(\mathcal{D})$ is itself a symmetric multilinear functor.

We will use the following generalization of the previous definition.

**Definition 7.3.** Let $\mathcal{C}, \mathcal{D}$ be categories acted on the left by $\Sigma_n$ (or more generally any group $G$). Then a functor $F : \mathcal{C} \to \mathcal{D}$ is called symmetric if there are natural transformations $\sigma F \xrightarrow{\mu_\sigma} F \circ \sigma$

satisfying $\mu_{\sigma \sigma'} = \mu_\sigma \sigma' \circ \sigma \mu_{\sigma'}$.

Furthermore, given symmetric functors $F, G$, we define a natural transformation between them to be a natural transformation $\eta : F \to G$ compatible with $\mu_{\eta, F}, \mu_{\eta, G}$ in the obvious way.

**Remark 7.4.** Notice that given $\Sigma_n$-categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$ and symmetric functors $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ the composite $G \circ F$ is also a symmetric functor with $\mu_{\sigma, G \circ F}$ the composite $\sigma \circ G \circ F \xrightarrow{\mu_{\sigma, G \circ F}} G \circ \sigma \circ F \xrightarrow{G \circ \mu_\sigma, F} G \circ F \circ \sigma$

**Proposition 7.5.** Suppose $\mathcal{C}$ is a cofibrantly generated model category acted on the left by a group $G$.

Then the functorial factorizations can be chosen compatible with the $G$ action.

**Proof.** Letting $I, J$ be the generating cofibrations and trivial cofibrations for $\mathcal{C}$, one extends these. Namely, $I^G = \{(g g(f)) : g \in G, f \in I\}$, and analogously for $J^G$. Performing the Quillen small object arguments for these sets will then yield the desired invariant factorizations.

**Remark 7.6.** We hence assume that the fibrant/cofibrant replacements performed in such a category are $G$ invariant.

We now construct a first avatar of $\mathcal{F}$, which we call $\mathcal{F}'$.

Notice that $\text{Sp}(\mathcal{D}, \Sigma^n)$ has an obvious action by the group $\Sigma_n$ induced by its action.
on \((S^1)^n\). We now define:

\[ F': Sp(C, \Sigma)^n \to Sp(D, \Sigma^n) \]  
(7.7)

\[ ((X^i_1)_{i\in \mathbb{N}}, \ldots, (X^i_n)_{i\in \mathbb{N}}) \mapsto (F(X^1_i, \ldots, X^n_i))_{i\in \mathbb{N}} \]  
(7.8)

with structure maps for \(F'(((X^i_1)_{i\in \mathbb{N}}, \ldots, (X^i_n)_{i\in \mathbb{N}})\) being given by the composite

\[(S^1)^n \wedge F(X^i_1, \ldots, X^n_i) \to F(S^1 \wedge X^i_1, \ldots, S^1 \wedge X^n_i) \to F(X^1_{i+1}, \ldots, X^n_{i+1})\]

where the first map comes from the multilinear structure of \(F\), and uses the sphere coordinates in the obvious fixed order.

**Proposition 7.9.** If \(F\) is a symmetric functor \(C^n \to D\), then so is \(F': Sp(C, \Sigma)^n \to Sp(D, \Sigma^n)\) (now taking into account that both the source and target are categories with a \(\Sigma_n\) action).

**Proof.** We require maps \(\sigma \circ F' \xrightarrow{\nu_\sigma} F' \circ \sigma\). Since the \(\Sigma_n\) action on \(Sp(D, \Sigma^n)\) does not change the underlying sequence of the spectrum, but merely twists the structure maps, such maps are immediately obtained levelwise by applying the \(\nu_{\sigma,F}\). That this is compatible with the structure maps of the spectra is precisely guaranteed by the referred twisting. \(\square\)

**Proposition 7.10.** \(\Omega^\infty: Sp(D, \Sigma^n) \to D\) is a symmetric functor.

**Proof.** We need only show that \(\Omega^\infty\) itself (i.e. the non derived version) is symmetric, since fibrant replacements are chosen symmetric.

We now need to produce maps \(\Omega^\infty \xrightarrow{\mu_\sigma} \Omega^\infty \circ \sigma\). But both sides, on a spectrum \((X_i)_{i\in \mathbb{N}}\), are given as colimits \(\lim Map((S^1)^n, X_i)\), the only difference being that the structure maps are twisted by the \(\Sigma_n\) action. That action then gives a map between the direct systems, and hence the required natural transformation. \(\square\)

**Remark 7.11.** The above statement would be true replacing \((S^1)^n\) by any other \(\Sigma_n\) simplicial pointed set.
We now prove it is $F$ equivalent to $\Omega^\infty \circ \tilde{F}' \circ (\Sigma^\infty)^{\times n}$.

**Proposition 7.12.** Let $F'$ be a multi-simplicial and multilinear homotopy functor.

There is a natural zig zag of equivalences of symmetric functors

$$F \sim \tilde{\Omega}^\infty \circ \tilde{F}' \circ (\Sigma^\infty)^{\times n}$$

*Proof.* Standard tricks allow us to just reduce to comparing $F$ and $\Omega^\infty \circ \tilde{F}' \circ (\Sigma^\infty)^{\times n}$ (namely, one can replace $\tilde{F}'$ so as to take fibrant values in the projective level model structure and replace $F$ to match its 0-th level functor) over cofibrant values.

$\Omega^\infty \circ \tilde{F}' \circ (\Sigma^\infty)^{\times n}$ is given by a colimit of functors $Map(S^{ni}, F(S^i \wedge X^1, \ldots, S^i \wedge X^n))$, of which $F$ is the 0-th functor. This yields the natural transformation, which is easily seen to be symmetric. That it is also a w.e. is just an immediate consequence of the multilinearity of $F$. \qed

**Remark 7.13.** A certain amount of care must be taken when reading the previous result. This is because it is not immediately clear that the functor $F'$ is actually homotopical with respect to stable equivalences on the spectra categories. Indeed, a priori, the only type of equivalences that are obviously preserved by $\tilde{F}'$ are levelwise equivalences of spectra.

The next result provides conditions in which this holds.

Before stating we point out, however, that in the previous result $\tilde{\Omega}^\infty$ can be defined using either a stable fibrant replacement or a mere levelwise fibrant replacement (for some fixed levelwise model structure). The reason is that, as the proof of the result shows, the image of $\tilde{F}' \circ \tilde{\Sigma}\infty$ is already an $\Omega$-spectrum up to level equivalence (or, in other words, the spaces of the spectrum already have the right homotopy type).

In other words, no problems are caused by viewing $Sp(D)$ as equipped with the stable model structure.

**Proposition 7.14.** Let $F'$ be a multilinear multihomotopical functor $C^n \to D$ as before, and suppose further that $F$ preserves filtered homotopy colimits (in each variable). Then $\tilde{F}'$ sends tuples $(f_1, \cdots, f_n)$, where the $f_i$ are either stable w.e.s between
cofibrant objects or identities, to stable w.e.s., and hence \( F' \circ Q^n \) is a homotopical functor with respect to the stable model structures.

Furthermore, \( F' \circ Q^n \) is a multilinear functor.

**Proof.** First notice that any diagrams in a spectra category that are hocolim/holim diagrams with respect to levelwise w.e.s are also hocolim/holim diagrams with respect to stable w.e.s.

Hence \( F' \) preserves level direct hocolims, and sends level homotopy pushout diagrams to level homotopy pullback diagrams. But since its target is stable (we show later in the chapter that \( Sp(D, \Sigma^n) \) is appropriately equivalent to \( Sp(D, \Sigma) \)) the latter are stable homotopy pushout diagrams.

Now by the 2 out of 3 property one need only show that \( F \) sends the localization maps \( X \to X_{loc} \) whenever \( X \) is cofibrant (this is because w.e.s between local objects are always levelwise).

But by Proposition 2.16 those localization maps are always transfinite composition of pushouts of the maps in \( \mathbb{A}S \). As these transfinite compositions and pushouts are in fact homotopy transfinite compositions and homotopy pushouts (in the level structure on \( Sp(C, \Sigma^1)^{\times n} \)), and \( F \) preserves these (when regarding \( Sp(D, \Sigma^n) \) as having the stable model structure), it suffices to show that \( F \) sends maps that are coordinatewise in \( \mathbb{A}S \) or identities to w.e.s. But this is now obvious from the description of \( \mathbb{A}S \), because those are always maps that are w.e.s in high enough degrees.

For the second part, one needs to show that pushout diagrams are sent to pullback diagrams (or equivalently pushout diagrams). But since any stable pushout diagram is stably equivalent to a level pushout diagram this is now obvious from the previous discussion.

\( \Box \)

We have now essentially proven for \( F' \) the desired properties for \( F \). The main difference between the two is their target categories, \( Sp(D, \Sigma^n) \) and \( Sp(D, \Sigma) \). But we claim that the two are Quillen equivalent (through a zig zag), as \( \Sigma_n \) model categories (the latter having a trivial \( \Sigma_n \) action), and in a way compatible with the \( \Omega^\infty \) functors,
and from this the result follows.

To see why this is plausible, notice that, topologically, $S^n$ decomposes, as a $\Sigma_n$-object, into $S^1 \wedge S^\emptyset$, with the action on $S^1$ being trivial and $S^\emptyset$ being the reduced regular representation. But then, since inverting $S^1 \wedge \bullet$ also inverts $S^\emptyset \wedge \bullet$ (notice that the presence of a $\Sigma_n$ action determines the action on the category of spectra, but is irrelevant for the model category structure itself), plausibility follows.

We now set to prove this. As a first step we need to replace the simplicial $S^n$, which lacks any decomposition as above, with something more amenable. That this can be done follows from the following (particular case of a) theorem of Hovey:

**Theorem 7.15.** Let $C$ be a simplicial model category for which spectra can be defined, as described in 2.4.

Suppose $f: A \rightarrow B$ a w.e. of pointed simplicial sets. Then, provided that the domains of the generating cofibrations of $C$ are cofibrant or that the induced natural transformation of functors $X \wedge \bullet \rightarrow Y \wedge \bullet$ is a pointwise w.e., $f$ induces a Quillen equivalence

$$Sp(C, A) \simeq Sp(C, B)$$

**Proof.** This is a particular case of Theorem 5.5 in [20]. It is worth noting that in the proof Hovey uses Proposition 2.3 from that paper, which supposes the involved categories to be left proper cellular. However, careful examination of the proofs shows that left properness and cellularity are only used to conclude the existence of the left Bousfield localization model structure, and hence the result still holds provided one knows those to exist.

**Remark 7.16.** The Quillen equivalence above is also compatible with the $\Omega^\infty$s. This is clear for the right adjoint (which is a forgetful functor) from the obvious map

$$\lim Map(B^\Lambda^i, X_i) \rightarrow \lim Map(A^\Lambda^i, X_i)$$

being a w.e. when $(X_i)$ is a projective fibrant spectrum. The result also follows for the left adjoint since (it's derived functor) is the inverse of the (derived functor) of the right adjoint.

Using the above result we are then allowed to replace $S^n$ by a $\Sigma_n$ w.e. simplicial set of the form $S^1 \wedge S^\emptyset$, possibly resorting to a ziz zag of $\Sigma_n$ w.e. $S^n \leftarrow C \rightarrow F \leftarrow S^1 \wedge S^\emptyset$. 66
We will now be finished by proving the following:

**Proposition 7.17.** Suppose $\mathcal{D}$ is in the finitely generated case described in [20], so that stable w.e. in the spectra categories are determined at the level of the $\Omega^{\infty-n}$ functors.

Then there is a $\Sigma_n$ Quillen equivalence

$$Sp(\mathcal{D}, S^1) \rightleftharpoons Sp(\mathcal{D}, S^1 \wedge S^n)$$

with left adjoint given by

$$L((X_i)_{i \in \mathbb{N}}) = ((S^\beta)^{\wedge i} \wedge X_i)_{i \in \mathbb{N}}$$

and right adjoint given by

$$R((Y_i)_{i \in \mathbb{N}}) = (Map((S^\beta)^{\wedge i}, Y_i))_{i \in \mathbb{N}}$$

**Proof.** First we show this is a Quillen adjunction. For the projective model structures we clearly have a Quillen adjunction, since the left adjoint carries generating cofibrations/triv. cofibrations to cofibrations/triv. cofibrations (namely, a generating cofib/triv cofib of the form $F_n(f)$ is sent to $F_n(f \wedge (S^\beta)^{\wedge n})$).

Similarly, the localizing set of maps $F_{n+1}(X \wedge S^1) \to F_n X$ is sent to $F_{n+1}(X \wedge S^1 \wedge (S^\beta)^{\wedge n+1}) \to F_n(X \wedge (S^\beta)^{\wedge n})$, also a stable equivalence. Hence one has indeed a Quillen adjunction.

We now turn to proving that this is indeed a Quillen equivalence.

As usual, consider a cofibrant $S^1$ spectrum $X$, a fibrant $S^1 \wedge S^\beta$ spectrum $Y$, so that we want to show a map $LX \to Y$ is w.e. iff the map $X \to RY$ is.

We now use our hypothesis that w.e. are determined at the $\tilde{\Omega}^{\infty-n}$ level. For $LX \to Y$ this yields maps (where $\tilde{\Omega}$ denotes true homotopical loops)

$$\text{hocolim}_i \tilde{\Omega}^{(1+\beta)}((S^\beta)^{i+n} \wedge X_{i+n}) \to \text{hocolim}_i \tilde{\Omega}^{(1+\beta)}Y_{i+n}$$
while for \( X \to RY \) it yields maps

\[
\text{hocolim}_i \tilde{\Omega}^n X_{i+n} \to \text{hocolim}_i \tilde{\Omega}^i \text{Map}((S^i)^{\wedge i}, Y_{i+n}) = \text{hocolim}_i \tilde{\Omega}^{i(1+\bar{p})} Y_{i+n}.
\]

But since this last map factors through the obvious map

\[
\text{hocolim}_i \Omega^n X_{i+n} \to \text{hocolim}_i \Omega^{i(1+\bar{p})} ((S^i)^{i+n} \wedge X_{i+n}),
\]

it remains to show that these maps are w.e. and, without loss of generality, to do so for \( i = 0 \).

Next one notices that the \( \Sigma_n \) actions are irrelevant as far as detecting w.e.s, so that we will be done if we prove the analogous result for the analogous adjunction of model categories\(^3\) given by the obvious analogous formulae:

\[
\text{Sp}(D, S^1) \rightleftarrows \text{Sp}(D, S^1 \wedge (S^1)^{n-1})
\]

In order words, it now remains to prove the map

\[
\text{hocolim}_i \tilde{\Omega}^i X_i \to \text{hocolim}_i \tilde{\Omega}^{i(n-1)}((S^{n-1})^i \wedge X_i) = \text{hocolim}_i \tilde{\Omega}^i \tilde{\Omega}^{(n-1)i} \Sigma^{(n-1)i} X_i
\]

is a w.e.. At this point a certain amount of care must be taken with the suspension/loop coordinates. Henceforth assume suspension coordinates ordered left to right (and loop coordinates ordered inversely to make the notation compatible with applying adjunction unit maps).

The maps defining the left hocolim are of the form

\[
\begin{align*}
\tilde{\Omega}^i \tilde{\Omega}^{(n-1)i} \Sigma^{(n-1)i} X_i & \to \tilde{\Omega}^i \tilde{\Omega}^{(n-1)i} \tilde{\Omega}^n \Sigma^n \Sigma^{(n-1)i} X_i \\
& \to \tilde{\Omega}^i \tilde{\Omega}^{(n-1)i} \tilde{\Omega}^i (n-1, i, \ldots, 2) \Sigma^{(2, \ldots, n)} \Sigma^{(n-1)i} \Sigma X_i \quad (7.18) \\
& \to \tilde{\Omega}^{i+1} \tilde{\Omega}^{(n-1)i(i+1)} \Sigma^{(n-1)i(i+1)} X_{i+1}
\end{align*}
\]

where the first map is the unit map, and the second map moves the first coordinate in

\(^3\)To see this one may find a bifibrant \( S^\delta \rightrightarrows C \leftarrow (S^1)^{n-1} \) and apply 7.15.
the inner $\Sigma^n$ until it is adjacent to $X_i$ (and, by symmetry, moves the associated loop coordinate in the opposite direction), and the final map is induced by $\Sigma X_i \to X_{i+1}$ (we also perform some rewriting of terms, but no further shuffling).

By contrast, consider the hocolimit with the same terms, but with the reshufflings in the middle step removed, so that the outer suspension coordinate is the one used in the final map.

These two hocolimits are of course equivalent, since when they are computed using only the intermediate terms $\tilde{\Omega}^i \tilde{\Omega}^{(n-1)i} \tilde{\Omega}^n \Sigma^n \Sigma^{(n-1)i} X_i$, the direct (sub)systems are actually isomorphic.

Now notice that the unshuffled direct system can be identified with $\tilde{\Omega}^\infty$ for a new $S^n$-spectrum $(S^{(n-1)i} \wedge X_i)_{i \in \mathbb{N}}$ with structure maps $S^n \wedge S^{(n-1)i} \wedge X_i = S^{(n-1)(i+1)} \wedge S^1 \wedge X_i \to S^{(n-1)(i+1)} \wedge X_{i+1}$ (with no shuffling used).

It then suffices to show the map relating this last direct system with that for $\Omega^\infty$ of $(X_i)_{i \in \mathbb{N}}$ induces a w.e. on hocolims.

To see this consider the diagram

\[
\begin{array}{ccccccc}
X_0 & S^1 \wedge X_0 & S^2 \wedge X_0 & S^3 \wedge X_0 & S^4 \wedge X_0 & \ldots, \\
X_1 & S^1 \wedge X_1 & S^2 \wedge X_1 & S^3 \wedge X_1 & \ldots \\
X_2 & S^1 \wedge X_2 & S^2 \wedge X_2 & \ldots \\
& \vdots & & & & & \\
\end{array}
\]

where the lines represent the spaces of $S^1$-spectra, and the columns obvious maps between these spectra. Now consider applying $\tilde{\Omega}^n$ to the $n$-th column, so that one has a full diagram. Notice that the direct system for $\tilde{\Omega}^\infty$ of $(X_i)_{i \in \mathbb{N}}$ is the line of slope 1 in this diagram, while the direct system for $\tilde{\Omega}^\infty$ of $(S^{(n-1)i} \wedge X_i)_{i \in \mathbb{N}}$ is the line of slope $n$. But since tracking definitions shows the map induced between those direct systems is given by traveling horizontally in the diagram above, the result finally follows by cofinality of those lines in the diagram.
Part II

Goodwillie calculus in $\text{Alg}_\mathcal{O}$
Chapter 8

Basic definitions

The majority of the material in this chapter is adapted from [15], which should be consulted for details. We will only cover here the bare minimum necessary for the remainder of the paper.

8.1 Symmetric Spectra

Definition 8.1. The category $Sp^{\Sigma}$ of symmetric spectra is the category such that

- objects $X$ are sequences $X_n \in SSet_*^{\Sigma_n}$ (i.e., $X_n$ is a pointed simplicial set with a $\Sigma_n$ action), together with structure maps (compatible with the $\Sigma_m \times \Sigma_n$ action)

$$S^m \wedge X_n \to X_{m+n}$$

satisfying appropriate compatibility conditions.

- maps $f: X \to Y$ are sequences of $\Sigma_n$ maps $f_n: X_n \to Y_n$ compatible with the structure maps in the obvious way.

Also, we denote by $\wedge$ the standard monoidal structure on $Sp^{\Sigma}$ with unit $S$, the canonical symmetric spectrum such that $S_n = S^n$, the $n$-sphere.

Definition 8.2 (Free symmetric spectra). Let $H \subset \Sigma_m$ be a subgroup and $A \in SSet_*^H$. 

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The free spectrum \( F^H_m(A) \) generated by \( A \) is the symmetric spectrum with spaces

\[
(F^H_m(A))_n = \begin{cases} 
\Sigma_n \times_{\Sigma_{n-m} \times H} (S^{n-m} \wedge A) & \text{if } n \geq m \\
* & \text{if } n < m
\end{cases}
\]

with the natural maps.

Or, in other words, \( F^H_m \) is the left adjoint to the forgetful functor \( Sp^\Sigma \to SSet^H \).

## 8.2 Stable model structures on \( Sp^\Sigma \)

Our interest on \( Sp^\Sigma \) is as a model category for spectra. The stable w.e.s used for this are somewhat technical to define, as the actual definition resorts to defining injective \( \Omega \)-spectra first, hence we refer to [21] for the precise definition. For our purposes it is enough to view these stable equivalences as being those maps that induce isomorphisms on the stable homotopy groups \( \pi_n \).

However, while we will only be interested in model structures on \( Sp^\Sigma \) which use the stable w.e.s as the notion of w.e.s, there are multiple such model structures, and it will be useful for us to be aware of several of them, which we list in the following definition.

**Definition 8.3.** We have the following stable model structures on \( Sp^\Sigma \).

- **level cofibrations stable model structure,** where the cofibrations are levelwise cofibrations of the underlying symmetric sequences.

- **flat stable model structure,** with generating cofibrations given by

\[
F^H_m((\delta \Delta_k)_+) \to F^H_m((\Delta_k)_+)
\]

for \( m \geq 0, H \subseteq \Sigma_m \) any subgroup.

\footnote{The catch here is that, when working with symmetric spectra, one can't usually simply define \( \pi_n^S(X) = \lim_{m} \pi_{n+m}X_m \), unless \( X \) already satisfies some fibrancy type condition.}
• **positive flat stable model structure**, with generating cofibrations given by

\[ F_m^H((\delta\Delta_k)_+) \to F_m^H((\Delta_k)_+) \]

for \( m \geq 1 \), \( H \subset \Sigma_m \) any subgroup.

• **stable model structure**, with generating cofibrations given by

\[ F_m^*(((\delta\Delta_k)_+)) \to F_m^*((\Delta_k)_+) \]

for \( m \geq 0 \), and where \( * \subset \Sigma_m \) denotes the trivial subgroup.

• **positive stable model structure**, with generating cofibrations given by

\[ F_m^*((\delta\Delta_k)_+) \to F_m^*((\Delta_k)_+) \]

for \( m \geq 0 \), and where \( * \subset \Sigma_m \) denotes the trivial subgroup.

**Remark 8.4.** It is worth noticing the hierarchy of these model categories: the level cofibrations stable structure has the most cofibrations, followed by the flat stable model structure, followed by either the positive flat stable or the stable model structures (which refine the flat stable model structure in different ways), and followed finally by the positive stable model structure, which has the least cofibrations of all.

**Proposition 8.5.** The five model structures listed above are all left proper cellular model categories.

**Proof.** We recall that left properness means that the pushout of a w.e. along a cofibration is again a w.e.. Hence it suffices to prove this property for the model category structure with the most cofibrations, namely the levelwise cofibration model structure. This result is then Lemma 5.4.3 part (1) of [21].

Cofibrant generation of each of these model structures is proved in several papers: for the level cofibration model structure and the stable model structure this is proved in [21]; for the flat stable model structure and the positive flat stable model structure
this is proved in [30], and for the positive stable model structure this is proved in [24].

Cellularity of these model categories (see Definition A.1 of [20] for the definition) follows immediately from these being categories built out of simplicial sets.

Indeed, any set of objects $A$ will be compact with respect to any set of level injections $K$ by choosing a regular ordinal $\gamma$ greater than the cardinality of all the simplicies appearing in $A$, as then a map from $a \in A$ into a relative $K$-cell complex will factor through the minimal complex containing the images of the simplices, and this subcomplex will have less than $\gamma$ cells.

Hence indeed the domains of $I$ are compact with respect to $I$, and the domains of $J$ small relative to the cofibrations (by a similar but easier argument). Finally, it is clear that the cofibrations are categorical monomorphisms, since they are always levelwise monomorphisms.

8.3 Operads and algebras

**Definition 8.6.** An (spectral) operad $O$ in $Sp^\Sigma$ is a sequence of “spectra of $n$-ary operations” $O(n) \in Sp^\Sigma$, for $n \geq 0$, together with

- $\Sigma_n$ actions on $O(n)$,

- multiplication maps

$$O(n) \wedge O(m_1) \wedge \cdots \wedge O(m_n) \to O(m_1 + \cdots + m_n)$$

and unit map

$$S \to O(1),$$

A little care is needed here because the results proven in [24] are for topologically based spectra instead of for simplicial ones, so one needs to adapt the arguments present there. Alternatively, it is fairly straightforward to see that one has a positive level model structure in the simplicial case, and that this is a left proper cellular model category, hence the result can also be derived by applying the left Bousfield localization techniques of [18].
• together with associativity, unity and change of order of variables compatibility conditions.

Definition 8.7. The category $\text{Alg}_O$ of algebras over the operad $O$ is the category such that

- objects $X$ are symmetric spectra plus algebra maps

$$\mathcal{O}(n) \wedge X^\wedge n \rightarrow X$$

satisfying appropriate compatibility conditions.

- maps $f: X \rightarrow Y$ are maps $X \rightarrow Y$ of the underlying symmetric spectra which are compatible with the algebra structure maps.

Definition 8.8 (Free algebras). The free $O$-algebra functor (usually denoted $\mathcal{O}$) is the left adjoint to the forgetful functor $\text{Alg}_O \rightarrow Sp_\Sigma$.

It assigns to a symmetric spectrum $X$ the canonical algebra $\mathcal{O}(X)$ whose underlying spectrum is $\bigvee_{n=0}^{\infty} (\mathcal{O}(n) \wedge X^\wedge n)_{\Sigma_n}$.

We will also occasionally use the notation $O \circ X$ for this algebra, where $\circ$ is meant to evoke the composition product of symmetric sequences.

Notations 8.9. Notice that for any $O$-algebra $X$ it’s structural multiplication maps can be packaged into a map

$$\mathcal{O}(X) \overset{\mu}{\rightarrow} X.$$ 

Notice further that this is actually a map of $O$-algebras.

We will be using the following standard result.

Proposition 8.10. Let $O \rightarrow O'$ be a map of operads. Then there is an adjunction

$$\xymatrix{ \text{Alg}_O \ar[r]^-\mathcal{O}' & \text{Alg}_{O'} \ar[l]_\text{forget} }$$

where the functor $O' \circ -$ is defined by the natural coequalizer $\text{coeq}(O'OX \rightrightarrows O'X)$. 

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Remark 8.11. As a particular case of the previous result, consider the obvious map \( \mathcal{O}(1) \to \mathcal{O} \). Then each spectrum \( \mathcal{O}(n) \) has a natural left action of \( \mathcal{O} \) together with \( n \) different right actions of \( \mathcal{O}(1) \). Note that these actions are all compatible, and that the right actions are furthermore related by the symmetric group action. Altogether these make \( \mathcal{O}(n) \) into a \( (\mathcal{O}(1), \mathcal{O}(1)^m) \)-bimodule.

It is then easy to see that

\[
\mathcal{O}_{\mathcal{O}(1)} X = \bigvee_{n \geq 0} (\mathcal{O}(n) \wedge_{\mathcal{O}(1)^n} X^{\wedge n})_{\Sigma_n} = \bigvee_{n \geq 0} \mathcal{O}(n) \wedge_{\mathcal{O}(1)^n} X^{\wedge n}.
\]

Proposition 8.12. Let \( X \) be in \( \text{Alg}_\mathcal{O} \), and consider the undercategory \( (\text{Alg}_\mathcal{O})_X \) of \( \mathcal{O} \)-algebras under \( X \).

Then there exists an enveloping operad \( \mathcal{O}_X \) such that

\[
(\text{Alg}_\mathcal{O})_X = \text{Alg}_{\mathcal{O}_X}.
\]

More specifically, one has

\[
\mathcal{O}_X(n) = \text{coeq}\left( \prod_{m \geq 0} \mathcal{O}(n + m) \wedge_{\Sigma_m} (\mathcal{O} \circ X)^{\wedge m} \Rightarrow \prod_{m \geq 0} \mathcal{O}(n + m) \wedge_{\Sigma_m} X^{\wedge m} \right)
\]

with the two maps induced by the operad structure and the algebra structure.

Proof. This is really just Proposition 4.7 of [15] reinterpreted (and restricted from left modules to algebras, i.e., left modules concentrated in degree 0.).

Indeed, what that proposition shows is that the forgetful functor \( (\text{Alg}_\mathcal{O})_X \) has its left adjoint given by \( \mathcal{O}_X \circ \cdot \). The result then follows since the conditions of Beck's monadicity theorem are immediate. \( \square \)

---

3Notice that \( \mathcal{O}(1) \) is itself a monoid, and can hence be viewed as an operad concentrated in degree 1.

4Here \( \mathcal{O}(1)^m \) denotes the wreath product \( \Sigma_m \ltimes \mathcal{O}(1)^{\wedge m} = \bigvee_{\sigma \in \Sigma_m} \mathcal{O}(1)^{\wedge m} \). Multiplication in this ring spectrum is such that multiplying the \( \sigma \) component by the \( \tau \) component lands in the \( \sigma \tau \) component, and \( \sigma \) acts on the second \( \mathcal{O}(1)^{\wedge m} \) copy before multiplying those coordinatewise.

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8.4 Model structures on $\text{Alg}_O$

The following is a particular case of the main result of [15], combined with a result of the follow-up paper for the simplicial structure:

**Theorem 8.13.** Suppose $Sp^E$ is endowed with either the positive stable model structure or the positive flat stable model structure.

Then, for any operad $O$, the projective model structure\(^5\) on $\text{Alg}_O$ exists.

Furthermore, this is a simplicial model structure.

We will also require a somewhat explicit description of the simplicial tensoring and cotensoring.

First, recall that in $Sp^E$ one has tensoring and cotensoring given by

$$(K \otimes X)_n = K_+ \land X_n$$

and

$$\text{Map}(K, X)_n = (X_n)^K,$$

where $K \in SSet$ and $X \in Sp^E$.

In other words, this tensoring and cotensoring reflect pointwise the tensoring and cotensoring of $SSet_*$ over $SSet$. Note that $K \otimes X$ can also be described as $F_0^* K \land X$.

In $\text{Alg}_O$, the tensoring of a simplicial set $K$ and a $O$-algebra $X$ is then given by the (algebraic) coequalizer

$$K \otimes^{alg} X = \mathcal{O}(K \otimes \mathcal{O}(X)) \xrightarrow{\mathcal{O}(K \otimes \mu)} \mathcal{O}(K \otimes X)$$

where $\mathcal{O}(X) \xrightarrow{\mu} X$ is the algebra structure map and $K \otimes \mathcal{O}(X) \xrightarrow{\tau} \mathcal{O}(K \otimes X)$ it the map

$$\bigvee_{n=0}^\infty K \otimes (\mathcal{O}(n) \land X^\land n)_{\Sigma_n} \to \bigvee_{n=0}^\infty (K^\times n \otimes \mathcal{O}(n) \land X^\land n)_{\Sigma_n}$$

---

\(^5\)We recall that in a category $\text{Alg}_C(C)$ of algebras over some monad $C$ in $C$, the projective model structure (when it exists) on $\text{Alg}_C(C)$ is the one where w.e.s/fibrations are the maps which are underlying w.e.s/fibrations in $C$. 

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which is induced at each level by the diagonal maps $K \to K^{\times n}$.

As for the cotensoring $\text{Map}_{\text{alg}}(K, X)$, the underlying symmetric spectrum is just $\text{Map}(K, X)$, with the algebra structures maps

$$\mathcal{O}(n) \wedge \text{Map}(K, X)^n \to \text{Map}(K, X)$$

being the adjoints to the composite

$$K \otimes \mathcal{O}(n) \wedge \text{Map}(K, X)^n \to K^n \otimes \mathcal{O}(n) \wedge \text{Map}(K, X)^n \cong \mathcal{O}(n) \wedge (K \otimes \text{Map}(K, X))^n \to X$$

where the first map is induced by the diagonal $K \to K^n$ and the last one by the counits $K \otimes \text{Map}(K, X) \to X$ and algebra structure map $\mathcal{O}(n) \wedge X^{\times n} \to X$.

### 8.5 Spectra in spectra

Recall that our basic definitions and assumptions for categories of spectra on a model category where made in Section 2.4.

We note that the hypothesis of 2.17 hold when $Sp^\Sigma$ is given the flat stable model structure.

Indeed, in that case $Sp^\Sigma$ is almost finitely generated (see [20] Chapter 4 for the definition), and in fact even finitely generated: searching the proofs given in [21] one sees that the generating cofibrations are maps $F_m^H((\delta \Delta^n)_+) \to F_m^H((\Delta^n)_+)$, with compact domains and codomains, and that the generating trivial cofibrations are the maps $F_m^H((\Lambda^i_k)_+) \to F_m^H((\Delta_k)_+)$ plus the simplicial mapping cylinders of the maps $F_{n+1}^*(S^1)_+ \to F_n^*(S^0)_+$, which again have compact domains and codomains.

The remaining two conditions, that sequential and finite products commute, and that $\text{Map}(S^1, -)$ preserves sequential colimits, follow immediately from the fact that all these constructions are levelwise at the simplicial set level, where the statements are known to be true.

We notice however that the conclusion of 2.17 actually holds for any of the model
structures on $Sp^\Sigma$ we listed.

Indeed, it suffices for this to check that the w.e.s on $Sp(Sp^\Sigma)$ are exactly the same no matter what the base model category chosen on $Sp^\Sigma$. 6

Now notice that clearly the various possible levelwise model structures on $Sp(Sp^\Sigma)$ are Quillen equivalent, and share the same weak equivalences. But since furthermore the notion of stable fibrant objects (i.e. local) in $Sp(Sp^\Sigma)$ only depends on the base model structure on $Sp(\Sigma)$ up to a levelwise w.e., it indeed follows that all the stable model structures on $Sp(Sp^\Sigma)$ do have the same equivalences, as desired.

---

6 That these model structures always exist follows from Proposition 8.5 together with the first main theorem of [20].
Chapter 9

Spectra on $\text{Alg}_\mathcal{O}$

As shown in Part I understanding homogeneous functors to/from a (good) model category is closely related to understanding its stabilization. Hence, our goal in this chapter will be to understand the stable model category $\text{Sp}(\text{Alg}_\mathcal{O})$.

Recall first of all that the underlying case model structures on $\text{Alg}_\mathcal{O}$ are those described in 8.4, i.e., the projective model structures built out of any of the “positive” stable model structures in $\text{Sp}^\Sigma$. We shall have no need to distinguish between these two model structures in our results.

Second, recall that in order to stabilize a model category $\mathcal{C}$, $\mathcal{C}$ must either be a pointed model category, else one must first replace $\mathcal{C}$ by its category $\mathcal{C}_*$ of pointed objects and then stabilize that category. In our context, we notice that $\text{Alg}_\mathcal{O}$ is pointed precisely when $\mathcal{O}(0) = *$\footnote{We shall also refer to this condition by saying that $\mathcal{O}$ is non unital.}, hence we make that assumption for the rest of this chapter. We also point out that, by 8.12, $(\text{Alg}_\mathcal{O})_*$ is still always an operadic algebra category, hence our results still cover that case.

Our main result will be that, $\text{Sp}(\text{Alg}_\mathcal{O})$ is just $\text{Mod}_{\mathcal{O}(1)}(\text{Sp}^\Sigma)$, up to a zig zag of Quillen equivalences.

However, some subtleties must be handled beforehand, like proving that the model structure on $\text{Sp}(\text{Alg}_\mathcal{O})$ actually exists. The reason one can’t simply just use the theory developed in [20] is that $\text{Alg}_\mathcal{O}$ is rarely left proper.
Here's a sketch for why: it is easy (by appealing to universal properties), to check that in $Alg_{\mathcal{O}}$ it is $\mathcal{O}(X(A))^\mathcal{O}_X Y = \mathcal{O}_Y(A)$. As the map $X \rightarrow \mathcal{O}(X(A))$ is a cofibration whenever $A$ is cofibrant, one sees that left properness would require $X \rightarrow \mathcal{O}_X$ to be a homotopical construction. But now recall that $\mathcal{O}(X(n))$ is constructed as a quotient of $\bigwedge_{m \geq 0} \mathcal{O}(n+m) \wedge \mathcal{X}^m$, which is not expected to be a homotopical construction for general $\mathcal{O}$, and hence neither is $\mathcal{O}_X$. It is not too hard to craft specific examples where this can be seen explicitly.

In order to prove the existence of a model structure on $Sp(Alg_{\mathcal{O}})$ directly, we first notice that this category can also be regarded as $Alg_{\mathcal{O}_{sp}}(Sp(Sp^E))$, the algebras for a certain monad $\mathcal{O}_{sp}$ on the category $Sp(Sp^E)$. One then needs only verify the hypothesis for the standard result for the existence of model structures in algebra model categories, which is Lemma 2.3 in [28].

We point out that this is exactly what is done in [15], where the pushouts relevant to apply Lemma 2.3 in [28] are studied by means of an adequate filtration. And, in fact, our proof that [28] applies will consist of studying the analogous filtration adapted to the monad $\mathcal{O}_{sp}$.

**Proposition 9.1.** $Sp(Alg_{\mathcal{O}})$ is the category $Alg_{\mathcal{O}_{sp}}(Sp(Sp^E))$ of algebras for the monad $\mathcal{O}_{sp}$ given by

$$(\mathcal{O}_{sp}X)_n = \mathcal{O}(X_n)$$

with the structure maps given by the composite\(^2\)

$$S^1 \wedge \mathcal{O}(X_n) \rightarrow \mathcal{O}(S^1 \wedge X_{n+1}) \rightarrow \mathcal{O}(X_{n+1}).$$

**Proof.** The first task is to show that $\mathcal{O}$ is actually a monad (and indeed a functor).

It is useful for this to consider the following auxiliary structures.

$$\wedge_{sp} : Sp(Sp^E) \times Sp(Sp^E) \rightarrow Sp(Sp^E)$$

\(^2\)Here the first map is the map we called $\tau$ in 8.4.
\[(X \wedge_{sp} Y)_n = X_n \wedge Y_n,\]

\[\wedge : Sp^\Sigma \times Sp(Sp^\Sigma) \to Sp(Sp^\Sigma)\]

\[(X \wedge Y)_n = X \wedge Y_n,\]

where the structure maps for \(X \wedge_{sp} Y\) are the natural composite \(S^1 \wedge X_n \wedge Y_n \to (S^1)^\wedge \wedge X_n \wedge Y_n \cong S^1 \wedge X_n \wedge S^1 \wedge Y_n \to X_{n+1} \wedge Y_{n+1}\), with the first map induced by the diagonal map, and the structure maps for \(X \wedge Y\) given by natural composite \(S^1 \wedge X \wedge Y_n \cong X \wedge S^1 \wedge Y_n \to S^1 \wedge Y_{n+1}\).

It is then clear that \(\wedge_{sp}\) is a non unital symmetric monoidal structure\(^3\) on \(Sp(Sp^\Sigma)\).

Furthermore, \(\wedge : Sp^\Sigma \times Sp(Sp^\Sigma)\) behaves unitaly with respect to the unit of \(Sp^\Sigma\) and associatively with respect to both the monoidal structure\(^4\) \(\wedge\) on \(Sp^\Sigma\) and the monoidal structure \(\wedge_{sp}\) on \(Sp^\Sigma\).

Since furthermore each of these operations preserves colimits in each variable, it is then a formality to notice that non unital operads in \(Sp^\Sigma\) induce monads on \(Sp(Sp^\Sigma)\).

Obviously the monad associated to \(O\) is just what we called \(O_{sp}\).

It now remains to see that the categories \(Sp(Alg_O)\) and \(Alg_{O_{sp}}(Sp(Sp^\Sigma))\) are indeed the same.

First, we show the objects are the same. It is immediately clear that \(O_{sp}\) algebras \(X\) are made out of \(O\)-algebras \(X_n\) at each level. So it really only remains to see that having a map \(O_{sp}(X) \to X\) is equivalent to having maps\(^5\) \(S^1 \wedge_O X_n \to X_{n+1}\) of algebras. To see this first rewrite the structure maps of spectra in adjoint form\(^6\).

---

\(^3\)This monoidal structure may look strange and unfamiliar at first. There is a very good reason for this, namely the fact that \(\wedge_{sp}\) is always homotopically trivial, i.e., \(X \wedge_{sp} Y\) is always nullhomotopic. This fact plays a crucial role in the proof of the main result of this chapter.

\(^4\)We purposefully abuse notation here in using the symbol \(\wedge\) to denote two different operations. We believe this should not cause confusion.

\(^5\)Here we use \(\wedge_0\) to denote the pointed simplicial tensoring of \(Alg_0\).

\(^6\)I.e., \(X_n \to \text{Map}(S^1, X_{n+1})\) rather than \(S^1 \wedge X_n \to X_{n+1}\).
But it is now clear that both conditions are just the commutativity of the following diagram

\[
\begin{array}{ccc}
\mathcal{O}(X_n) & \longrightarrow & \mathcal{O}(\text{Map}(S^1, X_{n+1})) \\
\downarrow & & \downarrow \\
X_n & \longrightarrow & \text{Map}(S^1, X_{n+1})
\end{array}
\]

the difference being that the first condition concerns the squares obtained by omitting \(\mathcal{O}(\text{Map}(S^1, X_{n+1}))\) and the second those obtained by omitting \(\text{Map}(S^1, \mathcal{O}(X_{n+1}))\).

It remains only to see that maps in the two categories are the same, but this is clear: compatibility with spectra structure maps gives the same condition in both cases, and compatibility with the \(\mathcal{O}_{sp}\) algebra structures is the same as compatibility with the \(\mathcal{O}\)-algebra structures at all the levels.

Lemma 9.2. The class of maps in \(\text{Sp}(\text{Sp}^\Sigma)\) which are both levelwise monomorphisms stable equivalences is closed under pushouts, transfinite compositions and retracts.

Proof. Recall that weak equivalences in \(\text{Sp}(\text{Sp}^\Sigma)\) are detected as equivalences at the \(\text{hocolim}_{n} X_{n+k}\) level.

Now consider a pushout of a such a map. Levelwise all these pushouts are actually homotopy pushouts (since the level cofibration model structure on \(\text{Sp}^\Sigma\) is left proper), and since \(\text{Sp}^\Sigma\) is a stable, they are actually also levelwise homotopy pullbacks. But then applying \(\tilde{\Omega}^n\) turns such (levelwise) squares into homotopy pullbacks. But then it is obvious that the original pushout square is a homotopy pushout after applying any of the \(\tilde{\Omega}^{\infty-n}\) functor.

A similar easier argument deals with the case of transfinite compositions, and the statement for retracts is obvious.

\[\square\]

Theorem 9.3. The (monadic) projective model structures on

\[\text{Sp}(\text{Alg}_\mathcal{O}) \cong \text{Alg}_{\mathcal{O}_{sp}}(\text{Sp}(\text{Sp}^\Sigma))\]
based on either the positive stable or the positive flat stable model structures on $Sp^\Sigma$ exist (and are cofibrantly generated).

Proof. We need to verify the conditions of Lemma 2.3 of [28].

It is immediate that $O_{sp}$ commutes with filtered direct colimits, and the smallness conditions are obviously satisfied by adapting the argument made in the proof of Proposition 8.5.

It hence only remains to check that for $J$ a set of generating trivial cofibrations on $Sp(Alg)$, then the closure of $O_{sp}(J)$ under pushouts and transfinite compositions, consists of w.e.s.

Hence suppose now that suppose $X \to Y$ is in $J$, and consider any pushout diagram in $Sp(Alg)$ of the form

$$
\begin{array}{ccc}
O_{sp}(X) & \to & A \\
\downarrow & & \downarrow \\
O_{sp}(Y) & \to & B.
\end{array}
$$

By Lemma 9.2, we will be done provided we can show that the map $A \to B$ is a level monomorphism and an underlying stable equivalence in $Sp(Alg)$.

The strategy for this is to break the map $A \to B$ using the filtration analog to the one used in [15] (pages 17 and adjacent). Explicitly, we write $B$ as $\text{colim}(A_0 \to A_1 \to A_2 \to \ldots)$, where the successive $A_n$ are built as pushouts

$$
\begin{equation}
O_{sp,A}(t) \wedge \Sigma_t Q^t_{t-1} \to A_{t-1} \\
\downarrow & \downarrow \\
O_{sp,A}(t) \wedge \Sigma_t Y^\wedge t \to A_t.
\end{equation}
$$

A little care is needed in explaining the notation here. We do not actually define terms $O_{sp,A}(t)$ in this context because the colimits used for the analogous definition in [15] would make no sense, as they would involve both objects of $Sp^\Sigma$ and of $Sp(Sp^\Sigma)$ simultaneously\footnote{This is a reflection of the fact that the monoidal structure $\wedge_{sp}$ is non unital.}. Instead, we define expressions $O_{sp,A}(t) \wedge Z_1 \wedge_{sp} \ldots \wedge_{sp} Z_t$ as a whole.
by the coequalizers

\[ O_{sp, A}(t) \wedge Z^T = coeq(\prod_{p \geq 0} O(p+t) \wedge \Sigma_p (O A)^{\wedge p} \wedge Z^T) = \prod_{p \geq 0} O(p+t) \wedge \Sigma_p A^{\wedge p} \wedge Z^T), \] 

with the equalized maps corresponding to the algebra structure of \( A \) and the structure maps of the operad.

Additionally, as in [15], the symbol \( Q_{t-1}^f \) is meant to suggest the "union" (i.e. colimit) of the terms in the \( t \)-cube \((X \to Y)^{\wedge p,t} \) with the \( Y^{\wedge p,t} \) terminal vertex removed. Finally, the subscripts \( \wedge \Sigma_t \) are meant to denote that one takes coinvariants with respect to the obvious (diagonal) \( \Sigma_t \) action.

The existence of the relevant maps in 9.4, along with the existence of the relevant maps \( A_t \to B \), is then formally analogous to the treatment in [15], merely replacing the monad \( O \) by the monad \( O_{sp} \).

It then follows immediately that \( B \) is indeed the colimit of the \( A_t \), since these colimits are levelwise on the (outer) spectra coordinates, and our filtration just restricts to the one in [15] for each level.

Hence by Lemma 9.2 it suffices to show that the maps

\[ O_{sp, A}(t) \wedge \Sigma_t Q_{t-1}^f \to O_{sp, A}(t) \wedge \Sigma_t Y^{\wedge t} \]

are all level monomorphisms and stable equivalences. The first of these follows since cofibrations in \( Sp(Sp^\Sigma) \) are always levelwise cofibrations for the underlying chosen model structure on \( Sp^\Sigma \), and by applying Proposition 4.28 of [15]. Note that we do know that the generating trivial cofibrations of \( Sp(Sp^\Sigma) \) have cofibrant domains and codomains, even though we do not have an explicit description for these. Indeed, this follows from the localization theory of [18], namely Proposition 4.5.1 together with the fact that the generating cofibrations in \( Sp(Sp^\Sigma) \) are directly seen to have

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8 We use the shorthand \( Z^T = Z_1 \wedge_p \cdots \wedge_p Z_l \).

9 Though note that here we only ever need to use the first \( p \) partial composition products.

10 Though, again, one needs to view the expression \( O_{sp, A}(t) \wedge Q_{t-1}^f \) defined as a whole.

11 Notice that for this statement we are restricting to either of the positive model structures on \( Sp^\Sigma \).
cofibrant domains and codomains.

For the second part, we note first that, since we are in the case $\mathcal{O}(0) = *$, the equalizer in 9.5 splits into two parts: the part with $p = 0$, which is just $\mathcal{O}(t) \wedge Z^\wedge T$, and the part with $p \geq 1$. We deal with these summands separately.

When $p = 0$, the term for $t = 1$ is clearly a stable equivalence, as we are just smashing every level spectrum with $\mathcal{O}(1)$, and since $X$ and $Y$ are levelwise cofibrant\(^{12}\), this smash product is homotopically significant. Hence

$$\Omega^{\infty, n}(\mathcal{O}(1) \wedge X) = \text{hocolim}_k(\tilde{\Omega}^k(\mathcal{O}(1) \wedge X_{k+n}))$$

$$= \text{hocolim}_k(\mathcal{O}(1) \wedge \tilde{\Omega}^k X_{k+n})$$

$$= \mathcal{O}(1) \wedge \text{hocolim}_k(\tilde{\Omega}^k X_{k+n}) = \mathcal{O}(1) \wedge \Omega^{\infty, n} X$$

and likewise for $Y$, so clearly the map $\mathcal{O}(1) \wedge X \to \mathcal{O}(1) \wedge Y$ is a stable equivalence.

When $p = 0$ and $t > 1$ the map is a weak equivalence simply because the $\mathcal{O}(t) \wedge_{\Sigma_t} Q^t_{t-1}$ and $\mathcal{O}(t) \wedge_{\Sigma_t} Y^t$ spectra are actually nullhomotpic. Indeed, since stable equivalences are detected by the $\Omega^{\infty, n}$ functors, this will follow if we show that the structure maps of these spectra are null homotopic. However, the structure maps $\mathcal{O}(t) \wedge_{\Sigma_t} Y^t$ are constructed by first considering the map

$$S^1 \wedge (Y_n)^t \to S^t \wedge (Y_n)^t \cong (S^1 \wedge Y_n)^t \to (Y_{n+1})^t$$

and then applying the functor $\mathcal{O}(t) \wedge_{\Sigma_t} -$ . But clearly the map above is nullhomotopic, since it factors through a higher suspension, and it remains nullhomotopic after applying the functor $\mathcal{O}(t) \wedge_{\Sigma_t} -$ because the $(Y_n)^t$ are $\Sigma_t$-cofibrant by Proposition 4.28 of [15]. The exact same analysis works for $\mathcal{O}(t) \wedge_{\Sigma_t} Q^t_{t-1}$, provided one does know that the $(Q^t_{t-1})_n$ are $\Sigma_t$-cofibrant. Since unfortunately [15] does not quite prove this, we show this improved result in Appendix A (Corollary A.8).

It remains to deal with the summands for $p \geq 1$. This will again follow from the spectra in those summands being nullhomotopic. Indeed, we claim that the structure

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\(^{12}\)For either of the positive model structures on $Sp^\Sigma$. 

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maps for \((\mathcal{O}_{sp, A}(t) \wedge Y^t)_{p \geq 1}\) factor as

\[
S^1 \wedge (\mathcal{O}_{A_0}(t) \wedge Y_n^t)_{p \geq 1} \to S^2 \wedge (\mathcal{O}_{A_0}(t) \wedge Y_n^t)_{p \geq 1} \to (\mathcal{O}_{A_{n+1}}(t) \wedge Y_{n+1}^t)_{p \geq 1}.
\]

This follows immediately from the following commutative diagram between the equalizers defining these terms\(^1\).

Here the leftward horizontal maps are induced by the diagonal maps \(S^1 \to S^2\), and the rightward ones are the induced by the previously described spectra structure maps.

We notice that the group actions are trivial on these sphere coordinates. That the diagram commutes is essentially the remark that those rightward maps are themselves built out of diagonal maps. Hence it now follows that the \((\mathcal{O}_{sp, A}(t) \wedge Y^t)_{p \geq 1}\) spectrum is nullhomotopic, and the same argument works \((\mathcal{O}_{sp, A}(t) \wedge Y^t)_{p \geq 1}\) (notice that no form of \(\Sigma_i\)-cofibrancy is now required), thereby concluding the proof.

\[\square\]

**Corollary 9.8.** The induced adjunctions \(Sp(\text{Mod}_{\mathcal{O}[1]}) \rightleftarrows Sp(\text{Alg}_\mathcal{O})\) are Quillen equivalences.

**Proof.** We deal first with the case \(\mathcal{O}[1] = S\).

We need to show that, for \(X \in Sp(\mathcal{O}^S)\) cofibrant and \(Y \in Sp(\text{Alg}_\mathcal{O})\) fibrant, a map \(\mathcal{O}_{sp}X \to A\) is w.e. iff the adjoint map \(X \to A\) is. However, the second of these maps factors through the first as \(X \xrightarrow{\mu} \mathcal{O}X \to Y\), where \(\mu\) is the unit of the free-forget adjunction. Hence all that needs to be done is to show that \(\mu\) is a stable equivalence whenever \(X\) is fibrant. But this is now obvious from the proof of Theorem 9.3, which makes clear the fact that the spectra \(\mathcal{O}(n) \wedge Y_n^t\) are all nullhomotopic for \(n \geq 2\).

\(^1\)Here we write \(\Sigma\) for \(S^1 \wedge -\) and omit some \(\wedge\) signs to save space.
For general $O(1)$, the adjunction $Sp(Mod_{O[1]}) \rightleftarrows Sp(Alg_O)$ is just obtained by applying $O \circ O(1) -$ levelwise (see 8.11). Since $O \circ O(1) X = \bigwedge_{n \geq 1} O(n) \wedge X^{\wedge n}$, one needs only show that $O(n) \wedge X^{\wedge n}$ is nullhomotopic. But repeating the argument in the proof of Theorem 9.3 this will follow provided one knows that $X^{\wedge n}$ is $O(1)^{\infty}$-cofibrant when $X$ is $O(1)$-cofibrant. This follows from Proposition A.10.

\[\square\]

**Remark 9.9.** We point out that Theorem 9.3 could have itself been proven by regarding $Sp(Alg_O)$ as the algebras for a certain monad over $Sp(Mod_{O[1]})$.

This would entail studying colimits of the form

\[O \circ O(1) (X) \longrightarrow A \]

\[\downarrow \]

\[O \circ O(1) (Y) \longrightarrow B,\]

where $X$ and $Y$ are $O(1)$-modules, by producing a filtration analogous to the one described in Diagram 9.4. Indeed, the only difference between the normal construction and this one is that $\wedge$ symbols get replaced by relative $\wedge_{O(1)}$ symbols.

One would then obtain a sequence of $(O(1), O(1)^{\infty})$-bimodules $O_A^{O(1)} (n)$, forming a "relative enveloping operad". We point out, however, that $O_A^{O(1)} (n)$ is just $O_A (n)$ again.\(^{15}\) Indeed, it's easy to see that the $O_A (n)$ are themselves $(O(1), O(1)^{\infty})$-bimodules (either by direct analysis of the formula, or by using the canonical map of operads $O \to O_A$), and one sees that these sequences must match since they are both codifying left adjoints to forgetful functors.

**Remark 9.10.** Notice that in the commutative diagram of Quillen adjunctions (with

\[^{14}\text{Notice that the theory from [20] does ensure that a stable model structure on } Sp(Mod_{O[1]}) \text{ exists, since } Mod_{O[1]} \text{ is left proper.}\]

\[^{15}\text{Note that this is not quite obvious from the defining formulae.}\]
vertical adjunctions induced by the map of operads $\mathcal{O} \to \mathcal{O}(1))$

\[
\begin{array}{c}
\text{Alg}_\mathcal{O} \leftrightarrow \text{Sp(Alg}_\mathcal{O}) \\
\downarrow \quad \downarrow \\
\text{Mod}_{\mathcal{O}(1)} \leftrightarrow \text{Sp(Mod}_{\mathcal{O}(1)}
\end{array}
\]

both the lower and right adjunctions are Quillen equivalences. Indeed, that the lower adjunction is a Quillen equivalence follows from Theorem 5.1 of [20], since $\text{Mod}_\mathcal{O}(1)$ was stable to start with. The right adjunction is also the right adjunction in the diagram of adjunctions

\[
\text{Sp(Mod}_{\mathcal{O}(1)} \rightleftarrows \text{Sp(Alg}_\mathcal{O}) \rightleftarrows \text{Sp(Mod}_{\mathcal{O}(1})
\]

induced by the maps of operads $\mathcal{O}(1) \to \mathcal{O} \to \mathcal{O}(1)$. Since these maps compose to the identity in $\mathcal{O}(1)$, so do the adjunctions, so that the right adjunction will be a Quillen equivalence if the first is, and this we showed in Corollary 9.8.

It then follows that we can think of the adjunction

\[
\text{Alg}_\mathcal{O} \overset{\mathcal{O}(1)_{\text{forget}}}{\longrightarrow} \text{Mod}_{\mathcal{O}(1)}
\]

as the stabilizing adjunction for $\text{Alg}_\mathcal{O}$, with $\mathcal{O}(1)_{\text{forget}} - \text{playing the role of } \Sigma^\infty$ and $\text{forget} - \text{the role of } \Omega^\infty$, so that we will occasionally refer to these functors by those names.

We notice that both of these functors also go by other names in the literature. $\Sigma^\infty$ is often called the \textit{indecomposables} functor, since it is obtained from an algebra $X$ by killing elements that can be written as higher $n$-ary operations (i.e. $n \geq 2$). It is also customary, at least for some operads, to denote its derived functor by \textit{topological Andre-Quillen homology}. $\Omega^\infty$ is often called the \textit{trivial extension}, since its image consists of those algebras where the higher $n$-ary operations identically vanish.

Finally, notice that we do not at this point yet know if the model structure in
Algo we constructed fits the paradigm described in Section 8.5.

**Proposition 9.11.** The model structure defined by Theorem 9.3 coincides with the left Bousfield localized model structure as required by Definition 2.14.

**Proof.** We first show that the two model structures have the same cofibrations. Both sets of generating cofibrations are constructed based on the generating cofibrations $I$ of $Sp^E$. For the model structure of Theorem 9.3 the generating cofibrations are then $O_{sp}(\bigcup_n F_n I)$, while for the model structure in Definition 2.14 they are $\bigcup_n F_n O I$. It is straightforward to check that these sets match (indeed, this just repeats the analysis in Proposition 9.1).

Now repeating the analysis above it also follows immediately that the generating trivial cofibrations for the projective model structure on $Sp(AlgO)$ are trivial cofibrations for the model structure from Theorem 9.3, it follows formally that that model structure is a left Bousfield localization of the projective model structure on $Sp(AlgO)$. Since left Bousfield localizations are completely determined by the classes of local objects$^{16}$. By Proposition 3.2 of [20]$^{17}$, the local objects as defined by Definition 2.14 are levelwise fibrant $X_+ \in Sp(AlgO)$ such that the structure maps $X_n \to Map(S_1, X_{n+1})$ are weak equivalences. Since clearly these are the fibrant objects for the model structure of Theorem 9.3, we are done.

$\square$

---

$^{16}$Note that this is not the same as saying that a model structure is determined by the cofibrations and the cofibrant objects.

$^{17}$Note that though that the statement of that Proposition requires left properness, the remarks immediately after the proof point out that that condition is unnecessary when the domains of the generating cofibrations are themselves cofibrant, as is the case for $AlgO$. 

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Chapter 10

Homogeneous functors between \( \text{Mod}_A \) and \( \text{Mod}_B \)

Our goal in this chapter is to classify finitary homogeneous functors from \( \text{Mod}_A \) to \( \text{Mod}_B \). We start by recalling the analogous result for finitary homogeneous functors from \( \text{Sp}^\Sigma \) to \( \text{Sp}^\Sigma \) (this can be found, for instance, as Corollary 2.5 in [22]).

**Theorem 10.1** (Goodwillie). Let \( F: \text{Sp}^\Sigma \to \text{Sp}^\Sigma \) be a finitary homogeneous functor. Then there exists a spectrum \( \delta_n F \) with a \( \Sigma_n \) action such that \( F \) is homotopic to the functor \( X \mapsto (\delta_n F \wedge X^\wedge n)_{h\Sigma_n} \).

The purpose of chapter is to show the following obvious generalization.

**Theorem 10.2.** Let \( F: \text{Mod}_A \to \text{Mod}_B \) be a finitary homogeneous simplicial functor. Then there exists a \( (B, A^\wedge n) \)-bimodule \( \delta_n F \) with a \( \Sigma_n \) action interchanging the \( A \)-module structures such that such that \( F \) is homotopic to the functor \( X \mapsto (\delta_n F \wedge A^\wedge n \ X^\wedge n)_{h\Sigma_n} \).

Here we assume the functor is already simplicial for simplicity, and because that suffices for the purpose we have in mind.

Naturally the bimodule \( \delta_n F \) in the theorem is just meant to be \( F(A(S_*)) \), the value at the free module over the sphere spectrum. The hardest task is to prove that this object can actually be given the desired bimodule structure in a strict sense in general.
Recalling that a (strict) $A$-module structure on $X$ is just a map of (strict) ring spectra $A \to \text{End}(X)$ it is clear that this problem would be immediately solved if one were dealing with spectral functors rather than mere simplicial ones. But since a little introspection reveals that spectral functors and linear simplicial functors share many homotopical properties, the case for $n = 1$ would essentially be solved were one to show that the homotopy theories of such functors are equivalent, and this is the essential goal of the next section.

It should be noticed, however, that the case $n > 1$ does not really present a bigger challenge. Indeed, though $n$-homogeneous functors can not, of course, be made spectral themselves, they correspond to symmetric $n$-multilinear functors (when the target category is stable), and hence dealing with the $n > 1$ case is just a matter of generalizing the $n = 1$ result to show that the homotopy theories of multilinear multisimplicial symmetric functors and multispectral symmetric functors are equivalent.

10.1 Linear simplicial functors are spectral functors

In this section we shall be using several basic notions of enriched category theory, such representable functors and weighted enriched colimits, along with some basic results. We recommend [27] as a general reference for these.

Remark 10.3. Let $Sp^F$ be given the flat stable model structure, and consider the induced projective model structure on $\text{Mod}_A$, for $A$ any ring spectrum.

Then it follows from $Sp^F$ being a monoidal category that $\text{Mod}_A$ is a $Sp^F$ model category with mapping spectra $Sp_A(X, Y) = eq(Sp(A \wedge X, Y) \Rightarrow Sp(X, Y))$ and the obvious (underlying) spectral tensoring and cotensoring.

Notice that $\text{Mod}_A$ is also a simplicial model category by reduction of structure, with the simplicial mapping spaces induced given by $SSet_A(X, Y) = \Omega^\infty Sp_A(X, Y)$, and the obvious (underlying) simplicial tensoring and cotensoring.

We will have use for the following technical result concerning cofibrant/fibrant
Proposition 10.4. \( \text{Mod}_A \) with the model structure described above has simplicial cofibrant and fibrant replacement functors. Furthermore, it also has spectral fibrant replacement functors.

Proof. The statement about simplicial cofibrant and fibrant replacements is well known (see, for example, Theorem 13.5.2 in [27]), as one needs only to perform the enriched version of the Quillen small object argument, which then works because all simplicial sets are cofibrant.

The claim about the existence of a spectral fibrant replacement functor is more delicate precisely because not all spectra are cofibrant, but a careful analysis of the enriched small object argument in that case still provides a fibrant replacement.

Indeed, note that a general generating trivial cofibration for \( \text{Mod}_A \) has the form \( A \wedge Z \to A \wedge W \), for \( Z \to W \) a generating trivial cofibration of \( \text{Sp}^\Sigma \), and that the enriched argument requires, at the stage \( X_\beta \) for one to build \( X_{\beta+1} \) by gluing \( \text{Sp}^\Sigma(A \wedge Z, X_\beta) \wedge A \wedge W \) to \( X_\beta \) along \( \text{Sp}^\Sigma(A \wedge Z, X_\beta) \wedge A \wedge Z \) (for each generating trivial cofibration). Since the map \( \text{Sp}^\Sigma(A \wedge Z, X_\beta) \wedge A \wedge Z \to \text{Sp}^\Sigma(A \wedge Z, X_\beta) \wedge A \wedge W \) is still a trivial cofibration in the levelwise model structure (though, crucially, possibly not in \( \text{Mod}_A \)), it still does follow that the map \( X \to X_\infty \) is an equivalence, and it is formal to check that \( X_\infty \) is fibrant, proving the result. \( \square \)

Proposition 10.5. Let \( C \subset \text{Mod}_A \) be a small subcategory.

Consider the categories \( \text{Fun}_\text{SSet}(C, \text{Mod}_B) \) and \( \text{Fun}_\text{Sp}^\Sigma(C, \text{Mod}_B) \) of, respectively, enriched simplicial and enriched spectral functors from \( C \) to \( \text{Mod}_B \).

Then the projective model structures on both of these functor categories exist.

Furthermore, these model categories are simplicial and cofibrantly generated, and \( \text{Fun}_\text{Sp}^\Sigma(C, \text{Mod}_B) \) is also a spectral model category.

Proof. Letting \( I \) and \( J \) denote the sets of generating cofibrations and generating cofibrations of \( \text{Mod}_B \), one obtains the natural candidates for the generating cofibrations for the functor categories \( \text{Fun}_\text{SSet}(C, \text{Mod}_B) \) and \( \text{Fun}_\text{Sp}^\Sigma(C, \text{Mod}_B) \), the sets \( \coprod_{c \in \text{Ob}(C)} \text{SSet}(c, -) \wedge I, \text{SSet}(c, -) \wedge J \) and \( \text{Sp}^\Sigma(c, -) \wedge I, \text{Sp}^\Sigma(c, -) \wedge J \).
To see this defines a cofibrantly generated model category one needs to check the conditions of Theorem 2.1.19 of [19]. The only non-obvious condition is 4, the fact $J-cell \subset W \cap (I-cof)$ or, more specifically, proving $J-cell \subset W$. But this follows from the fact that the maps in $SSet(c, -) \wedge J$ and $Sp^\Sigma(c, -) \wedge J$ are levelwise monomorphisms and stable equivalences (since so is the smash of a stable trivial cofibration with any spectrum).

That these categories are simplicial is formal, with mapping spaces defined by

$$SSet(F, G) = eq\left( \prod_{c \in C} SSet(F(c), G(c)) \Rightarrow \prod_{c, c' \in C} SSet(F(c) \wedge SSet(c, c'), G(c')) \right)$$

with each map induced by adjointness using either the structure of either $F$ or $G$ as simplicial functors. It is also formal that one has simplicial tensoring and cotensorings, which are just pointwise, and it is then obvious that the model structure is simplicial (by verifying that condition using the cotensoring).

The proof that $Fun_{Sp^\Sigma}(C, Mod_B)$ is also a spectral model category is entirely analogous.

**Proposition 10.6.** There is a simplicial Quillen adjunction

$$Spf : Fun_{SSet}(C, Mod_B) \rightleftarrows Fun_{Sp^\Sigma}(C, Mod_B) : fyt$$

**Proof.** The right adjoint is the natural “restriction” of spectral functors to simplicial ones obtained by applying $\Omega^\infty$ to the maps

$$Sp^\Sigma_C(c, c') \xrightarrow{G(c, c')} Sp^\Sigma_{Mod}(G(c), G(c'))$$

that compose a spectral functor $G$.

The left adjoint is defined freely by its value on representable functors $SSet(c, -) \wedge X$, which one easily verifies are necessarily sent to $Sp^\Sigma(c, -) \wedge X$. Since any simplicial functor is canonically an enriched weighted colimit of representables, namely

$$F = colim_{(c, X) \in Mod_B}^{W}(SSet(c, -) \wedge X),$$
where the weight $W: C \times D^{op}$ is $W(c, x) = SSet(X, F(c))$, one then defines

$$Spf F = \text{colim}_{c \in C}^{W} (Sp^\Sigma(c, -) \wedge X).$$

It is worth noticing that the model categories we just defined are not the ones we are ultimately interested in, since Goodwillie Calculus deals only with homotopy functors, and since we want to restrict to those topological functors which happen to be 1-homogeneous.

In order to do this, we need to localize the model structures we just defined, and being able to do this is the main purpose of the following result.

**Lemma 10.7.** The model structures on $\text{Funs}_{Set}(C, Mod_B)$ and $\text{Funs}_{Sp^\Sigma}(C, Mod_B)$ are left proper cellular.

**Proof.** Recall that it was shown in the proof of 10.5 cofibrations in these categories are in particular pointwise monomorphic natural transformations.

Hence left properness follows immediately from the fact that in $Mod_B$ the pushouts of weak equivalences by monomorphisms are weak equivalences.

As for cellularity, the proof is essentially a repeat of the proof of 8.5, but with a minor wrinkle, which we explain now. Suppose given a natural transformation $F \xrightarrow{\tau} G$, where $G$ is a cellular functor. Since the generating cofibrations are pointwise monomorphisms, the argument from the proof of 8.5 applies to show that there is a small enough\(^1\) subcellular functor $\tilde{G}$ of $G$ such that the $\tau(c)$ factor uniquely as $F \xrightarrow{\tilde{\tau}(c)} \tilde{G} \xrightarrow{i(c)} G$. The wrinkle to verify is that these $\tilde{\tau}(c)$ assemble to an enriched natural transformation. This amounts to a straightforward diagram chase of diagrams of enriched mapping spaces, but this requires pointing out that the $i(c)$ are monomorphisms in the enriched sense, rather than just categorically.

\(\blacksquare\)

Lemma 10.7 means that, by the main Theorem of [18], one is free to localize these model structures, and hence to obtain appropriate model categories of homotopical,

\(^1\)The cardinal of cells being bounded by the union of the cardinals of all the simplices in the $F(c)$.
excisive, etc, functors. Doing this requires some technical care, however, since our source category $C$ is a small subcategory of $\text{Mod}_A$. Hence, for instance, demanding a functor to be homotopical is not something one would expect meaningful if $C$ is too small, so that w.e. objects can not actually be linked by weak equivalences in $C$.

Ensuring these type of problems do not occur is the goal of the following definition.

**Definition 10.8.** Let $C \subset \text{Mod}_A$ be a small full simplicial/spectral subcategory.

We say $C$ is **Goodwillie closed**, if $\ast \in C$, $C$ is closed under finite hocolimits, tensoring with finite spectra, a chosen spectral fibrant replacement functor (as is ensured to exist by Proposition 10.4), and a chosen simplicial cofibrant replacement functor.

Notice that any small $C$ has a small Goodwillie closure $\bar{C}$, since the closure conditions only ever add "as many" new objects as those already in $C$.

**Definition 10.9.** The **homotopical** model structures $\text{Fun}^h_{\text{SSet}}(C, \text{Mod}_B)$ and $\text{Fun}^h_{\text{Sp}}(C, \text{Mod}_B)$ are the left Bousfield localizations with respect to the maps

$$\text{SSet}(c, -) \wedge X \to \text{SSet}(c', -) \wedge X$$

induced by weak equivalences in $C$.

Notice that the fibrant objects for these model structures are just the levelwise fibrant homotopical functors.

**Remark 10.10.** Notice that since the left adjoint clearly sends fibrant objects to fibrant objects, the adjunction from 10.6 descends to a Quillen adjunction between the homotopical model structures

$$\text{Fun}^h_{\text{SSet}}(C, \text{Mod}_B) \rightleftarrows \text{Fun}^h_{\text{Sp}}(C, \text{Mod}_B).$$

**Definition 10.11.** The linear model structure

$$\text{Fun}^h_{\text{SSet}}(C, \text{Mod}_B)$$
is the further localization with respect to the maps

\[ \ast \to \text{SSet}(\ast, -) \land X, \]

and, for each hococoartesian square

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
\]

in \( C \), the maps \( \text{SSet}(B, -) \land X \circ_{\text{SSet}(D, -) \land X} \text{SSet}(C, -) \land X \to \text{SSet}(A, -) \land X \). (Here in both cases \( X \) ranges over the domains/codomains \( X \) of the generating cofibrations of \( \text{Mod}_B \).)

**Remark 10.12.** Notice that, by Proposition 3.2 of [20], which says that w.e.s in \( \text{Mob}_B \) can be detected by the mapping spaces out of such \( X \), it follows that the fibrant objects of \( \text{Fun}_{\text{SSet}}^{\text{lim}}(C, \text{Mod}_B) \) are precisely the levelwise fibrant, homotopy functors which are homotopically pointed\(^2\) and sending pushout squares to pullback squares.

**Lemma 10.13.** Suppose \( C \) is Goodwillie closed.

Then for \( F \) any homotopical spectral functor, the reduced topological functor \( \text{fgt} F \) is pointed and 1-excisive.

**Proof.** To see that \( \text{fgt} F \) is pointed, note that \( \ast \) is the only object in either \( C \) or \( \text{Mod}_B \) where the identity map and the null self map coincide. But since any functor preserves identity maps and spectral functors preserve null maps it follows that \( \text{fgt} F \) is pointed.

To check that such a functor is 1-excisive it suffices, by the construction of chapter 4, to check that the natural map\(^3\) \( F \to \Omega F \Sigma \) is a weak equivalence (here we are free to assume that \( F \) is pointwise bifibrant, as necessary). Since the target category is stable this is equivalent to showing that the adjoint map \( \Sigma F \to F \Sigma \) is a weak equivalence.

\(^2\)I.e., sending \( \ast \) to a contractible object.

\(^3\)Here we drop \( \text{fgt} \) for simplicity of notation.
We show this by proving the existence of left and right inverses up to homotopy. To do this, let $\Sigma^{-1}$ denote $F_i S^0$ (recall that $\Sigma$ and $\Omega$ are defined based on $F_0 S^1$). Notice that there is a natural w.e. $\Sigma^{-1} \Sigma \to S^0$.

To prove the existence of a left inverse is suffices to do so for $\Sigma^{-1} \Sigma F \to \Sigma^{-1} F \Sigma$, and this inverse provided by the diagram

$$\Sigma^{-1} \Sigma F \to \Sigma^{-1} F \Sigma \to F \Sigma^{-1} \Sigma \to F$$

where the second map follows from $F$ being a spectral functor, and the last map is induced by $\epsilon$. The full composite is then a weak equivalence since it too is induced by $\epsilon$.

The other side is analogous. To show a left inverse it suffices to do so for $\Sigma F \Sigma^{-1} \to F \Sigma \Sigma^{-1}$. This inverse is provided by the diagram

$$\Sigma \Sigma^{-1} F \to \Sigma F \Sigma^{-1} \to F \Sigma \Sigma^{-1} \to F$$

where the first map again follows by $F$ being a spectral functor, the last map from $\epsilon$, and again the full composite is a weak equivalence since it is induced by $\epsilon$.

\[ \Box \]

**Lemma 10.14.** Suppose $C$ is Goodwillie closed.

Then the fibrant replacement of $SSet(c,-) \land X$ in $Fun_{SSet}^{h,lin}(C, Mod_B)$, where $X$ is stable cofibrant, is, up to levelwise fibrant replacement, given by $fgt(Sp^\Sigma((c)^c,(\epsilon)^f)) \land X$, where $(\epsilon)^f$ denotes a (spectral) functorial fibrant replacement functor, and $(\epsilon)^c$ denotes any cofibrant replacement functor.

**Proof.** Since levelwise fibrant replacement does not affect the other fibrancy conditions in $Fun_{SSet}^{h,lin}(C, Mod_B)$ we will largely omit it so as to simplify notation.

First we note that $SSet(c,-) \land X$ a priori fails all fibrancy conditions, as it is neither homotopical, homotopically pointed, or 1-excisive. We deal with these conditions in succession.

First, we claim that the “homotopification” of $SSet(c,-) \land X$ is $SSet((c)^c,(\epsilon)^f) \land$
That the canonical map \( SSet(c, -) \land X \to SSet((c)^c, -) \land X \) is w.e. in the homotopical model structure is immediate since this is precisely one of the maps being localized. Hence one needs only deal with the case of \( c \) a cofibrant object, which we assume in the remainder of the proof. Now notice that there is a canonical natural transformation \( SSet(c, -) \land X \to SSet(c, (-)^f) \land X \), induced by the natural transformation \( id \to (-)^f \), and hence, since clearly \( SSet(c, (-)^f) \land X \) is homotopic, the claim will follow if we show that it is the universal homotopy functor with a map from \( SSet(c, -) \land X \), where universality is read in the homotopy category of functors. Hence let \( SSet(c, -) \land X \to F \) be a natural transformation with \( F \) a homotopy functor. Existence of the factorization then follows immediately from the fact that the map on the right in the natural diagram

\[
\begin{array}{ccc}
SSet(c, -) \land X & \longrightarrow & F \\
\downarrow & & \downarrow \sim \\
SSet(c, (-)^f) \land X & \longrightarrow & F \circ (-)^f
\end{array}
\]

is a levelwise equivalence. Similarly, uniqueness follows from the bottom left map in

\[
\begin{array}{ccc}
SSet(c, -) \land X & \longrightarrow & SSet(c, (-)^f) \land X \\
\downarrow & & \downarrow \sim \\
SSet(c, (-)^f) \land X & \sim & SSet(c, ((-)^f)^f) \land X \\
& & \longrightarrow F \circ (-)^f
\end{array}
\]

being a levelwise equivalence. Indeed, the claim is that the right upper map is determined by the upper composite. But for this is suffices for it to be determined by the lower composite, and this follows from knowing that the indicated maps are levelwise equivalences.

We next show that the homotopically pointed localization of \( SSet(c, (-)^f) \land X \) is \( SSet_*(c, (-)^f) \land X \), where we denote by \( SSet_*(-, -) \) the enrichment over pointed simplicial sets \( SSet_* \) of the simplicial pointed category \( C \). This may seem slightly confusing since \( SSet(-, -) \) and \( SSet_*(-, -) \) have the same underlying simplicial set, but the crucial point here is that the \( \land \) operations are different depending on
which context one is working in, so that \( SSet(c, (-)^l) \wedge X \) becomes reinterpreted as \( (SSet_*(c, (-)^l))_+ \wedge X \) in the pointed simplicial context. But now notice that the natural pushout diagram

\[
\begin{array}{ccc}
X \cong (SSet_*(\ast, (-)^l))_+ \wedge X & \longrightarrow & (SSet_*(c, (-)^l))_+ \wedge X \\
\downarrow & & \downarrow \\
\ast & \longrightarrow & SSet_*(c, (-)^l) \wedge X
\end{array}
\]

is in fact a levelwise homotopy pushout (this follows from the properties of the level cofibration model structure on spectra, since the top map is an injection), and hence it is now clear that \( \text{Map}(SSet(c, (-)^l) \wedge X, F) \cong \text{Map}(SSet_*(c, (-)^l) \wedge X, F) \) for any \( F \) a homotopically pointed functor.

Finally, it remains to show that the "1-excisification" of \( SSet_*(c, (-)^l) \wedge X \) is \( fgt(Sp^\Sigma(c, (-)^l)) \). Recall that, as in chapter 4, the linearization of a pointed (pointwise fibrant) simplicial functor \( F \) can be computed by \( \text{hocolim}_k(\Omega^k \circ F \circ \Sigma^k) \). Consider the natural morphism\(^4\)

\[
\text{hocolim}_k(\Omega^k \circ (SSet_*(c, (-)^l) \wedge X) \circ \Sigma^k) \rightarrow \text{hocolim}_k(\Omega^k \circ (Sp^\Sigma(c, (-)^l) \wedge X) \circ \Sigma^k).
\]

By Lemma 10.13, it suffices to show that this map is a weak equivalence. But up to equivalence this map can be rewritten as

\[
\text{hocolim}_k(\Omega^k \circ \Sigma^\infty SSet_*(c, (-)^l) \circ \Sigma^k) \wedge X \rightarrow \text{hocolim}_k(\Omega^k \circ Sp^\Sigma_*(c, (-)^l) \circ \Sigma^k) \wedge X,
\]

and it hence suffices to show that

\[
\text{hocolim}_k(\Omega^k \circ \Sigma^\infty SSet_*(c, (-)^l) \circ \Sigma^k) \rightarrow \text{hocolim}_k(\Omega^k \circ Sp^\Sigma_*(c, (-)^l) \circ \Sigma^k)
\]

is a weak equivalence, but this follows from the fact that a spectrum \( K \) can be described as \( \text{hocolim}(\Sigma^{\infty-k}(\Sigma^k K)) \), finishing the proof.

\(^4\)Here we omit the reduction \( fgt \) of spectral functors to simplicial functors for simplicity of notation.
Theorem 10.15. The adjunction

\[ \text{Fun}^{h,\text{lin}}_{\text{SSet}}(\text{Mod}^{\text{lin}}_A, \text{Mod}_B) \rightleftharpoons \text{Fun}^h_{\text{Sp}^E}(\text{Mod}^{\text{lin}}_A, \text{Mod}_B) \]

is a Quillen equivalence.

Proof. That this is indeed a Quillen adjunction is just a restating of Lemma 10.13.

Next we check that the (homotopy) unit of the adjunction is a w.e. By the small object argument for the formation of cofibrant replacements it suffices to prove the statement for functors built cellularly from the generating cofibrations \( SSet(c, -) \land X \to SSet(c, -) \land Y \). Obviously the left adjoint functor preserves homotopy colimits, but now notice that so does the right adjoint \( fgt \), since colimits of natural transformations are pointwise in the target. It hence suffices to show that the (homotopy) unit is a w.e. for functors of the form \( SSet(c, -) \land X \). But this is just Lemma 10.14 together with the fact that the homotopical replacement of \( Sp^E(c, -) \land X \) is \( Sp^E(((c)^c, (-)^f) \land X \), as is clear from the proof of that Lemma.

Finally, it suffices to check that the (right derived functor of the) right adjoint is conservative with respect to weak equivalences. But this is clear since in both categories weak equivalences between fibrant functors are given by pointwise weak equivalences, and clearly \( fgt \) preserves those.

The previous theorem naturally generalizes to the case of multilinear functors. We briefly indicate the changes that need to be made to the previous discussion to obtain that result.

Definition 10.16. Let \( C, D \) be categories. A symmetric \( n \)-multifunctor from \( C \) to \( D \) is a functor \( C^n \to D \) together with natural isomorphisms

\[ F \cong F \circ \sigma, \]

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where $\sigma$ denotes the natural action of a permutation on $C^n$, together with the obvious compatibility conditions.

**Theorem 10.17.** Suppose $C$ is Goodwillie closed.

Then there is a Quillen equivalence

$$\text{SymMFun}^h_{\text{SSet}}(\text{Mod}^\text{fin}_A, \text{Mod}_B) \rightleftarrows \text{SymMFun}^h_{\text{Sp}}(\text{Mod}^\text{fin}_A, \text{Mod}_B).$$

**Proof.** First notice that in the category $\text{MFun}_{\text{SSet}}(C, D)$ of multifunctors without symmetry constraints, the role of representable functors is played by the functors of the form $\text{SSet}(c_1, -) \wedge \cdots \wedge \text{SSet}(c_n, -) \wedge X$, for $(c_1, \ldots, c_n) \in C^n$. There is an obvious action of $\Sigma^n$ on such functors (permuting the $c_i$), and hence one sees that the analog of 10.5 showing the existence of a cofibrantly generated projective model structure follows by considering generating cofibrations $1 \wedge \text{SSet}(c_1, -) \wedge \cdots \wedge \text{SSet}(c_n, -) \wedge \Sigma_n \Sigma_n$ and generating trivial cofibrations $J \wedge \text{SSet}(c_1, -) \wedge \cdots \wedge \text{SSet}(c_n, -) \wedge \Sigma_n \Sigma_n$. The case of multispectral functors follows analogously.

The proof of the analog of Lemma 10.7, showing these are left proper cellular model structures, is entirely analogous, and the sets of localizing maps to obtain multihomotopical and multilinear model structures are induced from the $n = 1$ case by

$$S_{\text{mult}} = S_h \wedge \text{SSet}(c_2, -) \wedge \cdots \wedge \text{SSet}(c_2, -) \wedge \Sigma_n \Sigma_n$$

$$S_{\text{multin}} = S_{\text{fin}} \wedge \text{SSet}(c_2, -) \wedge \cdots \wedge \text{SSet}(c_2, -) \wedge \Sigma_n \Sigma_n.$$  

The analog of Proposition 10.6, the existence of an adjunction is perhaps a little less clear, since it may not be obvious how to canonically express a symmetric functor as a colimit of appropriate representables. Rather than proving such a colimit expression, we notice instead that since such a technique does work for the non symmetric functor categories $\text{MFun}$, and since the right adjoint in that case is obviously compatible with the $\Sigma_n$ action on $C^n$, then so is the left adjoint, and hence that left adjoint naturally sends symmetric functors to symmetric functors, providing the

---

5Here we are abusing notation by noting that $S_h$ and $S_{\text{fin}}$ are built out of functors of the form $X^{S^1 \text{Set}(c, -)}$, so that the notion of "permuting the $c_i" makes sense.
desired adjunction.

Finally, to finish the proof that this adjunction is indeed a Quillen equivalence one needs only the appropriate analogs of Lemmas 10.13 and 10.14, the proofs of which require no noteworthy alterations. □

10.2 Characterization of $n$-homogeneous finitary functors

In this section we finish the proof of Theorem 10.2, after a couple of additional Lemmas.

Here is a sketch of the main idea (though we caution the reader that some important technicalities are ignored in the present discussion). As mentioned before, it suffices to prove the associated result saying that any symmetric $G$ multilinear functor has the form

$$X_1, \ldots, X_n \mapsto \delta_n G \wedge_{A^n} X_1 \wedge \cdots \wedge X_n.$$

By a cofibrant replacement argument, one can essentially assume that the objects in the source category are cell complexes. Lemma 10.18 then says that any cell complex is the filtered hocolimit of its finite sucomplexes, meaning that $G$ can be completely recovered by its values on the category of finite complexes $C_0$. But then letting $C$ be a Goodwillie closure of $C_0$, Theorem 10.17 allows one to replace that restriction by a spectral functor $\tilde{G}$, so that $\tilde{G}(A, \ldots, A)$ then has a genuine $(B, A^n)$ module structure, and Lemma 10.19 finally provides the desired map $\tilde{G}(A, \ldots, A) \wedge_{A^n} X_1 \wedge \cdots \wedge X_n \to \tilde{G}(X_1, \ldots, X_n)$, and it is then easy to finish the proof.

We now prove the aforementioned Lemmas.

**Lemma 10.18.** Let $X$ be any cellular object of $\text{Mod}_A$ (based on any of the generating sets of cofibrations in Definition 8.3 other than those for the level cofibration model structure).
Let further $X^{\text{fin}}$ denote the category of finite subcomplexes of $X$ together with their inclusions.

Then one has a canonical equivalence

$$
hocolim_{C \in X^{\text{fin}}} C \sim X.
$$

Proof. We first show that $\text{colim}_{C \in X^{\text{fin}}} C \rightarrow X$ is an isomorphism. Since this colimit is filtered (as the union of finite subcomplexes is clearly a finite subcomplex), this is just equivalent to showing that any simplex of (any spectral level) of $X$ belongs to some finite subcomplex, which we further claim can be chosen minimal. Call such a simplex $s$.

Now recall that $X$ is presented as a transfinite colimit of monomorphisms

$$
* = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \ldots X
$$

with $X_{\beta+1}$ obtained from $X_\beta$ by attaching cells$^6$ $A \wedge F_n \Delta^m$ along their “boundary” $A \wedge F_n \partial \Delta^m$.

Notice that there must be a minimal $\beta + 1$ for which $s \in X_{\beta+1}$, and a single specific cell $e_s$ being attached to $X_\beta$ for which $s \in e_s$ (abusing notation). The claim now follows by induction on $\beta + 1$. Indeed, the “boundary” of $e_s$ has the form $A \wedge F_n \partial \Delta^m$, and a map out this boundary is determined completely by the image of finitely many simplices $y_0, \ldots, y_m$ and, by induction, those images are contained in minimal finite subcomplexes $C_1, \ldots, C_m$, and it is then clear that $C_1 \cup \cdots \cup C_m \cup \{e_s\}$ is the minimal finite subcomplex containing $s$. Notice that from such such a complex being minimal it then follows that the intersection of finite subcomplexes is still a finite subcomplex.

We have now proven that $\text{colim}_{C \in X^{\text{fin}}} C \rightarrow X$, so that the hocolim result will immediately follow from knowing that the identity diagram $X^{\text{fin}} \rightarrow \text{Mod}_A$ is pro-

$^6$Note that though we present the argument for the generating cofibrations based on the stable model structure on $Sp^2$, the argument applies to the other model structures with no major alterations.
jectively cofibrant. This amounts to showing that for any $C$ the canonical map $\text{colim}_{C \subseteq C'} C \to C$ is a cofibration. But this is clear since $\text{colim}_{C \subseteq C'} C'$ is ensured to be a subcomplex of $C$ due to the intersection of finite subcomplexes being a finite subcomplex.

Lemma 10.19. Let $G: C^n \to \text{Mod}_B$ be a multispectral functor, where $C \subseteq \text{Mod}_A$ is a full subcategory closed under the free module construction on objects and containing $A$, the free module on the sphere spectrum $S$. Then there is a natural transformation

$$G(A, \ldots, A) \wedge_{A^n} X_1 \wedge \cdots \wedge X_n \to G(X_1, \ldots, X_n)$$

induced by the "identity" map at $(A, \ldots, A)$.

Furthermore, this natural transformation is compatible with $\Sigma_n$ actions when $F$ is a symmetric functor.

Proof. We first notice that $G(A, \ldots, A)$ does indeed have $n$ commuting right module $A$ structures, which are induced by the composites

$$A \to Sp_A^\Sigma(A, A) \xrightarrow{G_i} Sp_B^\Sigma(G(A, \ldots, A), G(A, \ldots, A))$$

where the first map describes the right $A$-module structure of $A$ in $\text{Mod}_A$ and the second map corresponds to spectrality of $F$ in its $i$-th variable. For the remainder of the proof we deal only with the case $n = 1$ so as not to overbear the notation, but we note that the case $n > 1$ presents no further difficulties.

Now consider the diagram

$$G(A) \wedge A \wedge X \Rightarrow G(A) \wedge X \to F(A \wedge X) \to F(X),$$

where the first two maps correspond to the left $A$-module structure on $X$ and the right $A$-module structure on $G(A)$, the middle map is adjoint to $X \to Sp_A^\Sigma(A, A \wedge X) \to Sp_B^\Sigma(G(A), G(A \wedge X))$, and the final map is obtained by applying $G$ to the structure
multiplication map \( A \wedge X \to X \), which we note is a map in \( \text{Mod}_A \). Since \( F(A) \wedge_A X \) is the coequalizer of the first two maps we will be finished by showing that the top and bottom compositions coincide.

Consider first the composition corresponding to the left module structure on \( X \). By adjointness this corresponds to a composite

\[
A \wedge X \to X \to \text{Sp}_A^\Sigma(A, A \wedge X) \to \text{Sp}_B^\Sigma(G(A), G(A \wedge X)) \to \text{Sp}_B^\Sigma(G(A), G(X)),
\]

and by functoriality of \( F \) with respect to \( A \wedge X \to X \) this is the same as a composite

\[
A \wedge X \to X \to \text{Sp}_A^\Sigma(A, A \wedge X) \to \text{Sp}_B^\Sigma(A, X) \to \text{Sp}_B^\Sigma(G(A), G(X)).
\]

On the other hand, the composition corresponding to the right module structure on \( G(A) \) corresponds by adjointness to a composite

\[
A \wedge X \to \text{Sp}_A^\Sigma(A, A \wedge X) \to \text{Sp}_B^\Sigma(G(A), G(A \wedge X)) \to \text{Sp}_B^\Sigma(G(A), G(X)),
\]

which is rewritten using naturality of \( F \) with respect to \( X \to \text{Sp}_B^\Sigma(A, A \wedge X) \) as

\[
A \wedge X \to \text{Sp}_A^\Sigma(A, A \wedge X) \to \text{Sp}_B^\Sigma(G(A), G(X)),
\]

which can be further rewritten using naturality with respect to the multiplication map \( A \wedge X \to X \) as

\[
A \wedge X \to \text{Sp}_A^\Sigma(A, A \wedge X) \to \text{Sp}_A^\Sigma(A, X) \to \text{Sp}_A^\Sigma(A, X) \to \text{Sp}_B^\Sigma(G(A), G(X)).
\]

Looking at both composites one sees that they both factor as a map \( A \wedge X \to \text{Sp}_B^\Sigma(A, X) \), and winding these definitions one sees that these two maps are precisely the maps that must match for \( X \) to be an \( A \)-module, finishing the proof. \( \square \)

*Proof of Theorem 10.2.* It suffices to show that any symmetric \( G \) multilinear multi-
simplicial functor is equivalent to one of the form

\[ X_1, \ldots, X_n \mapsto \delta_n G \wedge_{A^n} X_1 \wedge \cdots \wedge X_n. \]

First one replaces \( G \) by \( G \circ Q^n \), where \( Q \) denotes the Quillen small object argument cofibrant replacement functor. Since \( Q(f) \) for \( f: X \to Y \) is always a cellular map, one has induced functors\(^7\) \( \text{hocolim}_{((QX)^{\text{fin}})^n} G \to \text{colim}_{((QX)^{\text{fin}})^n} G \to G \circ Q^n \), where \((QX)^{\text{fin}}\) denotes the category of finite subcomplexes of \( QX \). It then follows by Lemma 10.18 that the full composite is a weak equivalence for all \( X \), since it is a weak equivalence on finite complexes and both functors commute with filtered colimits.

Furthermore, it is clear that one can replace the restriction of \( G \) to the category of finite complexes by any other equivalent functor \( \bar{G} \) while still having a zig zag of weak equivalences between \( G \) and \( \text{hocolim}_{((QX)^{\text{fin}})^n} \bar{G} \).

Now let \( C \) be the smallest Goodwillie closed category that contains (a skeleton of) all finite complexes, and that is further closed under forming the free algebra over its objects. Then Theorem 10.17 provides an equivalent spectral functor \( \bar{G} \), and Lemma 10.19 then provides a map of functors \( G(A, \ldots, A) \wedge_{A^n} X_1 \wedge \cdots \wedge X_n \to G(X_1, \ldots, X_n) \).

While it is not necessarily obvious whether this natural transformation is a weak equivalence over the whole of \( C \), since it is so on \((A, \ldots, A)\) it is also so for any tuple with coordinates of the form \( A \wedge F_n \delta \Delta^m \) (as this is a suspension of \( A \) and both sides are linear functors), and hence also on any finite complexes.

It hence finally follows that \( F \) is equivalent to \( \text{hocolim}_{((QX)^{\text{fin}})^n} \bar{G} \), and since this last one is clearly (the left derived functor of)

\[ X_1, \ldots, X_n \mapsto \delta_n \bar{G} \wedge_{A^n} X_1 \wedge \cdots \wedge X_n, \]

the proof is concluded.

\( \square \)

\(^7\)Here we use for \( \text{hocolim}_{((QX)^{\text{fin}})^n} G \) the models \( B(\ast, (QX)^{\text{fin}}, G) \simeq N(-/(QX)^{\text{fin}}) \otimes (QX)^{\text{fin}} \otimes G \), as described in Theorem 6.6.1 of [27], and were we make a (simplicial) pointwise cofibrant replacement of \( G \) if necessary.
Chapter 11

Goodwillie Calculus in the $\text{Alg}_O$ categories

In this chapter we present our main results concerning Goodwillie Calculus as it relates to the $\text{Alg}_O$ categories. We point out that we will throughout deal only with the case where those categories are pointed or, equivalently, $\mathcal{O}(0) = \ast$.

In section 11.1 we finally assemble the results proven so far to characterize the Goodwillie tower of the identity in $\text{Alg}_O$ as being the homotopy completion tower studied in [17] and associated to the truncated operads $\mathcal{O}_{\leq n}$.

In section 11.2 we show that, when studying $n$-excisive functors either to or from $\text{Alg}_O$ one can equivalently study $n$-excisive functors to or from $\text{Alg}_{\mathcal{O}_{\leq n}}$. Combining this with section 11.1 this effectively says that the category $\text{Alg}_{\mathcal{O}_{\leq n}}$ can be recovered from the category $\text{Alg}_O$ purely in Goodwillie calculus theoretic terms. One might then wonder whether an arbitrary homotopical category $\mathcal{C}$ admits an analogous “ truncation” $\mathcal{C}_{\leq n}$, a question the author would like to examine in future work.

Finally, section 11.3 shows that for finitary functors between categories of the form $\text{Alg}_O$ one does have at least a weak version of the chain rule as proved for spaces and spectra (and conjectured in general) in [1].
11.1 The Goodwillie tower of $Id_{ Alg\mathcal{O}}$

In this section we characterize the Goodwillie tower of $Id_{ Alg\mathcal{O}}$. First we introduce some notation.

**Definition 11.1.** Let $\mathcal{O}$ be an operad such that $\mathcal{O}(0) = \ast$. Then the truncation operad $\mathcal{O}_{\leq n}$ is the operad\(^\dagger\) whose spaces are

$$
\mathcal{O}_{\leq n}(m) = \begin{cases} 
\mathcal{O}(m) & \text{if } m \leq n \\
\ast & \text{if } n < m 
\end{cases}
$$

Notice that there is a canonical map of operads $\mathcal{O} \rightarrow \mathcal{O}_{\leq n}$.

**Definition 11.2.** Notice that putting together Theorem 10.2, the results of Part I and Remark 9.10 it follows that any $n$-homogeneous finitary functor $F : Alg\mathcal{O} \rightarrow Alg\mathcal{O}$ has the form\(^\ddagger\)

$$
X \mapsto \Omega^n_\mathcal{O}(\delta_n F \wedge_{\mathcal{O}(1)^n} (\Sigma^n_\mathcal{O} X)^n)_{h\Sigma_n}
$$

where $\delta_n F$ is a $(\mathcal{O}(1), \mathcal{O}(1)^n)$-bimodule, which we call the $n$-derivative of $F$.

The following is the main theorem of this section.

**Theorem 11.3.** The Goodwillie tower of the identity for $Alg\mathcal{O}$ is given by the (left derived) truncation functors $\mathcal{O}_{\leq n} \circ_{\mathcal{O}} -$.

Furthermore, the $n$-derivative is $\mathcal{O}(n)$ itself with its canonical $\mathcal{O}(1), \mathcal{O}(1)^n$-bimodule structure.

We first show that the proposed Goodwillie tower is indeed composed of $n$-excisive functors. This will follow by combining the results of [17] with the main result of the previous chapter.

**Lemma 11.4.** The left derived functors of $\mathcal{O}_{\leq n} \circ_{\mathcal{O}} : Alg\mathcal{O} \rightarrow Alg\mathcal{O}$ are $n$-excisive.

\(^\dagger\)We notice that the fact that this forms an operad depends on the fact that $\mathcal{O}(0) = \ast$.

\(^\ddagger\)Here we use the notation of Remark 9.10.
Proof. In order to avoid the need to introduce cofibrant replacements everywhere we instead deal only with the restrictions of these functors to the cofibrant objects in $Mod_\mathcal{O}$.

The proof is by induction on $n$. The case $n = 1$ follows since $\mathcal{O}_{\leq 1} \circ \mathcal{O}$ is just $\Omega^\infty \Sigma^\infty$.

For the induction step, assume for the moment that $\mathcal{O}$ satisfies the cofibrancy condition needed to apply Theorem 4.21 of [17].

Then letting $\mathcal{O}_n$ denote the $\mathcal{O}$-bimodule with values $\mathcal{O}_n(n) = \mathcal{O}(n)$ and $\mathcal{O}_n(m) = *$ for $n \neq m$, one gets a hofiber sequence

$$\mathcal{O}_n \circ \mathcal{O} \rightarrow \mathcal{O}_{\leq n} \circ \mathcal{O} \rightarrow \mathcal{O}_{\leq (n-1)} \circ \mathcal{O}.$$  \hspace{1cm} \text{(11.5)}

Now notice that $\mathcal{O}_n \circ \mathcal{O} X = \mathcal{O}(n) \wedge_{\Sigma_n} (\mathcal{O}(1) \wedge \mathcal{O} X)^{\wedge n}$, so that this is indeed an $n$-homogeneous functor.

To finish this case, notice that since holimits in algebra categories are underlying, it suffices to prove that $\mathcal{O}_{\leq n} \circ \mathcal{O}$ is $n$-excisive as a functor landing in spectra, and the result for cofibrant operads now follows by induction since spectra are stable.

Now consider the case of a general $\mathcal{O}$. As remarked in [17], any such operad can be replaced by a suitably cofibrancy operad $\mathcal{O}'$, and Theorem 3.26 of that paper shows that the $\mathcal{O}_{\leq n} \circ \mathcal{O}$ and $\mathcal{O'}_{\leq n} \circ \mathcal{O'}$ functors correspond to each other via the $Alg_{\mathcal{O}} \rightleftarrows Alg_{\mathcal{O}'}$ Quillen equivalence. But since the property of a functor being excisive is not changed by transferring over Quillen equivalences (Section 2.5), the result for general $\mathcal{O}$ follows.

$\square$

Proof of Theorem 11.3. In order to avoid the need to introduce cofibrant replacements everywhere we instead deal only with the restrictions of these functors to the cofibrant objects in $Mod_\mathcal{O}$.

First note that since holimits in $Alg_\mathcal{O}$ are underlying one is free to just prove that the $\mathcal{O}_n \circ \mathcal{O}$ form the Goodwillie tower of the identity when viewed as functors landing in $Sp^\mathcal{O}$. Thanks to Lemma 11.4 it remains only to show that the natural map

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Id \to \mathcal{O}_{\leq n} \circ \mathcal{O} - induces an isomorphism on the homogeneous layers 1 through \(n\).

Now consider the natural left Quillen adjoints

\[
\begin{align*}
\text{Mod}_{\mathcal{O}(1)} \xrightarrow{\mathcal{O} \circ (1)} \text{Alg}_{\mathcal{O}} \xrightarrow{(1) \circ \mathcal{O} -} \text{Mod}_{\mathcal{O}(1)}
\end{align*}
\]

which, as noticed in Remark 9.10, compose to the identity. Now, since \(\mathcal{O} \circ (1) -\) is a left Quillen functor it preserves homotopy colimits and hence precomposition with it commutes with the process of forming Goodwillie towers and layers. On the other hand, by Part I and Remark 9.10 any such homogeneous functor factors through \(\mathcal{O}(1) \circ \mathcal{O} -\), and it hence now follows that one can just as well verify the theorem after precomposing with \(\mathcal{O} \circ (1) -\).

We have hence reduced the result to the claim that \(\coprod_{k=1}^{n} \mathcal{O}(n) \wedge_{\Sigma_n} X^n\) is the \(n\)-th excisive approximation to \(\coprod_{k=1}^{\infty} \mathcal{O}(n) \wedge_{\Sigma_n} X^n\). But this is a well known fact about analytic functors, finishing the proof.

\[\square\]

\textbf{Remark 11.6.} Notice that as a particular case of Theorem 11.3 it follows that for any truncated operad \(\mathcal{O}_{\leq n}\) the category of algebras \(\text{Alg}_{\mathcal{O}_{\leq n}}\) has the property that its identity functor is \(n\)-excisive.

\textbf{Remark 11.7.} Theorem 11.3 asserts that the \(n\)-stage of the Goodwillie tower for \(\text{Id}_{\text{Alg}_\mathcal{O}}\) is the monad associated to the adjunction

\[\text{Alg}_\mathcal{O} \subseteq \text{Alg}_{\mathcal{O}_{\leq n}},\]

hence describing it as a left Quillen functor, followed by \(\text{Id}_{\mathcal{O}_{\leq n}}\), followed by a right Quillen functor, and from this perspective the \(n\)-excisiveness of this composite is then a consequence of the \(n\)-excisiveness of \(\text{Id}_{\text{Alg}_{\mathcal{O}_{\leq n}}}\).

Now suppose that \(A\) is a \((\mathcal{O}', \mathcal{O})\)-bimodule (in symmetric sequences), and consider the associated functor

\[\text{Alg}_\mathcal{O} \xrightarrow{FA = A \circ \mathcal{O} -} \text{Alg}_{\mathcal{O}'}.\]

It is then not hard to see that the proof of Theorem 11.3 can (with a little extra
care about cofibrancy conditions) be adapted to show that the Goodwillie tower for $F_A$ is given by the functor $F_{A_{\leq n}}$, where $A_{\leq n}$ denotes the obvious truncated $(O', O)$-bimodule.

Notice that we then have a factorization

$$
\begin{array}{ccc}
Alg_{O} & \xrightarrow{F_{A_{\leq n}}} & Alg_{O'} \\
\downarrow_{O_{\leq n} \otimes O} & & \uparrow_{f_{st}} \\
Alg_{O_{\leq n}} & \xrightarrow{F_{A_{\leq n}}} & Alg_{O'_{\leq n}}
\end{array}
$$

associating to the $n$-excisive functor $F_{A_{\leq n}} : Alg_{O} \rightarrow Alg_{O'}$ an $n$-excisive functor $F_{A_{\leq n}} : Alg_{O_{\leq n}} \rightarrow Alg_{O'_{\leq n}}$ between algebras over the truncated operads. Perhaps more surprising is the fact that any finitary $n$-excisive functor admits a similar factorization. That is the content of the next section.

### 11.2 $n$-excisive finitary functors can be “truncated”

The objective of this section is to prove the following result.

**Theorem 11.8.** Let $Alg_{O} \xrightarrow{F} Alg_{O'}$ be a finitary $n$-excisive functor.

Then there is a finitary $n$-excisive $Alg_{O_{\leq n}} \xrightarrow{\tilde{F}} Alg_{O'_{\leq n}}$ such that one has a factorization (up to homotopy)

$$
\begin{array}{ccc}
Alg_{O} & \xrightarrow{F} & Alg_{O'} \\
\downarrow_{O_{\leq n} \otimes O} & & \uparrow_{f_{st}} \\
Alg_{O_{\leq n}} & \xrightarrow{\tilde{F}} & Alg_{O'_{\leq n}}
\end{array}
$$

(here $Q$ denotes a fixed cofibrant replacement functor)

Furthermore, any two such $\tilde{F}$ are equivalent.

The existence part of Theorem 11.8 is a fairly straightforward consequence of Theorem 11.3 and the following Lemma, which is a fairly direct adaptation of Proposition

---

3Here we abuse notation by also viewing $A_{\leq n}$ as a $(O_{\leq n}, O_{\leq n})$-bimodule, and by hence also denoting by $F_{A_{\leq n}}$ the functor associated to that $(O_{\leq n}, O_{\leq n})$-bimodule.
Lemma 11.9. Let $F, G$ be pointed simplicial homotopy functors between categories of the form $\text{Alg}_O$ (allowing for different operads for the source and target categories). Assume $F$ and $G$ are composable. Then

1. The natural map $P_n(FG) \to P_n((P_nF)G)$ is an equivalence.

2. If $F$ is finitary, then the natural map $P_n(FG) \to P_n(F(P_nG))$ is an equivalence.

Proof. The proof is a straightforward adaptation of that of Proposition 3.1 of [1], and we hence indicate only the main differences.

First, and as was usual in the proofs of Part I, note that one is free to restrict oneself to cofibrant objects and to assume that $F$ and $G$ take bifibrant values.

The proof of part (1) is then essentially unchanged, with the map $P_n(P_n(F)G) \to P_n(FG)$ built using the enrichment of these functors over pointed simplicial sets, and its key properties being retained.

For part (2), the claim that the proof from [5] works when the middle category is spectra is replaced with the claim that that proof works whenever that middle category is $\text{Mod}_A$, with the use of Proposition 6.10 in [14] replaced by its generalization in Proposition 3.27. The remainder of the proof equally follows, as after reducing to the case where $F$ is $n$-homogeneous one can also write $F\tilde{F}'\Sigma^\infty$, where $\Sigma^\infty$ denotes the $O(1)\circ O$ - functor, and since $\Sigma^\infty$ shares all the relevant properties of $\Sigma^\infty$, the result follows.

Lemma 11.10. Let $C$ denote the composite of (derived functors)

$$
\text{Alg}_{O<^n} \xrightarrow{f_*} \text{Alg}_O \xrightarrow{\Sigma_n \circ O \circ Q} \text{Alg}_{O<^n}.
$$

(here $Q$ denotes a cofibrant replacement functor)

Then the canonical counit map $C \to \text{Id}_{\text{Alg}_{O<^n}}$ becomes a weak equivalence after applying $P_n$.  

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Proof. Notice that since $fgt$ is conservative (i.e. it reflects w.e.s) and it commutes with forming $P_n$, one may has well prove the result after postcomposition with $fgt$.

But now consider the canonical composite

$$fgt \circ Q \rightarrow fgt \circ O_\leq \circ Q \circ fgt \rightarrow fgt.$$  

This full composite is a weak equivalence, and it hence suffices to check that the first map is a weak equivalence after applying $P_n$. But this follows by Lemma 11.9, since the first map is induced by the unit $IdO \rightarrow fgt \circ O_\leq \circ Q$, which is just the $n$-excisive approximation of $IdO$ by Theorem 11.3.

\[\square\]

Proof of Theorem 11.8. Then set $\tilde{F}'$ to be the composite

$$AlgO_{\leq n} \xrightarrow{fgt_{\leq n}} AlgO \xrightarrow{F} AlgO' \xrightarrow{O'_\leq \circ Q} AlgO'_{\leq n}$$

and let $\tilde{F} = P_n(\tilde{F}')$.

Since $fgt_{O'_{\leq n}}$ commutes with filtered hocolims and with holims, and since $O_\leq \circ Q$ commutes with hocolimits, one has (a zig zag of) an equivalence $fgt_{O'_{\leq n}} \circ P_n(\tilde{F}') \circ O_\leq \circ Q \sim P_n(fgt_{O'_{\leq n}} \circ \tilde{F}' \circ O_\leq \circ Q)$.

But this is then just $P_n(P_n(IdAlgO) \circ F \circ P_n(IdAlgO))$, and Lemma 11.9 then applies to show that, since $F$ was assumed finitary and $n$-excisive, this functor is just equivalent to $F$ itself, proving the existence of factorization.

For uniqueness, the claim is then that any such $\tilde{F}$ is weak equivalent to $P_n(O'_{\leq n} \circ O' \circ fgt_{O'_{\leq n}})$, and our hypothesis says that this is weak equivalent to $P_n(C \circ F \circ C)$, where $C$ is the composite appearing in Lemma 11.10. One then finishes the proof by applying Lemmas 11.9 and 11.10.

\[\square\]

Remark 11.11. Theorem 11.8 roughly proves an equivalence of homotopy categories

$$Ho(Fun^{fin,\leq n}_{AlgO, AlgO'}) \simeq Ho(Fun^{fin,\leq n}_{AlgO_{\leq n}, AlgO'_{\leq n}}).$$
(here the notation $Fun^{fin, \leq n}$ is meant to denote $n$-excisive finitary functors)

We note however that this should underlie an actual Quillen equivalence of model
categories $Fun^{fin, \leq n}(Alg_\mathcal{O}, Alg_\mathcal{O'})$ and $Fun^{fin, \leq n}(Alg_{\mathcal{O}_n}, Alg_{\mathcal{O}'_n})$.

Indeed, ignoring excisiveness for the moment, the $O_{\leq n} \circ \cdot \ fgt\mathcal{O}$ type adjunc-
tions should include a combined pre and post-composition adjunction between the
functor categories, and this adjunction descends to $n$-excisive functors, which one
should be able to view as a localization of the "model category of all functors". The
proof of Theorem 11.8 is then essentially checking that the unit and counit in this
hypothetical Quillen adjunction are weak equivalences.

Unfortunately it seems hard to construct appropriate $Fun^{fin, \leq n}$ model categories.
For instance, the techniques of section 10.1 essentially break down due to $Alg_\mathcal{O}$ gen-
erally not being left proper$^4$, and one hence does not have direct access to the local-
ization machinery of [18], and that we are current reduced to the weaker form of the
result presented here.

11.3 Proto chain rule

Throughout this section we assume that $\mathcal{O}$ satisfies the cofibrancy condition used in
[17]. Note that in that case $\mathcal{O} \circ \cdot$ is both an homotopy functor when restricted to
stable cofibrant spectra and sends stable cofibrant spectra to stable cofibrant spectra.
This then ensures that the functors appearing in Theorem 11.12 are homotopically
meaningful.

Our goal in this section is to provide some evidence that a result analogous to the
Chain Rule proved in [1] should also hold for functors between the $Alg_\mathcal{O}$ categories.

Firstly, recall that in that paper it was conjectured that for $\mathcal{C}$ a category in which
one can do Goodwillie Calculus one should expect that the derivatives$^5 \delta_*(Id_\mathcal{C})$ form
in some sense an operad, and one can hence view Theorem 11.3 as evidence of this,

---

$^4$In the rare case where $Alg_\mathcal{O}$ is indeed left proper, the main cases of this being when $\mathcal{O}$ is a
monoid (i.e. concentrated in degree 1) or an enveloping operad of the form $Com_\mathcal{A}$, we do however
expect such a treatment to be viable.

$^5$We note though that also part of this conjecture is that there is a sensible way to generally
define such objects in the first place.
since as a symmetric sequence the derivatives $\delta_* Id_{\text{Alg}_O}$ are just $O$ itself. Unfortunately we have no intrinsic construction of an operad structure on the $\delta_* Id_{\text{Alg}_O}$ with which to compare the operad structure on $O$, but we can offer at least some heuristic in that sense. Namely, given a homotopy functor $F: \text{Alg}_O \to \text{Alg}_{O'}$ consider the composite

$$Sp^\Sigma \xrightarrow{O \circ Q} \text{Alg}_O \xrightarrow{F} \text{Alg}_{O'} \xrightarrow{fgt} Sp^\Sigma.$$  

As was argued in the proof of Theorem 11.3 one can read the derivatives of $F$ as the derivatives of this composite\(^6\). Applying this when $F = Id_{\text{Alg}_O}$, and using the already known chain rule for functors in $Sp^\Sigma$, one gets a hypothetical operad structure map

$$\delta_* (Id_{\text{Alg}_O}) \circ \delta_* (Id_{\text{Alg}_O}) \simeq \delta_* (O \circ -) \circ \delta_* (O \circ -) \to \delta_* (O \circ -) \simeq \delta_* (Id_{\text{Alg}_O})$$

with the middle map\(^7\) induced by the natural map of functors $fgt \circ O \circ fgt \circ O \to fgt \circ O$.

Taking this idea further one can also construct “module structures” over $O$ for the derivatives of other functors to or from $\text{Alg}_O$. For instance, in the case of $F$ a functor to $\text{Alg}_O$ one can write

$$\delta_* (Id_{\text{Alg}_O}) \circ \delta_* (F) \simeq \delta_* (O \circ -) \circ \delta_* (fgt \circ F \circ O' \circ -) \to \delta_* (fgt \circ F \circ O' \circ -) \simeq \delta_* (F),$$

where the middle map is induced by the natural transformation $fgt \circ O \circ fgt \circ F \to fgt \circ F$. The case of functors from $\text{Alg}_O$ is similar.

One has the following result, which according to the previous remarks can be viewed as a weak version of the chain rule. We note that this is very closely in form (and proof) related to Theorem 16.1 of [1].

\(^6\)Strictly speaking the argument used in that proof precomposed with the maps $\text{Mod}_O(1) \xrightarrow{O \circ (1)^n} \text{Alg}_O$ and $\text{Alg}_O \xrightarrow{fgt} \text{Mod}_O(1) \xrightarrow{O}$, so now we are actually disregarding the $(O (1), (O (1)^n))$-bimodule structures.

\(^7\)Technically speaking this map actually requires “inverting” the equivalence.
Theorem 11.12. Consider finitary homotopy functors

\[ Alg_{\mathcal{O}} \xrightarrow{F} Alg_{\mathcal{O}} \xrightarrow{G} Alg_{\mathcal{O}'} \]

Then the canonical maps

1. \[ |P_n B_\bullet (fgt \circ G \circ \mathcal{O}, fgt \circ \mathcal{O}, fgt \circ F \circ \mathcal{O}')| \xrightarrow{\sim} P_n(fgt \circ G \circ F \circ \mathcal{O}') \]

2. \[ |D_n B_\bullet (fgt \circ G \circ \mathcal{O}, fgt \circ \mathcal{O}, fgt \circ F \circ \mathcal{O}')| \xrightarrow{\sim} D_n(fgt \circ G \circ F \circ \mathcal{O}') \]

are equivalences.

Here the geometric realizations are taken in the homotopical sense (i.e. they are performed after making a Reedy cofibrant replacement).

Proof. The overall strategy of the proof mimics that of Theorem 16.1 of [1], and we hence focus on the points at which specific properties of \( Alg_\mathcal{O} \) need to be used.

Notice that part (2) will follow immediately once we know (1), as homotopy fibers commute with geometric realization.

We first deal with part (1). The first step is to notice that, by 11.9, the map \( G \to P_n G \) induces levelwise equivalences between the relevant augmented simplicial objects, so it suffices to prove the result assuming \( G \) is \( n \) - excisive.

Consider now a hofiber sequence of functors \( G' \to G \to G'' \), so that one wants to check that the result will follow for \( G \) if one knows it for both \( G' \) and \( G'' \). Notice that one then obtains an associated levelwise hofiber sequence of augmented simplicial objects. But since both weak equivalences and (homotopically meaningful) geometric realizations in \( Alg_\mathcal{O} \) are computed in \( Sp^\mathcal{E} \), the result follows immediately from noting that the hofiber sequence of augmented simplicial objects is also an underlying hocofiber sequence.

We have now hence reduced to proving the claim in the case that \( G \) is a finitary \( n \)-homogeneous functor. At this point notice that \( P_n \) actually commutes with the geometric realization (since geometric realizations commute with both filtered homcolimits and the punctured cube holimits), so it sufficed to check that in this case one has an equivalence as in (1) with the \( P_n \) removed. Recall that \( G \) has the form
$X \mapsto \text{triv}_{\mathcal{O}''}(\delta_n G \wedge_{\Sigma_n} TAQ_{\mathcal{O}}(X)^{\wedge n})$, where $\delta_n G$ is a $(\mathcal{O}''(1), \mathcal{O}(1)^n)$-bimodule. This means that in the terms of the simplicial augmented object one has $\text{fgt} \circ \text{triv}_{\mathcal{O}''}$ on the leftmost side, and since this composite is just forgetting from $\mathcal{O}''(1)$-modules to $Sp$, one sees that the homotopy coinvariants $h\Sigma_n$ commute with the whole construction, and can hence be removed, so that one can deal instead with a functor of the form $X \mapsto \text{triv}_{\mathcal{O}''}(\delta_n G \wedge TAQ_{\mathcal{O}}(X)^{\wedge n})$. But in this case the augmented simplicial object being discussed is naturally the diagonal of a $n$ multisimplicial object\footnote{By considering the multilinear functor associated to $G$, $X_1, \cdots, X_n \mapsto \delta_n G \wedge TAQ(X_1, \cdots, X_n)$.}, so that one can replace the geometric realization by a $n$-multigeometric realization. Since this amounts to moving the $\delta_n G \wedge -$ out of the whole thing, we have reduced to the case $G = TAQ$. But $TAQ$ is a left Quillen functor, hence commuting with the (homotopical) realization, and one can hence take $G$ as the identity. The result now follows by Theorem 1.8 of [16], saying that any $\mathcal{O}$-algebra is canonically the (homotopical) realization of its bar construction.
Appendix A

Cofibrancy of $Q^n_S(i)$

When proving Theorem 9.3 we used the fact that $Q^t_{i-1}(i)$ (in [15] notation) is a projective cofibrant $\Sigma_t$ object whenever $i: X \to Y$ is a cofibration between cofibrant objects in the positive flat stable model structure in $Sp^{E}$. This result is very closely related to Proposition 4.28 in [15] (though we focus our attention on mere symmetric spectra rather than symmetric sequences). Indeed, both of the results above would follow immediately from showing that $Q^t(i) \to Q^{t'}(i), t \leq t'$ is a projective $\Sigma_t$-cofibration. Unfortunately, though the proof presented in [15] proves many instances and consequences of this last result, it does not quite prove the full result, the proof of which will be the goal of this section.

We start with some notation.

**Definition A.1.** Let $i: I \to Sp^{E}$ be any diagram.

We then let $i^{\wedge n}$ denote the "cubical" diagram

$$i^{\wedge n}: I^{\times n} \xrightarrow{i^{\times n}} (Sp^{E})^{\times n} \xrightarrow{\Delta} Sp^{E}.$$

Now let $S \subset I^{\times n}$ be any subset which is symmetric with respect to the obvious $\Sigma_n$ action on $I^{\times n}$. We denote

$$Q^n_S(i) = colim_S(i^{\wedge n}).$$
Notice that since $\wedge$ is a symmetric model structure and $S$ is a symmetric set, this spectrum inherits a $\Sigma_n$ action.

**Remark A.2.** Notice that what is denoted by $Q^n(i)$ in [15] gets reinterpreted in this notation as $Q^n_S(i)$ where $i = X \to Y$ is a viewed as a map $(0 \to 1) \to Sp^\Sigma$ and $S$ is the set of objects of $(0 \to 1)^\times$ of tuples using at most $t$ 1's.

Notice that the maps $Q^n(i) \to Q^n_{t'}(i), t \leq t'$ from [15] get reinterpreted as the maps $Q^n_S(i) \to Q^n_{S'}(i), S \subset S'$ for $S, S'$ any $\Sigma_n$ symmetric downward closed\(^1\) subsets.

We shall need the following fact, which is used in the proofs in [15].

**Proposition A.3.** Let $i: X \to Y$ be a pushout of one of the generating cofibrations for the positive flat stable model structure in $Sp^\Sigma$. Assume further that $X, Y$ are themselves cofibrant.

Then the maps $Q^n(i) \to Q^n_{t'}(i), t \leq t'$ are projective $\Sigma_n$-cofibrations.

**Proof.** This is implicitly proved in [15], while proving Proposition 4.28 of that paper (note that this proof is found in section 6). Namely, it is proven explicitely that $Q^n_{t-1}(i) \to Y^\Lambda^t$ is a cofibration, and the lower cases $Q^n_{t}(i) \to Q^n_{t+1}(i), 0 \leq k \leq t + 2$ follow by induction from the case $Q^n_{k+1}(i) \to Y^\wedge k+1$ and formula (4.14) in that paper. □

Now recall that any cofibration in $Sp^\Sigma$ is a retract of a transfinite composition of maps as in Proposition A.3, which should hence be thought of as a sort of “base case”. This means that to prove the kind of result we want it now essentially remains to show that the composite of maps satisfying the conclusion of A.3 also satisfies it. This is the reason for the generality in Definition A.1: proving this result is best done by considering diagrams based on the category $(0 \to 1 \to 2)$.

The following is the main result of this section.

**Proposition A.4.** Let $i: (0 \to 1 \to 2) \to Sp^\Sigma$ be a diagram of cofibrations between cofibration objects $Z_0 \xrightarrow{f_0} Z_1 \xrightarrow{f_1} Z_2$ for the positive flat stable model structure.

\(^1\)We say that $S$ is downward closed if for any object $x \in S$ and map $y \to x$, so is $y \in S$. 
Suppose further that for the $f_i$ one has that the maps $Q^n_t(f_i) \to Q^n_{t'}(f_i), t \leq t'$ are projective $\Sigma_n$-cofibrations.

Now let $S \subset S' \subset (0 \to 1 \to 2)^{\times n}$ be any symmetric downward closed subsets. Then the map

$$Q^n_S(i) \to Q^n_{S'}(i)$$

is a projective $\Sigma_n$-cofibration.

Proof. Without loss of generality it suffices to consider the case where $S'$ is obtained from $S$ by adding as little as possible, namely a single orbit, the orbit of a point $e = \{2\}^{\times m_2} \times \{1\}^{\times m_1} \times \{0\}^{\times m_1}$. Now letting $S_e$ be the tuples under and different from $e$ (note that our hypothesis implies $S_e \subset S$), one sees that one has a pushout diagram

$$\begin{array}{ccc}
\Sigma_n \times_{\Sigma_{m_2} \times \Sigma_{m_1} \times \Sigma_{m_0}} Q^n_{S_e}(i) & \longrightarrow & Q^n_{S}(i) \\
\downarrow & & \downarrow \\
\Sigma_n \times_{\Sigma_{m_2} \times \Sigma_{m_1} \times \Sigma_{m_0}} Z_2^{m_2} \wedge Z_1^{m_1} \wedge Z_0^{m_0} & \longrightarrow & Q^n_{S'}(i).
\end{array}$$

The result will hence follow if one shows that the left hand map is a $\Sigma_n$-cofibration, and for this it suffices to show that $Q^n_{S_e}(i) \to Z_2^{m_2} \wedge Z_1^{m_1} \wedge Z_0^{m_0}$ is a $\Sigma_{m_2} \times \Sigma_{m_1} \times \Sigma_{m_0}$-cofibration.

Now notice that $S_e$ decomposes as $S_2^2 \cup_{S_2^1} S_1^1$, where $S_2^2$ is the set of tuples where one "reduces" at least one of the coordinates with a 2, and $S_1^1$ the set where one reduces at least one coordinate with a 1. $S_2^{2,1}$ is the intersection, where one performs both reductions. Since these conditions are independent, and $\wedge$ preserves colimits in each variable, it then follows that

$$Q^n_{S_2^2}(i) = Q^{m_2}_{m_2-1}(f_2) \wedge Z_1^{m_1} \wedge Z_0^{m_0},$$

$$Q^n_{S_1^1}(i) = Z_2^{m_2} \wedge Q^{m_1}_{m_1-1}(f_1) \wedge Z_0^{m_0},$$

$$Q^n_{S_2^{2,1}}(i) = Q^{m_2}_{m_2-1}(f_2) \wedge Q^{m_1}_{m_1-1}(f_1) \wedge Z_0^{m_0}.$$
One the concludes that the map $Q^n_{S^e}(i) \to Z_2^{m_2} \wedge Z_1^{m_1} \wedge Z_0^{m_0}$ is given by smashing $Z_0^{m_0}$ with the pushout product of the maps $Q^{m_2}_{m_2-1}(f_2) \to Z_2^{m_2}$ and $Q^{m_1}_{m_1-1}(f_1) \to Z_1^{m_1}$. Since these are all appropriately cofibrant, the result finally follows from the fact that the functor $(Sp^E)^{\Sigma_{m_2}} \times (Sp^E)^{\Sigma_{m_1}} \times (Sp^E)^{\Sigma_{m_0}} \xrightarrow{\Delta} (Sp^E)^{\Sigma_{m_2} \times \Sigma_{m_1} \times \Sigma_{m_0}}$ is a Quillen trifunctor.

\[ \square \]

**Remark A.5.** We notice that in the previous proof we needed to know that when $Z_0$ positive is flat stable cofibrant then $Z_0^{m_0}$ is a projective $\Sigma_{m_0}$-cofibrant object. This of course follows from Proposition 4.28 in [15], but this is a bit redundant, since the whole purpose of this section is to provide a refinement of that proof.

Rather, we notice that since knowing the $\Sigma_n$-cofibrancy of $A^n$ is the only obstacle to applying the results in this section to a positive stable cofibration $A \to B$, the desired $\Sigma_{m_0}$-cofibrancy of $Z_0^{m_0}$ follows by first running the full argument of this section to the map $* \to Z_0$.

**Remark A.6.** We notice that nothing about the previous proof relies too much on using diagrams indexed by $0 \to 1 \to 2$. Indeed, it should be clear how to generalized it to diagrams indexed by $0 \to 1 \to 2 \to 3$ or other finite linear indexing categories. And, by adding a little transfinite argument, it shouldn't be hard to generalize the proof to categories indexing “transfinite compositions”, or even entirely general direct categories\(^3\), provided one then assumes that $i$ is a projective cofibrant diagram.

However, since we have no actual use for such generality (and the notation would become more involved), we restrict ourselves to this simpler case, which suffices for our purposes.

**Corollary A.7.** Let $Z_0 \xrightarrow{f_1} Z_1 \xrightarrow{f_2} Z_2$ be as in A.4.

---

\(^3\)See Definition 2.6
Then the following diagram is a projective cofibrant diagram of $\Sigma_n$-objects$^4$.

\[
\begin{array}{c}
Q^n_t(f_1) \longrightarrow Q^n_t(f_2f_1) \\
\downarrow \quad \quad \quad \downarrow \\
Q^n_{t+1}(f_1) \longrightarrow Q^n_{t+1}(f_2f_1)
\end{array}
\]

Proof. This is a direct consequence of A.4, because all objects in this square can be identified with $Q^n_S(i)$ for some $S$. For $Q^n_k(f_1)$ this is $S^1_k$, the subset of tuples with no 2's and with at least $n - k$ 0's, while for $Q^n_k(f_2f_1)$ it is $S^2_k$, the subset of tuples with at least $n - k$ 0's.

The result then follows immediately from noticing that $S^1_{t+1} \cap S^2_t = S^1_t$ and $S^1_{t+1} \cup S^2_t \subset S^2_{t+1}$.

Corollary A.8. Let $A \to B$ be any cofibration between cofibrant objects for the positive flat stable model structure in $Sp^\Sigma$.

Then the maps $Q^n_t(f) \to Q^n_{t+1}(f)$ are $\Sigma_n$-cofibrations between $\Sigma_n$-cofibrant objects.

Proof. As usual, a retract argument allows one to reduce to the case of a (transfinite) cellular cofibration

\[A = A_0 \overset{i_0}{\to} A_1 \overset{i_1}{\to} A_2 \overset{i_2}{\to} \ldots \to \text{colim} A_i.\]

Denote by $i_\infty$ the full map $A \to \text{colim} A_i$.

Proposition A.7 then implies that

\[
\begin{array}{c}
Q^n_t(i_0) \longrightarrow Q^n_t(i_1i_0) \longrightarrow Q^n_t(i_2i_1i_0) \longrightarrow \ldots \longrightarrow Q^n_t(i_\infty) \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
Q^n_{t+1}(i_0) \longrightarrow Q^n_{t+1}(i_1i_0) \longrightarrow Q^n_{t+1}(i_2i_1i_0) \longrightarrow \ldots \longrightarrow Q^n_{t+1}(i_\infty)
\end{array}
\]

is a projective $\Sigma_n$-cofibrant diagram$^5$, and the result now follows. \qed

$^4$Explicitly, this means that all objects are $\Sigma_n$-cofibrant, that all maps are $\Sigma_n$-cofibrations, and that so is the map $Q^n_t(f_2f_1) \cup_{Q^n_t(f_1)} Q^n_{t+1}(f_1) \to Q^n_{t+1}(f_2f_1)$.

$^5$One needs for this to know that $Q^n_t$ commutes with transfinite composition. This follows directly from the fact that the $X \to X^n$ functors do.
The following result is not strictly necessary for our purposes, but it is of independent interest, as was pointed to the author by David White. Namely, we want to know how much of the previous results holds for cofibrations $A \rightarrow B$ when the objects aren’t cofibrant.

**Proposition A.9.** Let $A \xrightarrow{f} B$ be a positive flat stable cofibration in $Sp^X$. Then $Q^n_{n-1}(f) \rightarrow B^n$ is a $\Sigma_n$-cofibration.

**Proof.** The proof is just a repeat of the results in this section. We merely list the relevant differences that occur from dropping the cofibrancy hypothesis on the objects.

First, Proposition A.3 now holds only for the case $Q^n_{n-1}(i) \rightarrow Y^n$. Second, Proposition A.4 holds only when $S'$ is obtained from $S$ by adding points that have no $0$ coordinates, but this still allows one to conclude in Proposition A.7 that the vertical maps and the map $Q^n_{n-1}(f_2f_1) \cup Q^n_{n-1}(f_1)(Z_1)^n \rightarrow (Z_2)^n$ are $\Sigma_n$-cofibrations. The result then follows. 

**Proposition A.10.** Let $\mathcal{O}(1)$ be a ring spectrum, and $X \xrightarrow{i} Y$ a (projective) $\mathcal{O}(1)$-positive flat cofibration.

Then $Q^n_{n-1} \rightarrow Y^n$ is a $\mathcal{O}(1)^{ln}$-cofibration.

Furthermore, if $X$ is also $\mathcal{O}(1)$-positive flat cofibrant, then the maps $Q^n_i \rightarrow Q^n_{i'}$ are all $\mathcal{O}(1)^{ln}$-cofibrations.

**Proof.** It is fairly clear how to adapt the arguments in this section. The less obvious point is maybe whether the functor $\text{Mod}_{\mathcal{O}(1)^{ln}_2} \times \text{Mod}_{\mathcal{O}(1)^{ln}_1} \times \text{Mod}_{\mathcal{O}(1)^{ln}_0} \xrightarrow{\sim} \text{Mod}_{\Sigma_2 \times \Sigma_1 \times \mathcal{O}(1)^{ln}_2 + \Sigma_1 + \mathcal{O}(1)^{ln}_0}$ in indeed a Quillen trifunctor, but as usual it suffices to check it on generating cofibrations, for which this is obvious. 

\[\Box\]
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