On the Regular Slice Spectral Sequence

by

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Abstract

In this thesis, we analyze a variant of the slice spectral sequence of [HHR] (or SSS) called the regular slice spectral sequence (or RSSS). This latter spectral sequence is defined using only the regular slice cells. We show that the regular slice tower of a spectrum is just the suspension of the slice tower of the desuspension of that spectrum. Hence, many results for the RSSS are equivalent to corresponding results for the SSS. However, the RSSS has many multiplicative properties that the SSS lacks. Also, the slice towers that have been computed prior to this thesis happen to coincide with the corresponding regular slice towers. Hence, we find the RSSS to be much better behaved than the SSS. We give a comprehensive study of its basic properties, including multiplicative structure, Toda brackets, interaction with the norm functor of [HHR], vanishing lines and preservation of various kinds of extra structure. We identify a large portion of the first page of the spectral sequence algebraically by relating the RSSS to the homotopy orbit and homotopy fixed point spectral sequences, and determine the edge homomorphisms. We also give formulas for the slice towers of various families of spectra, and give several sample computations. The regular slice tower for equivariant complex K-theory is used to prove a special case of the Atiyah-Segal completion theorem. We also prove two conjectures of Hill from [Hil] concerning the slice towers of Eilenberg MacLane spectra, as well as spectra that are concentrated over a normal subgroup.

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Contents

Chapter 0

Introduction

The slice spectral sequence is a tool that originated in motivic homotopy theory (see [Voe]). An analogous construction was later introduced into equivariant stable homotopy theory for finite groups by Hill, Hopkins and Ravenel ([HHR]) in their solution of the Kervaire invariant problem. The first example of an equivariant slice spectral sequence, for the $K\mathbb{R}$ spectrum, was previously given by Dugger ([Dug]), though it was not called a slice spectral sequence at the time. This spectral sequence gives a simple proof of real Bott periodicity. Hence, the slice spectral sequence has already proven itself to be a valuable tool for studying stable homotopy theory, but has not been much studied in its own right.

In this thesis, we give a comprehensive study of the basic properties of a variant of the slice spectral sequence (SSS) called the regular slice spectral sequence (RSSS). The regular slice spectral sequence is defined using only the regular slice cells. We will see in Section I.3 that the regular slice constructions are simply the slice constructions conjugated by a suspension. Hence, results for the SSS can be deduced very easily from corresponding results for the RSSS. Also, in favorable cases such as $K\mathbb{R}$ and the various spectra constructed from $MU\mathbb{R}$ in the solution of the Kervaire invariant problem, the two spectral sequences coincide. Furthermore, the RSSS is much better behaved than the SSS. It has multiplicative structure which the SSS lacks, as well as a form of duality which results in a certain symmetry about 0, while the SSS is symmetric about -1. Hence we work almost exclusively with the RSSS.

In Chapter I we obtain many basic and useful results on the RSSS. After reviewing localizing subcategories and introducing the RSSS, we discuss multiplicative pairings, connecting homomorphisms, and the interaction of the slice filtration with the norm functor of [HHR]. Section I.6 is devoted to proving a conjecture of Hill (Conjecture 4.11 of [Hil]) on the slice towers of spectra that are concentrated over a normal subgroup. The result is stated there as Corollary I.6.5. Next we show that there is an equivariant version of Brown-Comenetz duality in the RSSS. This is used in subsequent sections to obtain "dual" results of many statements. For example, we give a determination of the spectra that are $< n$ for nonpositive n in terms of the vanishing of certain homotopy groups, which then immediately determines the spectra that are $> n$ for nonnegative n in similar terms. This is used in Section I.9 to show that a certain map from the RSSS to the homotopy fixed point spectral sequence (HFPSS) is an isomorphism in a large range on the E_2 page, thus identifying a large portion of the E_2 page algebraically. Duality then implies that a certain map from the homotopy orbit spectral sequence (HOSS) to the RSSS is an isomorphism in a certain range. In this section we also identify the edge homomorphisms of the RSSS, as well as the product structure on (most of) the $t - s$ axis and the "mixed products" of elements in the HOSS range with elements in the HFPSS range.

In Chapter II we give an alternative description of the slice filtration in terms of families of subgroups of order less than a given integer (which we call *order families*). This leads to a kind of formula for the slice tower of an arbitrary spectrum. Each stage is expressed in terms of a finite composite of functors, each of which is a cofiber of a natural map given by an explicit formula. We also obtain explicit formulas for the slice towers of Eilenberg MacLane spectra, as well as one half of the slice towers of free and cofree spectra. All of this gives insight about how subgroups "resonate" within the slice tower with frequencies equal to their orders. In Section II.5 we explain why the behavior of the RSSS changes when one crosses lines of slope one less than the order of a subgroup. We also give a partial, iterative description of the first page of the RSSS which suggests that there can be no general, algebraic formula for all of the entries.

In Chapter III we show that various forms of extra structure are preserved by the slice tower construction, under suitable assumptions. For example we show that the stages of the tower for a ring spectrum can be constructed uniquely in the homotopy category of ring spectra, provided that the ring spectrum in question is (-1)-connected. There are similar results for commutative rings and for modules. In Section III.5 we analyze how the slice filtration interacts with homological localization and acyclization, giving a criterion on homology theories which guarantees that the slice tower of a local spectrum consists of local spectra.

Chapter IV is devoted to proving that Massey products in the RSSS for an A_∞ ring spectrum converge to Toda brackets. The approach is to use model theory to prove that slice towers of rings can be constructed (essentially uniquely) as ring objects in a symmetric monoidal category of towers. These slice towers must have various good, point-set level properties. Hence, this chapter is the most technically demanding in this thesis. Some useful material is given in the Appendix.

In Chapter V we give several computations to illustrate the results of the preceding chapters. In Section V.2 we give computations of the RSSS for Eilenberg MacLane spectra in dimensions ± 1 (the case of dimension 1 confirms a conjecture of Hill from [Hil]), as well as the the slice towers for dimensions ± 2. We also compute the RSSS in all dimensions when the group is cyclic of prime order, and give a partial computation when the group is cyclic of order p^2 (p prime). Next, we give sample computations for certain free and cofree spectra. These last two computations shed some light on the behavior of the E_2 page of the RSSS outside of the region where it coincides with the HFPSS (or the HOSS). In Section V.4 we give an updated treatment of the RSSS for $K\mathbb{R}$, which leads to a simple and elegant proof of real Bott periodicity. Then in Section V.5 we determine the slice tower for equivariant complex K-theory when the group is cyclic. This leads to a very short proof of the Atiyah-Segal completion theorem for cyclic groups of prime order.

One important, philosophical question that arises in studying the slice spectral sequence is "What's so special about regular representations?" In fact, many of the most important results about the RSSS, no matter how technical, can be seen to derive from the following list of properties.

(i) ρ_G contains precisely one copy of the trivial representation.

(ii) If $f : G \to G'$ is an isomorphism then $f^*\rho_{G'} \cong \rho_G$.

(iii) If $H \subseteq G$ then $\operatorname{Res}^G_H \rho_G \cong |G/H|\rho_H$.

(iv) If $H \subseteq G$ then $\operatorname{Ind}^G_H \rho_H \cong \rho_G$.

(v) If $N \trianglelefteq G$ then $(\rho_G)^N \cong \rho_{G/N}$.

(vi) If $\phi : G \to \Sigma_i$ is a homomorphism and $(V^{\oplus i})^\phi$ denotes the direct sum of i copies of V with diagonal action multiplied by the permutation action induced by ϕ, then

$$(\rho_G^{\oplus i})^\phi \cong i\rho_G.$$

Property (i) excuses us from considering suspensions of slice cells (Lemma I.3.8). Properties (ii) and (iii) ensure that the slice tower construction commutes with restriction to subgroups, and that it is multiplicative. Property (iv) is the reason that the norm functor multiplies slice filtration by index (Corollary I.5.8), while property (v) is the essence of the result on geometric fixed points (Theorem I.6.4). Property (vi) allows the construction of commutative ring slice towers (see Lemma IV.3.7). Thus, the answer to the above question is "quite a lot."

In this thesis our groups G, H, etc. are finite unless otherwise stated; restriction to a subgroup H is denoted by i^*_H. The homotopy category of genuine G-spectra (indexed on a complete G-universe) is denoted by Sp_G, and the real regular representation is denoted by ρ_G. We use S^V to denote the one-point compactification of the representation V, and indicate Mackey functors with and underline (e.g. $\underline{M}$).

Chapter I

Basic Results

1 Introduction

In this chapter we analyze the basic properties of a variant of the slice spectral sequence (SSS) of [HHR], which we call the *regular slice spectral sequence* (RSSS). The definition of the RSSS is very similar to that of the SSS; the difference is that one only uses regular slice cells. After reviewing *localizing subcategories* in Section 2, we define the RSSS in Section 3 and state its precise relationship with the SSS. In fact, the RSSS is simply the SSS conjugated by a suspension (see Corollary 3.2). Hence, many of the results about the RSSS can be easily translated into results about the SSS. However, the RSSS has multiplicative structure which the SSS lacks. Furthermore, in favorable cases that have been computed, such as $K\mathbb{R}$ (see [Dug] or Section V.4) and the various spectra constructed from $MU\mathbb{R}$ in the solution of the Kervaire invariant problem (see [HHR]), the two spectral sequences coincide. Thus, we generally find the RSSS to be much better behaved than the SSS.

In Section 4 we construct multiplicative pairings in the RSSS, including composition products. We also introduce a kind of connecting homomorphism associated to cofiber sequences. In Section 5, we analyze the interaction of the regular slice filtration with the *norm* functor from [HHR]. In Section 6 we state and prove a conjecture of Mike Hill on the slice tower of a spectrum which is *concentrated over* a normal subgroup (Conjecture 4.11 of [Hil]) as Corollary 6.5.

In Section 7 we show that there is a version of Brown-Comenetz duality in the RSSS, which is seen in later sections to be a very potent tool for analyzing the RSSS. One can use this duality to derive a statement about the behavior of the RSSS in one half of the plane ($t - s < 0$ or $t - s > 0$) from a corresponding, dual statement in the other half. This process is on display in Section 8, in which we analyze the relationship between the connectivity of a spectrum and that of its slice tower, proving the "efficiency" of the RSSS. In that section we also determine the $\underline{E}_2$ page on certain vanishing lines and give a determination of the spectra that are $> n$ for positive n in terms of homotopy groups, generalizing Proposition 4.45 of [HHR].

In Section 9, we identify a large portion of the E_2 page of the RSSS by relating it to the homotopy orbit and homotopy fixed point spectral sequences. We then show how to identify the structures from Section 4 in these regions. Thus, the RSSS is most powerful when one has an understanding of the nonequivariant homotopy groups of the spectrum. We also determine the *edge homomorphisms* of the RSSS.

2 Recollections About Localizing Subcategories

We will say that a full (nonempty) subcategory τ of Sp_G is *localizing* if

- a spectrum isomorphic to an object of τ is in τ,
- τ is closed under taking cofibers and extensions,
- τ is closed under wedge sums,
- τ is closed under retract, and
- τ is closed under well-ordered homotopy colimits.

If τ is localizing then it contains the trivial spectrum and is closed under suspension and arbitrary homotopy colimits. From this it follows that the last two conditions above are redundant. See [Far] for more. We will assume from now on that τ is generated by a set T of spectra.

Define $\tau \perp$ to be the full subcategory of spectra X such that $[Y, X] = 0$ for all $Y \in \tau$ (or equivalently, for Y an iterated suspension of a generator). We write $\tau \perp X$ in place of $X \in \tau \perp$. Note that

- a spectrum isomorphic to an object of $\tau \perp$ is in $\tau \perp$,
- $\tau \perp$ is closed under desuspension and taking fibers and extensions,
- $\tau \perp$ is closed under products, and contains the trivial spectrum,
- $\tau \perp$ is closed under retract and well-ordered homotopy limits, and
- if τ has a set of compact generators then $\tau \perp$ is closed under wedge sums and directed homotopy colimits.

Recall from [Far] that the inclusion $\tau \perp \subseteq Sp_G$ has a left adjoint $P^{\tau\perp}$. The construction is the familiar one of iteratively attaching null-homotopies for maps from suspensions of generators. This must be iterated transfinitely in general, but may be iterated only countably many times if the generators are ω-small. Letting

$$P_\tau X := Fib(X \to P^{\tau\perp}X),$$

we have a functorial fiber sequence

$$P_\tau X \to X \to P^{\tau\perp}X$$

where the first map above is the terminal map to X from a member of τ and the second is the initial map from X to a member of $\tau\ \perp$. From this it follows that $X \in \tau$ if and only if $[X, Y] = 0$ whenever $\tau \perp Y$, so the two subcategories τ and $\tau \perp$ determine one another. The map $P_\tau X \to X$ is also characterized by

- $P_\tau X \in \tau$, and
- $[\Sigma^k Y, P_\tau X] \to [\Sigma^k Y, X]$ is surjective for all $k \geq 0$ and injective for all $k \geq -1$ and all generators $Y \in T$.

Thus we can construct $P_\tau X$ by starting with a wedge of suspensions of generators and then iteratively killing the kernel on suspensions (by $k \geq -1$) of generators. Hence, an object X is in τ if and only if X has a transfinite filtration with successive quotients that are wedges of suspensions of generators; if the generators are ω-small we may require the filtration to be finite or an ω-sequence. In what follows, we will generally find the P_τ to be more useful than the $P^{\tau\perp}$.

3 The Regular Slice Spectral Sequence

Just as in [HHR], we define the *slice cells of dimension* k to be the spectra $G_+ \wedge_H S^{n\rho_H}$ for $n|H| = k$ and $G_+ \wedge_H S^{n\rho_H - 1}$ for $n|H| - 1 = k$. Similarly, we define the *regular slice cells* to be the slice cells of the first type listed above. Let τ_k (resp. τ'_k) be the localizing category generated by the slice cells (resp. regular slice cells) of dimension $\geq k$. We will write τ_n^G, etc. if there is more than one group under consideration. The following facts are elementary:

- τ_n and τ'_n are closed under induction and restriction, and thus under smashing with (-1)-connected spectra,
- $\tau_n \subseteq \tau_{n-1}$, $\tau'_n \subseteq \tau'_{n-1}$,
- $\tau'_n \subseteq \tau_n, \tau_n \subseteq \tau'_{n-(|G|-1)}$,
- $S^{\rho_G} \wedge \tau_n \cong \tau_{n+|G|}, S^{\rho_G} \wedge \tau'_n \cong \tau'_{n+|G|}$.

See [HHR] and [Hil] for the basic arguments and results on the slice filtration. Less obvious is the following crucial fact.

Proposition 3.1. *For all* n *we have* $\Sigma\tau_n \cong \tau'_{n+1}$.

Proof. Since Σ is an equivalence of triangulated categories, it suffices to show that the suspension of a slice cell of dimension k is in τ'_{k+1} and that the desuspension of a regular slice cell of dimension k is in τ_{k-1}. The only nontrivial part is showing the inclusion $G_+ \wedge_H S^{n\rho_H+1} \in \tau'_{n|H|+1}$. We prove this by induction on $|G|$; the result is trivial for the trivial group. Thus we may assume the result for all proper subgroups of G. Since induction preserves the regular slice filtration, we may assume that $H = G$. Now take the cofiber sequence

$$S(\rho_G - 1)_+ \to S^0 \to S^{\rho_G - 1}$$

and smash with $S^{n\rho_G+1}$ to obtain a cofiber sequence as below.

$$S(\rho_G - 1)_+ \wedge S^{n\rho_G+1} \to S^{n\rho_G+1} \to S^{(n+1)\rho_G}$$

The spectrum on the left is built out of induced cells, and so the induction hypothesis implies that it is in $\tau'_{n|G|+1}$. The spectrum on the right is in $\tau'_{(n+1)|G|} \subseteq \tau'_{n|G|+1}$. Thus the middle spectrum is in $\tau'_{n|G|+1}$, as required. □

Remark: Amusingly, this implies that τ_n is generated by the *irregular slice cells* of dimension $\geq n$.

Corollary 3.2. *For all n we have natural isomorphisms*

$$\Sigma P_{\tau_n} \cong P_{\tau'_{n+1}} \Sigma.$$

Corollary 3.3. *For all n we have inclusions*

$$\Sigma \tau_n \subseteq \tau_{n+1}, \quad \Sigma \tau'_n \subseteq \tau'_{n+1}.$$

The proofs are immediate. Next we note that under certain circumstances, the slice and regular slice filtrations of a spectrum coincide.

Proposition 3.4. *If $P_{\tau_n}X \in \tau'_n$ then $P_{\tau_n}X = P_{\tau'_n}X$.*

The proof is immediate, considering the universal property which characterizes $P_{\tau'_n}X$. We can also give a criterion in terms of the slices.

Proposition 3.5. *If $P^k_k X \in \tau'_k$ for all k then $P_{\tau_n}X \in \tau'_n$ for all n.*

Proof. Since $P_{\tau_{n+|G|-1}}X \in \tau'_n$, the spectrum $P_{\tau_n}X$ has a finite filtration

$$* \to P_{\tau_{n+|G|-1}}X \to \dots \to P_{\tau_{n+1}}X \to P_{\tau_n}X$$

such that the successive cofibers are in τ'_n. □

This brings us to the following important point.

Remark: The regular slice spectral sequence (or RSSS) is easily seen to have multiplicative pairings (see Section 4), and its cells are self-dual. Thus, we will observe below a kind of duality or symmetry about 0 in the RSSS, while the slice spectral sequence (or SSS) is symmetric about -1 (see Section 7). Furthermore, in favorable cases such as $K\mathbb{R}$ (see [Dug] or Section V.4) and the various spectra constructed from $MU\mathbb{R}$ in the solution of the Kervaire invariant problem (see [HHR]), the two spectral sequences coincide. Thus, in many cases it may be more fruitful to work with the RSSS than the SSS. Note however that in cases where the two coincide, we are guaranteed different vanishing lines of slope $(|G|-1)$. For $t-s<0$ the SSS guarantees a stronger vanishing line, while for $t-s>0$ the RSSS does.

We can use Proposition 3.1 to quickly derive a few more basic facts, using what is known about the τ_n.

Proposition 3.6. *The category τ'_k consists of the $(k-1)$-connected spectra for $k=0,1$. The category τ'_2 consists of the connected spectra X such that $\pi_1^e X = 0$. For $k \geq 0$ and all $H \subseteq G$ we have $G/H_+ \wedge S^k \in \tau'_k$.*

The last statement is proved by induction on k, using Corollary 3.3 above. Finally, since we have $\tau_0 = \tau'_0$ and $S^{\rho_G} \wedge \tau_n \cong \tau_{n+|G|}$, $S^{\rho_G} \wedge \tau'_n \cong \tau'_{n+|G|}$, we conclude:

Corollary 3.7. *For all n the categories $\tau_{n|G|}$ and $\tau'_{n|G|}$ coincide.*

From now on, we work with the RSSS instead of the SSS. To simplify things, we drop the prime from our notation. Results for the SSS can be easily deduced from what follows by applying Proposition 3.1.

If $X \in \tau_n$ we write $X \geq n$ or $X > n-1$, and if $\tau_n \perp X$ we write $X < n$ or $X \leq n-1$. We write P_n in place of P_{τ_n} and P^{n-1} in place of $P^{\tau_n \perp}$, so that we have functorial fiber sequences as below.

$$P_n X \to X \to P^{n-1} X$$

For convenience, we will henceforth use the terms *slice cell*, *slice*, and *slice tower* in the sense of the RSSS (that is, we drop the qualifier "regular" from these expressions). We use the notation $\underline{E}(X)$ for the RSSS of X, which can be thought of as a spectral sequence of Mackey functors, or as a Mackey functor of spectral sequences, indexed as below.

$$\underline{E}_2^{s,t} := \underline{\pi}_{t-s}(P_t^t X)$$

Since we will usually be interested in the homotopy groups of the G-fixed points of X (we obtain the H-fixed points by first restricting the action to H and then taking fixed points), we use $E(X)$ to denote $\underline{E}(X)(G/G)$. We note that, in moving from the SSS to the RSSS, we have shifted the vanishing lines slightly: we now have no nonzero groups strictly below the line $s = (|G| - 1)(t - s)$ for $(t - s) < 0$ and no nonzero groups strictly above this line for $(t - s) > 0$.

Finally, we note the following easy fact, whose proof is an induction on the order of G as in the proof of Proposition 3.1.

Lemma 3.8. *A spectrum X is $< n$ if and only if $[\hat{S}, X] = 0$ for all slice cells $\hat{S}$ of dimension $\geq n$.*

In [HHR], the SSS is conceived of as the spectral sequence of a tower of fibrations $\{P^n X\}$ whose inverse limit is X. However, we prefer to think of the RSSS as coming from an increasing filtration $\{P_n X\}$ on X. We can identify the successive cofibers of the P_n with the successive fibers of the P^n as follows. Consider the composite map shown below.

$$P_n X \to X \to P^n X$$

This factors uniquely through the map $P_n X \to Cofib(P_{n+1}X \to P_n X)$. The map $Cofib(P_{n+1}X \to P_n X) \to P^n X$ then factors uniquely through

$$Fib(P^n X \to P^{n-1}X) \to P^n X.$$

Hence we get a canonical isomorphism

$$Cofib(P_{n+1}X \to P_n X) \xrightarrow{\cong} Fib(P^n X \to P^{n-1}X). \tag{3.9}$$

Of course, one gets equivalent spectral sequences from the two methods (see, for example, Section 5 of [Bou]), though one must take care with signs to identify them exactly. We define the *slice filtration* on the homotopy groups of X by

$$F^s\underline{\pi}_t X := im(\underline{\pi}_t P_{s+t}X \to \underline{\pi}_t X) \tag{3.10}$$

$$= ker(\underline{\pi}_t X \to \underline{\pi}_t P^{s+t-1}X). \tag{3.11}$$

Then the RSSS converges to

$$\underline{E}_\infty^{s,t}X \quad \cong \quad F^s\underline{\pi}_{t-s}X/F^{s+1}\underline{\pi}_{t-s}X.$$

We denote the SSS by $\underline{\tilde{E}}(X)$ and note that we have a natural map from the regular slice tower to the (irregular) slice tower, and thus a natural map of spectral sequences

$$\underline{E}(X) \to \underline{\tilde{E}}(X) \tag{3.12}$$

which is an isomorphism in certain cases (see the remark after Proposition 3.5). It is a nonequivariant isomorphism, and we shall see in Section 9 that it is actually an isomorphism on a large portion of the $\underline{E}_2$ page. We also have by Corollary 3.3 a *suspension map* from the suspension of the slice tower to the slice tower of the suspension (shifted by one), and hence a map of spectral sequences

$$\underline{E}^{s,t}(X) \xrightarrow{\Sigma} \underline{E}^{s,t+1}(\Sigma X) \tag{3.13}$$

which is also a nonequivariant isomorphism. (Actually, this map commutes with the differentials up to a sign of (-1) when we suspend on the left.) A word on sign conventions: if $A \to B$ is a map of spectra, we take the cofiber to be $I \wedge A \cup_A B$, where $I = [0,1]$ is the unit interval with basepoint 1, and we always take ΣX to mean $S^1 \wedge X$, not $X \wedge S^1$.

4 Multiplicative Pairings and Connecting Homomorphisms

In this section we show that the RSSS has multiplicative pairings with familiar properties, as well as a kind of connecting homomorphism for cofiber sequences. We begin with a basic observation.

Lemma 4.1. *If $\hat{S}$ and $\hat{T}$ are slice cells of dimension n and m, respectively, then $\hat{S} \wedge \hat{T}$ is a wedge of slice cells of dimension $n+m$.*

We then obtain the following (compare Proposition 4.25 of [HHR]).

Corollary 4.2. *If $X \geq n$ and $Y \geq m$ then $X \wedge Y \geq n+m$.*

We omit the proofs, which are easy. Next, we recall how to obtain pairings of spectral sequences. Let X and Y be two spectra. Choose explicit (cofibrant) models for the P_nX and the P_mY, as well as for the maps $P_nX \to P_{n-1}X$ and $P_mY \to P_{m-1}Y$. Then we can choose maps as below,

$$\theta_X : \operatorname*{hocolim}_{n \to -\infty} P_nX \xrightarrow{\cong} X$$
$$\theta_Y : \operatorname*{hocolim}_{m \to -\infty} P_mY \xrightarrow{\cong} Y$$

where we construct the homotopy colimits explicity as telescopes. The above maps need not be unique up to homotopy; however, this will not matter. We now denote by $\overline{P}_nX$ the partial telescope up to P_nX, and similarly for Y. Of course, $\overline{P}_nX$ is equivalent to P_nX, and similarly for Y. Next we consider the spectrum

$$(\operatorname*{hocolim}_{n \to -\infty} P_nX) \wedge (\operatorname*{hocolim}_{m \to -\infty} P_mY) \tag{4.3}$$

with the smash product filtration, described as follows. Consider the grid-shaped diagram depicted below.

$$
\begin{tikzcd}
& \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \\
\cdots & P_{i-1}X \wedge P_{j+1}Y \arrow[l] \arrow[d] & P_iX \wedge P_{j+1}Y \arrow[l] \arrow[d] & P_{i+1}X \wedge P_{j+1}Y \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & P_{i-1}X \wedge P_jY \arrow[l] \arrow[d] & P_iX \wedge P_jY \arrow[l] \arrow[d] & P_{i+1}X \wedge P_jY \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & P_{i-1}X \wedge P_{j-1}Y \arrow[l] \arrow[d] & P_iX \wedge P_{j-1}Y \arrow[l] \arrow[d] & P_{i+1}X \wedge P_{j-1}Y \arrow[l] \arrow[d] & \cdots \arrow[l] \\
& \cdots & \cdots & \cdots &
\end{tikzcd}
$$

Let K_n denote the homotopy colimit of the part of this diagram which is on or above the $i + j = n$ diagonal. Then the smash product 4.3 is filtered by the K_n. We need another simple observation.

Lemma 4.4. *The spectrum K_n is $\geq n$.*

Proof. It is easy to see that K_n can be given a countable filtration $\{Z_k\}$ such that $Z_0 = \overline{P}_nX \wedge \overline{P}_0Y$ (for example) and each Z_{k+1} is of the form

$$Z_k \cup_{\overline{P}_{i+1}X \wedge \overline{P}_jY} (\overline{P}_iX \wedge \overline{P}_jY) \quad \text{or} \quad Z_k \cup_{\overline{P}_iX \wedge \overline{P}_{j+1}Y} (\overline{P}_iX \wedge \overline{P}_jY)$$

for some i and j with $i + j = n$. The result now follows easily from Corollary 4.2. $\square$

It follows from this that the map

$$\theta_X \wedge \theta_Y : (\operatorname*{hocolim}_{n \to -\infty} P_nX) \wedge (\operatorname*{hocolim}_{m \to -\infty} P_mY) \to X \wedge Y$$

induces a unique map of towers $\{K_n\} \to \{P_n(X \wedge Y)\}$. We can then find maps to complete the diagrams below.

$$\begin{tikzcd}
K_{n+1} \arrow[r] \arrow[d] & K_n \arrow[r] \arrow[d] & K_n/K_{n+1} \arrow[r] \arrow[d, dotted] & \Sigma K_{n+1} \arrow[d] \\
P_{n+1}(X \wedge Y) \arrow[r] & P_n(X \wedge Y) \arrow[r] & P_n^n(X \wedge Y) \arrow[r] & \Sigma P_{n+1}(X \wedge Y)
\end{tikzcd}$$

These maps are unique, since $[\Sigma K_{n+1}, P_n^n(X \wedge Y)] = 0$. Thus we have arrived at a pairing of spectral sequences, as below.

$$\underline{E}(X) \otimes \underline{E}(Y) \to \underline{E}(X \wedge Y) \tag{4.5}$$

Of course, we have an isomorphism

$$K_n/K_{n+1} \cong \bigvee_{i+j=n} (P_i^i X \wedge P_j^j Y)$$

so we see that the map in question is determined by its restrictions to the spectra

$$(\overline{P}_i X \wedge \overline{P}_j Y)/((\overline{P}_i X \wedge \overline{P}_{j+1} Y) \cup_{\overline{P}_{i+1} X \wedge \overline{P}_{j+1} Y} (\overline{P}_{i+1} X \wedge \overline{P}_j Y))$$

for $i + j = n$. Now since $P_n^n(X \wedge Y) \leq n$ and

$$(\overline{P}_i X \wedge \overline{P}_{j+1} Y) \cup_{\overline{P}_{i+1} X \wedge \overline{P}_{j+1} Y} (\overline{P}_{i+1} X \wedge \overline{P}_j Y) \quad > \quad n,$$

these restrictions are determined by the maps indicated below.

$$\overline{P}_i X \wedge \overline{P}_j Y \to P_n(X \wedge Y)$$

However, these maps fit into a commutative diagram as below,

$$\begin{tikzcd}
\overline{P}_i X \wedge \overline{P}_j Y \arrow[r] \arrow[d] & (\operatorname{hocolim}_{k \to -\infty} P_k X) \wedge (\operatorname{hocolim}_{l \to -\infty} P_l Y) \arrow[d, "\theta_X \wedge \theta_Y"] \\
P_n(X \wedge Y) \arrow[r] & X \wedge Y
\end{tikzcd}$$

and while the right vertical map may not be unique, the composite along the top and right sides is. It then follows from the universal property of P_n that the pairing 4.5 is uniquely determined. Similar techniques can be used to prove the other parts of the following theorem.

Theorem 4.6. *There is a natural, associative and unital system of pairings*

$$\underline{E}(X) \otimes \underline{E}(Y) \to \underline{E}(X \wedge Y)$$

on the RSSS which converge to the smash product pairings on homotopy groups. The differentials interact with the products as follows: for any $r \geq 2$ and any $u \in E_r^{s,t}(X)$ and $v \in E_r^{s',t'}(Y)$ we have

$$d_r(u \cdot v) = d_r u \cdot v + (-1)^{t-s} u \cdot d_r v.$$

These pairings are also commuative, in the sense that

$$v \cdot u = (-1)^{(t-s)(t'-s')} \tau_*(u \cdot v)$$

where $\tau : X \wedge Y \xrightarrow{\cong} Y \wedge X$ is the twist map.

Remark: The unital property means that a certain element in $E_*^{0,0}(S^0)$ acts as a multiplicative unit when we make the identification $S^0 \wedge X \cong X$. Meanwhile, multiplication by a certain element in $E_*^{0,1}(S^1)$ induces the suspension map 3.13.

Using naturality of the RSSS, we obtain the following.

Corollary 4.7. *The RSSS of a ring spectrum R is a spectral sequence of differential graded algebras, which is (graded) commutative if R is, and converges to the associated graded homotopy ring of R for the slice filtration.*

We may map an arbitrary spectrum A into the slice tower for X, obtaining a spectral sequence that we denote by $E(A, X)$. Of course, we have $E(S^0, X) = E(X)$. This spectral sequence may or may not converge to $[A, X]_*$; it clearly does converge (conditionally) when A is compact or (more generally) bounded below. When using the pairings on these spectral sequences, the user must beware that additional factors may arise in the commutativity formula, and these may not even be ± 1.

Alternatively, the spectral sequence $E(F(A,X))$ *does* converge to $[A,X]_*$. Now, we have composition and evaluation pairings as below,

$$F(Y,Z) \wedge F(X,Y) \to F(X,Z)$$
$$F(X,Y) \wedge X \to Y$$

so we get *composition product* pairings on the RSSS

$$E(F(Y,Z)) \otimes E(F(X,Y)) \to E(F(X,Z))$$
$$E(F(Y,Z)) \otimes E(X,Y) \to E(X,Z)$$

that satisfy associative and unital properties.

Finally, we consider connecting homomorphisms. Let

$$A \to B \to C \to \Sigma A$$

be a cofiber sequence. Using $\underline{\tilde{E}}$, etc. to refer to the SSS, we have by Corollary 3.2 a map of spectral sequences

$$\delta : \underline{E}^{s,t}(C) \to \underline{E}^{s,t}(\Sigma A) \cong \underline{\tilde{E}}^{s,t-1}(A) \tag{4.8}$$

which we call a *connecting homomorphism* (actually, it commutes with the differentials up to a sign of (-1)). Now at first sight the composition product and connecting homomorphism may appear much less useful and computable than the corresponding structures in the Adams Spectral Sequence (see [Rav]). In fact, we don't even have an exact sequence of E_2 pages in the above. However, in Section 9 we will give an algebraic description of a large portion of the E_2 page of the (R)SSS, and use it to describe the smash and composition products. We will also give an algebraic description of the connecting homomorphism in this range.

5 The Norm Functor

In this section we determine how the norm functor of [HHR] interacts with the slice filtration. Hence we work with the category of orthogonal G-spectra $Sp^{\mathscr{O}}_G$. The result reflects the following basic fact about slice cells, which is Proposition 4.7 of [HHR]. (To be more precise, by "slice cell" we mean the most obvious choice of model for a slice cell in $Sp^{\mathscr{O}}_G$.)

Proposition 5.1. *If $\hat{S}$ is a wedge of H-slice cells of dimension d and $H \subseteq G$ then $N^G_H \hat{S}$ is a wedge of G-slice cells of dimension $d|G/H|$.*

The proof uses the simple fact that induction of representations maps regular representations to regular representations.

We need some technical facts about the norm. We follow closely the proof of Proposition B.36 of [HHR]. However, we can not simply quote the result there for the following reason: induction from subgroups does not preserve cofibrations. To get around this problem, we refer to [Sto], wherein the author constructs alternative model structures such that cofibrations *are* preserved by induction. These are called "$\mathbb{S}$ model structures." They are very simple to define: one simply enlarges the generating (acyclic) cofibrations by inducing up the classical (acyclic) cofibrations from all subgroups (alternatively, one pulls back along the collection of restriction functors). There are also positive versions of these model structures. For more, see Section A.4.

As in [HHR], we let J be a finite G-set, and denote by $\mathcal{B}_J G$ its translation category. We denote by $Sp^{\mathcal{B}_J G}$ the diagram category in orthogonal spectra, which we call *the category of equivariant J-diagrams.* Choosing points t from each orbit and letting H_t denote their stabilizers, we have an equivalence of categories as below.

$$Sp^{\mathcal{B}_J G} \cong \prod_t Sp^{\mathscr{O}}_{H_t}$$

We give the diagram category the model structure corresponding to the product of the positive stable $\mathbb{S}$ model structures under this equivalence. We have an indexed smash product functor

$$N^J : Sp^{\mathcal{B}_J G} \to Sp_G^{\mathcal{O}}$$

which is the norm when $J = G/H$. We can now state the corrected version of Proposition B.36 of [HHR].

Proposition 5.2. *Let J be a finite G-set. If $X \to Y$ is a cofibration in $Sp^{\mathcal{B}_J G}$ then the indexed smash product*

$$N^J X \to N^J Y$$

is an h-cofibration. It is a positive $\mathbb{S}$-cofibration if X is cofibrant.

The proof in [HHR] works almost unaltered after making these corrections.

The $\mathbb{S}$ model structure is more convenient for our purposes, since the slice cells are defined using induction. Hence, we pull the positive stable $\mathbb{S}$ model structure back to get a model structure on commmutative ring spectra. Then for any subgroup H of G, we have a Quillen pair as below.

$$comm_H \underset{N_H^G}{\overset{\mathrm{Res}_H^G}{\rightleftarrows}} comm_G$$

If one uses the classical model structures, one must use the model structure on $comm_H$ determined by the levels that are restrictions of G-representations (and not all H-representations) in order to get a Quillen pair. It then follows as in Proposition B.42 of [HHR] that the norm functor preserves weak equivalences of positive $\mathbb{S}$-cofibrant spectra. We strengthen this slightly with the following.

Lemma 5.3. *The norm functor preserves weak equivalences between $\mathbb{S}$-cofibrant spectra.*

Proof. Let $X \to Y$ be a weak equivalence, with X and Y being $\mathbb{S}$-cofibrant. Then consider the diagram below.

$$\begin{tikzcd}
S^{-1} \wedge S^1 \wedge X \arrow[r] \arrow[d] & S^{-1} \wedge S^1 \wedge Y \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}$$

Now S^1 and S^{-1} are cofibrant, hence flat by Proposition 7.3 of [MM], and so the vertical maps above are weak equivalences by Lemma 4.5 of [MM]. Since the bottom horizontal map is a weak equivalence by assumption, so is the top horizontal map. Also, the spectra on the top line are positive $\mathbb{S}$-cofibrant. We apply the norm functor to obtain the diagram below.

$$\begin{tikzcd}
S^{-\operatorname{Ind}_H^G \mathbb{R}} \wedge S^{\operatorname{Ind}_H^G \mathbb{R}} \wedge N_H^G X \arrow[r] \arrow[d] & S^{-\operatorname{Ind}_H^G \mathbb{R}} \wedge S^{\operatorname{Ind}_H^G \mathbb{R}} \wedge N_H^G Y \arrow[d] \\
N_H^G X \arrow[r] & N_H^G Y
\end{tikzcd}$$

The vertical maps are weak equivalences by the same reasoning as before, while the top horizontal map is a weak equivalence because the norm functor preserves weak equivalences of positive $\mathbb{S}$-cofibrant objects. Thus, the bottom horizontal map is a weak equivalence. $\square$

Corollary 5.4. *The norm functor is left derivable, and its derived functor can be computed by taking* $\mathbb{S}$*-cofibrant replacements.*

We now return to the notation from the beginning of this section, and note that Proposition 5.2 is still true if we drop the qualifier "positive" from the statement and our definitions.

Corollary 5.5. *The norm functor on equivariant J-diagrams is left derivable, and its derived functor can be computed by taking (nonpositive) cofibrant replacements.*

Proof. Let $X \to Y$ be a weak equivalence of cofibrant equivariant J-diagrams, so that $X_t \to Y_t$ is a weak equivalence of $\mathbb{S}$-cofibrant H_t-spectra for each t. Now $N^J X$

is isomorphic to

$$\bigwedge_t N_{H_t}^G X_t$$

and similarly for Y, so the result follows from Lemma 5.3 and the fact that the $N_{H_t}^G X_t$, etc. are $\mathbb{S}$-cofibrant, hence, flat. $\square$

Next we analyze the effect of the norm on cofibers.

Lemma 5.6. *If $A \to X$ is a map of equivariant J-diagrams then the diagram $N^J Cofib(A \to X)$ has a finite filtration by h-cofibrations which begins with $N^J X$ such that the successive quotients are finite wedges of spectra of the form*

$$G_+ \wedge_L \left(N^{i_L^* J_0} X \wedge N^{i_L^* J_1}(S^1 \wedge A)\right)$$

where L is the stabilizer of some sets J_0 and $J_1 \neq \emptyset$ with $J_0 \coprod J_1 = J$.

For the proof, we refer the reader to the proof of Proposition B.36 of [HHR].

We can now prove the main results in this section. For the statement, we will say that an equivariant J-diagram X corresponding to the collection $\{X_t\}$ of H_t-spectra is $\geq n$ if $X_t \geq n$ for all t. This does not depend on the choice of t, as the slice filtration is clearly preserved by the equivalences of categories of spectra induced by isomorphisms of groups.

Theorem 5.7. *If X is a cofibrant equivariant J-diagram and $X \geq n$ then*

$$N^J X \geq n|J|.$$

Proof. We proceed by induction on the order of J; the result is trivial if J has one element. Hence we may assume that the result has been proven for all finite groups and all sets smaller than J. We may then assume that J consists of one orbit, choosing a single $t \in J$. Since $X_t \geq n$, we may assume by Corollary 5.4 that X_t is built from the trivial spectrum by attaching generating acyclic $\mathbb{S}$-cofibrations and coning off suspensions (by $k \geq -1$) of slice cells of dimension $\geq n$. Now attaching generating acyclic cofibrations does not affect the homotopy groups of the norm by Lemma 5.3, so we are reduced to the following situation. Supposing X_t is $\mathbb{S}$-cofibrant and $\geq n$, $N_{H_t}^G X_t \geq n|J|$, $\hat{S}$ is a slice cell of dimension $\geq n$ and $k \geq -1$ we must show that any pushout

$$\begin{tikzcd} \Sigma^k \hat{S} \arrow[r] \arrow[d] & C(\Sigma^k \hat{S}) \arrow[d] \\ X_t \arrow[r] & Y \end{tikzcd}$$

satisfies $N_{H_t}^G Y \geq n|J|$. For this we apply Lemma 5.6 with $A = \Sigma^k \hat{S}$. Since restriction of group action preserves the slice filtration and $\mathbb{S}$-cofibrations, the induction hypothesis on J implies that $N^{i_L^* J_0} X$ is $\geq n|J_0|$ and $\mathbb{S}$-cofibrant. Also, the L spectrum $N^{i_L^* J_1}(S^1 \wedge A)$ is $\mathbb{S}$-cofibrant and is equivalent by Proposition 5.1 to a wedge of L-slice cells of dimension $\dim(\hat{S})|J_1| \geq n|J_1|$ smashed with a permutation representation sphere S^W (W may be zero). It follows that the smash product of these two is $\mathbb{S}$-cofibrant and $\geq n|J_0| + n|J_1| = n|J|$, and thus the quotients in Lemma 5.6 are $\geq n|J|$. □

Letting $J = G/H$ we obtain the following. (From now on, we use N_H^G to denote the derived norm functor on the homotopy category of spectra in all of our statements.)

Corollary 5.8. *If X is an H-spectrum, with $H \subseteq G$ and $X \geq n$, then*

$$N_H^G X \geq n|G/H|.$$

We now apply this to cofiber sequences.

Theorem 5.9. *Let H be a subgroup of G, and let*

$$A \to X \to C$$

be a cofiber sequence in the homotopy category of H-spectra. If $A \geq n$ and $X \geq m$

then

$$Cofib(N_H^G X \to N_H^G C) \quad \in \quad \Sigma\tau_{\min(n,m)|G/H|+\max(n-m,0)}.$$

Proof. We can model $A \to B$ by a map of $\mathbb{S}$-cofibrant spectra, and model C by the usual cofiber. Then Lemma 5.6 implies that the map $N_H^G X \to N_H^G C$ is an h-cofibration and gives filtration quotients for the corresponding derived cofiber. The L-spectrum $N^{i_L^* J_0} X$ is $\geq m|J_0|$ and is $\mathbb{S}$-cofibrant. The L-spectrum $N^{i_L^* J_1}(S^1 \wedge A)$ is isomorphic to $(N^{i_L^* J_1} A) \wedge S^W$ for some nonzero permutation representation W, and thus is $\mathbb{S}$-cofibrant and contained in $\Sigma\tau_{n|J_1|}$. Thus the smash product is $\mathbb{S}$-cofibrant and contained in $\Sigma\tau_{m|J_0|+n|J_1|}$. The result follows, since J_1 is nonempty. $\square$

The following is immediate.

Corollary 5.10. *If X is an H-spectrum, with $H \subseteq G$, then for each $n \in \mathbb{Z}$ there is a unique map*

$$N_H^G P_n X \to P_{n|G/H|}(N_H^G X)$$

such that the following diagram commutes.

$$\begin{tikzcd} N_H^G P_n X \arrow[r] \arrow[dr] & P_{n|G/H|}(N_H^G X) \arrow[d] \\ & N_H^G X \end{tikzcd}$$

This map is natural in X.

We also obtain a corresponding map for slices.

Corollary 5.11. *If X is an H-spectrum, with $H \subseteq G$, then for each $n \in \mathbb{Z}$ there is a unique map*

$$N_H^G P_n^n X \to P_{n|G/H|}^{n|G/H|}(N_H^G X)$$

such that the following diagram commutes.

$$\begin{tikzcd} N_H^G P_n X \arrow[r] \arrow[d] & P_{n|G/H|}(N_H^G X) \arrow[d] \\ N_H^G P_n^n X \arrow[r] & P_{n|G/H|}^{n|G/H|}(N_H^G X) \end{tikzcd}$$

This map is natural in X.

Proof. By applying Theorem 5.9 to the cofiber sequence below,

$$P_{n+1}X \to P_n X \to P_n^n X$$

we see that the cofiber of the map

$$N_H^G P_n X \to N_H^G P_n^n X$$

is in $\Sigma\tau_{n|G/H|+1}$. Thus, the cofiber and fiber of this map are $\geq n|G/H| + 1$. □

There is yet another natural map when the spectrum is bounded below.

Corollary 5.12. *If X is an H-spectrum, with $H \subseteq G$, then if $X \geq m$ and $n \geq m$ there is a unique map*

$$N_H^G P^n X \to P^{m|G/H|+n-m}(N_H^G X)$$

such that the following diagram commutes.

$$\begin{tikzcd} N_H^G X \arrow[r] \arrow[dr] & N_H^G P^n X \arrow[d] \\ & P^{m|G/H|+n-m}(N_H^G X) \end{tikzcd}$$

This map is natural in X.

Proof. Apply Theorem 5.9 to the cofiber sequence below.

$$P_{n+1}X \to X \to P^n X$$

□

These maps are related as follows.

Proposition 5.13. *If X is an H-spectrum, with $H \subseteq G$, then if $X \geq n$ the following diagram commutes.*

$$\begin{tikzcd} N_H^G P_n^n X \arrow[r] \arrow[d] & N_H^G P^n X \arrow[d] \\ P_{n|G/H|}^{n|G/H|} N_H^G X \arrow[r] & P^{n|G/H|} N_H^G X \end{tikzcd}$$

Proof. It suffices to prove that the two composites are the same after precomposing with the map

$$N_H^G P_n X \to N_H^G P_n^n X.$$

Using the fact that the diagrams of the form below commute,

$$\begin{tikzcd} P_j Y \arrow[r] \arrow[d] & Y \arrow[d] \\ P_j^j Y \arrow[r] & P^j Y \end{tikzcd}$$

the rest is an easy exercise in diagram chasing. □

An application to the slice filtration is given by the following.

Proposition 5.14. *If a map $f \in [A, X]$ of H-spectra is in the image of $[A, P_n X]$, and $H \subseteq G$, then $N_H^G f$ is in the image of $[N_H^G A, P_{n|G/H|} N_H^G X]$.*

This follows directly from Corollary 5.10. Finally, we have some results about norming elements of the RSSS using the map from Corollary 5.11.

Proposition 5.15. *If X is an H-spectrum, with $H \subseteq G$, then for each $n \in \mathbb{Z}$ and $r \geq 2$ there is a unique map filling in the diagram below.*

$$\begin{tikzcd}
N_H^G P_n X \arrow[r] \arrow[d] & P_{n|G/H|} N_H^G X \arrow[d] \\
N_H^G (P_n X / P_{n+r-1} X) \arrow[r, dotted] & P_{n|G/H|} N_H^G X / P_{n|G/H|+r-1} N_H^G X
\end{tikzcd}$$

This map is natural in X, and makes the following diagram commute.

$$\begin{tikzcd}
N_H^G (P_n X / P_{n+r-1} X) \arrow[r] \arrow[d] & P_{n|G/H|} N_H^G X / P_{n|G/H|+r-1} N_H^G X \arrow[d] \\
N_H^G P_n^n X \arrow[r] & P_{n|G/H|}^{n|G/H|} N_H^G X
\end{tikzcd}$$

Proof. Consider the following cofiber sequence.

$$P_{n+r-1} X \to P_n X \to P_n X / P_{n+r-1} X$$

By Theorem 5.9, the fiber and cofiber of the map

$$N_H^G P_n X \to N_H^G (P_n X / P_{n+r-1} X)$$

are $\geq n|G/H| + r - 1$. The result now follows easily. $\square$

The following is immediate.

Corollary 5.16. *Let A and X be H-spectra, with $H \subseteq G$. If $f : A \to P_n^n X$ represents an element of $E_r(A, X)$, then $N_H^G f : N_H^G A \to P_{n|G/H|}^{n|G/H|} N_H^G X$ represents an element of $E_r(N_H^G A, N_H^G X)$.*

Next we show that permanent boundaries norm to permanent boundaries.

Proposition 5.17. *Let A and X be H-spectra, with $H \subseteq G$. If $f : A \to P_n^n X$ represents zero in $E_{r+2}(A, X)$, then $N_H^G f$ represents zero in $E_{r|G/H|+2}(N_H^G A, N_H^G X)$.*

Proof. Since f is hit by a d_{r+1} differential, we can fill in the commutative diagram below.

$$\begin{tikzcd}
A \arrow[dr, dashed] \arrow[drr, "f"] \arrow[ddr, "0"'] & & \\
& P_nX \arrow[r] \arrow[d] & P_n^nX \\
& P_{n-r}X &
\end{tikzcd}$$

It follows that we can fill in the commutative diagram below.

$$\begin{tikzcd}
N_H^GA \arrow[dr, dashed] \arrow[drr, "N_H^Gf"] \arrow[ddr, "0"'] & & \\
& P_{n|G/H|}N_H^GX \arrow[r] \arrow[d] & P_{n|G/H|}^{n|G/H|}N_H^GX \\
& P_{n|G/H|-r|G/H|}N_H^GX &
\end{tikzcd}$$

□

Next we recall how the norm functor interacts with sums. Let H and L be subgroups of G, and X an H-spectrum. Letting $H^c := cHc^{-1}$, we have

$$i_L^*N_H^GX \cong \bigwedge_{[c_jH]\in L\backslash G/H} N_{L\cap H^{c_j}}^L(i_{L\cap H^{c_j}}^*X^{c_j})$$

where X^{c_j} denotes the H^{c_j}-spectrum obtained from X by conjugating the H action by c_j. We now recall how to compute the norm of a sum of maps. We omit the proof.

Lemma 5.18. *Let h_0 and h_1 be maps of H-spectra from X to Y, with $H \subseteq G$. Let $\{f_i\}$ be a set of orbit representatives for $\{0,1\}^{G/H}$, and let L_i denote the stabilizer of f_i. Then we have the following.*

$$N_H^G(h_0+h_1) = \sum_i t_{L_i}^G\Big(\bigwedge_{[c_{ij}H]\in L_i\backslash G/H} N_{L_i\cap H^{c_{ij}}}^{L_i}\big(r_{L_i\cap H^{c_{ij}}}^{H^{c_{ij}}} h_{f_i(c_{ij}H)}^{c_{ij}}\big)\Big)$$

Remark: To obtain the above sum formula for the norm map on the E_2 page of the RSSS, we need the commutativity of the following diagram, where the top horizontal map is the restriction of the map from Corollary 5.11 and the bottom horizontal map is the smash product of these maps.

$$\begin{tikzcd}
i_L^* N_H^G P_n^n X \arrow[r] \arrow[d, "\cong"'] & i_L^* P^{n|G/H|}_{n|G/H|} N_H^G X \arrow[d, "\cong"] \\
\bigwedge_{[c_jH]\in L\backslash G/H} N^L_{L\cap H^{c_j}} i^*_{L\cap H^{c_j}} (P_n^n X)^{c_j} \arrow[d, "\cong"'] & P^{n|G/H|}_{n|G/H|} i_L^* N_H^G X \arrow[d, "\cong"] \\
\bigwedge_{[c_jH]\in L\backslash G/H} N^L_{L\cap H^{c_j}} i^*_{L\cap H^{c_j}} P_n^n (X^{c_j}) \arrow[d, "\cong"'] & P^{n|G/H|}_{n|G/H|} \Big(\bigwedge_{[c_jH]\in L\backslash G/H} N^L_{L\cap H^{c_j}} i^*_{L\cap H^{c_j}} X^{c_j} \Big) \\
\bigwedge_{[c_jH]\in L\backslash G/H} N^L_{L\cap H^{c_j}} P_n^n (i^*_{L\cap H^{c_j}} X^{c_j}) \arrow[r] & \bigwedge_{[c_jH]\in L\backslash G/H} P^{n|L/L\cap H^{c_j}|}_{n|L/L\cap H^{c_j}|} N^L_{L\cap H^{c_j}} i^*_{L\cap H^{c_j}} X^{c_j} \arrow[u, "\wedge"']
\end{tikzcd}$$

We leave the proof of commutativity to the interested reader.

From the above we obtain a norm map on the E_∞ page.

Corollary 5.19. *Let A and X be H-spectra, with $H \subseteq G$. The norm map*

$$E_2(A, X) \to E_2(N_H^G A, N_H^G X)$$

induces a well-defined map $E_\infty(A, X) \to E_\infty(N_H^G A, N_H^G X)$.

Proof. Let h_0 be a permanent cycle, and let h_1 be hit by a differential. Applying Lemma 5.18 and Proposition 5.17, we see that $N_H^G(h_0 + h_1) - N_H^G h_0$ is a sum of terms, each of which is hit by a boundary. $\square$

Of course, when $E(A, X)$ converges this is the associated graded map of the norm functor.

To proceed any farther, we must consider towers of spectra. Let $\mathbb{Z}$ denote the set of integers regarded as a category with one morphism from n to m when $n \geq m$. We now consider the category of towers of orthogonal G-spectra, $(Sp_G^{\mathcal{O}})^{\mathbb{Z}}$. We denote the free tower on X in level n by $X[n]$, and the constant tower in X by $const(X)$. Note that this category has a symmetric monoidal structure; for more we refer the reader to Section IV.4.

It is a simple matter to pull back the (positive) $\mathbb{S}$ model structure from all levels of the tower to obtain a monoidal model structure on $(Sp_G^{\mathcal{O}})^{\mathbb{Z}}$. Then, using Section A.4, we obtain a model structure on commutative ring towers. This allows us to use the techniques from the beginning of this section to prove that the norm functor on *towers of spectra* preserves weak equivalences between $\mathbb{S}$-cofibrant towers. It is also simple to prove that cofibrant towers are flat, and that the analogue of Proposition 5.2 holds for towers.

We define a tower X to be *slice-like* if $X_n \geq n$ for all $n \in \mathbb{Z}$. We require two more easy facts; we omit the proofs.

Lemma 5.20. *If X and Y are slice-like and $\mathbb{S}$-cofibrant, then so is $X \wedge Y$.*

Lemma 5.21. *Let $H \subseteq G$. For H-spectra X there are natural isomorphisms*

$$N_H^G(X[n]) \cong (N_H^G X)[n|G/H|]$$

for each $n \in \mathbb{Z}$.

Using the above and the techniques from the beginning of this section, we obtain the following result.

Corollary 5.22. *Let X be a tower of H-spectra, with $H \subseteq G$. If X is $\mathbb{S}$-cofibrant and slice-like then so is $N_H^G X$.*

We now obtain a Leibniz formula for the norm in the RSSS. For the statement, note that, letting $T(G:H) = G/H - \{H\}$, we have

$$i_H^* N_H^G X \cong X \wedge \Big(\bigwedge_{[c_jH] \in T(G:H)/H} N_{H \cap H^{c_j}}^H i_{H \cap H^{c_j}}^* X^{c_j} \Big)$$

since the identity coset is fixed by H.

Theorem 5.23. *Let A and X be H-spectra, with $H \subseteq G$. If $f : A \to P^n_n X$ survives to the E_r page then, by slight abuse of notation, we have*

$$d_r(N^G_H f) = t^G_H\Big(d_r f \wedge \Big(\bigwedge_{[c_j H] \in T(G:H)/H} N^H_{H \cap H^{c_j}} r^{H^{c_j}}_{H \cap H^{c_j}} f^{c_j}\Big)\Big).$$

Proof. Let X be a cofibrant and fibrant spectrum, and let $s(X)$ be a cofibrant and fibrant model for the slice tower of X, with a map $s(X) \to const(X)$. Let $A = \Sigma B$, with B cofibrant. We may regard this as $I \wedge B/B$, where $I = [0,1]$ is given the basepoint 0. The data f and $d_r f$ are then given by a diagram of the following form.

$$\begin{tikzcd}
B \arrow[dr, "h"'] \arrow[drr, "d_r f"] \arrow[dd] & & \\
& P_{n+r-1}X \arrow[r] \arrow[d] & P^{n+r-1}_{n+r-1}X \\
I \wedge B \arrow[r, "\bar{f}"] \arrow[d] & P_n X \arrow[r] & P^n_n X \\
\Sigma B \arrow[urr, "f"'] & &
\end{tikzcd}$$

Here, we use $P_m X$ to denote $s(X)_m$; since $s(X)$ is a cofibrant tower, all of its structure maps are cofibrations, so we may define $P^m_m X := P_m X / P_{m+1} X$ for all m. Now consider the norm of $\bar{f}$. The restriction to $\partial(N^G_H I) \wedge N^G_H B$ lifts to $(N^G_H s(X))_{n|G/H|+r-1}$, while the restriction to $\partial^2(N^G_H I) \wedge N^G_H B$ lifts to $(N^G_H s(X))_{n|G/H|+2(r-1)}$. Thus the map

$$\partial(N^G_H I) \wedge N^G_H B \to (N^G_H s(X))_{n|G/H|+r-1}/(N^G_H s(X))_{n|G/H|+r}$$

factors through the map

$$\partial(N^G_H I) \wedge N^G_H B \to (\partial(N^G_H I)/\partial^2(N^G_H I)) \wedge N^G_H B,$$

which is a cotransfer map smashed with $N^G_H B$. Thus we consider the face of $\partial(N^G_H I)$

corresponding to the identity coset. On this face, the H-map

$$\partial(N_H^G I) \wedge N_H^G B \to (N_H^G s(X))_{n|G/H|+r-1}$$

is clearly equal to

$$h \wedge \Big(\bigwedge_{[c_j H] \in T(G:H)/H} N^H_{H\cap H^{c_j}} r^{H^{c_j}}_{H\cap H^{c_j}} \bar{f}^{c_j} \Big).$$

This proves the formula in the spectral sequence for the tower $N_H^G s(X)$. Since this is a slicelike tower over $N_H^G X$, there is a unique map over $N_H^G X$ to the slice tower of $N_H^G X$ (in the category of towers in the *homotopy category* of spectra). The proof is finished by noting the commutativity of the following two diagrams (the smash products are indexed over $T(G:H)/H$).

$$\begin{tikzcd}
N_H^G P_n X \arrow[r] \arrow[d] & N_H^G P_n^n X \arrow[d] \\
(N_H^G s(X))_{n|G/H|} \arrow[r] & (N_H^G s(X))_{n|G/H|}/(N_H^G s(X))_{n|G/H|+1}
\end{tikzcd}$$

$$\begin{tikzcd}
i_H^* N_H^G s(X) \arrow[r] \arrow[d, "\cong"'] & i_H^* s(N_H^G X) \arrow[d, "\cong"] \\
s(X) \wedge \Big(\bigwedge_{[c_j H]} N^H_{H\cap H^{c_j}} i^*_{H\cap H^{c_j}} s(X)^{c_j} \Big) \arrow[d, "\cong"'] & s(i_H^* N_H^G X) \arrow[d, "\cong"] \\
s(X) \wedge \Big(\bigwedge_{[c_j H]} N^H_{H\cap H^{c_j}} i^*_{H\cap H^{c_j}} s(X^{c_j}) \Big) \arrow[d, "\cong"'] & s\Big(X \wedge \Big(\bigwedge_{[c_j H]} N^H_{H\cap H^{c_j}} i^*_{H\cap H^{c_j}} X^{c_j} \Big)\Big) \\
s(X) \wedge \Big(\bigwedge_{[c_j H]} N^H_{H\cap H^{c_j}} s(i^*_{H\cap H^{c_j}} X^{c_j}) \Big) \arrow[r] & s(X) \wedge \Big(\bigwedge_{[c_j H]} s(N^H_{H\cap H^{c_j}} i^*_{H\cap H^{c_j}} X^{c_j}) \Big) \arrow[u, "\wedge"']
\end{tikzcd}$$

In the bottom row of the second diagram above we have implicitly used the fact that, if $Y \to const(Z)$ is a map of towers with Y cofibrant and slice-like and Z fibrant, then it can be factored through a cofibrant and fibrant model for the slice tower of Z (much like any map from a CW complex to a space can be extended to a CW approximation). $\square$

Remark: There is one representation-theoretic subtlety that we glossed over in the proof above. Note that the G-fixed subspace of $\mathbb{R}[G/H]$ is one-dimensional; let W denote its orthogonal complement. We have an explicit isomorphism of H-representations $\mathbb{R}[T(G:H)] \xrightarrow{\cong} i_H^* W$ as below.

$$\sum_{cH \in T(G:H)} x_{cH}[cH] \mapsto \sum_{cH \in T(G:H)} x_{cH}[cH] - \Big(|G/H|^{-1} \sum_{cH \in T(G:H)} x_{cH}\Big) \sum_{cH \in G/H} 1[cH]$$

The class $d_r(N_H^G f)$ is a map with domain $S^W \wedge N_H^G B$, which Theorem 5.23 gives as a transfer of an H-map defined on $S^{\mathbb{R}[T(G:H)]} \wedge i_H^*(N_H^G B)$. We identify this last spectrum with $i_H^*(S^W \wedge N_H^G B)$ by using the isomorphism of H-representations above.

6 Geometric Fixed Points

Let $\mathcal{F}$ be a family of subgroups of G, and let $\mathcal{F}'$ denote its complement. As usual we denote by $E\mathcal{F}$ the universal $\mathcal{F}$-space and $\tilde{E}\mathcal{F}$ its unreduced suspension. Recall that a spectrum X is called *$\mathcal{F}'$-local* if one of the following equivalent conditions hold:

- $E\mathcal{F}_+ \wedge X \cong *$,
- $X \cong \tilde{E}\mathcal{F} \wedge X$,
- $i_H^* X \cong *$ for all $H \in \mathcal{F}$,
- $\pi_n^H X = 0$ for all n and all $H \in \mathcal{F}$.

The inclusion of the full subcategory of $\mathcal{F}'$-local spectra, which we denote by $Sp_G^{\mathcal{F}'}$, has a left adjoint, given by $\tilde{E}\mathcal{F} \wedge (-)$. We call this *$\mathcal{F}'$-localization.* Now let τ denote the localizing subcategory generated by a set T. We have the following general fact.

Theorem 6.1. *If τ is closed under $\mathcal{F}'$-localization, then for any $\mathcal{F}'$-local spectrum X, $P_\tau X$ and $P^{\tau\perp} X$ are $\mathcal{F}'$-local.*

Proof. The second statement follows from the first. Let $\tau^{\mathcal{F}'}$ denote the localizing subcategory generated by $\tilde{E}\mathcal{F} \wedge T$; by hypothesis, this is contained in τ. Also, any element of $\tau^{\mathcal{F}'}$ has a (possibly transfinite) filtration whose succesive cofibers are wedges of suspensions of elements of $\tilde{E}\mathcal{F} \wedge T$, so $\tau^{\mathcal{F}'}$ consists of $\mathcal{F}'$-local spectra. This implies that $\tau^{\mathcal{F}'} \subseteq \tilde{E}\mathcal{F} \wedge \tau$. Conversely, suppose that X has a (possibly transfinite) filtration whose succesive cofibers are wedges of suspensions of elements of T. Smashing this filtration with $\tilde{E}\mathcal{F}$, we obtain a filtration for $\tilde{E}\mathcal{F} \wedge X$ with succesive cofibers that are wedges of suspensions of elements of $\tilde{E}\mathcal{F} \wedge T$. Thus we have $\tilde{E}\mathcal{F} \wedge \tau \subseteq \tau^{\mathcal{F}'}$ as well, so that $\tau^{\mathcal{F}'} \cong \tilde{E}\mathcal{F} \wedge \tau \cong Sp_G^{\mathcal{F}'} \cap \tau$.

Now let X be $\mathcal{F}'$-local, and consider the map $P_{\tau^{\mathcal{F}'}} X \to X$. The spectrum $P_{\tau^{\mathcal{F}'}} X$ is in τ by the above. Furthermore, for any $Y \in \tau$ the map

$$[Y, P_{\tau^{\mathcal{F}'}} X] \to [Y, X]$$

is isomorphic to the map

$$[\tilde{E}\mathcal{F} \wedge Y, P_{\tau_{\mathcal{F}'}} X] \to [\tilde{E}\mathcal{F} \wedge Y, X]$$

since both X and $P_{\tau_{\mathcal{F}'}} X$ are $\mathcal{F}'$-local. The above map is an isomorphism; thus, $P_{\tau_{\mathcal{F}'}} X \to X$ satisfies the required universal property. That is, $P_\tau X \cong P_{\tau_{\mathcal{F}'}} X$, so it is $\mathcal{F}'$-local. □

Since the categories τ_n satisfy the above criterion, we immediately get the following:

Corollary 6.2. *The RSSS for an $\mathcal{F}'$-local spectrum is $\mathcal{F}'$-local.*

Remark: This Corollary can be proven much more simply by using the fact that the RSSS construction commutes with restriction functors; in fact, if $i_H^* X = 0$ then $i_H^* P_n X = 0$ for all n. However, we will need the more precise arguments given above in what follows.

Warning: The example of $G = \mathbb{Z}/2\mathbb{Z}$, $\mathcal{F} = \{e\}$, $X = K\mathbb{R}$ shows that taking slices does not commute with localization in general. In fact, $\tilde{E}\mathcal{F} \wedge X = *$, but the localizations of the slices of X are not zero (see [Dug] or Section V.4).

Now suppose that N is a normal subgroup of G, and let $\mathcal{F}[N]$ denote the family of subgroups that do not contain N. Recall that $Sp_G^{\mathcal{F}[N]'}$ is the category of spectra whose homotopy groups are *concentrated over* N, and that this is equivalent to the category of G/N-spectra (see [LMS]). The equivalence is given by the N-fixed point functor, which is equal to the N-geometric fixed point functor Φ^N on $Sp_G^{\mathcal{F}[N]'}$. Following Hill ([Hil]), we call the inverse equivalence *pullback*, and denote it by ϕ_N^*. We recall the following basic fact, which may be proved by noting that $G \wedge_H (-)$ and Φ^N are left adjoint functors whose right adjoints, i_H^* and ϕ_N^*, fit into a commutative square with $i_{H/N}^*$ and $\phi_N^* : Sp_{H/N} \to Sp_H$.

Lemma 6.3. *There are natural isomorphisms as below.*

$$\Phi^N(G_+ \wedge_H X) \cong \begin{cases} G/N_+ \wedge_{H/N} \Phi^N(X) & \text{if } H \supseteq N \\ * & \text{if } H \not\supseteq N \end{cases}$$

In particular, the homotopy groups of $\Phi^N X$ are just the (possibly) nonzero homotopy groups of X, when X is concentrated over N. We can now prove the following results.

Theorem 6.4. *After saturating in isomorphism classes, we have*

$$\phi_N^*(Sp_{G/N}) \cap \tau_m^G = \phi_N^*(\tau_{\lceil m/|N| \rceil}^{G/N}).$$

Proof. First, note that the above intersection is $Sp_G^{\mathcal{F}[N]'} \cap \tau_m = \tau_m^{\mathcal{F}[N]'}$. Now let T_m denote the set of slice cells of dimension $\geq m$. Thus $\tau_m^{\mathcal{F}[N]'}$ is equal to the localizing category generated by $\tilde{E}\mathcal{F}[N] \wedge T_m$. Since $Sp_G^{\mathcal{F}[N]'}$ is a triangulated subcategory of Sp_G which is closed under wedge sums, we can regard $\tau_m^{\mathcal{F}[N]'}$ as the localizing subcategory of $Sp_G^{\mathcal{F}[N]'}$ generated by $\tilde{E}\mathcal{F}[N] \wedge T_m$. Now, the geometric fixed point functor is an equivalence of triangulated categories, so we can immediately identify $\Phi^N(\tau_m^{\mathcal{F}[N]'})$ as the localizing subcategory of $Sp_{G/N}$ generated by $\Phi^N(T_m)$. To determine this category, we may begin by throwing out the elements of $\Phi^N(T_m)$ that are trivial. By Lemma 6.3, the nontrivial elements are

$$\Phi^N(G_+ \wedge_H S^{k\rho_H}) \cong G/N_+ \wedge_{H/N} S^{k\rho_{H/N}}$$

for $H \supseteq N, k|H| \geq m$. These are all slice cells, and the above slice cell of dimension $d = k|H/N|$ is in this set when $k|H| = d|N| \geq m$; that is, when $d \geq \lceil m/|N| \rceil$. Thus we have $\Phi^N(\tau_m^{\mathcal{F}[N]}) = \tau_{\lceil m/|N| \rceil}$. □

Corollary 6.5. *Let $X = \phi_N^*(Y)$ be concentrated over N. Then $P_m X = \phi_N^*(P_{\lceil m/|N| \rceil} Y)$. Thus the $k|N|$-slice of X is the pullback of the k-slice of Y, and all other slices of X are trivial.*

Proof. We know that $P_m X$ is concentrated over N. Thus $P_m X \to X$ is terminal among maps to X from spectra in $\phi_N^*(Sp_{G/N}) \cap \tau_m^G = \phi_N^*(\tau_{\lceil m/|N| \rceil}^{G/N})$. The second statement follows from the first. □

Remark: The above result was originally conjectured by Mike Hill in [Hil]; it is stated there (in terms of the *irregular* slice filtration) as Conjecture 4.11. Theorem 4.9

of that paper gives the correct upper bound for spectra that are pulled back from a quotient group, while Theorem 4.12 gives a non-optimal lower bound; the above corollary remedies this situation. Hill also gave proofs of the special cases where $N = G$ and where $[G : N] = 2$; see Theorem 6.14 and Corollary 4.14 of [Hil], respectively.

Corollary 6.6. *If X is concentrated over a nontrivial normal subroup then the suspension map 3.13 for X is zero.*

Proof. If X is concentrated over $N \neq e$ then so is ΣX. Hence by Corollary 6.5 the maps

$$\Sigma P_k^k X \to P_{k+1}^{k+1}(\Sigma X)$$

are zero, since there do not exist consecutive multiples of $|N|$. □

The following corollary will be useful for doing inductive proofs in later sections.

Corollary 6.7. *If X is concentrated over G and $(n-1)$-connected then $X \geq n|G|$.*

Proof. X^G is $(n-1)$-connected; hence, it is $\geq n$, so its lowest possible nonzero slice is the n-slice. By Theorem 6.4, the lowest possible nonzero slice of X is the $n|G|$-slice. □

7 Brown-Comenetz Duality

In this section we show that the RSSS has a type of duality which will be very useful in deducing statements about the spectral sequence in one half plane (left or right) from corresponding statements in the other half plane. Following Brown and Comenetz ([BC]), though with different notation, we begin by defining

$$A^\vee := Hom(A, \mathbb{Q}/\mathbb{Z})$$

for an arbitrary abelian group A. Next, for a G-spectrum X, we consider the functor

$$B \mapsto (\pi_0^G(X \wedge B))^\vee$$

sending G-spectra to abelian groups. Since $\mathbb{Q}/\mathbb{Z}$ is injective, this is clearly the zeroth functor of a cohomology theory; hence it is representable. That is, there is a spectrum $\tilde{X}$ such that there is a natural isomorphism as below.

$$[B, \tilde{X}] \cong (\pi_0^G(X \wedge B))^\vee \tag{7.1}$$

It is clear that $X \mapsto \tilde{X}$ is a contravariant functor, which we refer to as *dualization.* Letting $X = S^0$ and using the isomorphism $S^0 \wedge B \cong B$, we obtain

$$[B, \widetilde{S^0}] \cong (\pi_0^G(B))^\vee.$$

Substituting $X \wedge B$ for B in the above and using 7.1, we obtain

$$\begin{aligned} [B, \tilde{X}] &\cong (\pi_0^G(X \wedge B))^\vee \\ &\cong [X \wedge B, \widetilde{S^0}] \\ &\cong [B, F(X, \widetilde{S^0})] \end{aligned}$$

so that we have a natural isomorphism

$$\tilde{X} \cong F(X, \widetilde{S^0})$$

by Yoneda's Lemma. The following facts are then clear.

- Dualization is an additive functor.
- Dualization converts wedge sums into products.
- Dualization converts homotopy colimits into homotopy limits.
- Dualization preserves cofiber sequences.

We require the following basic lemma.

Lemma 7.2. *For strongly dualizable spectra B and arbitrary spectra X there is a natural isomorphism as below.*

$$[B, \tilde{X}] \cong [DB, X]^\vee$$

Proof. We have the following chain of isomorphisms.

$$\begin{aligned}
[B, \tilde{X}] &\cong (\pi_0^G(X \wedge B))^\vee \\
&= [S^0, X \wedge B]^\vee \\
&\cong [S^0 \wedge DB, X]^\vee \\
&\cong [DB, X]^\vee
\end{aligned}$$

□

We apply this to B of the form $G/H_+ \wedge S^n$ to obtain the following corollary.

Corollary 7.3. *We have natural isomorphisms as below.*

$$\underline{\pi}_n \tilde{X} = (\underline{\pi}_{-n} X)^\vee$$

Here, by the dual $\underline{M}^{\vee}$ of a Mackey functor $\underline{M}$ we mean the functor $(-)^{\vee} \circ \underline{M} \circ D$, where D is the (Spanier-Whitehead) duality functor on the Burnside Category.

Hence, the effect of dualization on homotopy groups is to dualize them and "turn them upside down." Next, we have an elementary, yet crucial observation.

Proposition 7.4. *The Spanier-Whitehead duals of the slice cells are the slice cells:*

$$D(G_+ \wedge_H S^{n\rho_H}) \cong G_+ \wedge_H S^{-n\rho_H}.$$

The proof is trivial. To obtain duality in the RSSS, we need a description of the spectra that are $\geq n$ that is dual to the description of the spectra that are $\leq -n$. We begin with the following.

Lemma 7.5. *If $X \geq n$ then*

$$[\Sigma^{-i}\hat{S}, X] = 0$$

for all slice cells $\hat{S}$ of dimension $< n$ and all $i \geq 0$.

Proof. From the construction of $P_n X \cong X$ we see that we can assume that X is built out of slice cells of dimension $\geq n$ and their suspensions. Now $\Sigma^{-i}\hat{S}$ is a compact spectrum, so it suffices to show that

$$[\Sigma^{-i}\hat{S}, \Sigma^{j}\hat{T}] = 0$$

for any slice cell $\hat{T}$ of dimension $\geq n$ and any $j \geq 0$. The above group is isomorphic to

$$[S^0, \Sigma^{i+j}(\hat{T} \wedge D\hat{S})].$$

Now by Lemma 4.1 and Proposition 7.4, $\hat{T} \wedge D\hat{S}$ is a wedge of slice cells of dimension $\dim(\hat{T}) - \dim(\hat{S}) > 0$ so it is 0-connected. Hence, the above group is zero. $\square$

We now arrive at our dual description.

Proposition 7.6. *A spectrum X is $\geq n$ if and only if $[\hat{S}, X] = 0$ for all slice cells $\hat{S}$ of dimension $< n$.*

Proof. One direction is given by Lemma 7.5. Hence, suppose that $[\hat{S}, X] = 0$ for all slice cells $\hat{S}$ of dimension $< n$, and consider the cofiber sequence shown below.

$$X \to P^{n-1}X \to \Sigma P_n X$$

Letting $\hat{S}$ be a slice cell of dimension $< n$, we consider the resulting exact sequence shown below.

$$[\hat{S}, X] \to [\hat{S}, P^{n-1}X] \to [\hat{S}, \Sigma P_n X]$$

The first group above is zero by assumption, and the last group is zero by Lemma 7.5, so we have that $[\hat{S}, P^{n-1}X] = 0$ for *all* slice cells $\hat{S}$. It then follows from Lemma 3.8 that $P^{n-1}X < k$ for all $k \in \mathbb{Z}$. However, each spectrum $G/H_+ \wedge S^m$ is $\geq k$ for some k, so all the homotopy groups of $P^{n-1}X$ must be zero; hence, $P^{n-1}X \cong *$. $\square$

By combining this with Lemma 7.2 and Proposition 7.4, and using the fact that an abelian group A is zero if and only if its dual $A^\vee$ is zero, we arrive at the following.

Theorem 7.7. *The following conclusions hold.*

(i) $X \geq n \Leftrightarrow \tilde{X} \leq -n$

(ii) $X \leq n \Leftrightarrow \tilde{X} \geq -n$

Theorem 7.8. *We have the following natural isomorphisms.*

(i) $P_n \tilde{X} \cong \widetilde{P^{-n}X}$

(ii) $P^n \tilde{X} \cong \widetilde{P_{-n}X}$

(iii) $P_n^n \tilde{X} \cong \widetilde{P_{-n}^{-n}X}$

Proof. When we dualize the cofiber sequence

$$P_nX \to X \to P^{n-1}X$$

we obtain a cofiber sequence

$$\widetilde{P^{n-1}X} \to \tilde{X} \to \widetilde{P_nX}$$

where the spectrum on the left is $> -n$ and the spectrum on the right is $\leq -n$. For the last part, we use 3.9. $\square$

All of this means that the dual of the tower $\{P_nX\}$ is the tower $\{P^n\tilde{X}\}$, so that the exact couple defining the RSSS dualizes in the following sense. Let there be an exact couple, as shown below.

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

We can dualize this by setting $\tilde{A} := A^\vee$, $\tilde{\imath} := i^\vee$, $\tilde{\jmath} := k^\vee$, etc. When we dualize the derived exact couple, what we get is not quite the derived exact couple of the dual; the difference is in which of the two maps on the sides is composed with $\tilde{\imath}^{-1}$. However, these last two exact couples have the same differential. Thus we see that dual exact couples define dual spectral sequences, and the RSSS for X defined by the P_nX dualizes to the RSSS for $\tilde{X}$ defined using the $P^n\tilde{X}$ (with appropriate sign conventions). Thus, dualization has the effect of turning the RSSS upside down and dualizing the groups and differentials. We sum all this up by writing the following.

Theorem 7.9. *Taking the RSSS commutes with dualization:*

$$\underline{E}(\tilde{X}) \quad \cong \quad \underline{E}(X)^\vee.$$

Furthermore, the slice filtration dualizes, in the sense that the sequence

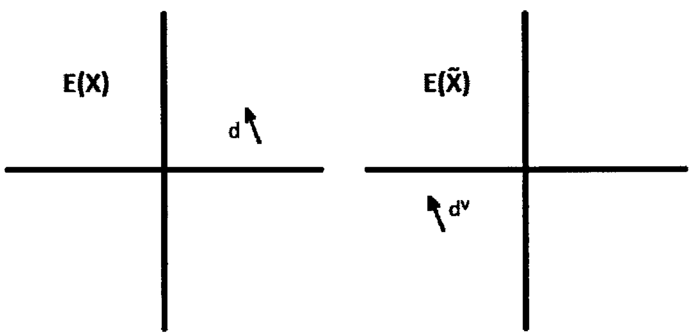
$$F^{1-s}\underline{\pi}_{-t}\tilde{X} \quad \to \quad \underline{\pi}_{-t}\tilde{X} \quad \to \quad \underline{\pi}_{-t}\tilde{X}/F^{1-s}\underline{\pi}_{-t}\tilde{X}$$

is canonically isomorphic to the dual of the sequence

$$F^{s}\underline{\pi}_{t}X \quad \to \quad \underline{\pi}_{t}X \quad \to \quad \underline{\pi}_{t}X/F^{s}\underline{\pi}_{t}X$$

and there is a canonical isomorphism

$$F^{-s}\underline{\pi}_{-t}\tilde{X}/F^{-s+1}\underline{\pi}_{-t}\tilde{X} \quad \cong \quad (F^{s}\underline{\pi}_{t}X/F^{s+1}\underline{\pi}_{t}X)^{\vee}.$$



Finally, we point out the following consequence of Proposition 7.6.

Corollary 7.10. *The categories τ_n are closed under taking arbitrary products and directed homotopy colimits.*

8 Generators and Vanishing Lines; Efficiency of the RSSS

In this section we give an alternative set of generators for the categories τ_n when $n \leq 1$. This leads to results on the relationship between connectivity and vanishing lines in the RSSS. We begin with the following elementary result.

Proposition 8.1. *If $n \geq 0$ and X is $(n-1)$-connected, then all of the spectra $P_m X$, $P^m X$, and $P_m^m X$ are $(\min(\lceil \frac{n+1}{|G|} \rceil, n) - 1)$-connected, and the spectra $P^m X$ are $(\min(\lceil \frac{n+1}{|G|} \rceil + 1, n) - 1)$-connected. If $n \leq 0$ and X is $(n+1)$-coconnected, then all of these spectra are $(\max(\lfloor \frac{n-1}{|G|} \rfloor, n) + 1)$-coconnected, and the spectra $P_m X$ are $(\max(\lfloor \frac{n-1}{|G|} \rfloor - 1, n) + 1)$-coconnected.*

Proof. For the first part, it suffices to prove the first statement for $P_m X$. Now $X \geq n$ by Corollary 3.3 and Proposition 3.6, so $P_m X \cong X$ when $m \leq n$; hence we may assume that $m > n$. Now if $\hat{S}$ is a slice cell of positive dimension, it is clear that its lowest possible nonzero homotopy group is in dimension $\lceil \dim(\hat{S})/|G| \rceil$, so the lowest possible nonzero homotopy group of $P_m X$ is in dimension $\lceil (n+1)/|G| \rceil$. The statement about the $P^m X$ follows from this by considering the cofiber sequences

$$X \to P^m X \to \Sigma P_{m+1} X$$

and using the fact that $P^m X \cong *$ when $m < n$. We may immediately deduce the second part from the first by using the results of Section 7 (though one can give a more elementary proof). □

We will derive further connectivity bounds from the following theorem.

Theorem 8.2. *If $n < 0$ then τ_n is generated by the spectra listed below.*

- $G/H_+ \wedge S^k$, $k \geq 0$, $H \subseteq G$
- $G/H_+ \wedge S^k$, $k < 0$, $|k||H| \leq |n|$

Furthermore, τ_0 is generated by the $G/H_+ \wedge S^k$ for $k \geq 0$, while τ_1 is generated by the $G/H_+ \wedge S^k$ for $k \geq 1$.

Proof. The last two statements follow from Proposition 3.6; hence, let $n < 0$. Let $\hat{S}$ be a slice cell of dimension $\geq n$. If $\dim(\hat{S}) \geq 0$ then $\hat{S}$ can be built out of the first type of spectra listed in the theorem. If not, let $\hat{S} = G_+ \wedge_H S^{-m\rho_H}$ with $m > 0$. Now $S^{m\rho_H}$ can be decomposed into cells of dimension k and type H/J for

$$k \leq \dim((S^{m\rho_H})^J \cong S^{m|H/J|}) = m|H/J|$$

(that is, $k|J| \leq m|H|$), so $G_+ \wedge_H S^{m\rho_H}$ can be decomposed into cells of dimension k and type G/J for $k|J| \leq m|H|$. Taking the Spanier Whitehead dual cell structure for $\hat{S}$, we see that $\hat{S}$ can be built from cells of dimension $k \leq 0$ and type G/J such that $|k||J| \leq |\dim(\hat{S})| \leq |n|$. It follows from all this that τ_n is contained in the localizing subcategory generated by the spectra listed in the statement. It remains to show that these spectra are actually contained in τ_n.

The spectra of the first type are contained in $\tau_0 \subseteq \tau_n$ by Proposition 3.6, so it remains to show that

$$G/H_+ \wedge S^{-k} \geq -k|H|$$

when $k \geq 0$. For this we simply note that $G/H_+ \wedge S^{-k} \cong G_+ \wedge_H (S^{-k\rho_H} \wedge S^{k(\rho_H - 1)})$. $\square$

Corollary 8.3. *If $n \leq 1$ and X is $(n-1)$-connected then so are all of the spectra $P_m X$, $P^m X$, and $P^m_m X$. If $n \geq -1$ and X is $(n+1)$-coconnected then so are all of the spectra $P_m X$, $P^m X$, and $P^m_m X$.*

Proof. The second statement follows from the first by the duality of Section 7. The cases $n = 0$ and $n = 1$ are covered by Proposition 8.1; hence, let $n < 0$, and suppose that X is $(n-1)$-connected. It suffices to show that $P_m X$ is $(n-1)$-connected for all m. This is automatic for $m \geq 0$, so suppose $m < 0$. We may construct $P_m X$ as follows. Let T_m denote the set of generators for τ_m given in Theorem 8.2. Note that

T_m is closed under suspension. Define

$$(P_mX)^{(0)} := \bigvee_{\substack{f:A\to X\\ f\neq 0}} A$$

with a map $(P_mX)^{(0)} \xrightarrow{p_0} X$ given by $p_0 = \vee_f f$, where the wedge runs over all nonzero maps to X from elements of T_m. Next, supposing we have constructed $(P_mX)^{(k)} \xrightarrow{p_k} X$, we fill in the diagram

$$\begin{tikzcd} \bigvee\limits_{\substack{f:A\to(P_mX)^{(k)}\\ f\neq 0, p_kf=0}} A \arrow[r, "\vee_f f"] & (P_mX)^{(k)} \arrow[r] \arrow[dr, "p_k"] & (P_mX)^{(k+1)} \arrow[d, dotted] \\ & & X \end{tikzcd}$$

to construct $(P_mX)^{(k+1)} \xrightarrow{p_{k+1}} X$, where the top row is a cofiber sequence and A runs over all elements of $\Sigma^{-1}T_m$. Then we let

$$P_mX = \varinjlim_{k\to\infty} (P_mX)^{(k)}$$

and prove that $(P_mX)^{(k)}$ is $(n-1)$-connected by induction on k. Firstly, if $A \in T_m$ then a map $A \to X$ can only be nonzero if $\dim(A) \geq n$, so we see that $(P_mX)^{(0)}$ is $(n-1)$-connected. For the inductive step, assume that $(P_mX)^{(k)}$ is $(n-1)$-connected. Then a map $A \to (P_mX)^{(k)}$ can only be nonzero if $\dim(A) \geq n$, so the cofiber of the map

$$(P_mX)^{(k)} \to (P_mX)^{(k+1)}$$

is a wedge of spheres of dimension $> n$. $\square$

As an immediate corollary to the last sentence of the above proof, we see that the map

$$\underline{\pi}_n(P_mX)^{(0)} \to \underline{\pi}_nX$$

is surjective. Furthermore, it is easy to see inductively that in forming $(P_mX)^{(k+1)}$ we only cone off spheres of dimension $\geq n+k$, so that the map

$$\underline{\pi}_n(P_mX)^{(1)} \to \underline{\pi}_n P_m X$$

is an isomorphism. Hence we introduce the following considerations. Denote the Burnside category of G by $\mathcal{B}(G)$, and denote the subcategory of orbits G/H by $\mathscr{O}_G$. Next, for any real number c we define subcategories

$$\mathscr{O}_G(c) := \{G/H \in \mathcal{B}(G) : |H| \leq c\}$$

and denote by $i_c : \mathscr{O}_G(c) \to \mathscr{O}_G$ the inclusions. Recall that a Mackey functor is an additive functor $\mathscr{O}_G^{op} \to Ab$. The restriction functor

$$i_c^* : Ab^{\mathscr{O}_G^{op}} \to Ab^{\mathscr{O}_G(c)^{op}}$$

has left and right adjoints given by additive left and right Kan extension, which we denote by $L(c)$ and $R(c)$, respectively. Now define filtrations on Mackey functors by the following:

$$\mathscr{F}^k\underline{M}(G/H) = \{x \in \underline{M}(G/H) : i_J^*x = 0 \quad \forall J \subseteq H, |J| \leq k\}$$

and $\mathscr{F}_k\underline{M}$ equals the sub-Mackey functor generated by the $\underline{M}(G/H)$ for $H \leq k$. We have the following.

Proposition 8.4. *If $n < 0$ and X is $(n-1)$-connected then for any $s \in \mathbb{Z}$ we have*

$$F^s\underline{\pi}_n X = \mathscr{F}_{(s+n)/n}\underline{\pi}_n X$$

and for $m \leq n$ we have

$$\underline{\pi}_n P_m X \cong L\Big(\frac{|m|}{|n|-1}\Big) i^*_{|m|/(|n|-1)} \mathscr{F}_{|m|/|n|}\underline{\pi}_n X$$

with the map $\underline{\pi}_n P_m X \to \underline{\pi}_n X$ being the counit of the adjunction $(L(\frac{|m|}{|n|-1}), i^*_{|m|/(|n|-1)})$ composed with the inclusion of $\mathscr{F}_{|m|/|n|}$. (In case $n = -1$, we take $|m|/0 = +\infty$.) Thus,

$$\underline{\pi}_n P^m_m X \cong L\Big(\frac{|m|}{|n|-1}\Big) i^*_{|m|/(|n|-1)} \frac{\mathscr{F}_{m/n}\underline{\pi}_n X}{\mathscr{F}_{(|m|-1)/|n|}\underline{\pi}_n X}$$

and so $\underline{E}_2^{s,s+n}(X) = 0$ unless n divides s.

Proof. In the first part, if $s > 0$ then both sides are clearly zero, so we assume $s \leq 0$. We then have

$$F^s \underline{\pi}_n X = im(\underline{\pi}_n (P_{s+n} X)^{(0)} \to \underline{\pi}_n X).$$

Now, only the wedge summands of dimension n in $(P_{s+n}X)^{(0)}$ contribute to $\underline{\pi}_n$, and these are of the form $G/H_+ \wedge S^n$ for $|n||H| \leq |s+n|$. The first part follows immediately. For the second part we must consider $\underline{\pi}_n (P_m X)^{(1)}$. Here, only the spheres of dimension n that we cone off have an effect on $\underline{\pi}_n$, so it is easy to see that we may describe this group in the following way. Let T denote the set of spheres of dimension n in T_m (the set of generators for τ_m given in Theorem 8.2), and let T' denote the set of spheres of dimension n whose suspension is in T_m. We let

$$Y = \bigvee_{\substack{f: A \to X \\ f \neq 0}} A$$

where A runs over T, and denote by $p : Y \to X$ the map $\vee_f f$. Then $\underline{\pi}_n P_m X$ is isomorphic to $\underline{\pi}_n$ of the cofiber of the map

$$\bigvee_{\substack{f: A' \to Y \\ f \neq 0, pf = 0}} A' \to Y$$

where A' runs over the elements of T'. Taking $\underline{\pi}_n$, we get the following diagram,

$$\begin{tikzcd}[column sep=small]
\displaystyle\bigoplus_{\substack{x\neq 0\in \underline{Y}(G/H)\\ p_*x=0\\ |n+1||H|\leq|m|}} [-,G/H] \arrow[r, "\oplus_x x"] & \underline{Y} = \displaystyle\bigoplus_{\substack{y\neq 0\in \underline{\pi}_n X(G/H)\\ |n||H|\leq|m|}} [-,G/H] \arrow[r] \arrow[dr, "p_*=\oplus_y y"'] & \underline{\pi}_n P_m X \arrow[d] \\
 & & \underline{\pi}_n X
\end{tikzcd}$$

in which the top line is a cokernel. The rest is easy category theory. To prove the last statement, we use this formula and the fact that the functors $L(c)$ and i_c^* are right exact. $\square$

In order to dualize this result, we need an algebraic lemma.

Lemma 8.5. *If $\underline{M}$ is a Mackey functor then for any k the dual of the sequence*

$$0 \to \mathscr{F}_k\underline{M} \to \underline{M} \to \underline{M}/\mathscr{F}_k\underline{M} \to 0$$

is canonically isomorphic to the sequence

$$0 \to \mathscr{F}^k\underline{M}^\vee \to \underline{M}^\vee \to \underline{M}^\vee/\mathscr{F}^k\underline{M}^\vee \to 0$$

while the dual of the sequence

$$0 \to \mathscr{F}^k\underline{M} \to \underline{M} \to \underline{M}/\mathscr{F}^k\underline{M} \to 0$$

is canonically isomorphic to the sequence

$$0 \to \mathscr{F}_k\underline{M}^\vee \to \underline{M}^\vee \to \underline{M}^\vee/\mathscr{F}_k\underline{M}^\vee \to 0.$$

That is, the filtrations $\mathscr{F}_k$ and $\mathscr{F}^k$ are dual.

Proof. The first sequence is the unique sequence with $\underline{M}$ in the middle such that the Mackey functor on the left is generated by the levels G/H for $H \leq k$, while the Mackey functor on the right is zero in these levels. Meanwhile, the second sequence is the unique sequence with $\underline{M}^\vee$ in the middle such that the Mackey functor on the left is zero in these levels, while the Mackey functor on the right satisfies $\mathscr{F}^k = 0$. Thus, we are reduced to proving the following implications.

$$\begin{aligned} \mathscr{F}_k\underline{M} = \underline{M} \quad &\Rightarrow \quad F^k\underline{M}^\vee = 0 \\ \mathscr{F}^k\underline{M} = 0 \quad &\Rightarrow \quad \mathscr{F}_k\underline{M}^\vee = \underline{M}^\vee \end{aligned}$$

This follows from the fact that $\mathscr{F}_k\underline{M} = \underline{M}$ precisely when the maps

$$\bigoplus_{\substack{J\subseteq H\\ |J|\le k}} \underline{M}(G/J) \xrightarrow{\oplus_J t_J^H} \underline{M}(G/H)$$

are surjective for all $H \subseteq G$, while $\mathscr{F}^k\underline{M} = 0$ precisely when the dual maps

$$\underline{M}(G/H) \xrightarrow{\times_J r_J^H} \bigtimes_{\substack{J\subseteq H\\ |J|\le k}} \underline{M}(G/J)$$

are injective for all $H \subseteq G$. □

Corollary 8.6. *If $n > 0$ and X is $(n+1)$-coconnected then for any $s \in \mathbb{Z}$ we have*

$$F^s\underline{\pi}_n X = \mathscr{F}^{(s+n-1)/n}\underline{\pi}_n X$$

and for $m \geq n$ we have

$$\underline{\pi}_n P^m X \cong R\Big(\frac{m}{n-1}\Big) i^*_{m/(n-1)}(\underline{\pi}_n X/\mathscr{F}^{m/n}\underline{\pi}_n X)$$

with the map $\underline{\pi}_n X \to \underline{\pi}_n P^m X$ being the quotient by $\mathscr{F}^{m/n}$ composed with the unit of the adjunction $(i^*_{m/(n-1)}, R(\frac{m}{n-1}))$. (In case $n = 1$, we take $m/0 = +\infty$.) Thus,

$$\underline{\pi}_n P^m_m X \cong R\Big(\frac{m}{n-1}\Big) i^*_{m/(n-1)} \frac{\mathscr{F}^{(m-1)/n}\underline{\pi}_n X}{\mathscr{F}^{m/n}\underline{\pi}_n X}$$

and so $\underline{E}_2^{s,s+n}(X) = 0$ unless n divides s.

Proof. By the duality of Section 7, Lemma 8.5 and Proposition 8.4, both of the sequences

$$0 \to F^s \underline{\pi}_n X \to \underline{\pi}_n X \to \underline{\pi}_n X / F^s \underline{\pi}_n X \to 0$$

and

$$0 \to \mathscr{F}^{(s+n-1)/n} \underline{\pi}_n X \to \underline{\pi}_n X \to \underline{\pi}_n X / \mathscr{F}^{(s+n-1)/n} \underline{\pi}_n X \to 0$$

dualize to the sequence

$$0 \to F^{1-s} \underline{\pi}_{-n} \tilde{X} \to \underline{\pi}_{-n} \tilde{X} \to \underline{\pi}_{-n} \tilde{X} / F^{1-s} \underline{\pi}_{-n} \tilde{X} \to 0.$$

Thus for each subgroup H of G, the maps $\pi_n^H X \to \mathbb{Q}/\mathbb{Z}$ that restrict to zero on each of the two filtrations are the same. However, if A and B are different subgroups of C, it is easy to show that there exists a map $C \to \mathbb{Q}/\mathbb{Z}$ which is zero on one of them but not on the other.

For the second part, we dualize the corresponding part of Proposition 8.4. Here the additive left Kan extension dualizes to the additive right Kan extension (and vice versa) essentially because the relevant categories have finitely many objects and finitely generated free *Hom* sets. We proceed as follows. Firstly, it follows from the first part that $\mathscr{F}^{m/n} \underline{\pi}_n X$ is the kernel of the map $\underline{\pi}_n X \to \underline{\pi}_n P^m X$. Next, Proposition 8.4 implies that when X is $(-n-1)$-connected, the truncated map $i^*_{m/(n-1)} \underline{\pi}_{-n} P_{-m} X \to i^*_{m/(n-1)} \underline{\pi}_{-n} X$ is injective (using that $i_c^* L(c) \cong Id$). It follows that, dually, the map $i^*_{m/(n-1)} \underline{\pi}_n X \to i^*_{m/(n-1)} \underline{\pi}_n P^m X$ is surjective under our assumptions. Combining these facts, we obtain an isomorphism

$$i^*_{m/(n-1)} \frac{\underline{\pi}_n X}{\mathscr{F}^{m/n} \underline{\pi}_n X} \xrightarrow{\cong} i^*_{m/(n-1)} \underline{\pi}_n P^m X.$$

Now the unit map

$$\underline{\pi}_n P^m X \to R\Big(\frac{m}{n-1}\Big) i^*_{m/(n-1)}(\underline{\pi}_n P^m X)$$

is an isomorphism, since the dual counit map is by Proposition 8.4, so the result follows immediately. For the last statement, we use the fact that the functors $R(c)$ and i^*_c are left exact. □

Next, we can obtain a special case of the above results without making any connectivity assumptions on X, as follows. If $n < 0$ then τ_n consists of $(n-1)$-connected spectra, so by the universal property characterizing P_n we can compute $P_n X$ by replacing X with $Post_n X$. We immediately obtain the following.

Corollary 8.7. *If $n < 0$ then for any G-spectrum X we have*

$$F^0 \underline{\pi}_n X = \mathscr{F}_1 \underline{\pi}_n X$$

and

$$\begin{aligned} \underline{\pi}_{-1} P_{-1} X &\cong \underline{\pi}_{-1} P^{-1}_{-1} X \cong \mathscr{F}_1 \underline{\pi}_{-1} X \\ \underline{\pi}_{-2} P_{-2} X &\cong \underline{\pi}_{-2} P^{-2}_{-2} X \cong L(2) i^*_2 \mathscr{F}_1 \underline{\pi}_{-2} X \\ \underline{\pi}_n P_n X &\cong \underline{\pi}_n P^n_n X \cong L(1) i^*_1 \underline{\pi}_n X \end{aligned}$$

for $n < -2$, with $\underline{\pi}_n P_n X \to \underline{\pi}_n X$ the evident structure maps.

We again have a dual result.

Corollary 8.8. *If $n > 0$ then for any G-spectrum X we have*

$$F^1 \underline{\pi}_n X = \mathscr{F}^1 \underline{\pi}_n X$$

and

$$\underline{\pi}_1 P^1 X \cong \underline{\pi}_1 P^1_1 X \cong \underline{\pi}_1 X / \mathscr{F}^1 \underline{\pi}_1 X$$
$$\underline{\pi}_2 P^2 X \cong \underline{\pi}_2 P^2_2 X \cong R(2) i_2^* \underline{\pi}_2 X / \mathscr{F}^1 \underline{\pi}_2 X$$
$$\underline{\pi}_n P^n X \cong \underline{\pi}_n P^n_n X \cong R(1) i_1^* \underline{\pi}_n X$$

for $n > 2$, with $\underline{\pi}_n X \to \underline{\pi}_n P^n X$ the evident structure maps.

Corollary 8.9. *For any G-spectrum X we have*

$$P^{-1}_{-1} X \cong \Sigma^{-1} H \mathscr{F}_1 \underline{\pi}_{-1} X.$$

Thus, the (-1)-slices are the Eilenberg MacLane spectra $\Sigma^{-1} H \underline{M}$ such that $\underline{M}$ is generated by $\underline{M}(G/e)$. Dually, we have

$$P^1_1 X \cong \Sigma H (\underline{\pi}_1 X / \mathscr{F}^1 \underline{\pi}_1 X).$$

Thus, the 1-slices are the Eilenberg MacLane spectra $\Sigma H \underline{M}$ such that all the restriction maps

$$\underline{M}(G/H) \xrightarrow{r_e^H} \underline{M}(G/e)$$

are injective (and hence all restrictions are injective). We also have

$$P^0_0 X \cong H \underline{\pi}_0 X$$

so that the 0-slices are the Eilenberg MacLane spectra in dimension zero.

Remark: The above results on the 0 and 1 (regular) slices are Propositions 4.19 and 4.47 of [HHR] (alternatively, Corollaries 2.11 and 2.12 and Theorem 2.13 of [Hil]).

As immediate corollaries to Theorem 8.2 and the duality of Section 7, we obtain the following, which will be used in the next section to identify part of the E_2 page of the RSSS algebraically.

Theorem 8.10. *The following conclusions hold.*

- *If $n < 0$ then $X \leq n$ if and only if X is 0-coconnected and*

$$\pi_{-k}^{H} X = 0$$

for all $k > 0$ with $k|H| < |n|$.

- *If $n \leq 0$ then $X < n$ if and only if X is 0-coconnected and*

$$\pi_{-k}^{H} X = 0$$

for all $k > 0$ with $k|H| \leq |n|$.

- *If $n > 0$ then $X \geq n$ if and only if X is 0-connected and*

$$\pi_{k}^{H} X = 0$$

for all $k > 0$ with $k|H| < n$.

- *If $n \geq 0$ then $X > n$ if and only if X is 0-connected and*

$$\pi_{k}^{H} X = 0$$

for all $k > 0$ with $k|H| \leq n$.

Remark: The above characterization of being > 1 is equivalent to Proposition 4.45 of [HHR]. Of course, being ≤ 0 and being ≥ 0 also have simple characterizations in terms of homotopy groups.

We now have the following collection of results:

- A spectrum is zero if and only if its (R)SSS is zero.
- A spectrum is n-connected ($n \leq 0$) if and only if its RSSS is.
- A spectrum is n-coconnected ($n \geq 0$) if and only if its RSSS is.

- A spectrum restricts to zero in a subgroup of G if and only if its (R)SSS does.

Thus, for example, one will not be forced to compute any of the negative columns in the SS past the first page if they are going to converge to zero anyway. We sum this up by saying that "*the (regular) slice spectral sequence is very efficient.*"

9 Relation to Homotopy Orbit and Fixed Point Spectral Sequences; Edge Homomorphisms

In this section we identify a large portion of the E_2 page of the (R)SSS algebraically and use this calculation to describe some of the structure defined in Section 4. First we provide some motivation. Let $n < -2$; according to Corollary 8.7, we have

$$\begin{aligned} E_2^{0,n}(X) &\cong (L(1)i_1^*\underline{\pi}_n X)(G/G) \\ &\cong Hom_{\mathcal{B}(G)^{op}}(G/e, G/G) \otimes_{Hom_{\mathcal{B}(G)^{op}}(G/e,G/e)} \underline{\pi}_n X(G/e) \\ &\cong \mathbb{Z} \otimes_G \underline{\pi}_n X(G/e) \\ &\cong (\underline{\pi}_n X(G/e))/G \end{aligned}$$

which is the same thing as on the E_2 page of the homotopy orbit spectral sequence (HOSS). Similarly, we have for $n > 2$

$$\begin{aligned} E_2^{0,n}(X) &\cong (R(1)i_1^*\underline{\pi}_n X)(G/G) \\ &\cong Hom_{Hom_{\mathcal{B}(G)^{op}}(G/e,G/e)\text{-Mod}}(Hom_{\mathcal{B}(G)^{op}}(G/G, G/e), \underline{\pi}_n X(G/e)) \\ &\cong Hom_G(\mathbb{Z}, \underline{\pi}_n X(G/e)) \\ &\cong (\underline{\pi}_n X(G/e))^G \end{aligned}$$

which is the same thing as on the E_2 page of the homotopy fixed point spectral sequence (HFPSS). We will show that there are such isomorphisms for many other entries on the E_2 page. In fact, we will obtain maps of spectral sequences

$$HOSS \to (R)SSS \to HFPSS$$

that induce the isomorphisms. For this we adopt the following notation. We use τ_n^R to refer to the localizing subcategories determined by the regular slice cells, τ_n^S for the irregular slice cells, τ_n^P for sphere spectra (the 'P' is for 'Postnikov'), and we note the following inclusions.

$$\tau_n^P \subseteq \tau_n^R \subseteq \tau_n^S \qquad (n \geq 0)$$
$$\tau_n^R \subseteq \tau_n^S \subseteq \tau_n^P \qquad (n \leq 0)$$

We have equality of all three when $n = 0$. Next, for each n we denote by τ_n^{min} the smallest of the three subcategories. Denoting $P_{\tau_n^{min}}$ by $\hat{P}_n$, etc. (and $P_{\tau_n^S}$ by $\tilde{P}_n$, etc.) we have a *mixed tower* $\{\hat{P}_n\}$ and natural maps of towers as below.

$$\begin{tikzcd} \{\hat{P}_n\} \arrow[r] \arrow[d] & \{P_n\} \arrow[r] & \{\tilde{P}_n\} \\ \{Post_n\} & & \end{tikzcd}$$

The vertical and left horizontal maps above are multiplicative maps of towers, since the filtrations τ_n^R and τ_n^P are multiplicative (see Section 4) and hence τ_n^{min} is as well. All of the maps are nonequivariant isomorphisms. It follows that when we apply the functors $(-) \wedge EG_+$ and $F(EG_+, -)$, we obtain isomorphisms. Of course, there are natural maps $Y \wedge EG_+ \to Y$ and $Y \to F(EG_+, Y)$, so we obtain the following.

Theorem 9.1. *There are natural maps of spectral sequences, as below.*

$$\begin{tikzcd} HOSS \arrow[r] \arrow[dr] & RSSS \arrow[r] \arrow[d] & HFPSS \\ & SSS \arrow[ur] & \end{tikzcd}$$

The maps in the top row are multiplicative.

We wish to know in what range of the E_2 page these maps are isomorphisms. For this we require some lemmas. Assuming G nontrivial, denote by $m(G)$ the order of the smallest nontrivial subgroup of G. Then we have the following.

Lemma 9.2. *If $n > 0$ then $S^n \wedge \tilde{E}G \geq nm(G)$.*

Proof. The spectrum $S^n \wedge \tilde{E}G$ is nonequivariantly contractible, so the smallest possible values of k and $|H|$ with $H \subseteq G$, $k > 0$ and $\pi_k^H(S^n \wedge \tilde{E}G) \neq 0$ are $k = n$ and

$|H| = m(G)$. The result now follows from Theorem 8.10. □

Next, since the dualization functor takes the form $F(-, \widetilde{S^0})$, we have the following easy lemma.

Lemma 9.3. *The Brown-Comenetz dual of the map $X \wedge EG_+ \to X$ is isomorphic to the map $\tilde{X} \to F(EG_+, \tilde{X})$.*

We can now identify a large portion of the E_2 page of the (R)SSS.

Theorem 9.4. *The following conclusions hold for any G-spectrum X.*

(i) The map $RSSS \to HFPSS$ is an isomorphism on $E_2^{s,t}$ for $t - s > 1$ and $t - s \geq \lfloor \frac{t}{m(G)} \rfloor + 2$. It is a monomorphism for $t - s > 0$ and $t - s \geq \lfloor \frac{t}{m(G)} \rfloor + 1$. The map on the SSS is an isomorphism for $t - s > 0$ and $t - s \geq \lfloor \frac{t+1}{m(G)} \rfloor + 1$. It is a monomorphism for $t - s \geq 0$ and $t - s \geq \lfloor \frac{t+1}{m(G)} \rfloor$.

(ii) The map $HOSS \to RSSS$ is an isomorphism on $E_2^{s,t}$ for $t - s < -1$ and $t - s \leq \lceil \frac{t}{m(G)} \rceil - 2$. It is an epimorphism for $t - s < 0$ and $t - s \leq \lceil \frac{t}{m(G)} \rceil - 1$. The map to the SSS is an isomorphism for $t - s < -2$ and $t - s \leq \lceil \frac{t+1}{m(G)} \rceil - 3$. It is an epimorphism for $t - s < -1$ and $t - s \leq \lceil \frac{t+1}{m(G)} \rceil - 2$.

Proof. Map the cofiber sequence

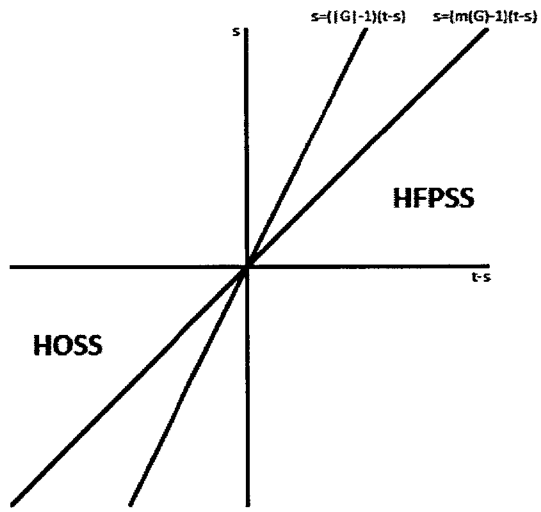
$$EG_+ \to S^0 \to \tilde{E}G$$

into $P_t^t X$ to obtain a cofiber sequence

$$F(\tilde{E}G, P_t^t X) \to P_t^t X \to F(EG_+, P_t^t X).$$

Now $\pi_{t-s}^G F(\tilde{E}G, P_t^t X) \cong [S^{t-s} \wedge \tilde{E}G, P_t^t X]$, so the first part for the RSSS follows from the long exact sequence of homotopy groups of the above cofiber sequence and Lemma 9.2. Using Lemma 9.3, we immediately obtain the second part for the RSSS by dualizing the first. Here, we have used the fact that $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator, and thus a map of abelian groups is injective (surjective) *if and only if* its dual is surjective (injective). For the SSS, we simply use the fact that the SSS is the RSSS conjugated by suspension. $\square$

If $n > 0$, then on the $s + (t - s) = n$ diagonal all lattice points except the first one strictly below the line of slope $(m(G) - 1)$ satisfy the condition in the above theorem; hence the E_2 pages of the RSSS and HFPSS coincide roughly below this line in the first quadrant. There is a symmetric statement about the RSSS and the HOSS above this line in the third quadrant.



We remark that if G is a cyclic group of prime order, then $m(G) = |G|$, so the above almost determines the entire E_2 page of the RSSS; the other groups are concentrated along the vanishing line and thus constitute 0% of the E_2 page. We distill some of the above information in the following corollary.

Corollary 9.5. *For any G-spectrum X we have isomorphisms as shown below.*

$$E_2^{s,t}(X) \cong H^s(G; \underline{\pi}_t X(G/e)) \qquad (s \geq 0, t - s > 0, t - s \geq \lfloor \frac{t}{m(G)} \rfloor + 2)$$
$$E_2^{s,t}(X) \cong H_{-s}(G; \underline{\pi}_t X(G/e)) \qquad (s \leq 0, t - s < 0, t - s \leq \lceil \frac{t}{m(G)} \rceil - 2)$$

In the first region the product of $x_1 \in E_2^{s_1,t_1}$ and $x_2 \in E_2^{s_2,t_2}$ is given by the usual product in cohomology times $(-1)^{s_1 t_2}$. In the second region the product is zero except *when* $s_1 = s_2 = 0$ *where it corresponds to the homology product shown below.*

$$[x_1] \cdot [x_2] = \sum_{g \in G} [x_1 \cdot (g \cdot x_2)] = \sum_{g \in G} [(g \cdot x_1) \cdot x_2]$$

Another consequence of Theorem 9.4 is that the map from the RSSS to the SSS is an isomorphism in a certain range.

Corollary 9.6. *The natural map from the RSSS to the SSS is an isomorphism on* $E_2^{s,t}$ *for* $t - s > 1$ *and* $t - s \geq \lfloor \frac{t}{m(G)} \rfloor + 2$*, and a monomorphism for* $t - s > 0$ *and* $t - s \geq \lfloor \frac{t}{m(G)} \rfloor + 1$*. It is also an isomorphism for* $t - s < -2$ *and* $t - s \leq \lceil \frac{t+1}{m(G)} \rceil - 3$*, and an epimorphism for* $t - s < -1$ *and* $t - s \leq \lceil \frac{t+1}{m(G)} \rceil - 2$.

Since the suspension maps for the HFPSS and HOSS are isomorphisms, we also obtain the following.

Corollary 9.7. *The suspension map*

$$E_2^{s,t}(X) \xrightarrow{\Sigma} E_2^{s,t+1}(\Sigma X)$$

is a monomorphism when $t > 0$, $t - s \geq \lfloor \frac{t}{m(G)} \rfloor + 1$ *and an isomorphism when* $t > 0$, $t - s > \lfloor \frac{t}{m(G)} \rfloor + 1$*. It is an epimorphism when* $t < 0$, $t - s \leq \lceil \frac{t}{m(G)} \rceil - 2$ *and an isomorphism* $t < 0$, $t - s < \lceil \frac{t}{m(G)} \rceil - 2$.

We can now identify the connecting homomorphism 4.8 in this range.

Corollary 9.8. *If* $s \geq 0$, $t - s > 0$ *and* $t - s \geq \lfloor \frac{t}{m(G)} \rfloor + 1$ *and*

$$A \to B \to C \to \Sigma A$$

is a cofiber sequence then the connecting homomorphism

$$\delta : E_2^{s,t}(C) \to \underline{\tilde{E}}_2^{s,t-1}(A)$$

coincides with the image under the functor $H^s(G; -)$ *of the connecting homomorphism* $\pi_t^e C \to \pi_{t-1}^e A$ *in the long exact sequence of homotopy groups. If* $s \leq 0$,

$t - s < 0$, and $t - s \leq \lceil \frac{t}{m(G)} \rceil - 1$ then it is the image under the functor $H_{-s}(G; -)$ of this map.

Using Corollary 9.5 we can identify the composition product in this range.

Corollary 9.9. *Under the line of slope $(m(G) - 1)$ in the first quadrant of the E_2 page, the composition product*

$$E(F(Y, Z)) \otimes E(F(X, Y)) \to E(F(X, Z))$$

coincides (up to the sign given in Corollary 9.5) with the product in $H^*(G; -)$ induced by the composition product of nonequivariant spectra

$$[Y, Z]_* \otimes [X, Y]_* \to [X, Z]_*.$$

Above this line in the third quadrant, the composition product on the E_2 pages is zero, except on the $t - s$ axis where it is induced by the homology product as in Corollary 9.5.

Next, we describe "mixed products" on the $t - s$ axis.

Proposition 9.10. *Let $u \in E_2^{0,n}(X_1)$ be the image of the equivalence class of $y \in \pi_n^e X_1$ in the HOSS, and let $v \in E_2^{0,m}(X_2)$ map to $z \in \pi_m^e X_2$ in the HFPSS. Then uv is the image of the equivalence class of yz in the HOSS, and maps to*

$$\Big(\sum_{g \in G} g \cdot y\Big) z$$

in the HFPSS (and similarly for vu). If instead $v \in E_2^{s,m}(X_2)$ with $s \neq 0$ then $uv = 0$ and $vu = 0$.

Proof. The second statement follows easily from the fact that the maps indicated by $HOSS \to RSSS$ and $RSSS \to HFPSS$ are multiplicative. For the first statement, let y be given by a map

$$G_+ \wedge S^n \to P_n^n X_1$$

so that u is given by the composite

$$S^n \to G_+ \wedge P^n_n X_1 \to P^n_n X_1$$

where the first map above is the adjoint of y. Next, let v be given by a map

$$S^m \to P^m_m X_2$$

so that the underlying nonequivariant map represents z. The product uv is then given by the composite along the top and right sides of the diagram below.

$$\begin{tikzcd}
S^{n+m} \cong S^n \wedge S^m \arrow[r, "y \wedge v"] \arrow[dr] & G_+ \wedge P^n_n X_1 \wedge P^m_m X_2 \arrow[r] \arrow[d] & P^n_n X_1 \wedge P^m_m X_2 \arrow[d] \\
& G_+ \wedge P^{n+m}_{n+m}(X_1 \wedge X_2) \arrow[r] & P^{n+m}_{n+m}(X_1 \wedge X_2)
\end{tikzcd}$$

The diagonal arrow is clearly adjoint to yz, so the result follows. For the last part, we simply note that if $s \neq 0$ then v is nonequivariantly zero, and apply the above argument. $\square$

We can also determine the "mixed products" with homology classes below the $t - s$ axis in terms of the product in Tate cohomology (see [GM]).

Proposition 9.11. *Let $x \in E_2^{s,t}(X_1)$ and $y \in E_2^{s',t'}(X_2)$ with $s < 0$. Suppose that x is the image of $u \in H_{-s}(G; \pi^e_t X_1) \cong \hat{H}^{s-1}(G; \pi^e_t X_1)$ and that y maps to the class $v \in \hat{H}^{s'}(G; \pi^e_{t'} X_2)$ in the Tate spectral sequence (or TSS). If $s' > |s|$ then $xy = 0$ and $yx = 0$. Otherwise, xy is the image under the map $HOSS \to RSSS$ of the Tate cohomology product $uv \in \hat{H}^{s+s'-1}(G; \pi^e_{t+t'}(X_1 \wedge X_2)) \to H_{-s-s'}(G; \pi^e_{t+t'}(X_1 \wedge X_2))$ times $(-1)^{st'}$, while yx is the image of the Tate product vu times $(-1)^{s'(t+1)}$.*

Proof. First let M_0 be a G-module, and let $\underline{M}$ be a Mackey functor such that $\underline{M}(G/e) = M_0$. We take the Tate cohomology of M_0 to be

$$\hat{H}^i(G; M_0) = \pi^G_{-i}(F(EG_+, H\underline{M}) \wedge \tilde{E}G)$$

and the group homology as below.

$$H_j(G;M_0) = \pi_j^G(H\underline{M} \wedge EG_+)$$

We define a map

$$\iota : \hat{H}^i(G;M_0) \to H_{-i-1}(G;M_0)$$

as follows: given the triangle

$$EG_+ \to S^0 \to \tilde{E}G \xrightarrow{\delta} \Sigma EG_+$$

and a map $f : S^{-i} \to F(EG_+, H\underline{M}) \wedge \tilde{E}G$, we let ιf be the unique map such that the composite

$$S^{-i} \cong S^1 \wedge S^{-i-1} \xrightarrow{\Sigma(\iota f)} S^1 \wedge H\underline{M} \wedge EG_+ \cong H\underline{M} \wedge S^1 \wedge EG_+$$

is equal to the composite below.

$$S^{-i} \xrightarrow{f} F(EG_+, H\underline{M}) \wedge \tilde{E}G \xrightarrow{Id\wedge\delta} F(EG_+, H\underline{M}) \wedge S^1 \wedge EG_+ \cong H\underline{M} \wedge S^1 \wedge EG_+$$

This map ι is an isomorphism when $i \leq -2$, a monomorphism when $i = -1$, and zero when $i \geq 0$. We then identify the group $\pi_{i-j}^G(F(EG_+, P_i^i Z) \wedge \tilde{E}G)$ with $\hat{H}^j(G; \pi_i^e Z)$ by identifying $P_i^i Z$ nonequivariantly with $S^i \wedge H\underline{\pi}_i Z$ and desuspending on the left, and similarly with $\pi_{i-j}^G(P_i^i Z \wedge EG_+)$ and group homology. We now use right multiplication by y and a map defined analagously to ι to obtain the commutative diagram below.

$$\begin{tikzcd}
\pi_{t-s+1}^G(F(EG_+, P_t^t X_1) \wedge \tilde{E}G) \arrow[r] \arrow[d, "\cdot y"'] & \pi_{t-s}^G(P_t^t X_1 \wedge EG_+) \arrow[d, "\cdot y"] \\
\pi_{t+t'+1-s-s'}^G(F(EG_+, P_{t+t'}^{t+t'}(X_1 \wedge X_2)) \wedge \tilde{E}G) \arrow[r] & \pi_{t+t'-s-s'}^G(P_{t+t'}^{t+t'}(X_1 \wedge X_2) \wedge EG_+)
\end{tikzcd}$$

The left vertical map above is the Tate product times $(-1)^{(s-1)t'}$, while the top and bottom horizontal maps are $(-1)^t\iota$ and $(-1)^{t+t'}\iota$, respectively, under the identifications we have made. The calculation of xy follows. If instead we multiply by y on the left, the left vertical map becomes the Tate product times $(-1)^{s't}$, while the diagram commutes up to the sign $(-1)^{t'-s'}$. $\square$

Finally, we describe the edge homomorphisms of the RSSS. The proof uses the simple fact that, if X is $(n-1)$-connected and EG is given its canonical cell structure with one zero-cell, then given an element x of $\pi_n^e X = [G_+ \wedge S^n, X]$, the element of $[S^n, X \wedge EG_+]$ corresponding to the equivalence class of x in $(\pi_n^e X)/G$ is given by the composite below.

$$S^n \xrightarrow{Dx} X \wedge G_+ \cong X \wedge EG_+^{[0]} \to X \wedge EG_+$$

Proposition 9.12. *If $n < -2$ ($n < -1$ if $m(G) > 2$) then the composite map*

$$(\pi_n^e X)/G \cong E_2^{0,n}(X) \to E_\infty^{0,n}(X) \cong F^0\pi_n^G X \subseteq \pi_n^G X$$

is induced by the transfer $t_e^G : \pi_n^e X \to \pi_n^G X$. Dually, if $n > 2$ ($n > 1$ if $m(G) > 2$) then the composite

$$\pi_n^G X \to \pi_n^G X/F^1\pi_n^G X \cong E_\infty^{0,n}(X) \subseteq E_2^{0,n}(X) \cong (\pi_n^e X)^G$$

is induced by the restriction $r_e^G : \pi_n^G X \to \pi_n^e X$.

Proof. We may use a natural zig-zag to relate X to its n'th Postnikov section. Then we use the fact that edge homomorphisms are natural for maps of spectral sequences. The statement then reduces to identifying π_0^G of the maps $H\underline{M} \wedge EG_+ \to H\underline{M}$ and $H\underline{M} \to F(EG_+, H\underline{M})$ for arbitrary Mackey functors $\underline{M}$. $\square$

Remark: One can similarly describe the edge homomorphisms for $n = \pm 1, \pm 2$ using the formulas given in Corollaries 8.7 and 8.8.

We will algebraically identify further structure on the RSSS in Chapter IV.

Chapter II

Order Families and Formulas for the Slice Tower

1 Introduction

In this chapter we give formulas for the slice towers of various classes of spectra. The slice cells will move to the background, and we will see that the families of subgroups of order less than a given integer (which we dub *order families*) play a fundamental role. In Section 2 we express each stage of the slice tower of an arbitrary spectrum in terms of a finite composite of maps, each of which is the cofiber of a map involving Postnikov section functors and universal spaces for these families. This suggests that subgroups "resonate" with different frequencies in the slice tower, according to their orders. In Section 3 we give two exact formulas for the slice towers of Eilenberg MacLane spectra, one for positive dimensions and one for negative dimensions. In Section 4 we give a formula for the positive part of the slice tower of a cofree spectrum, as well as a dual formula for the negative part of the slice tower of a free spectrum. We give simplifications of these formulas when the group is cyclic of prime power order. In Chapter V we will apply these formulas to gain some intuition about the general behavior of the RSSS outside of the region where it coincides with the HFPSS (or the HOSS). In Section 5 we explain why the behavior of the E_2 page changes when one crosses lines of slope one less than the order of a subgroup. We also give a partial, iterative description of the E_2 page when the group is cyclic of prime power order which suggests that there can be no general algebraic formula for the entire E_2 page.

2 Formulas for Arbitrary Spectra

In this section we determine a kind of formula for the slice tower of an arbitrary G-spectrum X, in terms of universal spaces, Postnikov section functors and cofibers. We begin by defining our fundamental families of subgroups, which we call *order families*, as below.

$$\mathcal{F}_i := \{H \subseteq G : |H| < i\}$$

Firstly, note that since we always have

$$i_H^* E\mathcal{F} \cong E(\mathcal{F} \cap H)$$

for any family $\mathcal{F}$ and any subgroup H, we obtain

- $i_H^* E\mathcal{F}_i \cong *$ if $|H| < i$,
- $i_H^* E\mathcal{F}_i \cong E\mathcal{P}_H$ if $|H| = i$,
- $E\mathcal{F}_1 = \emptyset$, and
- $E\mathcal{F}_{|G|+1} \cong *$,

where we use $\mathcal{P}$ to denote the family of all proper subgroups. Of course, we also have $\mathcal{F}_i \subseteq \mathcal{F}_{i+1}$ for all i. We begin by determining an alternative set of generators to the slice cells.

Proposition 2.1. *The localizing subcategory τ_n is generated by the spectra*

$$G/H_+ \wedge S^k \wedge \tilde{E}\mathcal{F}_{|H|}$$

for $H \subseteq G$ and $k|H| \geq n$.

Proof. First we prove that these spectra are in τ_n. We proceed by induction on $|G|$;

the result is trivial for the trivial group. We have

$$\begin{aligned} G/H_+ \wedge S^k \wedge \tilde{E}\mathcal{F}_{|H|} &\cong G_+ \wedge_H (S^k \wedge \tilde{E}\mathcal{P}) \\ &\cong G_+ \wedge_H (H/H_+ \wedge S^k \wedge \tilde{E}\mathcal{F}_{|H|}) \end{aligned}$$

so it suffices to do the case $H = G$. However, the spectrum $S^k \wedge \tilde{E}\mathcal{P}$ is isomorphic to $S^{k\rho_G} \wedge \tilde{E}\mathcal{P}$, so it is $\geq k|G| \geq n$. Next, we must show that τ_n is generated by these spectra. We again proceed by induction on $|G|$; the result is again trivial for the trivial group. Let τ denote the localizing subcategory generated by these spectra; we must show that $\tau_n \subseteq \tau$. Hence let $X \in \tau_n$, and consider the cofiber sequence below.

$$E\mathcal{P}_+ \wedge X \to X \to \tilde{E}\mathcal{P} \wedge X$$

The spectrum on the left is built out of spectra of the form $G_+ \wedge_H (i_H^* X)$ for $H \subsetneq G$, so by the induction hypothesis it is in the localizing subcategory generated by spectra of the form

$$G_+ \wedge_H (H/J_+ \wedge S^k \wedge \tilde{E}\mathcal{F}_{|J|}) \cong G/J_+ \wedge S^k \wedge \tilde{E}\mathcal{F}_{|J|}$$

for $H \subsetneq G$, $J \subseteq H$ and $k|J| \geq n$. Thus we have $E\mathcal{P}_+ \wedge X \in \tau$. Next, since $X \geq n$, X has a filtration with successive quotients that are wedges of suspensions of slice cells of dimension $\geq n$. Smashing this filtration with $\tilde{E}\mathcal{P}$, we obtain a filtration for $\tilde{E}\mathcal{P} \wedge X$ with successive quotients that are wedges of suspensions of spectra of the form $\tilde{E}\mathcal{P} \wedge \hat{S}$, where $\hat{S}$ is a non-induced slice cell of dimension $\geq n$ (since $\tilde{E}\mathcal{P} \wedge Y$ is contractible if Y is induced). These slice cells are of the form $S^{k\rho_G}$ for $k|G| \geq n$, so to show that $\tilde{E}\mathcal{P} \wedge X \in \tau$ it suffices to observe again that

$$\tilde{E}\mathcal{P} \wedge S^{k\rho_G} \cong \phi_G^* \Phi^G S^{k\rho_G} \cong \tilde{E}\mathcal{P} \wedge S^k.$$

□

Next we show that the map $X \to P^{n-1}X$ can be factored into maps related to these families. Let $\mathcal{T}_n^{(i)}$ denote the localizing subcategory generated by the spectra

$$G/H_+ \wedge S^k \wedge \tilde{E}\mathcal{F}_{|H|}$$

for $H \subseteq G$, $|H| = i$ and $k|H| \geq n$, and denote by $P^{n-1,i}$ the corresponding localization functor. By Proposition 2.1, $\mathcal{T}_n^{(i)} \subseteq \mathcal{T}_n$. We can now prove our factorization.

Proposition 2.2. *There is a natural isomorphism of functors as below.*

$$P^{n-1} \cong P^{n-1,|G|}P^{n-1,|G|-1}...P^{n-1,2}P^{n-1,1}$$

Proof. Let X be a G-spectrum, and consider the composite indicated below.

$$X \to P^{n-1,1}X \to P^{n-1,2}P^{n-1,1}X \to ... \to P^{n-1,|G|}P^{n-1,|G|-1}...P^{n-1,2}P^{n-1,1}X$$

To show that this is a model for $X \to P^{n-1}X$, it suffices to show that the cofiber is in $\Sigma\mathcal{T}_n$ and that the target spectrum is $< n$. For the first part, the cofiber has a finite filtration with successive cofibers of the form $Cofib(Y \to P^{n-1,j}Y)$. These spectra are in (resp.) $\Sigma\mathcal{T}_n^{(j)}$, and hence are in $\Sigma\mathcal{T}_n$. For the second part, by Proposition 2.1 it suffices to show that there are no nonzero maps from a spectrum of the form $G_+ \wedge_H (S^k \wedge \tilde{E}\mathcal{P})$ to the target when $H \subseteq G$ and $k|H| \geq n$. This automatically holds for the stage

$$P^{n-1,|H|}P^{n-1,|H|-1}...P^{n-1,2}P^{n-1,1}X$$

so we consider the successive cofibers of the sequence past this point. Each of these cofibers is in $\Sigma\mathcal{T}_n^{(j)}$ for some $j > |H|$. Since such spectra (and their desuspensions) restrict to zero in subgroups of order $< j$, the result follows by induction on j. $\square$

We now identify the colocalization functors at the $\mathcal{T}_n^{(i)}$. Denoting these colocalization functors by $P_{n,i}$, we have the following.

Proposition 2.3. *The colocalization functors at the $\tau_n^{(i)}$ are given by*

$$P_{n,i}X \cong (E\mathcal{F}_{i+1})_+ \wedge Post_{\lceil n/i \rceil} F(\tilde{E}\mathcal{F}_i, X)$$

where the natural map $P_{n,i}X \to X$ is given by the composite of the evident natural maps $E\mathcal{F}_+ \wedge Y \to Y$, $Post_m Y \to Y$ and $F(\tilde{E}\mathcal{F}, Y) \to Y$.

Proof. First we must show that this spectrum is in $\tau_n^{(i)}$. By using a cellular filtration for $E\mathcal{F}_{i+1}$, this reduces to showing that spectra of the form

$$G/H_+ \wedge Post_{\lceil n/i \rceil} F(\tilde{E}\mathcal{F}_i, X)$$

for $|H| \leq i$ are in $\tau_n^{(i)}$. Now the above spectrum is zero if $|H| < i$, since $i_H^* \tilde{E}\mathcal{F}_i \cong *$ in this case, so we may assume that $|H| = i$. In this case the above spectrum is of the form

$$G_+ \wedge_H (Post_{\lceil n/i \rceil} Y),$$

where Y restricts to zero in all proper subgroups of H. It follows that $Post_{\lceil n/i \rceil} Y$ can be built out of spectra of the form $S^k \wedge \tilde{E}\mathcal{P}_H$ for $k \geq \lceil n/i \rceil$; that is, $ki = k|H| \geq n$. To finish the proof, we must show that the maps

$$[G_+ \wedge_H (S^k \wedge \tilde{E}\mathcal{P}), (E\mathcal{F}_{i+1})_+ \wedge Post_{\lceil n/i \rceil} F(\tilde{E}\mathcal{F}_i, X)] \to [G_+ \wedge_H (S^k \wedge \tilde{E}\mathcal{P}), X]$$

are isomorphisms when $|H| = i$ and $k \geq n/i$, and monomorphisms when $|H| = i$ and $k + 1 \geq n/i$. Since $i_H^* E\mathcal{F}_{i+1} \cong *$ for such H, the above maps are then isomorphic to

$$[S^k \wedge \tilde{E}\mathcal{P}, Post_{\lceil n/i \rceil} F(\tilde{E}\mathcal{P}, i_H^* X)] \to [S^k \wedge \tilde{E}\mathcal{P}, i_H^* X].$$

If $k \geq n/i$ then by the universal property of $Post_*$ the above map is isomorphic to

$$[S^k \wedge \tilde{E}\mathcal{P}, F(\tilde{E}\mathcal{P}, i_H^* X)] \to [S^k \wedge \tilde{E}\mathcal{P}, i_H^* X],$$

which is clearly an isomorphism since the map $\tilde{E}\mathcal{P} \to \tilde{E}\mathcal{P} \wedge \tilde{E}\mathcal{P}$ is. If instead $k+1 \geq n/i$ but $k < n/i$, then we have

$$\begin{aligned}[S^k \wedge \tilde{E}\mathcal{P}, Post_{\lceil n/i \rceil} F(\tilde{E}\mathcal{P}, i_H^* X)] &\cong [S^k, Post_{\lceil n/i \rceil} F(\tilde{E}\mathcal{P}, i_H^* X)] \\ &= 0\end{aligned}$$

since $Post_{\lceil n/i \rceil} F(\tilde{E}\mathcal{P}, i_H^* X)$ is concentrated over H. □

Combining the above propositions, we arrive at the following theorem.

Theorem 2.4. *The assignment to X of the cofiber of the natural map*

$$(E\mathcal{F}_{i+1})_+ \wedge Post_{\lceil n/i \rceil} F(\tilde{E}\mathcal{F}_i, X) \to X,$$

which we denote by $P^{n-1,i}X$, extends to a functor, and this extension is unique such that the maps

$$X \to P^{n-1,i}X$$

form a natural transformation $Id \to P^{n-1,i}$. The composite functor

$$P^{n-1,|G|} \circ P^{n-1,|G|-1} \circ \ldots \circ P^{n-1,2} \circ P^{n-1,1}$$

with the composite natural transformation is uniquely isomorphic under Id to P^{n-1}.

Remark: Cofibers of natural maps are not generally functorial.

Next note that, if there do not exist subgroups of order i, then the set of generators of $\tau_n^{(i)}$ is empty; hence, the localization map

$$X \to P^{n-1,i}$$

is an isomorphism, and we may omit $P^{n-1,i}$ from the composition. Hence, if we define

$s_1, ..., s_r$ to be the orders of the subgroups of G in increasing order, we have that

$$P^{n-1} \cong P^{n-1,s_r} P^{n-1,s_{r-1}} ... P^{n-1,s_2} P^{n-1,s_1}.$$

Next we have the following easy facts, which are apparent from the explicit form of the colocalization functors.

Lemma 2.5. *The functors $P^{n-1,i}$ commute with restriction to subgroups.*

Lemma 2.6. *If X restricts to zero in subgroups of order i then $X \cong P^{n-1,i}X$.*

We now obtain a corollary that seems to generalize Corollary I.6.5.

Corollary 2.7. *Let m be the smallest order of a subgroup K such that i_K^*X is nontrivial. Then the n-slice of X is zero unless n is divisible by the order of a subgroup K such that i_K^*X is nontrivial and $|K| > m$ or n is divisible by m and $\pi_{n/m}^K X \neq 0$ for some subgroup K with $|K| = m$.*

Proof. By Lemmas 2.5 and 2.6, we may compute $P^{n-1}X$ using only the $P^{n-1,i}$ such that there is a subgroup K of order i with i_K^*X nontrivial. Now the n-slice of X is the fiber of the map $P^{(n+1)-1}X \to P^{n-1}X$, so we see from the explicit form of the colocalization functors $P_{*,j}$ that the n-slice is zero unless $\lceil \frac{n+1}{i} \rceil \neq \lceil \frac{n}{i} \rceil$ for one of these values of i. This condition is equivalent to i dividing n. Now suppose that n is not divisible by any of these orders except m. Since X restricts to zero in subgroups of order less than m, $X \cong F(\tilde{E}\mathcal{F}_m, X)$ so that $P_{*,m}X \cong (E\mathcal{F}_{m+1})_+ \wedge Post_{\lceil */m \rceil}X$. It follows that $P^{n-1,m}X = P^{n,m}X$ unless n is divisible by m and $\pi_{n/m}^K X \neq 0$ for some subgroup K of order m. □

As a sample application of this, we give the following.

Corollary 2.8. *If G is a p-group and X is a G-spectrum which is nonequivariantly contractible, then the n-slice of X is zero unless n is divisible by p.*

Finally, we give an alternative proof of part of Corollary I.6.5.

Corollary 2.9. *If X is concentrated over a normal subgroup N then the n-slice of X is zero unless n is divisible by $|N|$.*

Proof. A subgroup K can only satisfy $i_K^* X$ nontrivial if $K \supseteq N$ and hence, $|N|$ divides $|K|$. The result now follows from Corollary 2.7. $\square$

3 Formulas for Eilenberg MacLane Spectra

In this section we find formulas for the slice towers of arbitrary Eilenberg MacLane spectra in dimensions other than 0 and ± 1. We begin with positive dimensions; let $k \geq 2$, and let $\underline{M}$ be an arbitrary Mackey functor. We will give a formula for $P^{n-1}(\Sigma^k H\underline{M})$. Since

$$k \leq \Sigma^k H\underline{M} \leq k|G|$$

we may restrict ourselves to n such that $k < n \leq k|G|$. Fix a value of $n \in \mathbb{Z}$. Now, for $2 \leq j \leq |G|$ we define a functor $\mathscr{D}_j$ by

$$\mathscr{D}_j X := \begin{cases} Post_{\lceil n/j \rceil + 1} F((E\mathcal{F}_j)_+, X) & \text{if } \lceil n/j \rceil + 1 \leq k \\ X & \text{if } \lceil n/j \rceil + 1 > k \end{cases}$$

Note that, when X is $\min(\lceil n/j \rceil, k-1)$-connected, we have a natural map

$$X \to \mathscr{D}_j X$$

so that, when X is $(k-1)$-connected, we can form a composite map

$$X \to \mathscr{D}_{|G|} \mathscr{D}_{|G|-1} \ldots \mathscr{D}_2 X.$$

We can now determine the slice tower.

Theorem 3.1. *If $k \geq 2$ and $\underline{M}$ is an arbitrary Mackey functor, then we have a natural isomorphism*

$$P^{n-1}(\Sigma^k H\underline{M}) \cong \mathscr{D}_{|G|} \mathscr{D}_{|G|-1} \ldots \mathscr{D}_2 \Big(\Sigma^k H \Big(R \Big(\frac{n-1}{k-1} \Big) i^*_{(n-1)/(k-1)} \underline{M} / \mathscr{F}^{(n-1)/k} \underline{M} \Big) \Big).$$

Proof. If $n \leq k$ or $n > k|G|$ the result is true by inspection; hence, suppose that $k < n \leq k|G|$. Firstly, by Corollary I.8.3 we have that $P^{n-1}(\Sigma^k H\underline{M})$ is $(k+1)$-

coconnected. Furthermore, we have by Corollary I.8.6 that

$$\underline{\pi}_k P^{n-1}(\Sigma^k H\underline{M}) \cong R\Big(\frac{n-1}{k-1}\Big) i^*_{(n-1)/(k-1)}\underline{M}/\mathscr{F}^{(n-1)/k}\underline{M}.$$

Thus we have

$$\Sigma^k H\Big(R\Big(\frac{n-1}{k-1}\Big) i^*_{(n-1)/(k-1)}\underline{M}/\mathscr{F}^{(n-1)/k}\underline{M}\Big) \cong Post_k P^{n-1}(\Sigma^k H\underline{M}).$$

It is now easy to see that the map

$$\Sigma^k H\underline{M} \to \mathscr{D}_{|G|}\mathscr{D}_{|G|-1}\ldots\mathscr{D}_2\Big(\Sigma^k H\Big(R\Big(\frac{n-1}{k-1}\Big) i^*_{(n-1)/(k-1)}\underline{M}/\mathscr{F}^{(n-1)/k}\underline{M}\Big)\Big)$$

restricts in any subgroup to the corresponding map for that subgroup. Hence we proceed by induction on $|G|$; the result is trivial for the trivial group, so we assume G nontrivial. Denoting the spectrum in the statement by Y and the cofiber of the map $\Sigma^k H\underline{M} \to Y$ by C, we have by the induction hypothesis that

$$i^*_H Y < n \quad \text{and}$$
$$i^*_H \Sigma^{-1} C \geq n$$

for all proper subgroups H of G. Thus, to show that $Y < n$ it suffices by Proposition 2.1 to show that

$$[S^m \wedge \tilde{E}\mathcal{F}_{|G|}, Y] = 0$$

when $m|G| \geq n$. For this, first suppose that $\lceil \frac{n}{|G|} \rceil + 1 \leq k$. Then Y is of the form $Post_{\lceil n/|G|\rceil+1} F((E\mathcal{F}_{|G|})_+, Z)$ for some spectrum Z. By mapping $S^m \wedge \tilde{E}\mathcal{F}_{|G|}$ into the cofiber sequence

$$\Sigma^{-1} Post^{\lceil n/|G|\rceil} F((E\mathcal{F}_{|G|})_+, Z) \to Y \to F((E\mathcal{F}_{|G|})_+, Z)$$

we obtain an exact sequence where the first group is zero (since $m+1 > \lceil \frac{n}{|G|} \rceil$) and the third group is zero since $\tilde{E}\mathcal{F}_{|G|} \wedge (E\mathcal{F}_{|G|})_+ \cong *$. Hence, suppose that $\lceil \frac{n}{|G|} \rceil + 1 > k$. Then $\lceil \frac{n}{j} \rceil + 1 > k$ for all j, so that we have

$$Y = \Sigma^k H\Big(R\Big(\frac{n-1}{k-1}\Big) i^*_{(n-1)/(k-1)} \underline{M} / \mathscr{F}^{(n-1)/k} \underline{M}\Big).$$

Now we have $n > (k-1)|G|$ by assumption, so $\frac{n-1}{k-1} \geq |G|$. Thus, the above simplifies to

$$Y = \Sigma^k H(\underline{M} / \mathscr{F}^{(n-1)/k} \underline{M}).$$

We have $k - 1 < \frac{n}{|G|} \leq k$, so that $\lceil \frac{n}{|G|} \rceil = k$, and it suffices to show that

$$[S^m \wedge \tilde{E}\mathcal{P}, Y] = 0$$

for $m \geq k$, which reduces to showing that

$$[\tilde{E}\mathcal{P}, H(\underline{M} / \mathscr{F}^{(n-1)/k} \underline{M})] = 0.$$

It is easily seen that, for any Mackey functor $\underline{N}$, we have

$$\begin{aligned} [\tilde{E}\mathcal{P}, H\underline{N}] &\cong \{x \in \underline{N}(G/G) : i^*_H x = 0 \quad \forall H \subsetneq G\} \\ &= \mathscr{F}^{|G|-1} \underline{N}, \end{aligned}$$

so it follows that $Y < n$ since $\frac{n-1}{k} < |G| \Rightarrow \mathscr{F}^{|G|-1}(\underline{M} / \mathscr{F}^{(n-1)/k} \underline{M}) = 0$.

Finally, we must complete the proof that $\Sigma^{-1} C \geq n$. Since C is clearly 1-connected, it suffices by Theorem I.8.10 and the inductive hypothesis to show that

$$[S^m, \Sigma^{-1} C] = 0$$

when $m|G| < n$. That is, we must show that $[S^m, C] = 0$ when $m \leq \lceil \frac{n}{|G|} \rceil$. When $\lceil \frac{n}{|G|} \rceil + 1 \leq k$, this is clear since $\Sigma^k H\underline{M}$ and Y are both $(\lceil \frac{n}{|G|} \rceil)$-connected. Otherwise, we have as before that

$$Y = \Sigma^k H(\underline{M}/\mathscr{F}^{(n-1)/k}\underline{M})$$

and $\lceil \frac{n}{|G|} \rceil = k$, so we are reduced to the case $m = k$. Then we have

$$[S^k, C] \cong coker\Big(\underline{M}(G/G) \to (\underline{M}/\mathscr{F}^{(n-1)/k}\underline{M})(G/G)\Big) = 0.$$

□

Remark: It is unclear at present how this formula relates, if at all, to the general one from Section 2. It seems not to follow directly from it.

We now quickly derive the dual result for $k \leq -2$. Fix a value of $n \in \mathbb{Z}$. For $2 \leq j \leq |G|$ we define a functor $\tilde{\mathscr{D}}_j$ by

$$\tilde{\mathscr{D}}_j X := \begin{cases} Post^{\lfloor n/j \rfloor - 1}(E\mathcal{F}_j)_+ \wedge X & \text{if } \lfloor n/j \rfloor - 1 \geq k \\ X & \text{if } \lfloor n/j \rfloor - 1 < k \end{cases}$$

When X is $\max(\lfloor n/j \rfloor, k+1)$-coconnected, we have a natural map

$$\tilde{\mathscr{D}}_j X \to X$$

so that, when X is $(k+1)$-coconnected, we can form a composite map

$$\tilde{\mathscr{D}}_{|G|}\tilde{\mathscr{D}}_{|G|-1}...\tilde{\mathscr{D}}_2 X \to X.$$

We can now state the dual result.

Theorem 3.2. *If $k \leq -2$ and $\underline{M}$ is an arbitrary Mackey functor, then we have a natural isomorphism*

$$P_{n+1}(\Sigma^k H\underline{M}) \cong \tilde{\mathscr{D}}_{|G|}\tilde{\mathscr{D}}_{|G|-1}...\tilde{\mathscr{D}}_2\Big(\Sigma^k H\Big(L\Big(\frac{n+1}{k+1}\Big)i^*_{(n+1)/(k+1)}\mathscr{F}_{(n+1)/k}\underline{M}\Big)\Big).$$

Proof. Denote the spectrum in the statement by Y, and let $X = \Sigma^k H\underline{M}$. By Theorem 3.1, the duality of Section I.7 and the results of Section I.8, we have

$$\tilde{Y} \cong P^{-n-1}\tilde{X}$$

so that $\tilde{Y} < -n$, and hence $Y > n$. It follow that there is a unique map $Y \to P_{n+1}X$ such that the diagram below commutes.

$$\begin{tikzcd} Y \arrow[dr] \arrow[dd] & \\ & X \\ P_{n+1}X \arrow[ur] & \end{tikzcd}$$

By Theorem 3.1, the dual of this map is an isomorphism; hence, it is as well. $\square$

We will give some example computations in Section V.2 using these results. We end this section by drawing a consequence for the slices of Eilenberg MacLane spectra.

Corollary 3.3. *If $|k| \geq 2$ and $n \neq k$ then the n-slice of any Eilenberg MacLane spectrum in dimension k is zero unless n is divisible by the order of some nontrivial subgroup of G. The k-slice is zero if the group π_k^e is zero.*

Proof. The statement for negative k follows by duality from the statement for positive k; hence, let $k \geq 2$. Then if $n < k$, the n-slice is automatically zero since Eilenberg MacLane spectra in dimension k are $\geq k$. Hence, suppose $n > k$. Denoting $\Sigma^k H\underline{M}$ by X, the n-slice of X is the fiber of the map $P^{(n+1)-1}X \to P^{n-1}X$. Inspecting the proof of Theorem 3.1, we see that the same inductive proof works assuming we use the $\mathcal{F}_i$'s for i ranging over any subset of the numbers $2, ..., |G|$ that includes the orders of all the nontrivial subgroups of G; hence, we may use only the i's which occur as the orders of nontrivial subgroups of G. Then, inspecting the definitions of the functors $\mathscr{D}_i$, we see that they do not change when n is replaced by $n + 1$ unless n is divisible by such an integer. Hence, it remains to consider the expression

$$R\Big(\frac{n-1}{k-1}\Big)i^*_{(n-1)/(k-1)}\underline{M}/\mathscr{F}^{(n-1)/k}\underline{M}.$$

First of all, $\mathscr{F}^{(n-1)/k} = \mathscr{F}^{\lfloor (n-1)/k \rfloor}$ and $\lfloor \frac{n-1}{k} \rfloor = \lceil \frac{n}{k} \rceil - 1$, so we see that this doesn't change when n is replaced by $n+1$ unless n is divisible by k and there is a subgroup of order $\frac{n}{k} > 1$ (since $\mathscr{F}^{n/k-1} = \mathscr{F}^{n/k}$ otherwise). The same reasoning applied to $R(\frac{n-1}{k-1})i^*_{(n-1)/(k-1)}$, but with k replaced by $k-1$, shows that this part of the formula also does not change when n is replaced by $n+1$ unless n is divisible by the order of a nontrivial subgroup of G. For the second part, note that the k-slice equals $P^{(k+1)-1}$, so it is a functor of $\underline{M}/\mathscr{F}^1\underline{M}$, which is given as follows.

$$G/H \mapsto im\Big(\underline{M}(G/H) \xrightarrow{r_e^H} \underline{M}(G/e)\Big)$$

This is clearly zero when $\underline{M}(G/e)$ is. □

Remark: The first part of the above follows directly from Corollary 2.7; in fact, that result gives a slightly stronger statement.

Corollary 3.4. *If* $|k| \geq 2$, $n \neq k$ *and* G *is a* p*-group then the* n*-slice of any Eilenberg MacLane spectrum in dimension* k *is zero unless* n *is divisible by* p.

4 Formulas for Free and Cofree Spectra

Recall that a G-spectrum X is called *cofree* if the map

$$X \to F(EG_+, X)$$

is an isomorphism. Similarly, X is called *free* if the map

$$EG_+ \wedge X \to X$$

is an isomorphism (this is equivalent to having a cell structure with only free G-cells). In this section we give a formula for the positive part of the slice tower of a cofree spectrum. We also give a dual formula for the negative part of the slice tower of a free spectrum. We begin with cofree spectra.

Theorem 4.1. *For $n > 0$ and cofree spectra X there is a natural isomorphism*

$$P_n X \cong Post_{\lceil n/|G| \rceil} F((E\mathcal{F}_{|G|})_+, ... Post_{\lceil n/2 \rceil} F((E\mathcal{F}_2)_+, Post_n X)...).$$

Proof. Let Y denote the spectrum in the statement; we must first provide a map $Y \to X$. There is a natural zig-zag of maps relating Y to $Post_n X$, and these maps are all clearly nonequivariant isomorphisms. Hence we take our map $Y \to X$ to be the composite indicated below.

$$Y \to F(EG_+, Y) \cong F(EG_+, Post_n X) \to F(EG_+, X) \cong X$$

It is easy to see that this map restricts in any subgroup of G to the corresponding map for that subgroup. Hence, we proceed by induction on $|G|$; the result is trivial for the trivial group. Letting C denote the cofiber of $Y \to X$, we may therefore assume that

$$i_H^* Y \geq n \quad \text{and}$$
$$i_H^* C < n$$

for any proper subgroup H of G. Then by Proposition 2.1, to show that $C < n$ it suffices to show that

$$[S^m \wedge \tilde{E}\mathcal{P}, C] = 0$$

for $m|G| \geq n$. For this we map $S^m \wedge \tilde{E}\mathcal{P}$ into the cofiber sequence

$$X \to C \to \Sigma Y$$

to obtain an exact sequence

$$[S^m \wedge \tilde{E}\mathcal{P}, X] \to [S^m \wedge \tilde{E}\mathcal{P}, C] \to [S^m \wedge \tilde{E}\mathcal{P}, \Sigma Y]$$

where the first group is zero since $X \cong F(EG_+, X)$ and $EG_+ \wedge \tilde{E}\mathcal{P} \cong *$. Thus, it suffices to show that the last group is zero. For this we note that Y is of the form $Post_{\lceil n/|G| \rceil} F(E\mathcal{P}_+, Z)$ and map $S^m \wedge \tilde{E}\mathcal{P}$ into the cofiber sequence

$$Post^{\lceil n/|G| \rceil - 1} F(E\mathcal{P}_+, Z) \to \Sigma Y \to F(E\mathcal{P}_+, \Sigma Z)$$

to obtain an exact sequence where the first group is zero since $m \geq \lceil \frac{n}{|G|} \rceil$ and the last group is zero since $E\mathcal{P}_+ \wedge \tilde{E}\mathcal{P} \cong *$. To complete the proof that $Y \geq n$, it suffices by Theorem I.8.10 and the inductive hypothesis to show that

$$[S^m, Y] = 0$$

for $m < \lceil \frac{n}{|G|} \rceil$. This is trivial, since Y is clearly $(\lceil \frac{n}{|G|} \rceil - 1)$-connected. $\square$

To dualize this formula, we use the following easy fact.

Lemma 4.2. *A G-spectrum is free if and only its Brown-Comenetz dual is cofree.*

We now derive the dual result for free spectra.

Theorem 4.3. *For $n < 0$ and free spectra X there is a natural isomorphism*

$$P^n X \cong Post^{\lfloor n/|G| \rfloor}((E\mathcal{F}_{|G|})_+ \wedge ...Post^{\lfloor n/2 \rfloor}((E\mathcal{F}_2)_+ \wedge Post^n X)...).$$

Proof. Let Y denote the spectrum in the statement. As in the proof of Theorem 4.1, Y is related by a natural zig-zag of maps to $Post^n X$, and these maps are nonequivariant isomorphisms. Hence we obtain a map as below.

$$X \cong EG_+ \wedge X \to EG_+ \wedge Post^n X \cong EG_+ \wedge Y \to Y$$

By Lemma 4.2 and Theorem 4.1, we have $\tilde{Y} \cong P_{-n}\tilde{X}$. Thus we have $\tilde{Y} \geq -n$, so that $Y \leq n$. It follows that there is a unique map $P^n X \to Y$ such that the diagram

$$\begin{tikzcd}
 & Y \\
X \arrow[ur] \arrow[dr] & \\
 & P^n X \arrow[uu]
\end{tikzcd}$$

commutes. By Theorem 4.1, the dual of this map is an isomorphism; hence, it is as well. $\square$

We now derive consequences for the positive slices of cofree spectra and the negative slices of free spectra.

Corollary 4.4. *If X is cofree and $n > 0$ then the n-slice of X is zero unless n is divisible by the order of a nontrivial subgroup of G or $\pi_n^e X \neq 0$. If X is free and $n < 0$ then the n-slice of X is zero unless n is divisible by the order of a nontrivial subgroup of G or $\pi_n^e X \neq 0$.*

Proof. The second statement follows from the first by duality. Hence, let X be cofree and $n > 0$. The result is trivial for the trivial group, so assume G nontrivial. Inspecting the proof of Theorem 4.1, we see that the same inductive argument works if we only use the $\mathcal{F}_i$'s for i ranging over a subset of $2, ..., |G|$ that contains the orders of all the nontrivial subgroups of G. Hence, we may use only the $\mathcal{F}_i$'s for i the order of a nontrivial subgroup of G. Now the n-slice of X is the cofiber of the map $P_{n+1}X \to P_nX$, so we see from the explicit formulas for these spectra that they can only be different when n is divisible by such a value of i or when $F((E\mathcal{F}_j)_+, Post_{n+1}X)$ is not isomorphic to $F((E\mathcal{F}_j)_+, Post_nX)$, where j is the order of the smallest nontrivial subgroup of G. Now we have

$$\mathcal{F}_j = \{H \subseteq G : |H| < j\} = \{e\}$$

so that $E\mathcal{F}_j = EG$, and so the cofiber of the map

$$F((E\mathcal{F}_j)_+, Post_{n+1}X) \to F((E\mathcal{F}_j)_+, Post_nX)$$

is isomorphic to

$$F(EG_+, Post_n^nX) \cong F(EG_+, \Sigma^n H\underline{\pi}_nX),$$

which is nontrivial if and only if $\pi_n^eX \neq 0$. $\square$

Remark: The above also follows directly from Corollary 2.7.

We give a sample application of this below.

Corollary 4.5. *Let G be a p-group. If X is cofree and $n > 0$ then the n-slice of X is zero unless n is divisible by p or π_n^eX is nonzero. If X is free and $n < 0$ then the n-slice of X is zero unless n is divisible by p or π_n^eX is nonzero.*

We conclude this section by giving simplifications of the formulas for free and cofree spectra when $G = C_{p^m}$ for some prime p. Here we have

$$\mathcal{F}_{p^k} = \{e, C_p, ..., C_{p^{k-1}}\} = \mathcal{F}_{p^{k-1}+1}.$$

We require some lemmas.

Lemma 4.6. *The inclusion*

$$(E\mathcal{F}_{p^k+1})^{C_{p^k}} \to E\mathcal{F}_{p^k+1}$$

is a homotopy equivalence of G-spaces, and for any $j \leq k$, $(E\mathcal{F}_{p^k+1})^{C_{p^k}}$ is homotopy equivalent to $E\mathcal{F}_{p^{k-j}+1}$ as a G/C_{p^j}-space.

One proves this simply by checking fixed point sets. For the next lemma, we note that, when N is a normal subgroup of G, both of the spectra X^N and

$$X^{hN} \cong F(EG_+, X)^N$$

have the structure of G/N-spectra. We have the following simple fact.

Lemma 4.7. *For any $j \geq k$ there is a natural isomorphism*

$$F((E\mathcal{F}_{p^k+1})_+, X)^{C_{p^j}} \cong (X^{C_{p^k}})^{h(C_{p^j}/C_{p^k})}$$

of G/C_{p^j}-spectra.

Proof. By Lemma 4.6 we may assume that $E\mathcal{F}_{p^k+1}$ has trivial C_{p^k} action. Let our G-spectra be indexed on a complete G-universe $\mathcal{U}$, and let

$$i : \mathcal{U}^{C_{p^k}} \to \mathcal{U}$$

be the inclusion of universes. We have the following.

$$F((E\mathcal{F}_{p^k+1})_+, X)^{C_{p^j}} \cong (F((E\mathcal{F}_{p^k+1})_+, X)^{C_{p^k}})^{C_{p^j}/C_{p^k}}$$

Abusing notation slightly, we then have

$$\begin{aligned} F((E\mathcal{F}_{p^k+1})_+, X)^{C_{p^k}} &\cong F((E\mathcal{F}_{p^k+1})_+, i^*X)^{C_{p^k}} \\ &\cong F((E\mathcal{F}_{p^k+1})_+, (i^*X)^{C_{p^k}}) \\ &= F(E(G/C_{p^k})_+, X^{C_{p^k}}) \end{aligned}$$

as G/C_{p^k}-spectra indexed on $\mathcal{U}^{C_{p^k}}$, where we have used the trivial action of C_{p^k} on $E\mathcal{F}_{p^k+1}$ in the second line, and Lemma 4.6 on the third line. Combining the above equations, the result is immediate. $\square$

Applying this iteratively to the expression in Theorem 4.1, the following is immediate.

Corollary 4.8. *If $G = C_{p^m}$, $n > 0$ and X is cofree then $(P_nX)^G$ is naturally isomorphic to the spectrum below.*

$$Post_{\lceil n/p^m \rceil}(Post_{\lceil n/p^{m-1} \rceil}(\ldots Post_{\lceil n/p \rceil}(Post_n X)^{hC_p}\ldots)^{h(C_{p^{m-1}}/C_{p^{m-2}})})^{h(C_{p^m}/C_{p^{m-1}})}$$

To obtain the dual version of this, we require two more lemmas.

Lemma 4.9. *If N is a normal subgroup of G, then for G-spectra X and G-spaces A with trivial N action, there is a natural isomorphism*

$$A \wedge X^N \cong (A \wedge X)^N$$

of G/N-spectra (where the first A above is regarded as a G/N-space).

The proof is easy, using a cellular filtration for A to reduce to the case of orbits, and then Spanier-Whitehead duality. Returning to $G = C_{p^m}$, we obtain the following by applying this to a model of $E\mathcal{F}_{p^k+1}$ with trivial C_{p^k} action.

Lemma 4.10. *For any $j \geq k$ there is a natural isomorphism*

$$((E\mathcal{F}_{p^k+1})_+ \wedge X)^{C_{p^j}} \cong (X^{C_{p^k}})_{h(C_{p^j}/C_{p^k})}$$

of G/C_{p^j}-spectra.

We now state the dual version of Corollary 4.8.

Corollary 4.11. *If $G = C_{p^m}$, $n < 0$ and X is free then $(P^n X)^G$ is naturally isomorphic to the spectrum below.*

$$Post^{\lfloor n/p^m \rfloor}(Post^{\lfloor n/p^{m-1} \rfloor}(...Post^{\lfloor n/p \rfloor}(Post^n X)_{hC_p}...)_{h(C_{p^{m-1}}/C_{p^{m-2}})})_{h(C_{p^m}/C_{p^{m-1}})}$$

We will apply this formula in Section V.3 to gain some intuition about the behavior of the RSSS outside of the region where it coincides with the HFPSS (or the HOSS).

5 Order Families and Phase Changes; Description of the First Page

In Section I.9 we observed that, roughly between the s axis and the line of slope $m(G)-1$, the E_2 page of the RSSS only depends upon the nonequivariant homotopy groups of the spectrum (with their G-actions). When one crosses this line, the behavior of the E_2 page changes. In fact, one may observe such "phase transitions" at each line of slope one less than the order of a subgroup of G. In this section, we attempt to shed some light on these phase transitions. We begin with an easy fact.

Lemma 5.1. *If $k > 0$ and $2 \leq i \leq |G|$ then*

$$S^k \wedge E\tilde{\mathcal{F}}_i \quad \geq \quad ki.$$

The proof is easy, using Theorem I.8.10. The following is immediate.

Corollary 5.2. *If $n > 0$ and $k > \lfloor \frac{n}{i} \rfloor$ then the map*

$$\pi_k^G P_n^n X \to [S^k \wedge (E\mathcal{F}_i)_+, X]$$

is a monomorphism; it is an isomorphism if $k > \lfloor \frac{n}{i} \rfloor + 1$. Duallly, if $n < 0$ then the map

$$\pi_k^G((E\mathcal{F}_i)_+ \wedge X) \to \pi_k^G X$$

is an epimorphism when $k < \lceil \frac{n}{i} \rceil$ and an isomorphism when $k < \lceil \frac{n}{i} \rceil - 1$.

Corollary 5.3. *If a map $X \to Y$ of G-spectra is an isomorphism when restricted to subgroups of order $< i$, then when $t-s > 0$ the map*

$$E_2^{s,t}(X) \to E_2^{s,t}(Y)$$

is an isomorphism when $t - s \geq \lfloor \frac{t}{i} \rfloor + 2$ and a monomorphism when $t - s \geq \lfloor \frac{t}{i} \rfloor + 1$.

If $t - s < 0$ then it is an isomorphism when $t - s \leq \lceil \frac{t}{i} \rceil - 2$ and an epimorphism when $t - s \leq \lceil \frac{t}{i} \rceil - 1$.

By letting one the above spectra be trivial, we obtain the following.

Corollary 5.4. *If X restricts to zero in subgroups of order $< i$ then $E_2(X)$ is zero under the line of slope $i - 1$ in the first quadrant and above this line in the third quadrant.*

We now give a partial, iterative description of the E_2 page when $G = C_{p^m}$. When $m = 1$, we know most of the E_2 page by the results of Section I.9, so we know most of the (nonequivariant) homotopy groups of $(P_n^n X)^{C_p}$ in terms of the homotopy groups of X. Next, we have that

$$\begin{aligned} \pi_{t-s}^{C_{p^2}} P_t^t X &\cong \pi_{t-s}^{C_{p^2}}(F((E\mathcal{F}_{p+1})_+, P_t^t X)) \\ &\cong \pi_{t-s}((P_t^t X)^{C_p})^{h(C_{p^2}/C_p)} \end{aligned}$$

roughly under the line of slope $p^2 - 1$ in the first quadrant. These groups may be computable using a homotopy fixed point spectral sequence. One can then continue in this manner all the way up to the G-fixed point homotopy groups. Since there are generally differentials and nontrivial extensions in homotopy fixed point spectral sequences, it seems unlikely that one can determine a formula for the entire E_2 page of the RSSS; topology gets in the way. Similar considerations apply with homotopy orbit spectral sequences in the third quadrant. We will see this process at work in Section V.2, and we will see echoes in Section V.3.

Chapter III

Preservation Properties of Slice Towers

1 Introduction

In this chapter we show that the regular slice constructions preserve certain kinds of extra structure on spectra. In Section 2 we prove that the slice tower of a module spectrum is a tower of module spectra, when the ring is (-1)-connected. In Section 3 we prove that the P^n's of an algebra spectrum over a commutative ring spectrum form a tower of algebra spectra, provided the spectrum itself (not the coefficient ring) is (-1)-connected. In Section 4 we prove the same thing for commutative algebras. These statements have up-to-homotopy versions which, for the SSS, are stated as Corollary 4.31 of [HHR]. In Section 5 we analyze how the slice filtration is related to homological localization and acyclization. We give a criterion for the slice tower of a local spectrum to consist of local spectra (Theorem 5.7).

2 Preservation of Module Structure

In this section we prove that, under a simple connectivity assumption, the slice tower of a module spectrum over a ring spectrum is a tower of modules. We work throughout with S_G-modules, and let k denote a ring spectrum (in the strict sense). We denote by k-*Mod* the model category of modules over k. Letting $n \in \mathbb{Z}$ we define $Ho(k\text{-}Mod)^{<n}$ to be the full subcategory of k-modules whose underlying spectra are $< n$. Next we recall a construction of P^{n-1}; let X be an S_G-module. Then we can construct $P^{n-1}X$ as the colimit of a sequence

$$X = Y_0 \to Y_1 \to Y_2 \to ...$$

where for each $j \geq 0$ we have a pushout diagram as below

$$\begin{tikzcd}
\vee_\alpha F_S(\hat{S}_\alpha) \arrow[r] \arrow[d] & \vee_\alpha C(F_S(\hat{S}_\alpha)) \arrow[d] \\
Y_j \arrow[r] & Y_{j+1}
\end{tikzcd}$$

where the $\hat{S}_\alpha$ are slice cells of dimension $\geq n$ and there is at least one summand for each homotopy class of maps from each slice cell to Y_j. Finally we recall the standard adjunction show below.

$$\begin{tikzcd}
\mathscr{M}_G \arrow[r, shift right, "k\wedge(-)"'] & k\text{-Mod} \arrow[l, shift right, "forget"']
\end{tikzcd}$$

We are now ready to prove the following theorem.

Theorem 2.1. *For any associative ring spectrum k and any $n \in \mathbb{Z}$, the inclusion of $Ho(k\text{-}Mod)^{<n}$ into $Ho(k\text{-}Mod)$ has a left adjoint, which we denote by kP^{n-1}. Denoting the fiber of $X \to kP^{n-1}X$ by kP_nX, we have a functorial fiber sequence*

$$kP_nX \to X \to kP^{n-1}X$$

in $Ho(k\text{-}Mod)$. If k is (-1)-connected, the above fiber sequence forgets to

$$P_n X \to X \to P^{n-1} X$$

in $Ho(\mathscr{M}_G)$.

Proof. For the construction of kP^{n-1}, one merely mimics the construction of P^{n-1} given above, replacing $\hat{S}$ with $k \wedge \hat{S}$ etc. To see that kP_n and the above fiber sequence are functorial, we simply note that, for any X and Y,

$$[kP_n X, \Sigma^{-1} kP^{n-1} Y] \cong [kP_n X, kP^{n-1} Y] \cong 0$$

since $kP^{n-1}Y < n$ and $kP_n X$ is in the localizing subcategory generated by the free k-modules on the slice cells of dimension $\geq n$. Now suppose that k is (-1)-connected. The fiber sequence in the statement forgets to a fiber sequence of S_G-modules, so we need only show that $kP_n X \geq n$, or equivalently that

$$Cofib(X \to kP^{n-1} X) \in \Sigma \tau_n.$$

This cofiber has a filtration with successive quotients that are wedges of objects of the form $k \wedge \Sigma F_S(\hat{S})$, where $\hat{S}$ is a slice cell of dimension $\geq n$. Since k is (-1)-connected and this is a derived smash product (Lemma IV.3.3), this is in $\Sigma \tau_n$. $\square$

Remark: The functor kP_n is the right adjoint of the inclusion of the localizing subcategory generated by the free k-modules on the slice cells of dimension $\geq n$; this may not be equal to the full subcategory of k-modules whose underlying spectra are $\geq n$.

Remark: There is an up-to-homotopy version of the last statement of the theorem. Assuming that k is a (-1)-connected homotopy ring spectrum, one can easily deduce from (the RSSS version of) Lemma 4.29(i) of [HHR] and the fact that smashing with k preserves the τ_n that the towers $\{P_n X\}$ and $\{P^n X\}$ are towers of k-module spectra when X is a k-module spectrum.

Corollary 2.2. *If $k' \to k$ is a map of (-1)-connected ring spectra and X is a k-module and $n \in \mathbb{Z}$, then restriction of scalars sends the map*

$$X \to kP^{n-1}X$$

in $Ho(k\text{-}Mod)$ to the map

$$X \to k'P^{n-1}X$$

in $Ho(k'\text{-}Mod)$. Similarly for P_n.

Proof. Denote restriction of scalars by $R^k_{k'}$. Then since $R^k_{k'}kP^{n-1}X < n$, we can find a (unique) map in $Ho(k'\text{-Mod})$ to complete the diagram below.

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & k'P^{n-1}X \arrow[d, dotted] \\ & R^k_{k'}kP^{n-1}X \end{tikzcd}$$

By Theorem 2.1, the dotted arrow above forgets to the dotted arrow below

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & P^{n-1}X \arrow[d, dotted] \\ & P^{n-1}X \end{tikzcd}$$

in $Ho(\mathscr{M}_G)$. Thus, it is an isomorphism. $\square$

Corollary 2.3. *If k is a (-1)-connected ring spectrum and X is a k-module and $n \in \mathbb{Z}$, then for any subgroup H of G the map*

$$X \to kP^{n-1}X$$

in $Ho(k\text{-}Mod)$ restricts to the map

$$i_H^*X \to (i_H^*k)P^{n-1}i_H^*X$$

in $Ho((i_H^*k)\text{-}Mod)$. *Similarly for* P_n.

Proof. As above, using the fact that i_H^* commutes with P^{n-1}. $\square$

3 Preservation of Algebra Structure

We now prove an analogous theorem for algebras over a commutative ring. Let k denote a commutative ring spectrum, and denote by $assoc_k$ the model category of associative k-algebras. Now this category is not pointed, since the initial and terminal objects are not the same, so we can not form fibers or cofibers and thus we will not be able to construct the P_n from the P^{n-1}. Also note that, if R is a ring spectrum whose underlying spectrum is < 0, then the unit map $S^0 \to R$ will be null (up to homotopy), implying that R is trivial. Thus we can only hope to obtain the positive part of the slice tower in the category of algebras. Similar considerations, as well as an attempt to mimic the theorem proven below, imply that we can not hope to construct *any* of the P_n in the category of algebras (except P_0); thus, we are left with constructing P^{n-1} for $n > 0$.

Let $Ho(assoc_k)^{<n}$ denote the full subcategory of k-algebras whose underlying spectra are $< n$. Before stating our theorem, we recall a description of (certain) pushouts in $assoc_k$. Let $A \to B$ be a generating (acyclic) q-cofibration of S_G-modules, and denote by $k\mathbb{A}$ the free k-algebra functor. Then the pushout

$$\begin{array}{ccc} k\mathbb{A}(A) & \longrightarrow & k\mathbb{A}(B) \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

in $assoc_k$ is the same as the pushout

$$\begin{array}{ccc} \mathbb{A}(A) & \longrightarrow & \mathbb{A}(B) \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

in the category of rings. Thus Y can be written as a colimit of Y_i such that $Y_0 = X$

and there are pushouts

$$\begin{tikzcd} X^{\wedge i+1} \wedge \partial_A B^{\wedge i} \arrow[r] \arrow[d] & X^{\wedge i+1} \wedge B^{\wedge i} \arrow[d] \\ Y_{i-1} \arrow[r] & Y_i \end{tikzcd}$$

in the category of S_G-modules for all $i > 0$ (see [SS]). We also require two technical lemmas.

Lemma 3.1. *If k is a flat commutative ring spectrum then cofibrant k-algebras are flat.*

Proof. Let X be a cofibrant k-algebra. We may assume that X is a $k\mathbb{A}(I)$-cell. Then, by the above, X has a transfinite filtration $\{X_\alpha\}$ such that X_0 is k and $X_{\alpha+1}$ is a colimit of a sequence of h-cofibrations with successive quotients of the form

$$X_\alpha^{\wedge i+1} \wedge (B/A)^{\wedge i}$$

where $A \to B$ is a generating cofibration. Now k is flat by assumption, and $(B/A)^{\wedge i}$ is flat by Lemma IV.3.3, so the result follows by transfinite induction. □

Lemma 3.2. *Cofibrant commutative ring spectra are flat.*

Proof. Let X be a cofibrant commutative ring spectrum. Denoting the free commutative ring spectrum functor by $\mathbb{C}$, we may assume that X is a $\mathbb{C}(I)$-cell. Then X has a transfinite filtration $\{X_\alpha\}$ such that X_0 is the sphere spectrum and $X_{\alpha+1}$ is a colimit of a sequence of h-cofibrations with successive quotients of the form $X_\alpha \wedge (B/A)^{\wedge i}/\Sigma_i$, where $A \to B$ is a generating cofibration. Now the sphere spectrum is certainly flat, and $(B/A)^{\wedge i}/\Sigma_i$ is flat by Lemma IV.3.3, so the result follows by transfinite induction. □

Theorem 3.3. *For any $n \in \mathbb{Z}$ the inclusion of $Ho(assoc_k)^{<n}$ into $Ho(assoc_k)$ has a left adjoint, which we denote by $k\mathbb{A}P^{n-1}$. If $n \leq 0$ then this functor is zero, while if*

$n > 0$ and X is (-1)-connected the map

$$X \to k\mathbb{A}P^{n-1}X$$

forgets to the map

$$X \to P^{n-1}X$$

in $Ho(\mathscr{M}_G)$.

Proof. First we reduce to the case where k is a cofibrant commutative ring spectrum. Let $k' \to k$ be a cofibrant replacement of k in the category of commutative ring spectra. Then the "restriction of scalars" functor from $assoc_k$ to $assoc_{k'}$ induces an equivalence of homotopy categories which is the identity on underlying spectra (for proof, see Section A.3). Thus we may assume that k is cofibrant.

Let X be a cofibrant k-algebra; we now mimic the construction of P^{n-1} by applying the free k-algebra functor to the slice cells and their cones. We obtain a cofibrant k-algebra $k\mathbb{A}P^{n-1}X$ which is easily seen to have the required universal property. Now supposing that $n > 0$, we must prove the last statement. By coning off one slice cell at a time, we obtain a transfinite filtration $\{X_\alpha\}$ of $k\mathbb{A}P^{n-1}X$ by h-cofibrations such that $X_0 = X$ and there are pushout diagrams

$$\begin{tikzcd} k\mathbb{A}(F_S(\hat{S})) \arrow[r] \arrow[d] & k\mathbb{A}(F_S(C\hat{S})) \arrow[d] \\ X_\alpha \arrow[r] & X_{\alpha+1} \end{tikzcd}$$

in the category of k-algebras, where $\hat{S}$ is a slice cell of dimension $\geq n$. Thus the map $X \to k\mathbb{A}P^{n-1}X$ is an h-cofibration, so it suffices to show that

$$k\mathbb{A}P^{n-1}X/X \in \Sigma\tau_n.$$

The above quotient inherits a transfinite filtration by h-cofibrations with successive quotients of the form

$$X_\alpha^{\wedge i+1} \wedge F_S(\Sigma\hat{S})^{\wedge i}$$

where $\hat{S}$ is a slice cell of dimension $\geq n$ and $i > 0$. Now X_α is flat by Lemmas 3.2 and 3.1, so we see by transfinite induction that X_α is (-1)-connected. Applying transfinite induction again and using this connectivity, the result follows easily. □

The following corollaries are proven in the same manner as Corollaries 2.2 and 2.3.

Corollary 3.4. *If X is a (-1)-connected k-algebra and $k' \to k$ is a map of commutative ring spectra and $n \in \mathbb{Z}$, then restriction of scalars sends the map*

$$X \to k\mathbb{A}P^{n-1}X$$

in $Ho(assoc_k)$ to the map

$$X \to k'\mathbb{A}P^{n-1}X$$

in $Ho(assoc_{k'})$.

Corollary 3.5. *If X is a (-1)-connected k-algebra and $n \in \mathbb{Z}$, then for any subgroup H of G the map*

$$X \to k\mathbb{A}P^{n-1}X$$

in $Ho(assoc_k)$ restricts to the map

$$i_H^*X \to (i_H^*k)\mathbb{A}P^{n-1}i_H^*X$$

in $Ho(assoc_{(i_H^*k)})$.

Remark: There is an up-to-homotopy version of the last statement of the theorem. If X is a (-1)-connected homotopy ring spectrum, Corollary 4.31 of [HHR] (adapted to the RSSS) says that $\{P^n X\}$ is a tower of homotopy ring spectra.

In the next section we prove an analogous theorem in the commutative case.

4 Preservation of Commutative Algebra Structure

Again let k be a commutative ring spectrum, and let $Ho(comm_k)^{<n}$ denote the full subcategory of commutative k-algebras whose underlying spectra are $< n$. We denote by $k\mathbb{C}$ the free commutative k-algebra functor. Similar remarks to the ones at the beginning of the last section apply here. We recall a description of pushouts in the category of commutative k-algebras. If $A \to B$ is a generating (acyclic) cofibration of S_G-modules and we have a pushout diagram

$$\begin{tikzcd} k\mathbb{C}(A) \arrow[r] \arrow[d] & k\mathbb{C}(B) \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

in the category of commutative k-algebras, then Y can be written as the colimit of Y_i such that $Y_0 = X$ and there are pushout diagrams

$$\begin{tikzcd} X \wedge (\partial_A B^{\wedge i})/\Sigma_i \arrow[r] \arrow[d] & X \wedge (B^{\wedge i})/\Sigma_i \arrow[d] \\ Y_{i-1} \arrow[r] & Y_i \end{tikzcd}$$

in the category of S_G-modules for all $i > 0$. We require a modification of Lemma IV.3.7.

Lemma 4.1. *If $\hat{S}$ is a slice cell of dimension $n > 0$ and $i > 0$ then*

$$F_S(\Sigma\hat{S})^{\wedge i}/\Sigma_i \quad \in \quad \Sigma\tau_n.$$

Proof. Examining the proof of Lemma IV.3.7, we see that the above object is homotopy equivalent to $E_G\Sigma_{i+} \wedge_{\Sigma_i} F_S(\Sigma\hat{S})^{\wedge i}$, and that this is built out of spectra of the form

$$G_+ \wedge_H (\hat{W} \wedge S^V)$$

where $\hat{W}$ is a wedge of slice cells of dimension ni and V is a permutation representation of dimension i. Now induction preserves the slice filtration and commutes with suspension, so we need only note that V contains a nonzero fixed vector, so that S^V is a suspension of another representation sphere. $\square$

Theorem 4.2. *For any $n \in \mathbb{Z}$ the inclusion of $Ho(comm_k)^{<n}$ into $Ho(comm_k)$ has a left adjoint, which we denote by $k\mathbb{C}P^{n-1}$. If $n \leq 0$ then this functor is zero, while if $n > 0$ and X is (-1)-connected the map*

$$X \to k\mathbb{C}P^{n-1}X$$

forgets to the map

$$X \to P^{n-1}X$$

in $Ho(\mathscr{M}_G)$.

Proof. As in the proof of Theorem 3.3, for the most part. We again proceed to the case where $n > 0$ and X is (-1)-connected. In this case our successive quotients are of the form

$$X_\alpha \wedge F_S(\Sigma\hat{S})^{\wedge i}/\Sigma_i$$

where $\hat{S}$ is a slice cell of dimension $\geq n$ and $i > 0$. By Lemma IV.3.3(iv), this is a derived smash product. Then by Lemma 4.1 and transfinite induction, X_α is (-1)-connected for all α. Applying Lemma 4.1 and transfinite induction again, we see that the above object is in $\Sigma\tau_n$. $\square$

The following corollaries are proven in the same manner as Corollaries 3.4 and 3.5.

Corollary 4.3. *If X is a (-1)-connected commutative k-algebra and $k' \to k$ is a map of commutative ring spectra and $n \in \mathbb{Z}$, then restriction of scalars sends the map*

$$X \to k\mathbb{C}P^{n-1}X$$

in $Ho(comm_k)$ to the map

$$X \to k'\mathbb{C}P^{n-1}X$$

in $Ho(comm_{k'})$.

Corollary 4.4. *If X is a (-1)-connected commutative k-algebra and $n \in \mathbb{Z}$, then for any subgroup H of G the map*

$$X \to k\mathbb{C}P^{n-1}X$$

in $Ho(comm_k)$ restricts to the map

$$i_H^*X \to (i_H^*k)\mathbb{C}P^{n-1}i_H^*X$$

in $Ho(comm_{(i_H^*k)})$.

Corollary 4.5. *If X is a (-1)-connected commutative k-algebra and $n \in \mathbb{Z}$, then the map*

$$X \to k\mathbb{C}P^{n-1}X$$

in $Ho(comm_k)$ forgets to the map

$$X \to k\mathbb{A}P^{n-1}X$$

in $Ho(assoc_k)$.

Remark: In the solution of the Kervaire invariant problem ([HHR]), the authors construct the slice tower of $MU\mathbb{R}$ as a tower of $MU\mathbb{R}$-modules. This is consistent with Section 2. However, the results in this section imply that the slice tower (that is, the P^n) can in fact be constructed as a tower of commutative $MU\mathbb{R}$-algebras.

Remark: There is an up-to-homotopy version of the last statement of the theorem. If X is a (-1)-connected homotopy commutative and associative ring spectrum, Corollary 4.31 of [HHR] (again adapted to the RSSS) says that $\{P^n X\}$ is a tower of homotopy commutative and associative ring spectra.

5 Homological Localization

In this section we study the interaction of the slice tower with homological localization. Recall from [Bou] that if E is an ordinary spectrum, there is a functorial fiber sequence

$$_EX \to X \to X_E$$

where the first map above is the terminal map to X from a spectrum that is E-acyclic, while the second map is the initial map from X to a spectrum that is E-local. The spectrum $_EX$ is called the *E-acyclization* of X, while X_E is called the *E-localization* of X. The same general theory works in the equivariant case. We begin by giving some special cases of localization at an equivariant spectrum.

Proposition 5.1. *Let $\mathcal{F}$ be a family of subgroups of G. Then we have natural isomorphisms*

$$X_{E\mathcal{F}_+} \cong F(E\mathcal{F}_+, X),$$
$$X_{\tilde{E}\mathcal{F}} \cong \tilde{E}\mathcal{F} \wedge X.$$

We omit the proof, which is easy. Next we examine localization at a non-equivariant spectrum, by which we mean the following. Denote by

$$i_* : Sp \to Sp_G$$

the functor which regards an ordinary spectrum as a naive G-spectrum with trivial action and then pushes forward to the complete G-universe (see [LMS]). Recall that i_* is a symmetric monoidal functor, and that it (in a sense) preserves cell structures. To examine localization at spectra of the form i_*E, we need the following lemma.

Lemma 5.2. *For any subgroup H of G we have a natural isomorphism*

$$(i_*E \wedge X)^H \cong E \wedge X^H.$$

We omit the proof.

Corollary 5.3. *The map*

$$X \to Y$$

is an i_*E-localization if and only if the map

$$X^H \to Y^H$$

is an E-localization for all subgroups H of G.

Proof. First suppose that $X \to Y$ is an i_*E localization. Then we have that the map $i_*E \wedge X \to i_*E \wedge Y$ is an equivalence, which means it induces equivalences on all the fixed point spectra, so by Lemma 5.2 the maps $X^H \to Y^H$ are all E-homology equivalences. Thus we must prove that all the Y^H are E-local; hence, let Z be E-acyclic. Then we have

$$[Z, Y^H] \cong [G_+ \wedge_H i_*Z, Y] = 0$$

since Y is i_*E-local and

$$\begin{aligned} i_*E \wedge (G_+ \wedge_H i_*Z) &\cong G_+ \wedge_H (i_*E \wedge i_*Z) \cong G_+ \wedge_H i_*(E \wedge Z) \\ &\cong G_+ \wedge_H i_*(*) \cong *. \end{aligned}$$

Conversely, suppose that all the fixed point maps are E-localizations. Then all the maps $E \wedge X^H \to E \wedge Y^H$ are isomorphisms, so by Lemma 5.2 the map $X \to Y$ is an (i_*E)-homology equivalence. It follows that we can find a map to complete the diagram below.

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d, dotted] \\ & X_{i_*E} \end{tikzcd}$$

Applying the fixed point functors, we get diagrams as below.

$$\begin{tikzcd} X^H \arrow[r] \arrow[dr] & Y^H \arrow[d, dotted] \\ & X^H_{i_*E} \end{tikzcd}$$

By assumption and by the first part of the corollary proven above, the two solid arrows are E-localizations; hence, the dotted arrow is an isomorphism. $\square$

Corollary 5.4. *If E is bounded below and X is $(n-1)$-connected then X_{i_*E} is $(n-1)$-connected.*

Proof. By Corollary 5.3, we have $X^H_{i_*E} \cong (X^H)_E$, so Theorem 3.1 of [Bou] implies that $X^H_{i_*E}$ is either a localization or a completion of X^H at a set of primes. It is a standard fact that localization and completion at sets of primes preserve connectivity (see Propositions 2.4 to 2.6 of [Bou]). $\square$

We now state our main results in this section.

Theorem 5.5. *The following statements are equivalent.*

(i) If X is $(n-1)$-connected for any $n \in \mathbb{Z}$ then so is X_E.

(ii) If X is (-1)-connected then so is X_E.

(iii) If $X \geq n$ for any $n \in \mathbb{Z}$ then so is X_E.

Proof. For the equivalence of (i) and (ii), we note that localization commutes with (de-)suspension. For the equivalence of (ii) and (iii), we slighty reformulate Proposition I.7.6 as follows: $X \geq n \Leftrightarrow X \wedge D\hat{S}$ is 0-connected for all slice cells $\hat{S}$ of dimension $< n \Leftrightarrow X \wedge D\hat{S} \wedge S^{-1}$ is (-1)-connected for all slice cells $\hat{S}$ of dimension $< n$. The result now follows from the basic fact that smashing with strongly dualizable spectra commutes with localization. $\square$

Corollary 5.6. *The classes τ_n are closed under (i_*E)-localization for any nonequivariant spectrum E that is bounded below, and are closed under $\tilde{E}\mathcal{F}$-localization for any family $\mathcal{F}$ of subgroups of G.*

Theorem 5.7. *The following statements are equivalent.*

(i) X is E-local $\Rightarrow$ so are the P_nX, P^nX, and P_n^nX.

(ii) E-localization preserves connectivity.

Proof. First assume (i) holds. We may take "connectivity" to mean (-1)-connectivity; hence, suppose X is (-1)-connected. Then by assumption, the spectrum $P_0(X_E)$, which is the connective cover of X_E, is E-local. Hence, it follows from the universal properties of localizations and connective covers that there are unique maps that complete the successive diagrams below.

$$\begin{tikzcd} X \arrow[r, dotted] \arrow[dr] & P_0(X_E) \arrow[d] \\ & X_E \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & X_E \arrow[d, dotted] \\ & P_0(X_E) \end{tikzcd}$$

We then have the following commutative diagram.

$$\begin{tikzcd} & X_E \arrow[d] \\ X \arrow[ur] \arrow[r] \arrow[dr] & P_0(X_E) \arrow[d] \\ & X_E \end{tikzcd}$$

Since the outer triangle commutes, the vertical composite must be the identity; hence, X_E is a retract of $P_0(X_E)$ so it is (-1)-connected. Conversely, suppose that (ii) holds, and let X be E-local. It suffices to show that the P_nX are E-local. Fix $n \in \mathbb{Z}$. Since

X is E-local, we can find a unique map to complete the diagram below.

$$\begin{tikzcd} P_nX \arrow[r] \arrow[d] & X \\ (P_nX)_E \arrow[ur, dotted] & \end{tikzcd}$$

Next, by Theorem 5.5 the spectrum $(P_nX)_E$ is $\geq n$, so there is a unique map that completes the diagram below.

$$\begin{tikzcd} (P_nX)_E \arrow[r] \arrow[d, dotted] & X \\ P_nX \arrow[ur] & \end{tikzcd}$$

We then have the following commutative diagram.

$$\begin{tikzcd} P_nX \arrow[d] \arrow[dr] & \\ (P_nX)_E \arrow[r] \arrow[d] & X \\ P_nX \arrow[ur] & \end{tikzcd}$$

Since the outer triangle commutes, the vertical composite must be the identity; hence, P_nX is a retract of $(P_nX)_E$ so it is E-local. $\square$

The following corollary follows directly from the above results.

Corollary 5.8. *If X is local or complete at a set of primes, then so are the P_nX, P^nX, and P_n^nX.*

There are "dual" results to the above theorems and corollaries, where one replaces localization with acyclization, connectivity with coconnectivity, and P_n with P^n. The proofs are then identical after making these replacements and reversing the directions of the arrows, so we will not record them. We state the results below.

Theorem 5.9. *The following statements are equivalent.*

(i) If X is $(n+1)$-coconnected for any $n \in \mathbb{Z}$ then so is ${}_EX$.

(ii) If X is $(+1)$-coconnected then so is ${}_EX$.

(iii) If $X \leq n$ for any $n \in \mathbb{Z}$ then so is ${}_EX$.

Corollary 5.10. *The classes $\tau_n \perp$ are closed under the acyclizations corresponding to localization and completion at sets of primes, and are closed under $E\mathcal{F}_+$-acyclization for any family $\mathcal{F}$ of subgroups of G.*

Remark: It is clearly also true that $\tau_n \perp$ is closed under $E\mathcal{F}_+$-localization.

Theorem 5.11. *The following statements are equivalent.*

(i) X is E-acyclic $\Rightarrow$ so are the P_nX, P^nX, and P_n^nX.

(ii) E-acyclization preserves coconnectivity.

Corollary 5.12. *If the completion of X at a set of primes is zero, then the same holds for the P_nX, P^nX, and P_n^nX.*

Finally, we consider localization at a set of primes K. This can be constructed as a homotopy colimit of a sequence, wherein each map is multiplication by an element of K and each element of K occurs infinitely many times. Hence, it is elementary that both τ_n and $\tau_n \perp$ are closed under localization at K for any $n \in \mathbb{Z}$. Now localization at K preserves cofiber sequences as well, so the following result is immediate.

Proposition 5.13. *The functors P_n, P^n, and P_n^n commute with localization at K for every $n \in \mathbb{Z}$.*

We caution the reader that localization usually does not preserve the slice tower. In fact, at present, we know of no other cases where it does; it may be that there are none.

Chapter IV

Model Categories for Slice Towers

1 Introduction

In this chapter we prove that Massey products in the RSSS for a ring spectrum converge, under certain assumptions, to Toda brackets in the homotopy ring. There are also Toda brackets on the first page of the spectral sequence that, while more mysterious in general, can be computed in a certain range as Massey products for a fictional E_1 page. These results are very similar to corresponding results for the Adams Spectral Sequence in ordinary homotopy theory. However, not having a convenient, canonical model for slice towers, it is necessary to develop some theoretical machinery in order to achieve this.

In Section 2, we begin this development by constructing model structures such that cofibrant replacements give the necessary colocalization functors. For this, it is necessary to have a model category where objects are fibrant. Since we also require a point-set level smash product, we work with equivariant S-modules. In Section 3 we prove several technical lemmas which we need in later sections to obtain slice towers with good point-set level properties. A discussion of towers of spectra is contained in Section 4. In Section 5 we prove that the slice tower of a ring spectrum can be constructed as a point-set level ring tower; similar results for commutative ring spectra and module spectra are obtained in Sections 6 and 7. These results may be of independent interest, though the proofs use model theory and are thus not constructive.

In Section 8 we discuss pairings of slice towers and generalize the theory to arbitrary associative systems of pairings of spectra. Finally, the results on Toda brackets are derived in Section 9 from this theoretical apparatus.

The reader is strongly encouraged to refer to the Appendix while reading technical proofs in this chapter.

A quick word on notation: if G is a compact Lie group and $i > 0$ we will denote by $\mathcal{F}_G[i]$ the family of subgroups of $G \times \Sigma_i$ that have trivial intersection with Σ_i, and by $E_G\Sigma_i$ the universal $\mathcal{F}_G[i]$-space.

2 Bousfield Colocalizations of the q Model Structure

Fix $n \in \mathbb{Z}$. We seek a model structure on G-spectra such that the maps

$$P_n X \to X$$

are cofibrant replacements. We begin with Lewis-May spectra, defining three classes of maps as below:

- $\mathcal{W} = P_n$-equivalences,
- $\mathcal{F} =$ q-fibrations,
- $\mathcal{C} =$ maps that have the LLP with respect to $\mathcal{W} \cap \mathcal{F}$.

The following facts are immediately apparent:

- the class of weak equivalences is contained in $\mathcal{W}$,
- the acyclic q-cofibrations are contained in $\mathcal{C}$, and
- $\mathcal{C}$ is contained in the class of q-cofibrations, which is contained in the class of h-cofibrations.

Thus, we are attempting to *colocalize* the q model structure (see [Hir]). Standard model theoretic arguments give the following lemma.

Lemma 2.1. *$\mathcal{C} \cap \mathcal{W}$ is the class of ayclic q-cofibrations.*

All of the model axioms are now clear except the existence of factorizations into cofibrations followed by acyclic fibrations. Thus we must characterize the class $\mathcal{F} \cap \mathcal{W}$. We begin with the following.

Lemma 2.2. *If $i : A \to B$ is a q-cofibration, with A and B q-cofibrant and $\geq n$, then $i \in \mathcal{C}$.*

Proof. Let $p : X \to Y$ be a map in $\mathcal{F} \cap \mathcal{W}$. We must show that we can solve any lifting problem of the form shown below.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

Since p is a P_n-equivalence, and A and B are q-cofibrant and $\geq n$ (and all spectra are q-fibrant), it is immediate that we can find a lifting which makes the two triangles commute up to homotopy. Then using the fact that B is q-cofibrant and p is a q-fibration, we may deform the initial choice of lift so that the bottom triangle commutes strictly. We must now apply another homotopy to make the upper triangle commute strictly without destroying the commutativity of the lower triangle. We could obtain such a homotopy by solving a lifting problem of the form

$$\begin{tikzcd} B \cup_A A \wedge I_+ \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ B \wedge I_+ \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

where the bottom horizontal map is a constant homotopy and the left vertical map is the inclusion of the mapping cylinder of i, if we had a homotopy from the composite $A \to B \to X$ to the original map $A \to X$ such that the homotopy becomes constant after composing with p. Now the left vertical map in the above diagram is an acyclic q-cofibration, while the right vertical map is a q-fibration, so the above lifting problem can be solved. Thus, we are reduced to showing that there is a homotopy $A \wedge I_+ \to X$ from $A \to B \to X$ to $A \to X$ such that the composite $A \wedge I_+ \to X \to Y$ is a constant homotopy. Now these two maps from A to X project to the same map into Y, so this reduces to showing that two points in a common fiber of

$$X^A \to Y^A$$

can be connected by a path in that fiber. It is easy to see that the above map is a Serre fibration and a weak equivalence (since A is q-cofibrant and $\geq n$ and p is a P_n-equivalence), so every fiber is path connected. $\square$

Remark: The above proof does not work in the category of equivariant orthogonal spectra, since not all objects are fibrant.

We now make the necessary definitions to form our set of generating cofibrations.

Definition 2.3. A *slice$_n$ complex* is a spectrum X with a complete filtration

$$X_{-1} = * \subseteq X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots$$

and pushout diagrams

$$\begin{tikzcd} \vee_\alpha \Sigma^{i_\alpha} \hat{S}_\alpha \arrow[r] \arrow[d] & \vee_\alpha C(\Sigma^{i_\alpha} \hat{S}_\alpha) \arrow[d] \\ X_j \arrow[r] & X_{j+1} \end{tikzcd}$$

for each $j \geq -1$, where each $\hat{S}_\alpha$ is a slice cell of dimension $\geq n$ and each i_α is ≥ -1. We say that X is *finite* if it is built from finitely many (suspensions of) slice cells. A *slice$_n$ pair* is a pair of spectra (B, A), where B is a slice$_n$ complex and A is a subcomplex (in the obvious sense).

Observe that slice$_n$ complexes are q-cofibrant and $\geq n$. In fact, the construction of P_n produces slice$_n$ complexes, so every homotopy type which is $\geq n$ has a representative which is a slice$_n$ complex. By Lemma 2.2, inclusions of subcomplexes are in $\mathcal{C}$. Now let J be the usual set of generating acyclic q-cofibrations, and let

$$I_n := J \cup \{A \to B : (B, A) \text{ is a finite slice}_n \text{ pair}\}. \tag{2.4}$$

It is clear that the isomorphism classes of finite slice$_n$ pairs fit into a set, so we can take I_n to be a set. We can now characterize our acyclic fibrations.

Lemma 2.5. *A map $p : X \to Y$ is in $\mathcal{F} \cap \mathcal{W}$ if and only if it has the RLP with respect to I_n.*

Proof. We already have that p has the RLP with respect to I_n if it is in $\mathcal{F} \cap \mathcal{W}$, so suppose that p has the RLP with respect to I_n. Since I_n contains J, p must be a q-fibration. Thus we must show that p is a P_n-equivalence. This is equivalent to the statement that

$$p_* : [Z, X] \to [Z, Y]$$

is an isomorphism for all slice$_n$ complexes Z. Hence, let Z be a slice$_n$ complex. To show that p_* is surjective, it suffices to show that we can solve any lifting problem of the form shown below.

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ Z \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

Using Zorn's Lemma, we may choose a maximal lifting defined on a subcomplex Z' of Z. If $Z' \neq Z$, then there is some finite subcomplex B of Z not contained in Z'. Letting $A = B \cap Z'$, we see by maximality that the lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

has no solution. This is a contradiction, since $(A \to B) \in I_n$. For injectivity, let F be the fiber of p; it suffices to show that $[Z, F] = 0$ for all $Z \geq n$. However, this is equivalent to the statement that $F < n$, so it suffices to solve the lifting problems

$$\begin{tikzcd} \hat{S} \arrow[r] \arrow[d] & F \arrow[d] \\ C(\hat{S}) \arrow[r] \arrow[ur, dotted] & * \end{tikzcd}$$

where $\hat{S}$ is a slice cell of dimension $\geq n$. This is immediate, since the left vertical map is in I_n and the right vertical map is a pullback of p. $\square$

The small object argument now applies to obtain the model structure.

Theorem 2.6. *The category of Lewis-May G-spectra with $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{W}$ is a compactly generated, G-topological closed model structure, which we call the* slice$_n$ model structure. *The cofibrations are generated by I_n and the acyclic cofibrations are generated by J (I_n and J as above). Every object is fibrant, so this model structure is right proper. The cofibrant objects are the q-cofibrant spectra that are $\geq n$.*

Proof. We need to show that this model structure is G-topological and prove the last statement. Hence let i be a generating (acyclic) cofibration and i' a generating (acyclic) q-cofibration of G-spaces. Since the usual model structure is G-topological, the pushout product map $i\Box i'$ is a q-cofibration of q-cofibrant spectra which is acyclic if i or i' is. In the case that neither i nor i' is acyclic, the domain and codomain are easily seen to be $\geq n$, so the pushout product is in $\mathcal{C}$ by Lemma 2.2.

For the last statement, let X be q-cofibrant and $\geq n$. We must show that we can solve the lifting problems

$$\begin{tikzcd} * \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] \arrow[ur, dotted] & Z \end{tikzcd}$$

where $Y \to Z$ is a fibration and P_n-equivalence. Since X is q-cofibrant and $\geq n$, we can find a lift up to homotopy, which can then be deformed to a precise lift since $Y \to Z$ is a fibration. Conversely, suppose that X is cofibrant in the slice$_n$ model structure. Since slice$_n$-cofibrations are q-cofibrations, X is q-cofibrant. Now suppose $Y < n$; it suffices to show that $[X, Y] = 0$. Letting the interval $[0,1]$ have basepoint 0, the evaluation map $ev_1 : Y^{[0,1]} \to Y$ is a fibration and P_n-equivalence, so we can solve the lifting problem below.

$$\begin{tikzcd} * \arrow[r] \arrow[d] & Y^{[0,1]} \arrow[d, "ev_1"] \\ X \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

$\square$

Corollary 2.7. *The identity functor is a left Quillen functor from the slice$_n$ model structure to the usual model structure, and the functor P_n together with the natural transformation $P_n \to Id$ is the data of its left derived functor. All of these model structures have the same fibrations and acyclic cofibrations, and the slice$_n$-cofibrations are contained in the slice$_{n-1}$-cofibrations for all n.*

Remark: The slice$_n$ model structure is NOT stable. To see this, note that the suspension functor is ΣP_n, while the loop functor is again Ω. The composite $\Omega\Sigma P_n$ is actually naturally P_n-equivalent to the identity. However, consider the other composite $\Sigma P_n \Omega$. This produces spectra that are $\geq n$, so if it were naturally P_n-equivalent to the identity we would have $\Sigma P_n \Sigma^{-1} \cong P_n$. This would imply that τ_n was closed under inverse suspension. One can easily disprove this by considering Eilenberg MacLane spectra of sufficiently high dimension and desuspending them sufficiently many times, using the elementary connectivity bounds on τ_n.

Remark: The "periodicity" of the τ_n can be reformulated in terms of the slice$_n$ model structures. One can easily show that the suspension and loop functors by the regular representation form a Quillen equivalence relating slice$_n$ to slice$_{n+|G|}$.

We next pull back these model structures to the category of S_G-modules. Let F_S denote the free S_G-module functor, and U its right adjoint. Recall that U is naturally weakly equivalent to the forgetful functor.

Theorem 2.8. *The category of S_G-modules with P_n-equivalences, q-fibrations, and cofibrations determined by the LLP is a compactly generated, G-topological closed model structure, which we call the* slice$_n$ model structure *and denote by slice$_n(\mathscr{M}_G)$. The cofibrations are generated by $F_S(I_n)$ and the acyclic cofibrations are generated by $F_S(J)$ (I_n and J as above). Every object is fibrant, so this model structure is right proper. The cofibrant objects are the q-cofibrant S_G-modules that are $\geq n$. Every q-cofibration of q-cofibrant S_G-modules that are $\geq n$ is a cofibration.*

Proof. Much of this is immediate. The rest is proven by identical arguments to the case of Lewis-May spectra, so we will not repeat them. In fact, all of the statements about the classes of cofibrations, fibrations, and weak equivalences in this section are true verbatim for S_G-modules, and the above corollary and remarks apply as well. Of course, the functors F_S and U form a Quillen equivalence relating the two versions of the slice$_n$ model structure. $\square$

From now on we work with S_G-modules, so that we have a precise, point-set level smash product. In the next section, we obtain some key technical lemmas which will be used in later sections to obtain slice towers with good point-set level properties.

3 Some Technical Lemmas

First we require some "mixed pushout product axioms."

Lemma 3.1. *If i is a slice$_n$ cofibration and i' is a slice$_m$ cofibration then the pushout product $i\Box i'$ is a slice$_{n+m}$ cofibration which is trivial if i or i' is.*

Proof. Standard arguments reduce us to the case where i and i' are generating cofibrations which are not acyclic, and then the S_G-module version of Lemma 2.2 applies (recalling that a smash product of spectra that are $\geq n$ and $\geq m$, respectively, is $\geq n+m$). □

Lemma 3.2. *If i is a q-cofibration (of G-spaces or S_G-modules) and i' is an h-cofibration of S_G-modules then the pushout product $i\Box i'$ is an h-cofibration of S_G-modules.*

Proof. Denote the inclusion $0_+ \to [0,1]_+$ by j, and let $i : A \to B$, $i' : X \to Y$. We first assume that A and B are S_G-modules. The statement that $i\Box i'$ is an h-cofibration is equivalent to the statement that $i\Box i'\Box j$ is a coretraction. This is equivalent to the statement that any map on the domain of $i\Box i'\Box j$ can be extended to a map on the codomain, i.e. that $i\Box i'\Box j$ has the LLP with respect to all maps $Z \to *$. By a formal manipulation, this is equivalent to solving a lifting problem of the form shown below.

$$\begin{tikzcd}
B \cup_A A \wedge [0,1]_+ \arrow[r] \arrow[d, "i\Box j"'] & F(Y,Z) \arrow[d, "{F(i',Z)}"] \\
B \wedge [0,1]_+ \arrow[r] \arrow[ur, dotted] & F(X,Z)
\end{tikzcd}$$

The left vertical map is an acyclic q-cofibration, while the right vertical map is an h-fibration. Applying the functor U, we obtain an h-fibration of Lewis-May G-spectra. This is levelwise a pointed h-fibration of G-spaces, which is an unpointed h-fibration, which is a Serre fibration. Thus the right vertical map above is a q-fibration, so a lifting exists. For the case where A and B are G-spaces, we proceed as above, but replace the internal hom functor with the mapping space functor. □

Lemma 3.3. *The following conclusions hold for any compact Lie group G and S_G-modules indexed on a complete G-universe.*

(i) Cofibrant S_G-modules are flat.

(ii) Smashing with cofibrant G-spaces preserves weak equivalences.

(iii) If B is cofibrant then the quotient map $E_G\Sigma_{i+} \wedge_{\Sigma_i} B^{\wedge i} \to (B^{\wedge i})/\Sigma_i$ is a homotopy equivalence of S_G-modules.

(iv) Symmetric powers of cofibrant S_G-modules are flat.

Proof. For (i), the equivariant versions of the arguments in [EKMM] reduce us to showing that $F_S(\Sigma^\infty G/H_+) \cong G/H_+ \wedge F_S(S)$ is flat for any closed subgroup H of G. Let $X \to Y$ be a weak equivalence, so that $[A, X] \to [A, Y]$ is an isomorphism for any cofibrant A; we must show that

$$[A, F_S(\Sigma^\infty G/H_+) \wedge X] \to [A, F_S(\Sigma^\infty G/H_+) \wedge Y]$$

is an isomorphism, where we take "$[,]$" to mean homotopy classes of maps in the *naive* sense. Using Spanier-Whitehead duality (see [LMS] and [May1]) to move $F_S(\Sigma^\infty G/H_+)$ over to the other side, and using the facts that smash products of cofibrant S_G-modules are cofibrant and that the map $F_S(S) \wedge Z \to S \wedge Z \cong Z$ is a weak equivalence for any Z, we see that the above map is isomorphic to

$$[A \wedge D(F_S(\Sigma^\infty G/H_+)), X] \to [A \wedge D(F_S(\Sigma^\infty G/H_+)), Y],$$

which is an isomorphism. For (ii), we have that smashing with a cofibrant G-space B is weakly equivalent to smashing with $F_S(S) \wedge B$, which is cofibrant, so this case reduces to part (i). For (iii), let B be a cofibrant S_G-module and $i \geq 2$. We may assume that $B = F_S(Z)$ for some cofibrant spectrum Z, since B is homotopy equivalent to an S_G-module of this type. The map in question is

$$S \wedge_{\mathscr{L}} \left((E_G\Sigma_i \times \mathscr{L}(i)) \ltimes_{\Sigma_i} Z^{\overline{\wedge} i}\right) \to S \wedge_{\mathscr{L}} \left(\mathscr{L}(i) \ltimes_{\Sigma_i} Z^{\overline{\wedge} i}\right)$$

where the map $E_G\Sigma_i \times \mathscr{L}(i) \to \mathscr{L}(i)$ relevant to the half-smash product is the projection map. Now $\mathscr{L}(i)$ is a universal $\mathcal{F}_G[i]$-space and has the homotopy type of a $(G \times \Sigma_i)$-CW complex (by Lemma XI.1.6 of [EKMM]), so this map is a homotopy equivalence. However, we can take a homotopy inverse to be the identity in the $\mathscr{L}(i)$ factor, so it is not just a homotopy equivalence of $(G \times \Sigma_i)$-spaces, but a homotopy equivalence of $(\mathscr{L}(1) \rtimes (G \times \Sigma_i))$-spaces over $\mathscr{L}(i)$. For part (iv), we utilize the Quillen equivalence

$$Sp_G^{\mathscr{O}} \underset{\mathbb{N}}{\overset{\mathbb{N}^{\#}}{\rightleftarrows}} \mathscr{M}_G$$

from [MM], where $Sp_G^{\mathscr{O}}$ denotes the category of orthogonal G-spectra. Again let B be cofibrant; by part (iii) it suffices to show that $E_G\Sigma_{i+} \wedge_{\Sigma_i} B^{\wedge i}$ is flat. We may assume that $B = \mathbb{N}(Z)$ for some positive cofibrant orthogonal G-spectrum Z, since B is homotopy equivalent to an object of this type. Then we have

$$E_G\Sigma_{i+} \wedge_{\Sigma_i} B^{\wedge i} \cong \mathbb{N}(E_G\Sigma_{i+} \wedge_{\Sigma_i} Z^{\wedge i}),$$

so it suffices to show that $\mathbb{N}(E_G\Sigma_{i+} \wedge_{\Sigma_i} Z^{\wedge i})$ is cofibrant by part (i). In [Sto], the author constructs model structures on equivariant orthogonal spectra, called "$\mathbb{S}$ model structures," such that induction from subgroups preserves cofibrations. It is not difficult to show that $E_G\Sigma_{i+} \wedge_{\Sigma_i} Z^{\wedge i}$ is positive $\mathbb{S}$-cofibrant, and that the pair $(\mathbb{N}, \mathbb{N}^{\#})$ is also a Quillen pair when the classical (positive) stable model structure is replaced with the positive stable $\mathbb{S}$ model structure. The result follows immediately. $\square$

We will need the following facts in the section on commutative ring slice towers. For the statement of the next lemma in this section, we need a simple definition. For a map $A \to B$ and $i > 0$, denote by $\partial_A B^{\wedge i} \to B^{\wedge i}$ the "inclusion" of the "union of the images" of the maps $B^{\wedge j} \wedge A \wedge B^{\wedge i-j-1} \to B^{\wedge i}$. This map, in fact, is simply the iterated pushout product $(A \to B)^{\square i}$.

We can now state our next "mixed pushout product axiom."

Lemma 3.4. *If $A \to B$ is a q-cofibration and $Y \to Z$ is an h-cofibration and $i > 0$ then the pushout product*

$$(\partial_A B^{\wedge i} \to B^{\wedge i})/\Sigma_i \quad \square \quad (Y \to Z)$$

is an h-cofibration of S_G-modules.

Proof. First, the map in question is isomorphic to

$$\big((\partial_A B^{\wedge i} \to B^{\wedge i})\square(Y \to Z)\big)/\Sigma_i,$$

so it suffices to show that the map

$$(\partial_A B^{\wedge i} \to B^{\wedge i})\square(Y \to Z)$$

is a Σ_i-h-cofibration. Next, by the model axioms B is a retract under A of a cell in the generating q-cofibrations, so we may assume that B is of this form. We may take a transfinite filtration of B such that one cell is attached at a time. One then obtains a transfinite factorization of the above map such that each individual map is a pushout of a map obtained by applying an induction functor of the form

$$\Sigma_{i+} \wedge_{\Sigma_{i_1} \times \ldots \times \Sigma_{i_k}} (\, - \,)$$

to a map of the form

$$(Y \to Z)\square(\partial_{A_1} B_1^{\wedge i_1} \to B_1^{\wedge i_1})\square \ldots \square(\partial_{A_k} B_k^{\wedge i_k} \to B_k^{\wedge i_k})$$

where each $A_j \to B_j$ is a generating q-cofibration and $i_1 + \ldots + i_k = i$. Now induction and pushout preserve h-cofibrations, so we need only show that the above map is an h-cofibration. We let

$$(A_j \to B_j) = F_S\big(\Sigma^\infty_{V_j}\,(G/H_j \times S^{d_j-1} \to G/H_j \times D^{d_j})_+\big)$$

for each j, where the V_j are G-representations, the H_j are closed subgroups of G, and the d_j are non-negative integers. It is then easy to see that there is an S_G-module C (with $\Sigma_{i_1} \times ... \times \Sigma_{i_k}$-action) and a $(G \times \Sigma_{i_1} \times ... \times \Sigma_{i_k})$-representation W (with trivial G-action) such that the map in question is of the form

$$C \wedge \big((Y \to Z)\square(S(W)_+ \to D(W)_+)\big).$$

The map $S(W) \to D(W)$ is a q-cofibration of $(G \times \Sigma_{i_1} \times ... \times \Sigma_{i_k})$-spaces (see [Ill]), so the above map is an h-cofibration by Lemma 3.2 (it is immaterial that we are not indexing on a complete $(G \times \Sigma_{i_1} \times ... \times \Sigma_{i_k})$-universe). □

Lemma 3.5. *Let H be a finite group and $\phi : H \to \Sigma_i$ a homomorphism. For a real H-representation V, denote by $(V^{\oplus i})^\phi$ the direct sum of i copies of V, with H-action multiplied by the pullback of the permutation action along ϕ. Then for any $m > 0$ we have*

$$((m\rho_H)^{\oplus i})^\phi \cong im\rho_H.$$

Proof. For the above vector space, we choose a basis consisting of the usual basis vectors for ρ_H (corresponding to elements of H) in each copy. Then it is easy to see that $((m\rho_H)^{\oplus i})^\phi$ is a permutation representation such that each basis vector has trivial stabilizer. Thus, it must be a multiple of the regular representation. We obtain the result by a dimension count. □

Lemma 3.6. *Let Λ be a subgroup of $G \times \Sigma_i$ such that $\Lambda \cap (1 \times \Sigma_i) = 1$, and let X be a pointed $(G \times \Sigma_i)$-space. Then there is a subgroup H of G and a homomorphism $\phi : H \to \Sigma_i$ such that $\Lambda = \{(h, \phi(h)) : h \in H\}$ and*

$$(G \times \Sigma_i)/\Lambda_+ \wedge_{\Sigma_i} X \quad \cong \quad G_+ \wedge_H X^\phi$$

where X^ϕ is X with H-action multiplied by the pullback of the Σ_i-action along ϕ.

Proof. The above space is

$$((G \times \Sigma_i)/\Lambda_+ \wedge X)/\Sigma_i \cong ((G \times \Sigma_i)_+ \wedge_\Lambda X)/\Sigma_i$$
$$\cong (\Sigma_i \backslash G \times \Sigma_i)_+ \wedge_\Lambda X \cong G_+ \wedge_\Lambda X$$

where Λ acts on G via its projection onto H. The last space above can be described equivalently as $G_+ \wedge_H X^\phi$. □

Lemma 3.7. *If B is a q-cofibrant S_G-module, $B \geq n$ and $i \geq 1$ then*

$$(B^{\wedge i})/\Sigma_i \quad \geq \quad ni.$$

Proof. Firstly, B is homotopy equivalent to an S_G-module of the form $F_S(Z)$, where Z is a slice$_n$ complex, so we may assume that B is of this form. By attaching one slice cell at a time to form Z and using the proof of Lemma 3.4, we obtain a (transfinite) filtration of $(B^{\wedge i})/\Sigma_i$ by h-cofibrations with successive quotients of the form

$$\bigwedge_{j=1}^{k} (F_S(\Sigma^{m_j} \hat{S}_j)^{\wedge i_j})/\Sigma_{i_j}$$

where each m_j is at least 0, $\hat{S}_j$ is a slice cell of dimension $\geq n$ and $i_1 + ... + i_k = i$. By Lemma 3.3 this is a derived smash product, so it suffices to prove that

$$(F_S(\Sigma^m \hat{S})^{\wedge i})/\Sigma_i \quad \geq \quad ni$$

when $\hat{S}$ is a slice cell of dimension n. Next, Lemma 3.3(iii) implies that the above is weakly equivalent to $E_G\Sigma_{i+} \wedge_{\Sigma_i} F_S(\Sigma^m \hat{S})^{\wedge i}$, so by using a cellular filtration of $E_G\Sigma_i$ we are reduced to showing that

$$(G \times \Sigma_i)/\Lambda_+ \wedge_{\Sigma_i} F_S(\Sigma^m \hat{S})^{\wedge i} \quad \geq \quad ni$$

where Λ is a subgroup of $G \times \Sigma_i$ such that $\Lambda \cap (1 \times \Sigma_i) = 1$. Then for some subgroup H of G and some homomorphism $\phi : H \to \Sigma_i$ we have $\Lambda = \{(h, \phi(h)) : h \in H\}$. Next we reduce to the case that $n \geq 0$; suppose that $n < 0$. Choosing k large enough that $n + k|G| \geq 0$ and smashing the above spectrum with $S^{ik\rho_G}$, we see that it suffices to show that

$$(G \times \Sigma_i)/\Lambda_+ \wedge_{\Sigma_i} F_S(\Sigma^m \hat{S})^{\wedge i} \wedge S^{ik\rho_G} \quad \geq \quad (n + k|G|)i.$$

The above spectrum is isomorphic to

$$(S^{ik\rho_G} \wedge (G \times \Sigma_i)/\Lambda_+) \wedge_{\Sigma_i} F_S(\Sigma^m \hat{S})^{\wedge i}$$

with Σ_i acting trivially on $S^{ik\rho_G}$. By Lemma 3.5 we have

$$\begin{aligned} S^{ik\rho_G} \wedge (G \times \Sigma_i)/\Lambda_+ &\cong (G \times \Sigma_i)_+ \wedge_\Lambda (S^{ik|G/H|\rho_H}) \\ &\cong (G \times \Sigma_i)_+ \wedge_\Lambda (S^{(k|G/H|\rho_H^{\oplus i})^\phi}) \\ &\cong (G \times \Sigma_i)_+ \wedge_\Lambda ((S^{k|G/H|\rho_H})^{\wedge i})^\phi \\ &\cong (G \times \Sigma_i)/\Lambda_+ \wedge (S^{k\rho_G})^{\wedge i} \end{aligned}$$

where Σ_i permutes the smash factors in the final line above. Our spectrum can now be rewritten as

$$(G \times \Sigma_i)/\Lambda_+ \wedge_{\Sigma_i} F_S(\Sigma^m \hat{S} \wedge S^{k\rho_G})^{\wedge i}.$$

Since $\hat{S} \wedge S^{k\rho_G}$ is a slice cell of dimension $n + k|G| \geq 0$, we are reduced to the case where $n \geq 0$, so that

$$\hat{S} = \Sigma^\infty(G_+ \wedge_J S^{t\rho_J})$$

for some $t \geq 0$ and some subgroup J of G such that $t|J| = n$. Our spectrum is then isomorphic to

$$(G \times \Sigma_i)/\Lambda_+ \wedge_{\Sigma_i} \left(S \wedge_{\mathscr{L}} \Sigma^\infty \left(\mathscr{L}(i)_+ \wedge (S^m)^{\wedge i} \wedge (G_+ \wedge_J S^{t\rho_J})^{\wedge i}\right)\right)$$

which is weakly equivalent to the (Lewis-May) spectrum

$$\begin{aligned}&\Sigma^\infty\Big((G\times\Sigma_i)/\Lambda_+\wedge_{\Sigma_i}\big(\mathscr{L}(i)_+\wedge(S^m)^{\wedge i}\wedge(G_+\wedge_J S^{t\rho_J})^{\wedge i}\big)\Big)\\ &\quad\cong\Sigma^\infty\Big(G_+\wedge_H\big(\mathscr{L}(i)_+^{\phi}\wedge S^{m\phi}\wedge((i_H^*(G_+\wedge_J S^{t\rho_J}))^{\wedge i})^{\phi}\big)\Big)\end{aligned}$$

where on the second line we have used Lemma 3.6. Now since a complete G-universe $\mathcal{U}$ is a complete H-universe, the H-space $\mathscr{L}(i)^\phi = Isom((\mathcal{U}^{\oplus i})^\phi, \mathcal{U})$ is contractible. Hence, the above spectrum is homotopy equivalent to

$$G_+\wedge_H\left(S^{m\phi}\wedge\Sigma^\infty\left(((i_H^*(G_+\wedge_J S^{t\rho_J}))^{\wedge i})^{\phi}\right)\right).$$

The spectrum $\Sigma^\infty(((i_H^*(G_+\wedge_J S^{t\rho_J}))^{\wedge i})^{\phi})$ is easily seen to be a wedge of slice cells of dimension $it|J| = ni$, by the same reasoning as in the proof of Lemma 3.5, so the above spectrum is $\geq ni$, as required. $\square$

4 The Category of Towers

We seek a model structure on towers of S_G-modules such that cofibrant replacements of constant towers are slice towers. First we discuss the category of towers of S_G-modules.

Definition 4.1. Denote by $\mathbb{Z}$ the set of integers, regarded as a category with one morphism from n to m when $n \geq m$, as shown below.

$$\ldots \leftarrow (n-1) \leftarrow n \leftarrow (n+1) \leftarrow \ldots$$

We define the *category of towers* to be the diagram category $\mathscr{M}_G^{\mathbb{Z}}$ and the constant tower functor

$$const : \mathscr{M}_G \to \mathscr{M}_G^{\mathbb{Z}}$$

to be the functor which assigns to each S_G-module X the constant diagram at X. If X is an S_G-module and $n \in \mathbb{Z}$, we denote by $X[n]$ the free diagram in level n. That is,

$$X[n]_m = \begin{cases} X & \text{if } m \leq n \\ * & \text{if } m > n \end{cases}$$

with the structure maps being the identity of X when both objects are X. We denote by

$$ev_n : \mathscr{M}_G^{\mathbb{Z}} \to \mathscr{M}_G$$

the evaluation at n functor; this is right adjoint to the functor $(-)[n]$.

Since the category $\mathbb{Z}$ is symmetric monoidal, we obtain a closed symmetric monoidal structure on the category of towers, which we denote simply by $\wedge$. This smash product is not difficult to describe. The unit is $S[0]$. For any two towers X and Y, the S_G-module $(X \wedge Y)_n$ is the colimit of the staircase-shaped diagram

$$\begin{tikzcd}
\cdots \arrow[d] & & & \\
X_{i-1} \wedge Y_{j+1} & X_i \wedge Y_{j+1} \arrow[l] \arrow[d] & & \\
& X_i \wedge Y_j & X_{i+1} \wedge Y_j \arrow[l] \arrow[d] & \\
& & X_{i+1} \wedge Y_{j-1} & \cdots \arrow[l]
\end{tikzcd}$$

where $i + j = n$. Inspection yields the formula

$$(X \wedge A[n])_m \cong X_{m-n} \wedge A,$$

i.e. the effect of smashing with $A[n]$ is to shift by n and smash levelwise with A. It follows that

$$A[n] \wedge B[m] \cong (A \wedge B)[n+m].$$

Next we turn to model-theoretic considerations. We use the collection of adjoint pairs $\{((-)[n], ev_n)\}_{n \in \mathbb{Z}}$ to pull back our cofibrantly generated model structures to $\mathscr{M}_G^{\mathbb{Z}}$. Let I and J be the usual sets of generating q-cofibrations and acyclic q-cofibrations of S_G-modules. Most of the following theorem is now obvious.

Theorem 4.2. *The category of towers of S_G-modules with levelwise q-fibrations, levelwise weak equivalences and determined cofibrations is a compactly generated, closed, G-topological and monoidal model structure, which we call the* q model structure. *The cofibrations are generated by $\cup_{n \in \mathbb{Z}} I[n]$ and the acyclic cofibrations are generated by $\cup_{n \in \mathbb{Z}} J[n]$. (Acyclic) cofibrations are levelwise (acyclic) q-cofibrations and h-cofibrations. Every object is fibrant, so this structure is right proper. It is also stable.*

Proof. Most of this is standard; we must prove that this model structure is monoidal.

For the unit axiom, note that $F_S(S)[0] \to S[0]$ is a cofibrant replacement of the unit, and for any tower X, the level n component of $X \wedge (F_S(S)[0] \to S[0])$ is the map

$$X_n \wedge F_S(S) \to X_n \wedge S \cong X_n,$$

which is a weak equivalence by Proposition I.6.2 in [EKMM]. Finally, we must prove the pushout product axiom. Standard arguments reduce us to considering pushout products of the form $i[n]\Box j[m]$, where i and j are generating (acyclic) q-cofibrations. But this map is simply $(i\Box j)[n+m]$, so the result is trivial. $\square$

We obtain a different model structure by pulling back the collection of slice$_n$ model structures. Denote by I_n the set of generating slice$_n$-cofibrations given by 2.4, and define a map of towers $X \to Y$ to be a *P_*-equivalence* if $X_n \to Y_n$ is a P_n-equivalence for all n.

Theorem 4.3. *The category of towers of S_G-modules with levelwise q-fibrations, P_*-equivalences and determined cofibrations is a compactly generated, closed, G-topological and monoidal model structure, which we call the* slice model structure *and denote by slice$(\mathscr{M}_G^{\mathbb{Z}})$. The cofibrations are generated by $\cup_{n\in\mathbb{Z}} I_n[n]$ and the acyclic cofibrations are generated by $\cup_{n\in\mathbb{Z}} J[n]$. If $X \to Y$ is a cofibration then $X_n \to Y_n$ is a slice$_n$-cofibration for all n. Hence, if X is cofibrant, then X_n is q-cofibrant and $\geq n$ for all n. Every object is fibrant, so this structure is right proper.*

Proof. As above, for the most part. Note that, if $A \to B$ is a slice$_n$-cofibration, then it is a slice$_m$-cofibration for all $m \leq n$. The unit axiom is proved as before, noting that $F_S(S) \geq 0$, while the pushout product axiom follows from Lemma 3.1. $\square$

Remark: Of course, the identity functor is a left Quillen functor from the slice model structure to the q model structure.

Corollary 4.4. *If X is an S_G-module and T is a cofibrant replacement for const(X) in the slice model structure then T is a model for the slice tower of X.*

Next we define some conditions on towers that will be relevant in later sections.

Definition 4.5. Let X be a tower. We say that X is

- *slice-like* if $X_n \geq n$ for all n,
- *flat* if the functor $X \wedge (-)$ preserves levelwise weak equivalences,
- *h-flat* if the functor $X \wedge (-)$ preserves levelwise h-cofibrations,
- *h-cofibrant* if $X_{n+1} \to X_n$ is an h-cofibration for all n, and
- *nice* if it is slice-like, flat, and h-flat.

Observe that if a tower satisfies one of these conditions then so does any retract of it. Also, a P_*-equivalence of slice-like towers is a weak equivalence. We sum up the other basic properties of these conditions in the following propositions.

Proposition 4.6. *The following conclusions hold.*

(i) If $A \to B \to C$ is a cofiber sequence and two of the towers A, B and C are flat, then so is the third.

(ii) Flat objects are closed under smashing levelwise with q-cofibrant S_G-modules (or G-spaces).

(iii) A well-ordered colimit of h-cofibrations of flat objects is flat.

(iv) A smash product of flat objects is flat.

(v) A flat tower is levelwise flat.

(vi) Q-cofibrant towers are flat.

Proof. Most of this is obvious. For (i), we use the stability of the q model structure and the fact that naive cofiber sequences of S_G-modules are derived cofiber sequences. For (v), we smash with a map of the form $A[0] \to B[0]$, where $A \to B$ is a weak equivalence. For (vi), the previous parts reduce us to showing that $B[n]$ is flat when B is a q-cofibrant S_G-module, but smashing with this object simply shifts and smashes levelwise with B. □

Proposition 4.7. *The following conclusions hold.*

(i) A smash product of h-flat towers is h-flat.

(ii) If a tower is h-flat then it is h-cofibrant.

(iii) Q-cofibrant towers are h-flat.

Proof. The first part is immediate. For (ii), let $S[-1] \to S[0]$ be the map which is the identity in all negative levels; this is a levelwise h-cofibration. Smashing with an h-flat tower X, we may identify the resulting map in level n with $X_{n+1} \to X_n$. For the last part, let X be q-cofibrant; we may assume that X is a $\cup_n I[n]$-cell. Thus we may assume that X is the colimit of a transfinite sequence $\{X_\alpha\}$ such that

- $X_0 = *$,
- $X_\alpha \to X_{\alpha+1}$ is a pushout of an element of $\cup_n I[n]$, and
- if α is a limit element then $X_\alpha = \varinjlim_{\beta<\alpha} X_\beta$.

Now let $Y \to Z$ be a levelwise h-cofibration. To show that $X \wedge (Y \to Z)$ is a levelwise h-cofibration, it suffices to show that

- $X_0 \wedge (Y \to Z)$ is a levelwise h-cofibration, and
- the pushout product maps $(X_\alpha \to X_{\alpha+1})\square(Y \to Z)$ are levelwise h-cofibrations.

The first part above is trivial, since $X_0 = *$. For the second part, suppose that $X_\alpha \to X_{\alpha+1}$ is a pushout of $(A \to B)[n]$. Then $(X_\alpha \to X_{\alpha+1})\square(Y \to Z)$ is a pushout of $(A \to B)[n]\square(Y \to Z)$. This map in level m is simply $(A \to B)\square(Y_{m-n} \to Z_{m-n})$, which is an h-cofibration by Lemma 3.2. $\square$

Since we have already observed that slice-cofibrant towers are slice-like, we obtain the following.

Corollary 4.8. *Slice-cofibrant towers are nice.*

Proposition 4.9. *The following conclusions hold.*

(i) Slice-like towers are closed under taking cofibers and extensions.

(ii) Slice-like towers are closed under suspension and smashing levelwise with (-1)*-connected flat* S_G*-modules.*

(iii) Wedge sums of slice-like towers are slice-like.

(iv) A well-ordered colimit of h-cofibrations of slice-like towers is slice-like.

The proof is trivial.

Proposition 4.10. *If* X *and* Y *are levelwise flat and h-cofibrant, the smash product* $X \wedge Y$ *is weakly equivalent to the derived smash product. Hence, if* X *and* Y *are also slice-like, so is* $X \wedge Y$*.*

Proof. For each n, recall that $(X \wedge Y)_n$ can be described as the colimit of a certain staircase-shaped diagram. By considering partial staircases, we see that we can obtain $(X \wedge Y)_n$ (naturally) as the colimit of a sequence of maps, where the first object is $X_n \wedge Y_0$ (for example) and each map is either a pushout of $X_i \wedge (Y_{j+1} \to Y_j)$ or a pushout of $(X_{i+1} \to X_i) \wedge Y_j$ for some i and j with $i + j = n$. If X and Y are h-cofibrant, then all of these maps are h-cofibrations. Applying this to q-cofibrant replacements of X and Y as well, the conclusion follows easily. For the second part, since X and Y are slice-like we may take their cofibrant replacements to be slice-cofibrant. □

Corollary 4.11. *Smash products of nice towers are nice.*

5 Ring Slice Towers

We now turn to the construction of ring slice towers.

Definition 5.1. The category of *ring towers* is the category of monoids in $(\mathscr{M}_G^{\mathbb{Z}}, \wedge)$, and we denote it by *assoc*$(\mathscr{M}_G^{\mathbb{Z}})$. We denote by $\mathbb{A}$ the free ring tower functor, which is given by

$$\mathbb{A}(X) = S[0] \vee X \vee (X \wedge X) \vee (X \wedge X \wedge X) \vee ...$$

To do inductive proofs, we require the combinatorial description of pushouts (in the category of rings) of the form

$$\begin{tikzcd} \mathbb{A}(A) \arrow[r] \arrow[d] & \mathbb{A}(B) \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

given in [SS]. For $i > 0$ denote by $\partial_A B^{\wedge i} \to B^{\wedge i}$ the inclusion of the "union of the images" of the maps $B^{\wedge j} \wedge A \wedge B^{\wedge i-j-1} \to B^{\wedge i}$. This map is, in fact, simply $(A \to B)^{\square i}$. Then Y can be written as a colimit of objects Y_i, where $Y_0 = X$ and for each $i > 0$ we have a pushout diagram

$$\begin{tikzcd} X^{\wedge i+1} \wedge \partial_A B^{\wedge i} \arrow[r] \arrow[d] & X^{\wedge i+1} \wedge B^{\wedge i} \arrow[d] \\ Y_{i-1} \arrow[r] & Y_i \end{tikzcd}$$

in the original category (to be precise, we have permuted the smash factors along the top row from what is given in [SS], but this makes no difference to the underlying objects). Note that the above horizontal maps are h-cofibrations when $A \to B$ is a q-cofibration, and are strong deformation retracts when $A \to B$ is a generating acyclic q-cofibration. We now obtain model structures on the category of ring towers.

Theorem 5.2. *The category of ring towers with levelwise weak equivalences, levelwise q-fibrations and determined cofibrations is a compactly generated, closed and*

G-topological model structure, which we call the q model structure*. The cofibrations and acyclic cofibrations are generated by* $\cup_{n\in\mathbb{Z}}\mathbb{A}(I[n])$ *and* $\cup_{n\in\mathbb{Z}}\mathbb{A}(J[n])$*, respectively. The cofibrations are h-cofibrations. Every object is fibrant, so this model structure is right proper.*

Theorem 5.3. *The category of ring towers with* P_**-equivalences, levelwise q-fibrations and determined cofibrations is a compactly generated, closed and G-topological model structure, which we call the* slice model structure *and denote by* $slice(assoc(\mathscr{M}_G^{\mathbb{Z}}))$*. The cofibrations and acyclic cofibrations for this model structure are generated by* $\cup_{n\in\mathbb{Z}}\mathbb{A}(I_n[n])$ *and* $\cup_{n\in\mathbb{Z}}\mathbb{A}(J[n])$*, respectively. The cofibrations are h-cofibrations. Every object is fibrant, so this model structure is right proper.*

Proof. As above; we need only observe that a $\cup_n\mathbb{A}(J[n])$-cell is a levelwise weak equivalence, hence a P_*-equivalence. □

Since the sphere S_G-module is not cofibrant, cofibrant ring towers are not cofibrant towers. However, we have the following.

Proposition 5.4. *Cofibrant ring towers are flat and h-flat. Slice-cofibrant ring towers are also slice-like. Hence, slice-cofibrant ring towers are nice.*

Proof. Let X be a cofibrant ring tower; we may assume that X is a $\cup_n\mathbb{A}(I[n])$-cell. Let X have the transfinite filtration $\{X_\alpha\}$ such that one cell is attached at a time. We show flatness by transfinite induction; the initial stage (the unit $S[0]$) is obviously flat, so we proceed to the inductive step. By Proposition 4.6 and the combinatorial description of pushouts given above, it suffices to show that, for $A \to B$ a generating q-cofibration of S_G-modules and any $n \in \mathbb{Z}$ and $i > 0$, the successive cofiber $X_\alpha^{\wedge i+1} \wedge (B/A)^{\wedge i}[ni]$ is flat. This follows from Proposition 4.6 and the inductive hypothesis.

Next we prove h-flatness. Hence, let $Y \to Z$ be a levelwise h-cofibration. We proceed as in the proof of Proposition 4.7(iii). The initial stage (the unit $S[0]$) is clearly h-flat. For the inductive step, we must show that the pushout product map

$$\begin{aligned}\left(X_\alpha^{\wedge i+1} \wedge (\partial_A B^{\wedge i})[ni] \to X_\alpha^{\wedge i+1} \wedge B^{\wedge i}[ni]\right) \Box (Y \to Z) \\ \cong X_\alpha^{\wedge i+1} \wedge \left(\left((\partial_A B^{\wedge i})[ni] \to B^{\wedge i}[ni]\right) \Box (Y \to Z)\right)\end{aligned}$$

is a levelwise h-cofibration. By the inductive hypothesis, X_α is h-flat so this reduces to showing that

$$\left((\partial_A B^{\wedge i})[ni] \to B^{\wedge i}[ni]\right) \Box (Y \to Z)$$

is a levelwise h-cofibration, which follows from Lemma 3.2. Finally, suppose that X is slice-cofibrant. Then in the above setup we may take $A \to B$ to be either a strong deformation retract or the inclusion of a finite slice$_n$ pair. The initial stage (the unit $S[0]$) is clearly slice-like, since $S \geq 0$. For the inductive step, we must show that the successive cofibers $X_\alpha^{\wedge i+1} \wedge (B/A)^{\wedge i}[ni]$ are slice-like. If $A \to B$ is acyclic then this tower is contractible. Otherwise the conclusion follows from the flatness and h-flatness of X_α, the induction hypothesis (X_α is slice-like), and Proposition 4.10. □

Now note that

$$const(X) \wedge const(Y) \cong const(X \wedge Y),$$

so the constant tower functor is a symmetric monoidal functor. As a corollary to the above, we obtain ring tower models for slice towers of rings.

Theorem 5.5. *If X is an associative ring spectrum then a cofibrant replacement of $const(X)$ in $slice(assoc(\mathscr{M}_G^{\mathbb{Z}}))$ is a model for the slice tower of X, and is unique up to an (essentially) unique homotopy equivalence in the category of ring towers. Since $slice(assoc(\mathscr{M}_G^{\mathbb{Z}}))$ is cofibrantly generated, we can take cofibrant replacement to be functorial.*

Remark: On homotopy categories we may view this process as below.

$$Ho(assoc(\mathscr{M}_G)) \xrightarrow{const} Ho(slice(assoc(\mathscr{M}_G^{\mathbb{Z}}))) \xrightarrow{LId} Ho(assoc(\mathscr{M}_G^{\mathbb{Z}}))$$

For completeness, we note how to obtain analogous results for non-unital rings. If $\mathcal{C}$ is a pointed, closed symmetric monoidal category we denote by $assoc_0(\mathcal{C})$ the category of non-unital rings in $\mathcal{C}$. Then the forgetful functor $assoc(\mathcal{C}) \to assoc_0(\mathcal{C})$ has a left adjoint, given by $X \mapsto 1 \coprod X$. Note then that X is naturally a retract of $1 \coprod X$ in $\mathcal{C}$. From this it is trivial to deduce that, in our case, pushouts of generating (acyclic) cofibrations are h-cofibrations, which implies smallness. It is also trivial that pushouts of generating acyclic cofibrations are strong deformation retracts. Thus we get classical and slice model structures on $assoc_0(\mathscr{M}_G^{\mathbb{Z}})$, and this is related by a Quillen pair to $assoc(\mathscr{M}_G^{\mathbb{Z}})$. The non-unital version of Proposition 5.4 follows, since (slice-)cofibrant elements of $assoc_0(\mathscr{M}_G^{\mathbb{Z}})$ are retracts of (slice-)cofibrant elements of $assoc(\mathscr{M}_G^{\mathbb{Z}})$.

6 Commutative Ring Slice Towers

In this section we construct commutative ring slice towers. The development is similar to the case of associative rings, but is slightly more technical and difficult. Most of the work, however, is hidden in Section 3.

Definition 6.1. The category of *commutative ring towers* is the category of commutative monoids in $(\mathscr{M}_G^{\mathbb{Z}}, \wedge)$, and we denote it by *comm*$(\mathscr{M}_G^{\mathbb{Z}})$. We denote by $\mathbb{C}$ the free commutative ring tower functor, which is given by

$$\mathbb{C}(X) = S[0] \vee X \vee (X \wedge X)/\Sigma_2 \vee (X \wedge X \wedge X)/\Sigma_3 \vee ...$$

To do inductive proofs, we require a combinatorial description of pushouts (in the category of commutative rings) of the form

$$\begin{tikzcd} \mathbb{C}(A[n]) \arrow[r] \arrow[d] & \mathbb{C}(B[n]) \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where $A \to B$ is a generating (acyclic) q- or slice$_n$-cofibration. In this case Y can be written as a colimit of a sequence of towers Y_i with $Y_0 = X$ such that for each $i > 0$ we have a pushout diagram

$$\begin{tikzcd} X \wedge (\partial_A B^{\wedge i})/\Sigma_i[ni] \arrow[r] \arrow[d] & X \wedge (B^{\wedge i})/\Sigma_i[ni] \arrow[d] \\ Y_{i-1} \arrow[r] & Y_i \end{tikzcd}$$

in the category of towers. By Lemma 3.4 the top horizontal map is an h-cofibration, so the bottom horizontal map is an h-cofibration with quotient $X \wedge ((B/A)^{\wedge i})/\Sigma_i[ni]$. We can now obtain our model structures.

Theorem 6.2. *The category of commutative ring towers with levelwise weak equivalences, levelwise q-fibrations and determined cofibrations is a compactly generated, closed and G-topological model structure, which we call the* q model structure. *The* *cofibrations are generated by* $\cup_{n\in\mathbb{Z}}\mathbb{C}(I[n])$ *and the acyclic cofibrations are generated by* $\cup_{n\in\mathbb{Z}}\mathbb{C}(J[n])$. *Cofibrations are h-cofibrations. Every object is fibrant, so this model structure is right proper.*

For the proof, we need only remark that when $A \to B$ is a generating acyclic q-cofibration, B/A is contractible. Next we pull back the slice model structure to commutative rings.

Theorem 6.3. *The category of commutative ring towers with* P_**-equivalences, levelwise q-fibrations and determined cofibrations is a compactly generated, closed and* G*-topological model structure, which we call the* slice model structure *and denote by* $slice(comm(\mathscr{M}_G^{\mathbb{Z}}))$. *The cofibrations are generated by* $\cup_{n\in\mathbb{Z}}\mathbb{C}(I_n[n])$ *and the acyclic cofibrations are generated by* $\cup_{n\in\mathbb{Z}}\mathbb{C}(J[n])$. *Cofibrations are h-cofibrations. Every object is fibrant, so this model structure is right proper.*

For the proof, we note that $\cup_n\mathbb{C}(J[n])$-cells are in fact levelwise weak equivalences, so they are P_*-equivalences. As with associative ring towers, cofibrant commutative ring towers are not cofibrant towers, but we have the following.

Proposition 6.4. *Cofibrant commutative ring towers are flat and h-flat. Slice-cofibrant commutative ring towers are also slice-like. Hence, slice-cofibrant commutative ring towers are nice.*

Proof. Let X be a q- or slice-cofibrant commutative ring tower. We may assume that X is a $\cup_n\mathbb{C}(I[n])$-cell. We first prove flatness. Since the initial object (the unit $S[0]$) is flat, transfinite induction and Proposition 4.6 reduce us to showing that, if X_α is flat and $A \to B$ is a generating q-cofibration of S_G-modules, then for any $n \in \mathbb{Z}$ and $i > 0$ the tower $X_\alpha \wedge ((B/A)^{\wedge i})/\Sigma_i[ni]$ is flat. But Lemma 3.3(iv) implies that $((B/A)^{\wedge i})/\Sigma_i[ni]$ is flat, and smash products of flat towers are flat.

To prove that X is h-flat, let $Y \to Z$ be a levelwise h-cofibration. We prove that $X \wedge (Y \to Z)$ is a levelwise h-cofibration by transfinite induction, proceeding as in the proof of Proposition 4.7(iii). The initial stage (the unit $S[0]$) is obviously h-flat, so we proceed to the inductive step. Thus we must show that, if X_α is h-flat and $A \to B$

is a generating q-cofibration, then for any $n \in \mathbb{Z}$ and $i > 0$ the pushout product map

$$X_\alpha \wedge \left((Y \to Z)\Box\left((\partial_A B^{\wedge i})/\Sigma_i[ni] \to (B^{\wedge i})/\Sigma_i[ni]\right)\right)$$

is a levelwise h-cofibration. Since X_α is h-flat, this reduces to showing that the map

$$(Y \to Z)\Box\left((\partial_A B^{\wedge i})/\Sigma_i[ni] \to (B^{\wedge i})/\Sigma_i[ni]\right)$$

is a levelwise h-cofibration, which follows from Lemma 3.4.

Now let X be slice-cofibrant; it remains to show that X is slice-like. We again proceed by transfinite induction. The initial stage (the unit $S[0]$) is slice-like. Transfinite induction and Proposition 4.9 reduce us to showing that, if X_α is slice-like and $A \to B$ is a generating slice$_n$-cofibration for some $n \in \mathbb{Z}$ and $i > 0$ then the tower $X_\alpha \wedge ((B/A)^{\wedge i})/\Sigma_i[ni]$ is slice-like. This is contractible if $A \to B$ is acyclic, so we may assume that $A \to B$ is the inclusion of a finite slice$_n$ pair, so that $B/A \geq n$. Then since $((B/A)^{\wedge i})/\Sigma_i[ni]$ is h-cofibrant, flat, and slice-like by Lemmas 3.3 and 3.7, the result follows from Proposition 4.10. $\square$

As a corollary to the above, we obtain commutative ring tower models for slice towers of commutative rings.

Theorem 6.5. *If X is a commutative ring spectrum then a cofibrant replacement of $const(X)$ in $slice(comm(\mathscr{M}_G^{\mathbb{Z}}))$ is a model for the slice tower of X, and is unique up to an (essentially) unique homotopy equivalence in the category of commutative ring towers. Since $slice(comm(\mathscr{M}_G^{\mathbb{Z}}))$ is cofibrantly generated, we can take cofibrant replacement to be functorial.*

Remark: On homotopy categories we may view this process as below.

$$Ho(comm(\mathscr{M}_G)) \xrightarrow{const} Ho(slice(comm(\mathscr{M}_G^{\mathbb{Z}}))) \xrightarrow{LId} Ho(comm(\mathscr{M}_G^{\mathbb{Z}}))$$

The analogues of the above results for non-unital commutative rings also hold, the argument proceeding as in the remarks at then end of Section 5.

7 Module Slice Towers

Next we construct module slice towers over ring slice towers. If R is a ring tower, we denote by *R-Mod* the category of module towers over R. We pull back the model structures on towers to get model structures on R-Mod. As always with module categories we have an adjunction

$$\mathscr{M}_G^{\mathbb{Z}} \underset{R\wedge(-)}{\overset{forget}{\leftrightarrows}} R\text{-Mod}\,.$$

We refer to the left adjoint above as the *free R-module functor.* The following theorems are now obvious.

Theorem 7.1. *The category of R-modules with levelwise weak equivalences, levelwise q-fibrations and determined cofibrations is a compactly generated, closed and G-topological model structure, which we call the* q model structure. *The cofibrations are generated by $\cup_{n\in\mathbb{Z}} R\wedge I[n]$ and the acyclic cofibrations are generated by $\cup_{n\in\mathbb{Z}} R\wedge J[n]$. Cofibrations are h-cofibrations. Every object is fibrant, so this model structure is right proper.*

Theorem 7.2. *The category of R-modules with P_*-equivalences, levelwise q-fibrations and determined cofibrations is a compactly generated, closed and G-topological model structure, which we call the* slice model structure *and denote by slice(R-Mod). The cofibrations are generated by $\cup_{n\in\mathbb{Z}} R\wedge I_n[n]$ and the acyclic cofibrations are generated by $\cup_{n\in\mathbb{Z}} R\wedge J[n]$. Cofibrations are h-cofibrations. Every object is fibrant, so this model structure is right proper.*

As before, a $\cup_n R\wedge J[n]$-cell is a weak equivalence, so it is a P_*-equivalence. We now examine how good properties of the ring R result in good properties of R-modules.

Proposition 7.3. *If R is flat, then cofibrant R-modules are flat.*

Proof. As in the proof of Proposition 4.6(vi), this reduces to showing that for any cofibrant S_G-module B and $n\in\mathbb{Z}$ the tower $R\wedge B[n]$ is flat, but smash products of flat towers are flat, so this is immediate. □

Proposition 7.4. *If R is h-flat, then cofibrant R-modules are h-flat.*

Proof. As in the proof of Proposition 4.7(iii), this reduces to showing that, for any levelwise h-cofibration $Y \to Z$ and any generating cofibration $(A \to B)[n]$ the map

$$R \wedge ((Y \to Z)\square(A \to B)[n])$$

is a levelwise h-cofibration. Since R is h-flat, this follows from Lemma 3.2. $\square$

Proposition 7.5. *If R is slice-like, then slice-cofibrant R-modules are slice-like.*

Proof. By Proposition 4.9, this easily reduces to the statement that, if $A \to B$ is a generating slice$_n$-cofibration then $R \wedge (B/A)[n]$ is slice-like. In any case, $(B/A)[n]$ is a slice-cofibrant tower. Now let $R' \to R$ be a slice-cofibrant replacement of R in the category of towers; this map is a weak equivalence since R and R' are both slice-like. Then since $(B/A)[n]$ is flat, $R \wedge (B/A)[n]$ is weakly equivalent to $R' \wedge (B/A)[n]$. This last tower is slice-cofibrant; hence, it is slice-like. $\square$

Corollary 7.6. *If R is nice, then slice-cofibrant R-modules are nice.*

Corollary 7.7. *Let R be a ring S_G-module, and M an R-module. Suppose that $slice(R) \to const(R)$ is a slice-trivial fibration of ring towers with $slice(R)$ nice. Then a $slice(slice(R)$-Mod$)$-cofibrant replacement of $const(M)$ (which we denote by $slice(M)$) is a model for the slice tower of M, and is unique up to an (essentially) unique homotopy equivalence in the category of $slice(R)$-modules. Since the model category $slice(slice(R)$-Mod$)$ is cofibrantly generated, we can take cofibrant replacement to be functorial. We have commutative diagrams*

$$\begin{tikzcd} slice(R) \wedge slice(M) \arrow[r] \arrow[d] & slice(M) \arrow[d] \\ const(R) \wedge const(M) \arrow[r] & const(M) \end{tikzcd}$$

where the horizontal maps are the module action maps.

Remark: On homotopy categories we may view this process as below

$$Ho(R\text{-Mod}) \xrightarrow{const} Ho(slice(const(R)\text{-Mod})) \longrightarrow Ho(slice(slice(R)\text{-Mod}))$$

$$\xrightarrow{\quad LId \quad} Ho(slice(R)\text{-Mod})$$

where the second map is induced by "restriction of scalars" from $const(R)$ to $slice(R)$.

We remark that analogous results can be obtained for modules over non-unital rings. In fact, in any pointed, closed symmetric monoidal category $\mathcal{C}$, if R is a non-unital ring then the category of R-modules is isomorphic to the category of $1 \coprod R$ modules. Thus the free R-module functor in the non-unital case is given by $X \mapsto X \coprod R \otimes X$. It is now a simple matter to adjust the proofs in this section for the non-unital case.

8 Pairings of Slice Towers and Associators

In the next section we will obtain Toda brackets in the regular slice spectral sequences of ring and module spectra. We will see that Toda brackets in the E_r page for $r > 2$ are computed algebraically, as Massey products. Hence the uniqueness issues involve only the E_2 Toda brackets, which are rather more mysterious in general. We will establish uniqueness by using model theory.

We now explain the significance of the condition of being "nice," with a view toward Toda brackets. Let R be an associative ring S_G-module, and suppose that $slice(R) \to const(R)$ is a slice-trivial fibration of ring towers with $slice(R)$ nice. Since $slice(R)$ and all of its smash powers are nice, we can construct the slices of R as the quotients $slice(R)_n/slice(R)_{n+1}$, and similarly for the smash powers of the slice tower. Hence we get a map of cofiber sequences

$$\begin{tikzcd}
(slice(R) \wedge slice(R))_{n+1} \arrow[r] \arrow[d] & (slice(R) \wedge slice(R))_n \arrow[r] \arrow[d] & \bigvee_{i+j=n} \frac{slice(R)_i}{slice(R)_{i+1}} \wedge \frac{slice(R)_j}{slice(R)_{j+1}} \arrow[d] \\
slice(R)_{n+1} \arrow[r] & slice(R)_n \arrow[r] & \frac{slice(R)_n}{slice(R)_{n+1}}
\end{tikzcd}$$

for each $n \in \mathbb{Z}$, where the vertical maps are the multiplication maps and the right horizontal maps are *quotient* maps. It follows that we have an associative system of pairings

$$\frac{slice(R)_i}{slice(R)_{i+1}} \wedge \frac{slice(R)_j}{slice(R)_{j+1}} \to \frac{slice(R)_{i+j}}{slice(R)_{i+j+1}} \tag{8.1}$$

on the slices of R which induce the multiplicative structure. If R is commutative, we may assume that $slice(R)$ is commutative, and hence that the above pairings are commutative. (There is also a unital property, but we have no use for it.)

Now suppose that $R' \to const(R)$ is a slice-trivial fibration with R' slice-cofibrant, so that R' is also a model for the slice tower of R. By the model axioms we may find a map $R' \to slice(R)$ to complete the diagram below.

$$\begin{tikzcd} & slice(R) \arrow[d] \\ R' \arrow[r] \arrow[ur, dotted] & const(R) \end{tikzcd}$$

This map is unique up to homotopies that are constant when projected to $const(R)$. It is a weak equivalence and induces weak equivalences

$$\frac{R'_n}{R'_{n+1}} \to \frac{slice(R)_n}{slice(R)_{n+1}}$$

for all $n \in \mathbb{Z}$. These associative pairings allow us to define the E_2 Toda brackets, and simple arguments of the above sort will prove uniqueness and naturality. In the commutative case, the commutativity of the pairings on the slices will result in juggling formulas.

There are many other situations where we would hope to find Toda brackets. For example, if α and β are in the homotopy of a ring, and γ is in the homotopy of a module over that ring, we can form $\langle \alpha, \beta, \gamma \rangle$. There are many other cases, some of which we indicate schematically below.

- (right module, ring, ring)
- (ring_1, bimodule, ring_2)
- (ring_1, ring_2, module) with a map from ring_1 to ring_2
- (ring_1, ring_2, module over ring_1) with a map from ring_2 to ring_1

All of these cases can be dealt with separately using basic model category arguments as before. (For example, in the bimodule case we can use the fact that an (R_1, R_2)-bimodule is the same as a left module over $R_1 \wedge R_2^{op}$.) This case-by-case approach is important, as it imparts a sense of inevitability to the definitions of the Toda brackets. However, there is an elegant way to treat all cases simultaneously within a single framework. For this purpose we introduce the category of "associators."

In what follows, m will be a positive integer; denote by $Int(m)$ the set of (nonempty) intervals in $\{1, ..., m\}$. That is, $Int(m)$ consists of the sets $[i,j]$ for $1 \leq i \leq j \leq m$, where

$$[i,j] := \{i, i+1, ..., j\}.$$

If K_1 and K_2 are adjacent intervals (with the elements of K_1 less than the elements of K_2), we denote by $K_1 \cup K_2$ their union; this is again an element of $Int(m)$. We can now make our fundamental definition.

Definition 8.2. Let $(\mathcal{C}, \otimes)$ be a (possibly non-unital) monoidal category and suppose $m \geq 3$. An (m-fold) *associator* in $\mathcal{C}$ is a collection of objects of $\mathcal{C}$

$$\{X_K\}_{K \in Int(m)}$$

together with an associative system of maps for adjacent intervals

$$X_{K_1} \otimes X_{K_2} \to X_{K_1 \cup K_2}.$$

We denote the category of (m-fold) associators by $ASSOC_m(\mathcal{C})$.

Remark: One can also make the above definition for $m = 1, 2$ and for $m = \infty$, but these are not needed for our purposes.

Remark: Associators are used (although not by that name) as the framework to define matric Massey products in [May2].

Observe that, in all the cases where we desire Toda brackets, we actually have associators. For example, to define Toda brackets on the homotopy of a ring R, we use the obvious constant associator, as shown below.

$$K \in Int(m) \mapsto R$$

To define Toda brackets for the case (ring, ring, module), with ring R and module

M, we use the associator indicated below.

$$K \mapsto \begin{cases} R & \text{if } 3 \notin K \\ M & \text{if } 3 \in K \end{cases}$$

Hence, we seek a model structure on associators of towers such that cofibrant replacements of constant (in the tower variable) associators are objectwise slice towers. We continue to work with an arbitrary (possibly non-unital) monoidal category, which we further assume to be a cofibrantly generated monoidal model category that is closed in the monoidal sense. In this case it is clear that $ASSOC_m(\mathcal{C})$ has all small limits, directed colimits and reflexive coequalizers, and that these are formed objectwise. We require the following fact.

Proposition 8.3. *Under the above assumptions, the category of associators is cocomplete.*

Proof. We show that the category of associators is actually a category of algebras over a certain monad on $\mathcal{C}^{Int(m)}$. Since this monad will preserve reflexive coequalizers, it will be a purely formal fact that $ASSOC_m(\mathcal{C})$ has all small colimits (see for example Proposition II.7.4 of [EKMM]). Our monad, which we denote by

$$\mathbb{P} : \mathcal{C}^{Int(m)} \to \mathcal{C}^{Int(m)}$$

is given simply by

$$\mathbb{P}(X)(K) = \coprod_{K_1 \cup ... \cup K_r = K} X(K_1) \otimes ... \otimes X(K_r)$$

with obvious structure maps. The verification of the necessary details is straightforward. □

We wish to pull back the product model structure on $(\mathscr{M}_G^{\mathbb{Z}})^{Int(m)}$ to obtain a model structure on associators of towers. For this, we require "free associator functors" and a description of pushouts. For the following definition, we let 0 denote the initial object of $\mathcal{C}$, and note that $Y \otimes 0 \cong 0$ for any $Y \in \mathcal{C}$ since the monoidal structure is closed.

Definition 8.4. For any $X \in \mathcal{C}$ and $K \in Int(m)$ we denote by $F_K(X)$ the associator such that

$$F_K(X)(K') = \begin{cases} X & \text{if } K' = K \\ 0 & \text{if } K' \neq K \end{cases}$$

and call it the *free associator on* X *in level* K. We call the functor

$$X \mapsto F_K(X)$$

the *free associator functor* in level K, and denote it by F_K. This functor is left adjoint to the *evaluation at* K functor

$$ev_K : ASSOC_m(\mathcal{C}) \to \mathcal{C}$$

which is defined by

$$Y \to Y(K).$$

We must give a description of pushouts (in the category of associators) of the form

$$\begin{tikzcd} F_K(A) \arrow[r] \arrow[d] & F_K(B) \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \tag{8.5}$$

where $K \in Int(m)$ and $A \to B$ is a generating (acyclic) cofibration. We offer the following lemma without proof; it is an easy exercise.

Lemma 8.6. *Let $K = [i, j]$ and $K' = [k, l]$. The pushout of 8.5 is given by*

(i) $Y(K') = X(K')$ *if* $K' \not\supseteq K$, *and*

(ii) if $K' \supseteq K$ *then* $Y(K')$ *is the pushout shown below.*

$$\begin{tikzcd}
X([k, i-1]) \otimes A \otimes X([j+1, l]) \arrow[r] \arrow[d] & X([k, i-1]) \otimes B \otimes X([j+1, l]) \arrow[d] \\
X(K') \arrow[r] & Y(K')
\end{tikzcd}$$

(The intervals $[k, i-1]$ and $[j+1, l]$ might be empty; in this case we delete the corresponding factor from the expression.) The following theorem is now apparent.

Theorem 8.7. *The category of (m-fold) associators in $\mathscr{M}_G^{\mathbb{Z}}$ with objectwise weak equivalences, objectwise q-fibrations and determined cofibrations is a compactly generated, closed and G-topological model structure which we call the* q model structure. *The cofibrations are generated by $\cup_{K \in Int(m), n \in \mathbb{Z}} F_K(I[n])$ and the acyclic cofibrations are generated by $\cup_{K \in Int(m), n \in \mathbb{Z}} F_K(J[n])$. Cofibrations are objectwise h-cofibrations, and cofibrant associators are objectwise cofibrant. Every object is fibrant, so this model structure is right proper.*

For the proof, we note that a generating acyclic cofibration of S_G-modules is a strong deformation retract. Next we pull back the slice model structure.

Theorem 8.8. *The category of (m-fold) associators in $\mathscr{M}_G^{\mathbb{Z}}$ with objectwise P_*-equivalences, objectwise q-fibrations and determined cofibrations is a compactly generated, closed and G-topological model structure which we call the* slice model structure *and denote by slice($ASSOC_m(\mathscr{M}_G^{\mathbb{Z}})$). The cofibrations for this model structure are generated by $\cup_{K \in Int(m), n \in \mathbb{Z}} F_K(I_n[n])$ and the acyclic cofibrations are generated by $\cup_{K \in Int(m), n \in \mathbb{Z}} F_K(J[n])$. Cofibrations are objectwise h-cofibrations, and cofibrant associators are objectwise slice-cofibrant. Every object is fibrant, so this model structure is right proper.*

Of course, a $\cup_{K,n} F_K(J[n])$-cell is an objectwise weak equivalence, so it is an objectwise P_*-equivalence.

Theorem 8.9. *If $\{X(K)\}$ is an (m-fold) associator of S_G-modules, then a $slice(ASSOC_m(\mathscr{M}_G^{\mathbb{Z}}))$-cofibrant replacement of $\{const(X(K))\}$ (which we denote by $\{slice(X(K))\}$) is objectwise a model for the slice towers of the $X(K)$, and is unique up to an (essentially) unique homotopy equivalence in the category of associators of towers. Since $slice(ASSOC_m(\mathscr{M}_G^{\mathbb{Z}}))$ is cofibrantly generated, we may take cofibrant replacement to be functorial. We have commutative diagrams*

$$\begin{tikzcd} slice(X(K_1)) \wedge slice(X(K_2)) \arrow[r] \arrow[d] & slice(X(K_1 \cup K_2)) \arrow[d] \\ const(X(K_1)) \wedge const(X(K_2)) \arrow[r] & const(X(K_1 \cup K_2)) \end{tikzcd}$$

where the horizontal maps are the structure maps of the associators. We also obtain associative systems of pairings on the slices of the $X(K)$, as in 8.1.

Remark: On homotopy categories we can view this process as below.

$$Ho(ASSOC_m(\mathscr{M}_G)) \xrightarrow{const} Ho(slice(ASSOC_m(\mathscr{M}_G^{\mathbb{Z}}))) \xrightarrow{LId} Ho(ASSOC_m(\mathscr{M}_G^{\mathbb{Z}}))$$

Remark: If one applies this to the constant associator of a ring, one will (most likely) not obtain a constant associator of slice towers, so this approach would seem less compelling from a conceptual standpoint. However, the model axioms for the category of associators will imply that this approach yields the same Toda brackets.

9 Toda Brackets in the RSSS

In this section we prove that Massey products in the RSSS for an A_∞ ring spectrum converge to Toda brackets, under suitable hypotheses. On the E_2 page these "Massey products" come from a fictional E_1 page, so they are inherently more mysterious. However, we will show that they are given by the usual Massey products in group cohomology (up to a sign) in the region where the RSSS coincides with the HFPSS, and the model theory of the previous sections will give a compelling argument for the correctness of the definition. In the region where the RSSS coincides with the HOSS, the E_2 "Massey products" are in fact Massey products in group homology (up to a sign). These operations seem to be new, so we give an algebraic description and some preliminary results, including nontriviality.

Setting aside sign conventions for the moment, suppose that R is an A_∞ ring spectrum (an S_G-algebra). We begin with the following definition.

Definition 9.1. A *nice cover* of R is a nice ring tower $slice(R)$ together with a map

$$slice(R) \to const(R)$$

of ring towers which is a P_*-equivalence.

Hence, a nice cover of R gives a model for the slice tower of R. As remarked in the previous section, we obtain an associative system of pairings on the slices of R. Hence, still ignoring sign conventions, we may define the "Massey products" on the E_2 page of the RSSS for R as Toda brackets with respect to these pairings. It is easy to see that the resulting operations do not depend on the choice of nice cover. In fact, if we choose a particular slice-trivial fibration

$$R' \to const(R)$$

with R' slice-cofibrant, and similarly let

$$R'' \to slice(R)$$

be a slice-cofibrant replacement, we can fill in the diagram below.

$$\begin{tikzcd} R'' \arrow[r, dotted] \arrow[d] & R' \arrow[d] \\ slice(R) \arrow[r] & const(R) \end{tikzcd}$$

Of course, a weak equivalence of associators results in bijections on Toda brackets. This seems to be a very compelling way of defining the E_2 "Massey products." One can make similar definitions for the case of (ring, ring, module), as well as for the other examples given in the previous section, but this quickly becomes tiresome. Hence, we generalize to associators. For the following definition, let $\{X_J\}$ be an associator of S_G-modules.

Definition 9.2. A *nice cover* of $\{X_J\}$ is an associator of nice towers $\{slice(X_J)\}$ together with a map

$$\{slice(X_J)\} \to \{const(X_J)\}$$

of associators which is an objectwise P_*-equivalence.

We can then define the E_2 "Massey products" as Toda brackets associated to the resulting pairings on the slices. Using the same argument as above, this time in the category of associators with the slice model structure, it is easy to show that the choice of nice cover is irrelevant. Also, this approach yields the same Massey products as in the case of a ring spectrum, and in all other special cases. Massey products on the E_r page for $r > 2$ are defined algebraically, as usual. Requiring that these Massey products converge to Toda brackets necessitates certain sign conventions, which we now turn to.

Let I denote the unit interval $[0,1]$ with basepoint at 1. Let $\{X_J\}$ be an m-fold associator of S_G-modules, and let $x_i \in \pi^G_{k_i} X_{[i,i]}$ for $i = 1, ..., m$. Recall that we have a canonical associative and anti-commutative system of isomorphisms

$$S^p \wedge S^q \cong S^{p+q}$$

in the homotopy category. For each $n \in \mathbb{Z}$ let S_c^n be a cofibrant model for the n-sphere; we can then represent x_i by a map

$$S_c^{k_i} \xrightarrow{x_i} X_{[i,i]}.$$

Now supposing that the Toda bracket $\langle x_1, ..., x_m \rangle$ can be defined, we may construct maps of the form

$$\partial(S_c^{k_1} \wedge I \wedge S_c^{k_2} \wedge ... \wedge S_c^{k_{m-1}} \wedge I \wedge S_c^{k_m}) \to X_{[1,m]}$$

satisfying certain properties, including that the restriction to

$$S_c^{k_1} \wedge S^0 \wedge ... \wedge S^0 \wedge S_c^{k_m} \cong S_c^{k_1} \wedge ... \wedge S_c^{k_m}$$

coincides with $x_1 \cdot ... \cdot x_m$. To identify

$$\partial(S_c^{k_1} \wedge I \wedge S_c^{k_2} \wedge ... \wedge S_c^{k_{m-1}} \wedge I \wedge S_c^{k_m})$$

as a sphere, we use the orientation induced from the smash product orientation. Denote any such map by $\tilde{A}_{[1,m]}$. We define the Toda bracket

$$\langle x_1, ..., x_m \rangle$$

to be the set of homotopy classes represented by the maps $\tilde{A}_{[1,m]}$.

Letting $\overline{x}$ denote $(-1)^{|x|+1}x$, it is easy to see that $\langle x_1, x_2 \rangle = \overline{x}_1 x_2$, just as with Massey products. Next we examine the differentials in the RSSS. We describe the maps

$$\pi_j^G P_n^n Y \xrightarrow{\delta} \pi_{j-1}^G P_{n+1} Y$$

in the exact couple as follows. Let

$$P_{n+1}Y \xrightarrow{\theta_n^{n+1}} P_n Y \xrightarrow{p_n} P_n^n Y \xrightarrow{\delta_n} S^1 \wedge P_{n+1} Y$$

be the defining triangles and denote by

$$\theta_n^{n+r} : P_{n+r} Y \to P_n Y$$
$$\theta^n : P_n Y \to Y$$

the natural maps for any $r > 0$ and any n. If $x \in \pi_j^G P_n^n Y$ then we let $\delta(x)$ be the unique map

$$S^{j-1} \to P_{n+1} Y$$

such that the composite

$$S^j \cong S^1 \wedge S^{j-1} \xrightarrow{\Sigma(\delta(x))} S^1 \wedge P_{n+1} Y$$

is equal to $\delta_n x$. With this sign convention, the pairings in the RSSS satisfy

$$d_r(yz) = d_r(y)z + (-1)^{|y|} y d_r(z).$$

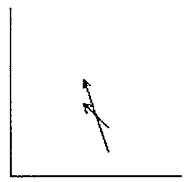
We must give one more definition before stating the convergence theorem. If

$$d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$$

is a differential in the RSSS, a *crossing differential* is a nonzero differential

$$d_{r'} : E_{r'}^{s',t'} \to E_{r'}^{s'+r',t'+r'-1}$$

where $s' < s$, $t' - s' = t - s$ and $s' + r' > s + r$. This situation is depicted below (the crossing differential is the longer one). If all these conditions hold except that $s' + r' = s + r$ instead, we shall call the differential *coterminal.*



We can now state and prove our convergence theorem. In order to simplify the statement, we treat the E_2 case along with the rest by invoking a fictional E_1 page.

Theorem 9.3. *Let $m \geq 3$ and let $\{X_J\}$ be an m-fold associator of S_G-modules. Let $r \geq 1$, and suppose that $y_1, ..., y_m$ are permanent cycles on the E_{r+1} pages of the RSSS's for the $X_{[i,i]}$ which converge to $x_1, ..., x_m$. Suppose further that the Toda bracket $\langle x_1, ..., x_m \rangle$ is strictly defined, and that there are no crossing differentials for the differentials which occur in the formation of defining systems for $\langle y_1, ..., y_m \rangle$. Then the following conclusions hold.*

(i) If $\langle y_1, ..., y_m \rangle$ is strictly defined, then it contains a permanent cycle which converges to an element of $\langle x_1, ..., x_m \rangle$.

(ii) If $\langle y_1, ..., y_m \rangle$ is defined, and $E_{r+1}^{s,t}(X_J)$ consists of permanent cycles for all values of s, t and J such that $E_r^{s,t}(X_J)$ is the source of a differential occuring in the formation of defining systems for this Massey product, then every element of it is a permanent cycle which converges to an element of $\langle x_1, ..., x_m \rangle$.

(iii) Suppose that the assumptions in (ii) hold, except that $\langle y_1, ..., y_m \rangle$ is not known to be defined. If the relevant differentials also have no coterminal differentials, then $\langle y_1, ..., y_m \rangle$ is strictly defined, so the conclusion of (ii) holds.

Proof. We begin by choosing a slice-trivial fibration of associators

$$\{slice(X_J)\} \to \{const(X_J)\}$$

with $\{slice(X_J)\}$ slice-cofibrant. First we prove (i) and (ii) for $r = 1$, so that the "Massey products" in question are in fact Toda brackets and the differentials in question are fictional d_1 differentials. Let the y_i be given by homotopy classes of maps

$$S_c^{k_i} \to P_{n_i}^{n_i} X_{[i,i]} = P_{n_i} X_{[i,i]} / P_{n_i+1} X_{[i,i]}$$

and choose maps

$$S_c^{k_i} \xrightarrow{x_i'} P_{n_i} X_{[i,i]}$$

such that $p_{n_i} x_i'$ represents y_i and $\theta^{n_i} x_i'$ represents x_i. We now prove (i) by constructing a choice of $\tilde{A}$ for $x_1', ..., x_m'$ which maps to one for $x_1, ..., x_m$ under $\theta^{n_1+...+n_m}$ and to one for $y_1, ..., y_m$ under $p_{n_1+...+n_m}$. This can be done by the following argument. While making the construction we will obtain homotopy classes of maps

$$S^a \xrightarrow{f} P_b X_J$$

representing Toda brackets of the x_i's for i in some proper subinterval J of $[1, ..., m]$; we must show that this is zero so that we can choose a null-homotopy. Firstly, since the Toda bracket $\langle x_1, ..., x_m \rangle$ is strictly defined we have that $\theta^b f = 0$. Hence we may let w be the smallest nonnegative integer such that $\theta_{b-w}^b f = 0$; we must show that $w = 0$. Hence, suppose $w > 0$. Then we may choose g such that $\delta g = \theta_{b-w+1}^b f$. Next, by assumption the "Massey product" $\langle y_1, ..., y_m \rangle$ is strictly defined, so we must have that $p_b f = 0$ and therefore we can choose h such that $\theta_b^{b+1} h = f$. Thus, $\delta g = \theta_{b-w+1}^{b+1} h$. Now since $\delta g \neq 0$ by assumption, g is not a permanent cycle, so there must be a value of $r \geq w+2$ such that $d_r(g) \neq 0$. This is a crossing differential for one of our fictional

d_1 differentials, so we have a contradiction, establishing (i). To prove (ii) we suppose given a choice of $\tilde{A}$ for $y_1, ..., y_m$ and construct a choice for $x'_1, ..., x'_m$ which maps to it up to homotopy. We begin with a choice of maps representing the x'_is and homotopies from the $p_{n_i}x'_i$ to the y_i. Suppose inductively that we have constructed a choice of $\tilde{A}_J$ for the x'_is with i in some proper subinterval J of $[1, ..., m]$, as well as a homotopy from $p_n\tilde{A}_J$ (for appropriate n) to the given value of $\tilde{A}_J$ for the y_i's. By the same argument as in case (i), this map is null homotopic, so we may choose a null-homotopy. The image of this null-homotopy under p_n may not coincide with the given null-homotopy for the y_i's, but our hypothesis guarantees that p_n induces a surjection on the relevant homotopy group, so that we can adjust our initial choice to account for the difference. Continuing in this way, we arrive at a choice of $\tilde{A}_{[1,...,m]}$ for the x'_is and a homotopy from its composite with p_n (for appropriate n) to the given choice of $\tilde{A}_{[1,...,m]}$ for the y_i's. Thus the case $r = 1$ is proven, so we may let $r \geq 2$. Now our differentials are no longer fictional, and the Massey products are defined purely algebraically, as in [May2]. We begin with (i), where we inductively construct the $\tilde{A}_J$ for the x'_is by dropping down the tower $r - 1$ places each time we find a null-homotopy. Letting $J = [j_0, ..., j_1]$ denote an arbitrary subinterval of $[1, ..., m]$, suppose inductively that we have constructed the $\tilde{A}_K$ for the x'_is for $\#K \leq \#J$ and that

$$p_*\tilde{A}_K$$

represents an element of the appropriate Massey product of the y_is for all such K. If J' is an interval containing J with $\#J' = \#J + 1$, and $\tilde{A}_J$ is represented by a homotopy class of maps

$$S^a \xrightarrow{f} P_b X_J$$

then we must first show that $\theta^b_{b-r+1} f$ is null. First, by hypothesis $p_b f$ represents an element of a Massey product of the y_is which is zero, and is thus zero on the E_{r+1}

page. This implies that there exists a map

$$S^a \xrightarrow{g} P_{b+1}X_J$$

such that $\theta^b_{b-r+1}(f + \theta^{b+1}_b g) = 0$. This implies that

$$\theta^{b+1}_w g = -\theta^b_w f$$

for all $w \leq b - r + 1$. Thus, if $\theta^b_{b-r+1} f$ is not null, there is a crossing differential which violates our hypotheses, since $\theta^b_w f$ must be zero for sufficiently small w. This allows us to construct $\tilde{A}_{J'}$; we must show that $p_* \tilde{A}_{J'}$ represents an element of the appropriate Massey product of the y_is. Letting $J' = [j_0, j_1]$ and $\tilde{A}_{J'}$ be represented by a map

$$S^a \cong \partial(S^{k_{j_0}}_c \wedge I \wedge ... \wedge I \wedge S^{k_{j_1}}_c) \xrightarrow{f} P_b X_{J'}$$

we have by construction that $p_b f$ is precisely null on the $(j_1 - j_0 - 2)$-skeleton of $I^{\wedge j_1 - j_0}$, and thus is equal to a sum of maps, one for each face of $I^{\wedge j_1 - j_0}$. By the inductive hypothesis, one sees that this element can be expressed as follows.

$$\begin{aligned}
(-1)^{k_{j_0}+1} y_{j_0} d_r^{-1}(p_* \tilde{A}_{[j_0+1, j_1]}) &+ \sum_{l=j_0+1}^{j_1-2} (-1)^{l-j_0+1+k_{j_0}+...+k_l} d_r^{-1}(p_* \tilde{A}_{[j_0, l]}) d_r^{-1}(p_* \tilde{A}_{[l+1, j_1]}) \\
&+ (-1)^{j_1-j_0+k_{j_0}+...+k_{j_1-1}} d_r^{-1}(p_* \tilde{A}_{[j_0, j_1-1]}) y_{j_1} \\
\in \overline{y}_{j_0} d_r^{-1}(\langle y_{j_0+1}, ..., y_{j_1} \rangle) &+ \sum_{l=j_0+1}^{j_1-2} \overline{d_r^{-1}(\langle y_{j_0}, ..., y_l \rangle)} d_r^{-1}(\langle y_{l+1}, ..., y_{j_1} \rangle) \\
&+ \overline{d_r^{-1}(\langle y_{j_0}, ..., y_{j_1-1} \rangle)} y_{j_1}
\end{aligned}$$

Thus (i) is proved. To prove (ii) we must show that we can choose our null-homotopies such that we obtain an arbitrary element of $d_r^{-1}(p_* \tilde{A}_J)$. Letting the element of $d_r^{-1}(\tilde{A}_J)$ induced by our null-homotopy be represented by a homotopy class $S^a \rightarrow P^b_b X_J$, this follows easily from the hypothesis that any element of $ker(d_r)$ in $\pi^G_a P^b_b X_J$ is a permanent cycle, and hence is in the image of

$$(p_b)_* : \pi_a^G P_b X_J \to \pi_a^G P_b^b X_J.$$

For (iii), the argument is mostly the same as above. While building defining systems one must show that $p_* \tilde{A}_J$ is zero on the E_{r+1} page for proper subintervals J. Since the Toda bracket $\langle x_1, ..., x_m \rangle$ is strictly defined, we know that $p_* \tilde{A}_J$ is a permanent cycle converging to zero, so if it is not zero on the E_{r+1} page then it must be hit by a d_s differential for some $s \geq r+1$. This would be a coterminal differential; the result follows. □

Remark: The theorem holds also for $m = 2$; it is merely the convergence theorem for products (up to a sign).

Remark: This theorem is similar to the Moss convergence theorem for the Adams spectral sequence (Theorem 1.2 of [Mos]).

Of course, these Massey products on the E_r pages for $r > 2$ satisfy all the usual algebraic properties of Massey products (see [May2]). This also holds for the E_2 "Massey products."

Proposition 9.4. *The E_2 "Massey products" in the RSSS satisfy all the associativity and commutativity properties ("juggling theorems") that Toda brackets do.*

Proof. We need only address the commutativity properties. If $\{Y_J\}$ is an associator in a symmetric monoidal category $\mathcal{C}$, define the *opposite associator* $\{Y_J\}^{op} = \{Y_J^{op}\}$ by $Y_{[i,j]}^{op} = Y_{[n+1-j,n+1-i]}$, with the pairings as below.

$$\begin{aligned} Y_{[i,j]}^{op} \wedge Y_{[j+1,k]}^{op} = Y_{[n+1-j,n+1-i]} \wedge Y_{[n+1-k,n-j]} &\xrightarrow[\cong]{\tau} Y_{[n+1-k,n-j]} \wedge Y_{[n+1-j,n+1-i]} \\ &\to Y_{[n+1-k,n+1-i]} = Y_{[i,k]}^{op} \end{aligned}$$

First we observe that if $\{slice(X_J)\}$ is a nice cover of $\{X_J\}$ then $\{slice(X_J)\}^{op}$ is a nice cover of $\{X_J\}^{op}$. This proves the analogue of Corollary 3.7 of [May2]. For the analogues of Propositions 3.8 and 3.9 of [May2], we use a cofibrant replacement in the slice model structure on non-unital commutative ring towers. $\square$

Next, we wish to identify the E_2 "Massey products" in the region where the RSSS coincides with the HFPSS.

Proposition 9.5. *Let $\{X_J\}$ be an m-fold associator of S_G-modules, and suppose that $y_i \in E_2^{s_i,t_i}(X_{[i,i]})$ with $t_i - s_i > 0$ and $s_i < (m(G)-1)(t_i - s_i)$. Then the E_2 "Massey product" $\langle y_1, ..., y_m \rangle$ is (strictly) defined if and only if the corresponding Massey product in group cohomology is, in which case the two coincide up to the sign*

$$(-1)^{\sum_{i=1}^m it_i + \sum_{i<j} s_i t_j}.$$

Proof. As before we begin with an associator of nice slice towers $\{slice(X_J)\}$. Choosing a model for EG, and denoting the slices of the X_J by

$$slice_n^n(X_J) := slice(X_J)_n / slice(X_J)_{n+1},$$

we then obtain maps of associators

$$\{slice_*^*(X_J)\} \to \{F(EG_+^{\#J}, slice_*^*(X_J))\}$$

that induce monomorphisms on all relevant homotopy groups, and *isomorphisms* on all relevant homotopy groups which parametrize null-homotopies. The first statement follows immediately. For the second part, we consider the associators of towers

$$\{F(EG_+^{\#J}/(EG^{\#J})_+^{[k-1]}, slice_*^*(X_J))\}$$

and the triangles

$$\begin{aligned} F(EG_+^{\#J}/(EG^{\#J})_+^{[k]}, slice_*^*(X_J)) &\to F(EG_+^{\#J}/(EG^{\#J})_+^{[k-1]}, slice_*^*(X_J)) \\ &\to F((EG^{\#J})_+^{[k]}/(EG^{\#J})_+^{[k-1]}, slice_*^*(X_J)) \end{aligned}$$

and proceed as in the proof of Theorem 9.3. We leave the details, which are tedious, to the interested reader. □

Remark: The requirement that $s_i < (m(G)-1)(t_i - s_i)$ is only to ensure that the map from the RSSS to the HFPSS is a monomorphism or isomorphism at the relevant entries; this may be the case even if this inequality fails.

In order to describe the E_2 "Massey products" in the region where the RSSS coincides with the HOSS, we first define Massey products in group homology algebraically. First let Y and Z be G-modules, and let $EG \to \mathbb{Z}$ denote a free $\mathbb{Z}[G]$-modules resolution of $\mathbb{Z}$ (with trivial action). We define a pairing of chain complexes as below.

$$(Y \otimes EG)/G \otimes (Z \otimes EG)/G \to ((Y \otimes Z) \otimes EG^{\otimes 2})/G$$
$$[a \otimes y] \otimes [b \otimes z] \mapsto [\sum_{g \in G} a \otimes gb \otimes y \otimes gz] = [\sum_{g \in G} ga \otimes b \otimes gy \otimes z]$$

This induces products in group homology. However, if x and y are homology classes that are not both of degree zero, we have $xy = 0$ since one can project off either of the two copies of EG. The product in degree zero is easily seen to be given by

$$[x][y] = [\sum_{g \in G} x(gy)] = [\sum_{g \in G} (gx)y].$$

Thus, the homology product is not very interesting. Note that it is the same product one finds in the RSSS (see Corollary I.9.5). Now let $\{X_J\}$ be an m-fold associator of G-modules. Since the above construction is clearly associative, we may apply it to obtain an associator of chain complexes as below.

$$\{(X_J \otimes EG^{\otimes \# J})/G\}$$

Using these we define Massey products in group homology. We can now identify the E_2 "Massey products" in the RSSS in the region where it coincides with the HOSS.

Proposition 9.6. *Let $\{X_J\}$ be an m-fold associator of S_G-modules, and suppose that $y_i \in E_2^{s_i,t_i}(X_{[i,i]})$. Suppose that all entries of the RSSS's involved in the formation of "defining systems" for $\langle y_1, ..., y_m\rangle$ coincide with the corresponding entries of the HOSS's. Then the E_2 "Massey product" $\langle y_1, ..., y_m\rangle$ is (strictly) defined if and only if the corresponding Massey product in group homology is, in which case the two coincide up to the sign*

$$(-1)^{\sum_{i=1}^{m} it_i + \sum_{i<j} s_i t_j}.$$

Proof. As in the proof of Proposition 9.5, this time using the associators of towers

$$\{slice^*_*(X_J) \wedge (EG^{\times \# J})^{[k]}_+\}$$

and the triangles below.

$$slice^*_*(X_J) \wedge (EG^{\times\#J})^{[k-1]}_+ \to slice^*_*(X_J) \wedge (EG^{\times\#J})^{[k]}_+ \\ \to slice^*_*(X_J) \wedge (EG^{\times\#J})^{[k]}/(EG^{\times\#J})^{[k-1]}$$

We again leave the details (including the sign difference) to the reader. □

Remark: It is possible to give a slightly more refined statement: it is only necessary that the maps $HOSS \to RSSS$ be *surjective* on the entries which are sources of (fictional d_1) differentials occuring in the formation of "defining systems."

Next, we give some preliminary results on these homology Massey products.

Proposition 9.7. *Let x, y and z be homology classes such that $xy = 0$ and $yz = 0$. Then the following conclusions hold.*

(i) $\langle x, y, z\rangle = 0$ *unless* $\deg(x) = \deg(y) = \deg(z) = 0$.

(ii) $\langle x, y, z\rangle$ *consists of a single element.*

(iii) If $x = [x_0]$, $y = [y_0]$ and $z = [z_0]$ are of degree zero and $\sum_{g\in G} g$ acts as zero on any of x_0, y_0 or z_0, then $\langle x, y, z\rangle = 0$.

Proof. For (i), suppose we are given cycles x, y and z representing homology classes. After forming a representative of $\langle x, y, z\rangle$ we may project off the first two factors of EG to $\mathbb{Z}$, so that $\langle x, y, z\rangle$ is represented by

$$x'd^{-1}(y'z),$$

where x' and y' denote the images of x and y, respectively, under the projection. If $\deg(x) > 0$ then $x' = 0$. If $\deg(y) > 0$ then $y' = 0$, so that the above is actually a homology product where not both elements are of degree zero; it is thus a boundary. Thus we must have $\deg(x) = \deg(y) = 0$ if we are to obtain a nonzero Massey product; a similar argument obtains $\deg(y) = \deg(z) = 0$ by projecting off the last two factors of EG instead. For (ii), we simply note that any two elements of $d^{-1}(y'z)$ differ by a cycle, and thus the resulting elements of $\langle x, y, z\rangle$ differ by a homology product where one element is of positive degree. For (iii), the above simplifies to

$$[x_0]d^{-1}([y_0](z_0 \otimes 1))$$

where we have taken $EG_0 = \mathbb{Z}[G]$. If $\sum_{g\in G} g$ acts as zero on y_0, we again have a homology product which must be a boundary, while if it acts as zero on x_0 we get the zero chain. A similar argument applies to z_0 if we instead project off the last two factors of EG instead of the first two. $\square$

The above proposition implies that, if we are to obtain a nonzero homology triple product, we may not use such simple G-modules as sign representations or the trivial action of C_p on $\mathbb{Z}/p\mathbb{Z}$. Thus the homology triple product will be zero in many familiar cases. We now give a constructive proof that the homology triple product is not identically zero.

Proposition 9.8. *There exist nonzero homology triple products when $G = C_2$.*

Proof. Let $G = C_2 = \{1, g\}$ and let $EG \to \mathbb{Z}$ be as below.

$$\mathbb{Z} \xleftarrow{\text{aug}} \mathbb{Z}[G] \xleftarrow{1-g} \mathbb{Z}[G] \xleftarrow{1+g} \mathbb{Z}[G] \xleftarrow{1-g} \dots$$

Thus, for any G-module M the chain complex $(M \otimes EG)/G$ is as below.

$$M \xleftarrow{1-g} M \xleftarrow{1+g} M \xleftarrow{1-g} \ldots$$

Letting our homology classes in degree zero be represented by x, y and z, we have the following.

$$\begin{aligned} [x][y] = 0 \quad &\Rightarrow \quad \exists s : x(y + gy) = (1 - g)s \\ [y][z] = 0 \quad &\Rightarrow \quad \exists t : (y + gy)z = (1 - g)t \end{aligned}$$

The homology triple product $\langle [x], [y], [z] \rangle$ is then represented by $(x + gx)t$. Thus we define a universal commutative G-ring R in which we can form a homology triple product, as below.

$$\begin{aligned} R := \mathbb{Z}[x, gx, y, gy, z, gz, t, gt, s, gs]/(x(y + gy) = (1 - g)s, (y + gy)z = (1 - g)t, \\ (x + gx)(y + gy) = 0, (y + gy)(z + gz) = 0) \end{aligned}$$

Note that these four relations are equivalent to the first two plus g times the first two. We can simplify this by using the first two relations to express gs and gt in terms of other variables, as well as changing basis as below.

$$\begin{aligned} u := x + gx, \quad & \overline{x} := gx \\ v := y + gy, \quad & \overline{y} := gy \\ w := z + gz, \quad & \overline{z} := gz \end{aligned}$$

We then obtain the following isomorphism, with given G-action.

$$R \quad \cong \quad \mathbb{Z}[u, \overline{x}, v, \overline{y}, w, \overline{z}, s, t]/(uv = 0 = vw)$$

$$gu = u, \quad g\overline{x} = u - \overline{x}, \quad gv = v, \quad g\overline{y} = v - \overline{y}, \quad gw = w, \quad g\overline{z} = w - \overline{z},$$

$$gs = s + v\overline{x}, \quad gt = t + v\overline{z}$$

We now show that ut is not in the image of $(1+g)$; suppose that it is. We set $(1+g)f = ut$ and deduce what we can about the coefficients of f in the obvious monomial basis for R. This task is made easier by the fact that g times a monomial is a sum of monomials of the same and higher degrees (times integers). Though it is tedious, the reader may check the following claims. First, the only monomials whose images have a ut term are $\overline{x}t$ and ut. We have $(1+g)(\overline{x}t) = ut - v\overline{xz}$ and $(1+g)(ut) = 2ut$. Thus the coefficient of $\overline{x}t$ in f must be odd. This introduces an odd coefficient times $v\overline{xz}$, which must be eliminated somehow. The only possible sources of such a term are as follows.

$$\begin{aligned}
(1+g)(s\overline{z}) &= sw - v\overline{xz} \\
(1+g)(v\overline{xz}) &= 2v\overline{xz} \\
(1+g)(\overline{xyz}) &= v\overline{xz} + w\overline{xy} + u\overline{yz} - uw\overline{y}
\end{aligned}$$

Now, the only other source of an sw term is $(1+g)(sw) = 2sw$, so $s\overline{z}$ must have an even coefficient in f. Thus, $\overline{xyz}$ must have an odd coefficient in f. However, this contributes odd coefficients times $w\overline{xy}$ and $u\overline{yz}$ to $(1+g)f$. The only other sources for these terms are as below.

$$\begin{aligned}
(1+g)(w\overline{xy}) &= 2w\overline{xy} - uw\overline{y} \\
(1+g)(u\overline{yz}) &= 2u\overline{yz} - uw\overline{y}
\end{aligned}$$

Thus we have a contradiction, and therefore a nonzero homology triple product. $\square$

Remark: At present, nothing more is known about these homology Massey products. It may be the case that one always needs all the elements to be of degree zero in order to obtain a nonzero Massey product, but this is not known. It is also unknown whether or not the indeterminacy of higher order Massey products is always zero. It would also be desirable to have more straightforward examples of nonzero Massey products than the one given above.

Next, we turn to the Leibniz formula for these Massey products.

Theorem 9.9. *Let $\{X_J\}$ be an m-fold associator of S_G-modules, with $x_i \in E_{r+1}(X_{[i,i]})$ such that $\langle x_1, ..., x_m\rangle$ is defined and each x_i survives to the E_s page. Suppose that, for each entry $E_r^{p,q}(X_J)$ which is the source of a differential occuring in the formation of defining systems for $\langle x_1, ..., x_m\rangle$ and each t satisfying $r < t < s$ we have that*

(i) $E_{r+s-t}^{p+t,q+t-1}(X_J)$ consists of permanent cycles,

(ii) there are no nonzero differentials hitting $E_k^{p+t,q+t-1}(X_J)$ for $k \geq t$, and

(iii) $E_\infty^{p+t,q+t-1}(X_J) = 0$.

Choose classes $y_i \in E_{r+1}(X_{[i,i]})$ surviving to the $d_s x_i$. Then for any $\alpha \in \langle x_1, ..., x_m\rangle$, α survives to the E_s page and there is an element of

$$\langle \begin{pmatrix} y_1 & \bar{x}_1 \end{pmatrix}, \begin{pmatrix} x_2 & 0 \\ y_2 & \bar{x}_2 \end{pmatrix}, ..., \begin{pmatrix} x_{m-1} & 0 \\ y_{m-1} & \bar{x}_{m-1} \end{pmatrix}, \begin{pmatrix} x_m \\ y_m \end{pmatrix}\rangle$$

which survives to $-d_s(\alpha)$.

Proof. First let $r > 1$. We alter the lifting procedure used in the proof of Theorem 9.3, as follows. Suppose a class $f : S_c^k \to P_n^n Z$ survives to the E_s page and that $d_s f$ is represented by $g : S_c^{k-1} \to P_{n+s-1}^{n+s-1} Z$. Letting I_0 denote the unit interval $[0,1]$ with basepoint 0, we can find a map

$$f' : (I_0 \wedge S_c^{k-1}, S_c^{k-1}) \to (P_n Z, P_{n+s-1} Z)$$

such that the composite of $f'|_{S_c^{k-1}}$ with p_{n+s-1} is homotopic to g and the induced map

$$S_c^k \cong I_0 \wedge S_c^{k-1}/S_c^{k-1} \to P_n Z/P_{n+s-1} Z \to P_n Z/P_{n+1} Z = P_n^n Z$$

is homotopic to f. Applying this to representatives for the x_i in $\pi_{k_i}^G P_{n_i}^{n_i} X_{[i,i]}$ and chosen representatives for the y_i we obtain maps

$$x_i' : (I_0 \wedge S_c^{k_i-1}, S_c^{k_i-1}) \to (P_{n_i} X_{[i,i]}, P_{n_i+s-1} X_{[i,i]})$$

which keep track of both the x_i and the y_i. We use such maps as our "lifts" for any given defining system for α. Recall that, each time we find a null-homotopy, we must drop down the tower by $(r-1)$ places. In our relative case, this corresponds to composing with the map of pairs below.

$$(P_nX_J, P_{n+s-1}X_J) \xrightarrow{(\theta^n_{n-(r-1)}, \theta^{n+s-1}_{n+s-r})} (P_{n-(r-1)}X_J, P_{n+s-r}X_J)$$

Note that $n-(r-1) < n < n+s-r < n+s-1$. Let a defining system for α be given by classes $a_J, \tilde{a}_J$ on the E_r page, beginning with the representatives for the x_i determined by the x'_i. We will lift these maps to maps from subspectra of spectra of the form

$$(I_0 \wedge S_c^{k_{j_0}-1}) \wedge I_1 \wedge \ldots \wedge I_1 \wedge (I_0 \wedge S_c^{k_{j_1}-1})$$

to the relevant slice towers, where $J = [j_0, j_1]$ is a subinterval of $[1, m]$ and I_1 denotes the unit interval $[0,1]$ with the basepoint 1. We will use the symbol ∂_s to denote the subspectrum where at least one of the I_0 coordinates is equal to 1, and ∂_r to denote the subspectrum where at least one of the I_1 coordinates is equal to 0. We will also use these symbols to denote restrictions of maps to these subspectra. We will denote our lifts of the $\tilde{a}_J$ by $\tilde{A}_J$. We begin with $\tilde{A}_{[i,i+1]}$ for $i \in [1, m-1]$; we simply take the maps corresponding to $x'_i \wedge x'_{i+1}$ under the isomorphisms below.

$$\partial_r((I_0 \wedge S_c^{k_i-1}) \wedge I_1 \wedge (I_0 \wedge S_c^{k_{i+1}-1})) \cong (I_0 \wedge S_c^{k_i-1}) \wedge (I_0 \wedge S_c^{k_{i+1}-1})$$

Clearly this lifts $\tilde{a}_{[i,i+1]} = \bar{x}_i x_{i+1} = (-1)^{k_i+1} x_i x_{i+1}$ so that $\partial_s \tilde{A}_{[i,i+1]}$ induces the following map.

$$(-1)^{k_i+1} y_i x_{i+1} - x_i y_{i+1} = -\overline{\begin{pmatrix} y_i & \bar{x}_i \end{pmatrix}} \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix}$$

Suppose inductively that we have constructed lifts $\tilde{A}_J$ of the $\tilde{a}_J$ for $\#J \leq i$ and lifts A_J of the a_J for $\#J < i$. For each interval J of size i we must drop $\tilde{A}_J$ down the tower $(r-1)$ places and find a null homotopy which lifts a_J. Suppose then that we are given a map of pairs as below.

$$(\tilde{A}_J, \partial_s \tilde{A}_J) : (I_0 \wedge S_c^{k-1}, S_c^{k-1}) \to (P_n X_J, P_{n+s-1} X_J)$$

First we claim that $\theta_{n+s-r}^{n+s-1} \partial_s \tilde{A}_J$ is null-homotopic. For this, we first note that $\tilde{a}_J$ is a permanent cycle; this immediately implies that $\theta_{n+1}^{n+s-1} \partial_s \tilde{A}_J \simeq \Sigma^{-1} \delta_n \tilde{a}_J \simeq 0$. Suppose inductively that $\theta_{n+l}^{n+s-1} \partial_s \tilde{A}_J$ is null-homotopic for some l with $0 < l < s-r$. Then a null homotopy can be given by a map of pairs

$$(I_0 \wedge S_c^{k-1}, S_c^{k-1}) \to (P_{n+l} X_J, P_{n+s-1} X_J)$$

representing an element of $E_{s-l}^{n+l, n+l+k}(X_J)$; hypothesis (i) then implies that it is a permanent cycle, so that $\theta_{n+l+1}^{n+s-1} \partial_s \tilde{A}_J$ is null-homotopic, as in the base case. Now choose a null-homotopy h of $\theta_{n+s-r}^{n+s-1} \partial_s \tilde{A}_J$ and consider the map below.

$$\tilde{A}_J \cup -h : \partial(I_0 \wedge I_0) \wedge S_c^{k-1} \to P_n(X_J)$$

Since $d_r a_J = \tilde{a}_J$, we can also choose a map of pairs

$$H : ((I_0 \wedge S^1) \wedge S_c^{k-1}, \partial(I_0 \wedge S^1) \wedge S_c^{k-1}) \to (P_{n-(r-1)} X_J, P_n X_J)$$

which lifts a_J such that $(\partial H \cup 0)$ and $(\tilde{A}_J \cup -h)$ induce homotopic maps to $P_n^n X_J$. It then follows that $(\tilde{A}_J \cup -h) - (\partial H \cup 0)$ is equal to $\theta_n^{n+1} f$ for some homotopy class $f : S^k \to P_{n+1} X_J$. Hypotheses (ii) and (iii) of the statement can now be used inductively to show that

$$f \in im\big(\theta_{n+1}^{n+s-r}\big) + ker\big(\theta_{n-(r-1)+1}^{n+1}\big),$$

so we can correct our initial choice of h so that

$$\theta^n_{n-(r-1)+1}(\tilde{A}_J \cup -h) \quad \simeq \quad \theta^n_{n-(r-1)+1}(\partial H \cup 0).$$

We can use a homotopy connecting these two maps and H to finally obtain a map

$$A_J : (I_0 \wedge I_0) \wedge S_c^{k-1} \to P_{n-(r-1)} X_J$$

lifting a_J. Using these maps to construct $\tilde{A}_L$ for intervals L of size $i+1$, we find that $\tilde{A}_L$ lifts $\tilde{a}_L$ by a similar calculation as in the proof of Theorem 9.3. We also see that $\partial_s A_J = -h$, which is an element of $-d_r^{-1}(\partial_s \tilde{A}_J)$ (abusing notation slightly by identifying a map with the element of the appropriate entry on the E_r page that it lifts). Again abusing notation in this fashion, we find on the E_r page that, for the interval $J = [j_0, j_1]$, we have the following.

$$\partial_s \tilde{A}_J = \sum_{l=j_0}^{j_1-1} (-1)^{1+|a_{[j_0,l]}|} (\partial_s A_{[j_0,l]} a_{[l+1,j_1]} + (-1)^{|a_{[j_0,l]}|} a_{[j_0,l]} \partial_s A_{[l+1,j_1]})$$

Here we have taken $A_{[i,i]} := x_i'$ so that $\partial_s A_{[i,i]} = y_i$. It is then an easy calculation that the matrices

$$\begin{pmatrix} \tilde{a}_J & 0 \\ -\partial_s \tilde{A}_J & -\overline{\tilde{A}}_J \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_J & 0 \\ \partial_s A_J & \bar{a}_J \end{pmatrix}$$

form a defining system for the matric Massey product in the statement (with first row omitted if $1 \in J$ and second column omitted if $m \in J$). This proves the theorem when $r > 1$. In the case $r = 1$ we can use a similar procedure to show that α survives to the E_s page and give a combinatorial description of $-d_s(\alpha)$. We believe that this can be identified as a member of the appropriate matric Toda bracket, but these arguments don't really belong in this thesis; note that the case $m = 2$ holds trivially. $\square$

Remark: The above theorem is analogous to Theorem 4.3 of [May2], though the hypotheses are slightly weaker and the conclusion slightly stronger.

The following, which is analogous to Corollary 4.4 of [May2], follows directly from the theorem and Proposition 2.10 of [May2].

Corollary 9.10. *Assume in addition to the hypotheses of Theorem 9.9 that each*

$$\langle \bar{x}_1, ..., \bar{x}_{k-1}, y_k, x_{k+1}, ..., x_m \rangle$$

is strictly defined. Assume further that these Massey products and the matric Massey product in the statement of Theorem 9.9 all have zero indeterminacy. Then

$$d_s\langle x_1, ..., x_m\rangle = -\sum_{k=1}^{m} \langle \bar{x}_1, ..., \bar{x}_{k-1}, y_k, x_{k+1}, ..., x_m \rangle.$$

Finally, we have a sort of combination of Theorems 9.3 and 9.9.

Theorem 9.11. *Let $\{X_J\}$ be an m-fold associator of S_G-modules, with $x_i \in E_{r+1}(X_{[i,i]})$ such that $\langle x_1, ..., x_m\rangle$ is defined and the x_i's are permanent cycles converging to z_i. Let n be given such that $1 \leq n \leq m-2$ and the*

$$\langle z_k, ..., z_{k+n}\rangle$$

are strictly defined. Further assume that the conditions of Theorem 9.3(ii) hold for these subproducts, and that the conditions of Theorem 9.9 hold for subintervals J such that $\#J \geq n+1$ and some $s > r$. Let $\alpha \in \langle x_1, ..., x_m\rangle$. Then α survives to the E_s page and there exist $y_k \in E_{r+1}(X_{[k,k+n]})$ converging to elements of $\langle z_k, ..., z_{k+n}\rangle$ such that there is an element of

$$\left\langle \begin{pmatrix} y_1 & \bar{x}_1 \end{pmatrix}, \begin{pmatrix} x_{2+n} & 0 \\ y_2 & \bar{x}_2 \end{pmatrix}, ..., \begin{pmatrix} x_{m-1} & 0 \\ y_{m-n-1} & \bar{x}_{m-n-1} \end{pmatrix}, \begin{pmatrix} x_m \\ y_{m-n} \end{pmatrix} \right\rangle$$

which survives to $d_s(\alpha)$. Here the bidegree of y_k is $(p+s-r, q+s-r)$ when (p,q) is the bidegree of $\langle x_k, ..., x_{k+n}\rangle$.

Proof. Suppose $r > 1$. Let $a_J, \tilde{a}_J$ be a defining system for α. We begin with the lifting procedure from the proof of Theorem 9.3, proceeding until we have defined lifts $\tilde{A}_J$

of $\tilde{a}_J$ for intervals J with $\#J \leq n+1$ and lifts A_J of a_J when $\#J < n+1$. We then switch to the relative case as in the proof of Theorem 9.9, regarding what we have so far as being maps of pairs with $\partial_s \mapsto *$. It is now obvious that $\partial_s A_J$ is the zero map unless $\#J \geq n+1$. Thus, using the notation from the proof of Theorem 9.9, we have the following in the E_r page for $J = [j_0, j_1]$ with $\#J \geq n+2$.

$$\begin{aligned}\partial_s \tilde{A}_J = & \sum_{l=j_0+n}^{j_1-1} (-1)^{1+|a_{[j_0,l]}|} \partial_s A_{[j_0,l]} a_{[l+1,j_1]} \\ & + \sum_{l=j_0}^{j_1-n-1} (-1)^{1+|a_{[j_0,l]}|} (-1)^{|a_{[j_0,l]}|} a_{[j_0,l]} \partial_s A_{[l+1,j_1]}\end{aligned}$$

Now, given an interval $J = [j_0, j_1]$ with $\#J \geq n+1$, let $J' := [j_0, j_1 - n]$ and $J'' := [j_0 + n, j_1]$ and note that $J \mapsto J'$ is a bijection from such intervals onto $Int(m-n)$. It is now an easy exercise to show that

$$J' \mapsto \begin{pmatrix} a_{J''} & 0 \\ -\partial_s A_J & \bar{a}_{J'} \end{pmatrix}, \begin{pmatrix} \tilde{a}_{J''} & 0 \\ \partial_s \tilde{A}_J & -\bar{\tilde{a}}_{J'} \end{pmatrix}$$

is a defining system for the matric Massey product in the statement (with first row omitted if $1 \in J'$ and second column omitted if $m - n \in J'$), as long as we set

$$y_k = -\partial_s A_{[k,k+n]}.$$

Recall from the proof of Theorem 9.9 that $d_r(y_k)$ is in the same bidegree as $d_s(\tilde{a}_{[k,k+n]})$, so the bidegree is as stated; it remains to show that y_k converges to an element of $\langle z_k, ..., z_{k+n} \rangle$. Regarding $A_{[k,k+n]}$ as a map

$$(I_0 \wedge I_0) \wedge S^d_c \to P_e X_{[k,k+n]}$$

we note that taking $-\partial_s$ corresponds to setting the second coordinate equal to 1. Since $A_{[k,k+n]}$ is trivial on $\partial^2(I_0 \wedge I_0) \wedge S^d_c$, we see that $\theta_e^{e+s-1}(-\partial_s A_{[k,k+n]}) \simeq \theta_e^{e+r-1}(\tilde{A}_{[k,k+n]})$ (wherein we regard these maps as non-relative since they are trivial on the boundaries of their domains). The result follows. Now suppose that $r = 1$. A similar argument again shows that α survives to the E_s page, and we can give a combinatorial description of $d_s(\alpha)$. Identifying this as an element of the appropriate matric Toda bracket again doesn't belong in this thesis. We note, however, that it is easy to identify it as the appropriate matrix product when $n = m - 2$. □

Remark: The above theorem is analogous to Theorem 4.5 of [May2], though the hypotheses are slightly weaker.

The following, which is analogous to Corollary 4.6 of [May2], follows directly from the theorem and Proposition 2.10 of [May2].

Corollary 9.12. *Assume in addition to the hypotheses of Theorem 9.11 that for each k there is just one y_k in the appropriate bidegree converging to an element of $\langle z_k, ..., z_{k+n}\rangle$ and that each*

$$\langle \bar{x}_1, ..., \bar{x}_{k-1}, y_k, x_{k+n+1}, ..., x_m\rangle$$

is strictly defined. Assume further that these Massey products and the matric Massey product in the statement of Theorem 9.11 all have zero indeterminacy. Then

$$d_s\langle x_1, ..., x_m\rangle = \sum_{k=1}^{m-n} \langle \bar{x}_1, ..., \bar{x}_{k-1}, y_k, x_{k+n+1}, ..., x_m\rangle.$$

Remark: The $r = 1$ cases of the above two theorems and corollaries require the definition of matric Toda brackets with respect to the smash product. A definition can be given, but is beyond the scope of this thesis. With such a definition, one could formulate and prove generalizations of all of the theorems in this section. In the case of the results identifying the E_2 "Massey products" in the HOSS and HFPSS, one might have to use shifted chain complexes to encode the sign difference, rather than having an overall sign difference.

Chapter V

Computations

1 Introduction

In this chapter we apply the theory of the preceding chapters to make some calculations of the RSSS for various types of spectra. In Section 2 we completely compute the RSSS for Eilenberg MacLane spectra in dimensions ± 1, and in arbitrary dimension when the group is cyclic of prime order. We also give a general formula for the slice tower in dimensions ± 2. The case of dimension 1 verifies a conjecture of Hill from [Hil]. In Section 3 we give a sample calculation for a cofree spectrum, where the group is cyclic of prime power order, as well as a dual calculation for a free spectrum. These calculations give a hint as to the general behavior of the RSSS outside the region where it coincides with the HFPSS (or the HOSS). Next, in Section 4 we give an updated treatment of Dugger's spectral sequence for the $K\mathbb{R}$ spectrum (see [Dug]), including the derivation of real Bott periodicity and the computation of some Toda brackets in KO_*. In Section 5 we determine the slice tower for equivariant complex K-theory when the group is cyclic. This is used to give a simple proof of the Atiyah-Segal completion theorem for cyclic groups of prime order.

2 Computations for Eilenberg MacLane Spectra

We begin with Eilenberg MacLane spectra in dimensions ± 1.

Theorem 2.1. *If $\underline{M}$ is a Mackey functor then we have*

$$P_n(\Sigma H\underline{M}) \cong \Sigma H \mathscr{F}^{n-1}\underline{M}$$

so that the only nonzero entries in the RSSS for $\Sigma H\underline{M}$ are

$$\underline{E}_2^{s,s+1} \cong \mathscr{F}^s\underline{M}/\mathscr{F}^{s+1}\underline{M}$$

and all the differentials are zero. Similarly, we have

$$P_n(\Sigma^{-1}H\underline{M}) \cong \Sigma^{-1}H\mathscr{F}_{-n}\underline{M}$$

so that the only nonzero entries in the RSSS for $\Sigma^{-1}H\underline{M}$ are

$$\underline{E}_2^{s,s-1} \cong \mathscr{F}_{-s+1}\underline{M}/\mathscr{F}_{-s}\underline{M}$$

and all the differentials are zero.

Proof. By Corollary I.8.3, for an Eilenberg MacLane spectrum X in dimension ± 1, all the spectra P_nX, P^nX, P_n^nX are Eilenberg MacLane spectra in dimension ± 1. Then the long exact sequences of homotopy groups for the cofiber sequences

$$P_nX \to X \to P^{n-1}X$$

imply that the maps

$$\underline{\pi}_{\pm 1}P_nX \to \underline{\pi}_{\pm 1}X$$

are injective. The result then follows from Proposition I.8.4 and Corollary I.8.6. $\square$

Remark: The result for dimension 1 was originally conjectured by Mike Hill. The result for cyclic groups of prime power order (dimension 1) is Theorem 5.8 of [Hil].

We now give the slice tower for dimensions ± 2.

Theorem 2.2. *If $\underline{M}$ is a Mackey functor then we have*

$$P^n(\Sigma^2 H\underline{M}) \cong \Sigma^2 H\left(R(n)i_n^*\left(\underline{M}/\mathscr{F}^{n/2}\underline{M}\right)\right),$$
$$P_n(\Sigma^{-2} H\underline{M}) \cong \Sigma^{-2} H\left(L(-n)i_{-n}^*\left(\mathscr{F}_{-n/2}\underline{M}\right)\right).$$

Proof. For dimension 2, Proposition I.8.1 and Corollary I.8.3 imply that the P^m's are Eilenberg MacLane spectra in dimension 2. The first statement then follows from Corollary I.8.6. The proof of the other statement is similar. □

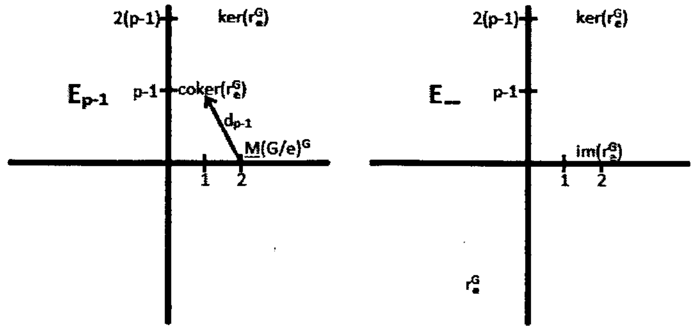
Now suppose $G = C_p$, with p an odd prime. We have the following.

$$\underline{\pi}_2 P^n(\Sigma^2 H\underline{M}) \cong \begin{cases} 0 & \text{if } n < 2 \\ R(1)i_1^*\underline{M} & \text{if } 2 \le n \le p-1 \\ \underline{M}/\mathscr{F}^1\underline{M} & \text{if } p \le n \le 2p-1 \\ \underline{M} & \text{if } n \ge 2p \end{cases}$$

It follows that the only nonzero slices (2, p and $2p$) satisfy

$$(P_2^2(\Sigma^2 H\underline{M}))^G \cong \Sigma^2 H\left(\underline{M}(G/e)^G\right)$$
$$(P_p^p(\Sigma^2 H\underline{M}))^G \cong \Sigma H\left(\underline{M}(G/e)^G/r_e^G\underline{M}(G/G)\right)$$
$$(P_{2p}^{2p}(\Sigma^2 H\underline{M}))^G \cong \Sigma^2 H\big(ker\big(\underline{M}(G/G) \xrightarrow{r_e^G} \underline{M}(G/e)\big)\big)$$

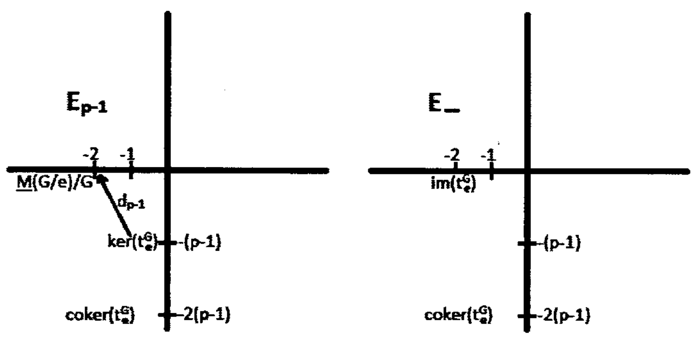
so that the RSSS for the fixed points is as below



and the slice filtration for $\underline{M}(G/G)$ reduces to

$$0 \to ker(r_e^G) \to \underline{M}(G/G) \to im(r_e^G) \to 0.$$

For $\Sigma^{-2}H\underline{M}$ a similar calculation yields the picture below,



with the slice filtration for $\underline{M}(G/G)$ reducing to

$$0 \to im(t_e^G) \to \underline{M}(G/G) \to \underline{M}(G/G)/im(t_e^G) \to 0.$$

For $p = 2$, the situation is even simpler. We have

$$\underline{\pi}_2 P^n(\Sigma^2 H\underline{M}) \cong \begin{cases} 0 & \text{if } n < 2 \\ \underline{M}/\mathscr{F}^1\underline{M} & \text{if } 2 \leq n \leq 3 \\ \underline{M} & \text{if } n \geq 4 \end{cases}$$

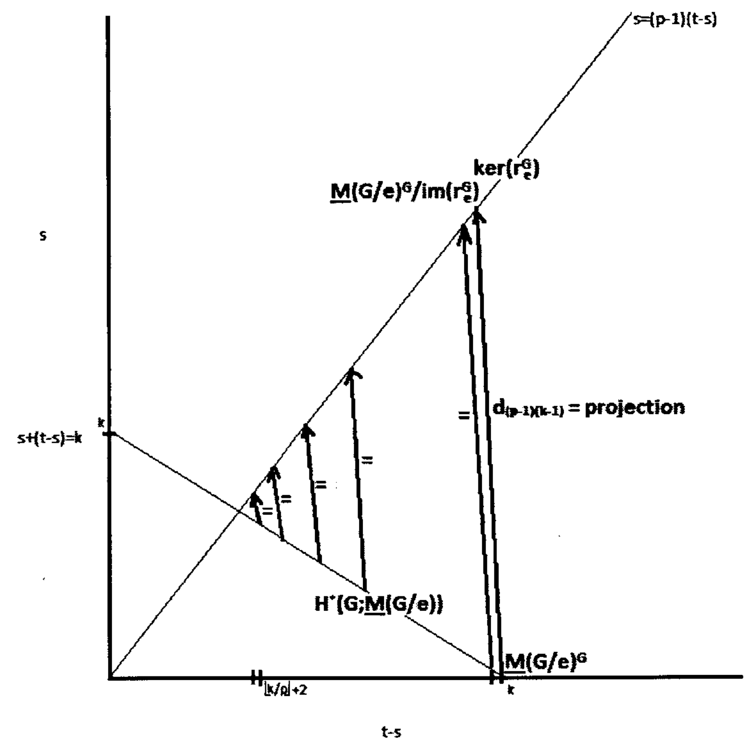
so that the only nonzero slices are

$$P_2^2(\Sigma^2 H\underline{M}) \cong \Sigma^2 H(\underline{M}/\mathscr{F}^1\underline{M})$$
$$P_4^4(\Sigma^2 H\underline{M}) \cong \Sigma^2 H\mathscr{F}^1\underline{M}$$

and there are no differentials. Dually, we have

$$P_{-2}^{-2}(\Sigma^{-2} H\underline{M}) \cong \Sigma^{-2} H\mathscr{F}_1\underline{M}$$
$$P_{-4}^{-4}(\Sigma^{-2} H\underline{M}) \cong \Sigma^{-2} H(\underline{M}/\mathscr{F}_1\underline{M})$$

with no differentials for dimension -2. Next we describe the RSSS for an Eilenberg MacLane spectrum in dimension > 2, again with G a cyclic group of prime order. This is relatively simple to do using the formulas given in Section II.3. However, it is much faster to use Corollary I.9.5. The RSSS for the fixed points is as below.



To verify this, we argue as follows. Theorem I.9.4 implies that the only groups on the E_2 page below the vanishing line are on the line $s + (t - s) = k$, and are as shown, except possibly the group in the $t - s = \lfloor \frac{k}{p} \rfloor + 1$ column. For each of these groups, there is a unique differential hitting the vanishing line, so all (but one) of the groups in the columns $t - s < k$ must annihilate in pairs, as shown. The unknown group at $t - s = \lfloor \frac{k}{p} \rfloor + 1, t = k$ must be zero, since it annihilates with a group in column $t - s = \lfloor \frac{k}{p} \rfloor$, which must be zero by a basic connectivity estimate (Proposition I.8.1). The differential emanating from $t - s = k, s = 0$ must be surjective. The rest follows from Corollary I.8.6. Of course, there is a dual description for $k < -2$, involving group homology, which we leave as an exercise for the reader.

We can now use the results of Section II.5 to compute most of the groups in the E_2 page when $G = C_{p^2}$. Again let $k > 2$, and let $\underline{M}$ be an arbitrary Mackey functor. By Corollary II.3.3 and basic connectivity estimates, the only nonzero slices of $\Sigma^k H\underline{M}$ other than the k-slice are the jp-slices for $k < jp \leq kp^2$. In fact, under the line of slope $p^2 - 1$ the homotopy groups of the fixed points of the slices are contained in the homotopy groups of

$$\left(\left(P_{jp}^{jp}(\Sigma^k H\underline{M}) \right)^{C_p} \right)^{h(G/C_p)},$$

which are zero unless $jp \leq kp$. Hence suppose that $\lfloor \frac{k}{p} \rfloor + 1 \leq j \leq k$. The homotopy fixed point spectral sequence for the above spectrum is then trivial, and so when $i \geq \lfloor \frac{j}{p} \rfloor + 2$ we obtain the following.

$$\pi_i^G P_{jp}^{jp}(\Sigma^k H\underline{M}) \cong \begin{cases} H^{j-i}\left(G/C_p; H^{k-j-1}(C_p; \underline{M}(G/e))\right) & \text{if } j < k-1 \\ H^{j-i}\left(G/C_p; \underline{M}(G/e)^{C_p}/im\left(r_e^{C_p}\right)\right) & \text{if } j = k-1 \\ H^{j-i}\left(G/C_p; ker\left(r_e^{C_p}\right)\right) & \text{if } j = k \end{cases}$$

Of course, similar formulas involving group homology can be given when $k < -2$. We will also see very similar formulas for certain free and cofree spectra in the next section, which will be derived in a very different way.

3 Example Computations for Free and Cofree Spectra

The goal of this section is to shed some light on the behavior of the RSSS outside of the region where it coincides with the HFPSS (or the HOSS). We do this by way of example computations for two types of spectra. Let $G = C_{p^2}$ (with p prime), and let $\underline{M}$ be a Mackey functor. First we let $k > 0$, and consider the spectrum

$$X := F(EG_+, \Sigma^k H\underline{M}).$$

By Corollary II.4.8, for $n > 0$ we have

$$(P_n^n X)^G \cong Cofib\big(Post_{\lceil (n+1)/p^2 \rceil}(Post_{\lceil (n+1)/p \rceil}(Post_{n+1} X)^{hC_p})^{h(G/C_p)} \to \\ Post_{\lceil n/p^2 \rceil}(Post_{\lceil n/p \rceil}(Post_n X)^{hC_p})^{h(G/C_p)}\big).$$

Now the map $\Sigma^k H\underline{M} \to X$ is a nonequivariant isomorphism, so we can replace X with $\Sigma^k H\underline{M}$ in the above formula. It follows from this formula that $(P_n^n X)^G \cong *$ when $n > k$; suppose that $0 < n < k$. Then

$$Post_{n+1}(\Sigma^k H\underline{M}) \cong Post_n(\Sigma^k H\underline{M}) \cong \Sigma^k H\underline{M}.$$

Also, the outermost Postnikov functors do not affect the homotopy groups in degrees $\geq \lfloor \frac{n}{p^2} \rfloor + 2$, so we see that in this range,

$$\pi_m^G P_n^n X \cong \pi_m Cofib\big((Post_{\lceil (n+1)/p \rceil}(\Sigma^k H\underline{M})^{hC_p})^{h(G/C_p)} \to \\ (Post_{\lceil n/p \rceil}(\Sigma^k H\underline{M})^{hC_p})^{h(G/C_p)}\big).$$

This is clearly zero unless n is divisible by p; hence suppose that $0 < n = pj < k$. Then the above reduces to

$$\pi_m^G P_n^n X \cong \pi_m \big(Post_j^j(\Sigma^k H\underline{M})^{hC_p}\big)^{h(G/C_p)}$$

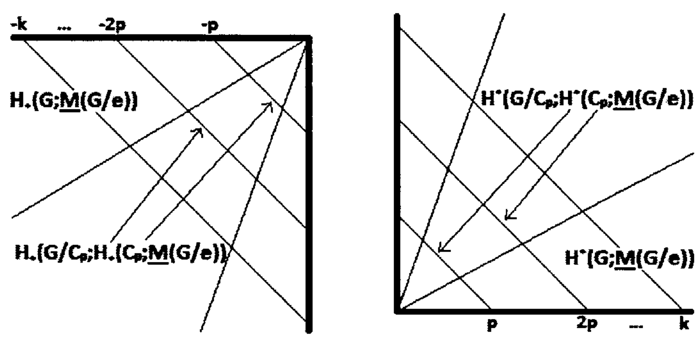
so that

$$\pi_m^G P_n^n X \cong H^{j-m}\big(G/C_p; H^{k-j}\big(C_p; \underline{M}(G/e)\big)\big) \tag{3.1}$$

for $m \geq \lfloor \frac{n}{p^2} \rfloor + 2$ and $0 < n = pj < k$. Of course, we also have

$$\pi_m^G P_k^k X \cong H^{k-m}(G; \underline{M}(G/e))$$

for $m \geq \lfloor \frac{n}{p} \rfloor + 2$ by Corollary I.9.5. Note that the newly calculated groups 3.1 are all in the region of the E_2 page between the lines of slope $p-1$ and p^2-1. We give a rough picture of our E_2 page below, along with the corresponding dual picture for $EG_+ \wedge \Sigma^{-k} H\underline{M}$, which we derive from Corollary II.4.11.



As one moves from the region below the line of slope $p-1$ to the region between this line and the line of slope p^2-1, one can observe the "phase transition" in the behavior of the groups on the E_2 page mentioned in Section II.5.

4 Dugger's Spectral Sequence for KR-Theory

In this section we give a treatment of the $K\mathbb{R}$ spectrum with its (regular) slice spectral sequence that updates the one given in [Dug]. We begin by quickly reviewing the construction of $K\mathbb{R}$. Throughout this section our group is $G = C_2$. First recall that, for a G space X, $K\mathbb{R}(X)$ is defined to be the Grothendieck group of "Real" vector bundles over X (see [Ati]). It is then easy to see that reduced $K\mathbb{R}$-theory is represented by the space $\mathbb{Z} \times BU$, where BU is constructed as usual from Grassmanians and has a G action coming from complex conjugation. Equivariantly, the usual cell structure becomes a G-CW(V) structure, with $V = \mathbb{C} \cong \mathbb{R}[G]$. Next, as is pointed out in Section 2 of [Ati], the Fourier series approach to complex Bott periodicity works essentially verbatim in the "Real" context. Thus we obtain a Bott element

$$S^{\mathbb{C}} \xrightarrow{\beta} \mathbb{Z} \times BU$$

and an associated Bott map

$$\mathbb{Z} \times BU \xrightarrow{\beta} \Omega^{\mathbb{C}}(\mathbb{Z} \times BU)$$

which is a weak equivalence. A priori, this map is only unique up to homotopy on finite subcomplexes. To show that it is unique up to homotopy, we proceed as follows. With a given choice of Bott map, we may construct an Ω-spectrum which is $\mathbb{Z} \times BU$ in each level that is a multiple of the regular representation. We call this spectrum $K\mathbb{R}$. We now filter $\mathbb{Z} \times BU$ by finite G-CW($\mathbb{C}$) complexes X_i, and examine the corresponding Milnor sequence, as below.

$$0 \to \varprojlim_i{}^1 \widetilde{K\mathbb{R}}(\Sigma X_i) \to [\mathbb{Z} \times BU, \mathbb{Z} \times BU] \to \varprojlim_i [X_i, \mathbb{Z} \times BU] \to 0$$

Thus to show uniqueness, it suffices to show that the first group above is zero. Now we may build up the X_i's by adding one cell at a time, and by Bott periodicity we have

$$\widetilde{K\mathbb{R}}(\Sigma S^{n\mathbb{C}}) \cong \pi_1(BO) \cong \mathbb{Z}/2\mathbb{Z},$$

so it is easy to see that the groups $\widetilde{K\mathbb{R}}(\Sigma X_i)$ are finite 2-groups. Thus the $\varprojlim^1$ term vanishes, since inverse systems of finite groups automatically satisfy the Mittag-Leffler condition. It follows that the Bott map, and therefore the spectrum $K\mathbb{R}$, is essentially unique. Very similar arguments can be used to give $K\mathbb{R}$ a canonical multiplication, making it a commutative ring spectrum. Then Bott periodicity can be restated as the existence of a unit β in $\pi_{\mathbb{C}}K\mathbb{R}$.

We now apply the (regular) slice spectral sequence to $K\mathbb{R}$. First we determine the slices. It is easy to determine that

$$\underline{\pi}_0 K\mathbb{R} \cong \underline{\mathbb{Z}},$$

since the groups and restrictions are obvious and the transfer is then determined by the relations that all Mackey functors satisfy. Next, by both nonequivariant and "Real" Bott periodicity we have

$$\begin{aligned}
\pi^e_{-1} K\mathbb{R} &= 0 \\
\pi^G_{-1} K\mathbb{R} &\cong [S^\sigma, BU] \\
&\cong \pi_1(BU, BO) = 0
\end{aligned}$$

where σ denotes the sign representation of G, and the equality on the third line follows by considering the structure of S^σ and the fact that $BU^G = BO$. We now obtain the slices (both regular and irregular)

$$\begin{aligned}
P^{-1}_{-1} K\mathbb{R} &\cong * \\
P^0_0 K\mathbb{R} &\cong H\underline{\mathbb{Z}}
\end{aligned}$$

from Corollary I.8.9 and Proposition I.3.1. By Bott periodicity and the periodicity of the slice filtration we then have

$$P_{2n-1}^{2n-1}K\mathbb{R} \cong *$$
$$P_{2n}^{2n}K\mathbb{R} \cong S^{n\mathbb{C}} \wedge H\mathbb{Z}$$

for all $n \in \mathbb{Z}$ (and similarly for the irregular slices). It follows that the maps

$$P_{2n}K\mathbb{R} \to P_{2n-1}K\mathbb{R}$$

are isomorphisms in both the regular and irregular contexts. Then Corollary I.3.7 implies that the regular and irregular slice constructions give the same result on $K\mathbb{R}$. We could immediately compute the groups in the E_2 page of the RSSS at this point, but we first determine the slice tower in order to understand the multiplicative structure. Since P_0 always yields the (-1)-connected cover, we let $k\mathbb{R}$ denote the connective cover of $K\mathbb{R}$, so that

$$P_0K\mathbb{R} = k\mathbb{R}.$$

Letting $n \in \mathbb{Z}$ and smashing with $S^{n\mathbb{C}}$, we obtain

$$S^{n\mathbb{C}} \wedge k\mathbb{R} \to S^{n\mathbb{C}} \wedge K\mathbb{R} \xrightarrow[\cong]{\beta} K\mathbb{R}$$

as a model for $P_{2n}K\mathbb{R}$. Then, regarding the Bott element as a map

$$S^{\mathbb{C}} \to k\mathbb{R}$$

we can complete the diagrams

$$\begin{tikzcd}
S^{n\mathbb{C}} \wedge k\mathbb{R} \arrow[r] \arrow[d, dotted] & K\mathbb{R} \arrow[d, "="] \\
S^{(n-1)\mathbb{C}} \wedge k\mathbb{R} \arrow[r] & K\mathbb{R}
\end{tikzcd}$$

with the maps

$$S^{n\mathbb{C}} \wedge k\mathbb{R} \cong S^{(n-1)\mathbb{C}} \wedge S^{\mathbb{C}} \wedge k\mathbb{R} \xrightarrow{\Sigma^{(n-1)\mathbb{C}}\beta} S^{(n-1)\mathbb{C}} \wedge k\mathbb{R}.$$

The multiplication in the tower is then given by the maps

$$\begin{aligned}(S^{n\mathbb{C}} \wedge k\mathbb{R}) \wedge (S^{m\mathbb{C}} \wedge k\mathbb{R}) &\xrightarrow[\cong]{1\wedge\tau\wedge 1} (S^{n\mathbb{C}} \wedge S^{m\mathbb{C}}) \wedge (k\mathbb{R} \wedge k\mathbb{R}) \\ &\longrightarrow S^{(n+m)\mathbb{C}} \wedge k\mathbb{R}\end{aligned}$$

where we have used the twist map and the multiplication map of $k\mathbb{R}$ above. It is now clear that the pairings on the slices are suspensions of the pairing on the 0-slice, which in turn is determined by the commutative diagram below.

$$\begin{array}{ccc} k\mathbb{R} \wedge k\mathbb{R} & \longrightarrow & k\mathbb{R} \\ \downarrow & & \downarrow \\ H\underline{\mathbb{Z}} \wedge H\underline{\mathbb{Z}} & \longrightarrow & H\underline{\mathbb{Z}} \end{array}$$

That is, the multiplication on the 0-slice is the obvious one, and it determines the pairings on all the other slices. Next we compute the groups on the E_2 page of the spectral sequence. We have

$$E_2^{s,t} \cong [S^{t-s}, S^{(t/2)\mathbb{C}} \wedge H\underline{\mathbb{Z}}]$$

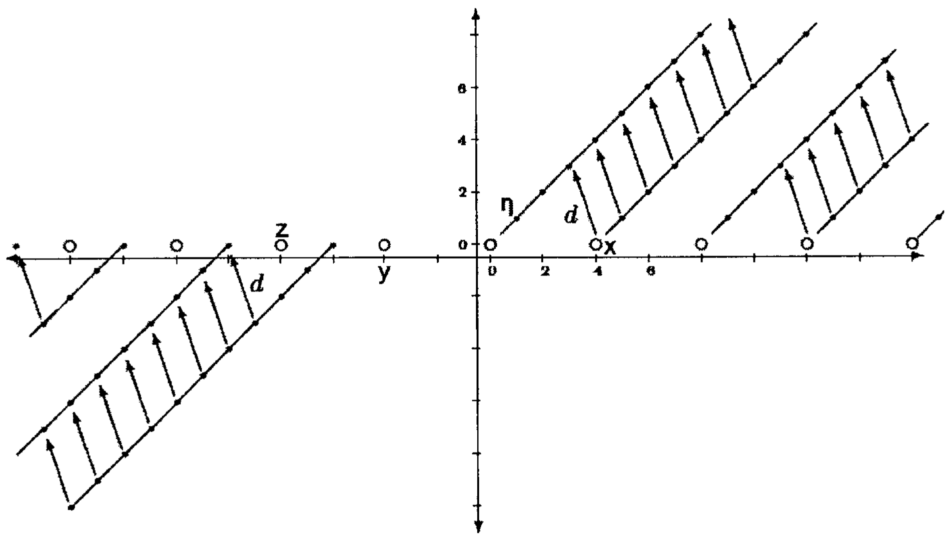
so that

$$E_2^{s,t} \cong \begin{cases} \tilde{H}_{t/2-s}(S^{(t/2)\sigma}; \underline{\mathbb{Z}}) & \text{if } s \geq 0, t-s \geq 0 \\ \tilde{H}^{s-t/2}(S^{(|t|/2)\sigma}; \underline{\mathbb{Z}}) & \text{if } s \leq 0, t-s \leq 0 \end{cases}$$

when t is even, all other groups being zero. To compute these groups, we filter $S^{n\sigma}$ by the subspaces $S^{k\sigma}$. It is easy to see that $S^{(k+1)\sigma}$ is obtained from $S^{k\sigma}$ by attaching a single free G-cell, and we obtain the (reduced) equivariant cell structure shown below (with attaching maps correct up to a sign).

$$* \leftarrow G \xleftarrow{1-g} G \xleftarrow{1+g} G \xleftarrow{1-g} \dots \xleftarrow{1-(-1)^n g} G$$

Here we use g to denote the nontrivial element in G. We obtain the following picture, which is taken from [Dug].



We have added labels for the generators of certain groups. The open circles represent copies of $\mathbb{Z}$, while the dots represent copies of $\mathbb{Z}/2\mathbb{Z}$. We will eventually deduce the pattern of differentials shown in the picture. We begin by noting that, since this spectral sequence is equal to the irregular slice spectral sequence, the map to the HFPSS in the first quadrant is an isomorphism on all the groups shown, except perhaps on the line of slope 1. Here it is a monomorphism (except possibly at $(0,0)$). We require an easy lemma, which we state without proof.

Lemma 4.1. *The nonequivariant homotopy groups of $K\mathbb{R}$ are isomorphic to $\mathbb{Z}$ in even degrees and are zero in odd degrees. The action is trivial in degrees divisible by 4, and is the sign action is degrees that are* 2 *mod* 4.

Now it is easy to see that the HFPSS has copies of $\mathbb{Z}/2\mathbb{Z}$ in these same places above the $t-s$ axis, so the map must be an isomorphism on the nonzero groups in the first quadrant (this is easily verified for $(0,0)$ as well). It follows that the first quadrant is isomorphic to the ring

$$\mathbb{Z}[x, \eta]/(2\eta).$$

The product on the $t - s$ axis in the third quadrant is given by the product in group homology. This is just the usual product on $\mathbb{Z}$ multiplied by 2, e.g. $y^2 = 2z$. Next, the map from the HOSS to the HFPSS induces the map

$$\mathbb{Z} \cong (\mathbb{Z})/G \xrightarrow{1+g} (\mathbb{Z})^G = \mathbb{Z}$$

at $t = -4n$ on the $t - s$ axis, which is multiplication by 2, so we have $xy = 2$. Also, Proposition I.9.10 implies that multiplication by x sends generators to generators on the negative $t-s$ axis. Next, to deduce the differentials, we consider the third column. It is easy to see that we have

$$\pi_3^G K\mathbb{R} \cong \pi_3 BO \cong \pi_2 O = 0$$

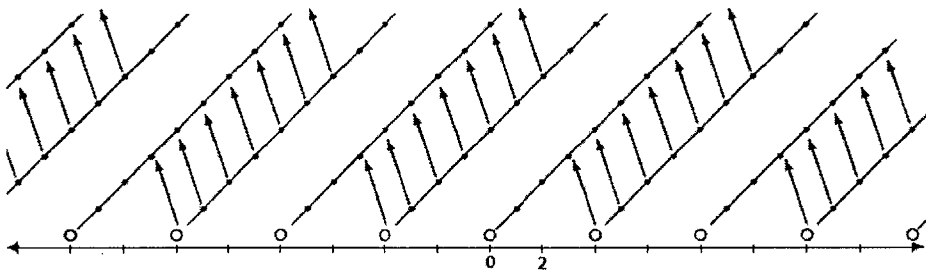
so the element η^3 must be hit by a d_3 differential as shown in the figure. The multiplicative structure then forces all of the other differentials shown in the first quadrant to exist. At this point we have already proven the space level version of real Bott periodicity (to do so, we need not have considered the third quadrant at all). To continue, we require another easy algebraic lemma, whose proof we omit.

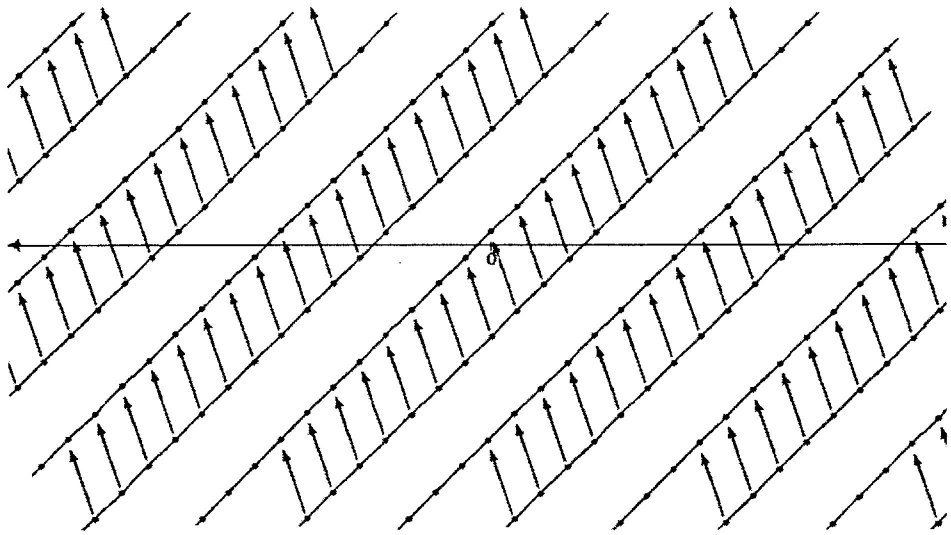
Lemma 4.2. *The Tate cohomology of G with coefficients in $\mathbb{Z}$ (with trivial action) is $\mathbb{Z}/2\mathbb{Z}$ in even degrees and zero in odd degrees. With coefficients in $\mathbb{Z}$ with sign action it is $\mathbb{Z}/2\mathbb{Z}$ in odd degrees and zero in even degrees. The evident pairings between these (nonzero) groups are all nonzero.*

Now since we have multiplicative maps of spectral sequences

$$RSSS \to HFPSS \to TSS$$

where *TSS* denotes the Tate spectral sequence (see [GM]), we can completely deduce the structures of the HFPSS and the TSS. They are as shown below.





Next, since we have a shifted map of spectral sequences $TSS \to HOSS$ (commuting with the differentials up to a sign of (-1)) which is an isomorphism on the E_2 page below $s = -1$ and a monomorphism on $s = -1$, we immediately obtain all of the other differentials shown in the figure. We can also finish the determination of the product structure on the E_2 page, as follows. The solid lines in the figure for the TSS are η-towers, and multiplication by the image of x is an isomorphism on the E_2 page, so multiplication by either x or η is an isomorphism when it maps one copy of $\mathbb{Z}/2\mathbb{Z}$ to another in the third quadrant of the RSSS, by Proposition I.9.11. The multiplicative structure is now completely determined. To finish the computation of $K\mathbb{R}_*$ it remains to resolve the extensions in degrees $-8n$. For these groups there are two possibilities: the extension is nontrivial and the group is isomorphic to $\mathbb{Z}$, or the group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. To prove that the former holds, it suffices to show that these groups are infinite cyclic. For this we argue as follows. Let λ denote the element of $\pi_8^G K\mathbb{R}$ represented by x^2 (nonequivariantly, this is β^4), and consider the diagram shown below.

$$\begin{tikzcd} k\mathbb{R} \arrow[r] \arrow[d] & K\mathbb{R} \\ k\mathbb{R}[\lambda^{-1}] & \end{tikzcd}$$

Since multiplication by λ induces isomorphisms

$$\lambda\cdot : \underline{\pi}_n K\mathbb{R} \xrightarrow{\cong} \underline{\pi}_{n+8} K\mathbb{R}$$

when $n \geq 0$, it is clear that both maps are connective covers. It follows that there is a natural isomorphism

$$[X, K\mathbb{R}] \cong [X, k\mathbb{R}[\lambda^{-1}]]$$

for X connective. Then for $n > 0$ we have isomorphisms

$$\begin{aligned}
[S^{-8n}, K\mathbb{R}] &\cong [S^{8n\sigma}, K\mathbb{R}] \\
&\cong [S^{8n\sigma}, k\mathbb{R}[\lambda^{-1}]] \\
&\cong [S^{8n+8n\sigma}, k\mathbb{R}[\lambda^{-1}]] \\
&\cong [S^{8n\mathbb{C}}, K\mathbb{R}] \\
&\cong [S^0, K\mathbb{R}] \cong \mathbb{Z},
\end{aligned}$$

where we have used "Real" Bott periodicity and the fact that $k\mathbb{R}[\lambda^{-1}]$ is 8-periodic. It follows that the extensions on the E_∞ page are nontrivial. In particular, there exists an element $w \in \pi_{-8}^G K\mathbb{R}$ such that $2w$ is represented by z. Then since $x^2 z = xy = 2$ we have $\lambda w = 1$; that is, λ is a unit. Letting ν denote the element of $\pi_4^G K\mathbb{R}$ represented by $2x$, we have the following isomorphism,

$$\pi_*^G K\mathbb{R} \cong \mathbb{Z}[\eta, \nu, \lambda, \lambda^{-1}]/(2\eta, \eta^3, \eta\nu, \nu^2 - 4\lambda)$$

with the restriction map $r_e^G : \pi_*^G K\mathbb{R} \to \pi_*^e K\mathbb{R}$ given by the following.

$$\lambda \mapsto \beta^4$$
$$\eta \mapsto 0$$
$$\nu \mapsto 2\beta^2$$

Now the homotopy ring of $(K\mathbb{R})^{hG}$ has the same description, with the generators of $\pi_*^G K\mathbb{R}$ mapping to the corresponding generators of $\pi_*(K\mathbb{R})^{hG}$, and the Tate spectrum of $K\mathbb{R}$ is trivial, so we have isomorphisms

$$(K\mathbb{R})_{hG} \cong (K\mathbb{R})^G \cong (K\mathbb{R})^{hG}.$$

Hence, $K\mathbb{R}$ is both free and cofree, and its G-fixed point spectrum is equivalent to the homotopy fixed point spectrum of complex K-theory with conjugation action (i.e. the real K-theory spectrum).

We conclude this section by using the results of Chapter IV to compute some Toda brackets in $\pi_*^G K\mathbb{R} \cong KO_*$. On the E_3 page of the RSSS, for any $m \in \mathbb{Z}$ we have $2m \cdot \eta = 0$ and $\eta \cdot \eta^2 = dx$. There are no crossing differentials, and the E_4 page is the E_∞ page, so we see that all elements of the set

$$2mx + 2m \cdot 2x\mathbb{Z}$$

are permanent cycles and converge to representatives of $\langle 2m, \eta, \eta^2 \rangle$. However, the indeterminacy of this Toda bracket is precisely $2m\nu\mathbb{Z}$, so in KO_* we have

$$\langle 2m, \eta, \eta^2 \rangle = (1 + 2\mathbb{Z})m\nu.$$

Next, consider the Massey product $\langle 2, \eta, 2, \eta \rangle$. It is easy to compute that this Massey product is $(1 + 2\mathbb{Z})x$ in the HFPSS, and hence in the RSSS by Proposition IV.9.5. Then by Theorem IV.9.11 there is an element $y \in E_2^{2,4}(K\mathbb{R})$ such that y converges to an element of $\langle 2, \eta, 2\rangle$ and

$$d_3 x = \eta y.$$

Thus we have $\langle 2, \eta, 2\rangle = \{\eta^2\}$. Finally, we show how to use Massey products to resolve the extension in $\pi^G_{-8} K\mathbb{R}$. Suppose the extension is trivial, and let w denote the nonzero element in $E_2^{-2,-10}$. By our assumption, w converges to multiples of z plus an element of order 2. It follows that if w converges to u, then $\eta^2 u\lambda = 0$. Hence, if we let t denote the nonzero element in $E_2^{0,-6}(K\mathbb{R})$, it suffices to show that $tx^2 = 0$ does not hold in $K\mathbb{R}_*$, even though it holds in the RSSS. For this we argue as follows. Abusing notation slightly, we have by Theorem IV.9.11 that

$$d_3\langle \eta, t, x^2\rangle = \eta(tx^2).$$

Since the indeterminacy of the above Massey product is precisely $2\mathbb{Z}x$, it then suffices to show that $\langle \eta, t, x^2\rangle$ maps to the nonzero element in that degree of the TSS. For this, we claim that the image of $\langle \eta, t, x^2\rangle$ in the TSS is actually just the product of η, x^2 and the Tate class corresponding to t. We regard the Tate construction as the cofiber $EG_+ \wedge (-) \to F(EG_+, -)$. We can regard the class t as coming from a homology class $t' : S_c^{-6} \to EG_+ \wedge P_{-6}^{-6} K\mathbb{R}$. Then since $\eta t' \simeq 0$, we may choose a null-homotopy of ηt coming from a null homotopy of $\eta t'$. If we use such a null-homotopy to construct an element of $\langle \eta, t, x^2\rangle$, then after coning off $EG_+ \wedge P_4^4 K\mathbb{R}$ we may replace it by the null-homotopy that simply travels up the cone at constant speed. We then obtain a map which is η times the Tate class corresponding to the product of t and x^2 induced by the pairing of the HOSS with the RSSS, which is the Tate product $\eta^{-1}x$.

5 Computations for Equivariant K-Theory

In this section we determine the slice tower for the equivariant complex K-theory spectrum KU_G, when $G = C_n$ is cyclic. First we consider the real and complex representations of G. Since G is abelian, the complex irreducibles are all one-dimensional. In fact, we can describe them as follows. Fix a generator g of G, and let σ denote the representation such that

$$g \mapsto e^{2\pi i/n}.$$

Then the irreducible complex representations of G form a cyclic group of order n generated by σ. Thus we have

$$\mathbb{C}[G] \cong \mathbb{C} + \sigma + \sigma^2 + ... + \sigma^{n-1}.$$

As real representations we have $\sigma^j \cong \sigma^{-j}$, while if n is even then $\sigma^{n/2}$ is the complexification of a real representation. Denote this real representation (the "sign" representation) by s. It follows easily that the real regular representation is as below.

$$\mathbb{R}[G] \cong \begin{cases} \mathbb{R} + \sigma + ... + \sigma^{(n-1)/2} & \text{if } n \text{ is odd} \\ \mathbb{R} + \sigma + ... + \sigma^{n/2-1} + s & \text{if } n \text{ is even} \end{cases}$$

The homotopy groups of KU_G are easily determined. We have $\underline{\pi}_0 KU_G \cong \underline{RU}$, the complex representation ring Mackey functor. That is, $\underline{RU}(G/H) = RU(H)$ is the Grothendieck ring of finite-dimensional complex H-representations. Of course, we have

$$RU(H) \cong \mathbb{Z}[\sigma_H]/(\sigma_H^{|H|} - 1).$$

The restrictions are induced by restriction of representations, while the transfer maps are induced by induction. If $H = C_m$ and we choose $g^{n/m}$ as our generator for H, then we have $i_H^* \sigma_G = \sigma_H$. Now KU_G has Bott elements (units) for each irreducible complex representation. Denote by β_j the Bott element

$$S^{\sigma^j} \to KU_G$$

for σ^j, so that the composite

$$S^0 \to S^{\sigma^j} \to KU_G$$

represents the element $1 - \sigma^j \in RU(G)$. Now it is easy to see that $\underline{\pi}_1 KU_G = 0$, so since we have a Bott isomorphism $S^{\mathbb{C}} \wedge KU_G \cong KU_G$ it follows that KU_G satisfies

$$\underline{\pi}_{2m} KU_G \cong \underline{RU}$$
$$\underline{\pi}_{2m+1} KU_G = 0$$

for all $m \in \mathbb{Z}$. Letting ku_G denote the connective cover of KU_G, we have the following theorem, which essentially says that the slice tower expresses KU_G as the localization of ku_G at the collection of Bott elements.

Theorem 5.1. *For $G = C_n$ and all $j \in \mathbb{Z}$ we have*

$$P_{2j} KU_G \cong P_{2j-1} KU_G \cong S^{\mathbb{C}+\sigma+\ldots+\sigma^{j-1}} \wedge ku_G$$

with the maps $P_{2j}KU_G \to KU_G$ given by multiplication by the appropriate Bott elements, and the maps $P_{2j}KU_G \to P_{2j-2}KU_G$ given by

$$S^{\mathbb{C}+\sigma+\ldots+\sigma^{j-1}} \wedge ku_G \cong S^{\mathbb{C}+\sigma+\ldots+\sigma^{(j-1)-1}} \wedge S^{\sigma^{j-1}} \wedge ku_G \xrightarrow{\Sigma^{\mathbb{C}+\sigma+\ldots+\sigma^{(j-1)-1}}(\beta_{j-1}\cdot)} S^{\mathbb{C}+\sigma+\ldots+\sigma^{(j-1)-1}} \wedge ku_G$$

Proof. We proceed by induction on the order of G; the result is trivial for the trivial group. Now multiplication by all the Bott elements gives an isomorphism

$$S^{\mathbb{C}[G]} \wedge KU_G \xrightarrow{\cong} KU_G,$$

so by the periodicity of the slice filtration it suffices to determine $P_l KU_G$ for l such that $0 \leq l \leq 2n-1$. Of course, we have $P_0 KU_G = ku_G$. Hence, choose j such that $1 \leq j \leq n$. First we prove that

$$S^{\mathbb{C}+\sigma+\ldots+\sigma^{j-1}} \wedge ku_G \quad \geq \quad 2j.$$

Since the above spectrum is clearly 0-connected, and is $\geq 2j$ when restricted to any proper subgroup by the induction hypothesis, it suffices by Theorem I.8.10 to show that π_k^G of this spectrum is zero when $0 < k < 2j/|G|$. However, we have $2j/|G| \leq 2$ so this condition is either vacuous or reduces to the case $k=1$. Since the spectrum in question is clearly 1-connected, it follows that it is $\geq 2j$. Of course, it is then also $\geq 2j-1$. We will be done if we can show that the cofiber of the map

$$S^{\mathbb{C}+\sigma+\ldots+\sigma^{j-1}} \wedge ku_G \to KU_G$$

is $< 2j-1$, since it is then also $< 2j$, so consider the cofiber sequence below.

$$S^{\mathbb{C}+\sigma+\ldots+\sigma^{j-1}} \wedge ku_G \to KU_G \to C \to \Sigma(S^{\mathbb{C}+\sigma+\ldots+\sigma^{j-1}} \wedge ku_G) \to \Sigma KU_G$$

We have that the restriction of C to any proper subgroup is $< 2j-1$ by the inductive hypothesis, so it suffices to show that

$$[S^{m\rho_G}, C] = 0$$

when $m|G| \geq 2j-1$. By the above cofiber sequence it suffices to show that

$$[S^{m\rho_G}, S^{\mathbb{C}+\sigma+\ldots+\sigma^{j-1}} \wedge ku_G] \to [S^{m\rho_G}, KU_G]$$

is surjective and that

$$[S^{m\rho_G-1}, S^{\mathbb{C}+\sigma+\ldots+\sigma^{j-1}} \wedge ku_G] \to [S^{m\rho_G-1}, KU_G]$$

is injective. First we suppose that $m \geq 2$. Then $m\rho_G \supseteq \mathbb{C}[G]$, so there is a representation V such that the above maps are isomorphic to the maps

$$[S^V, ku_G] \to [S^V, KU_G]$$
$$[S^{V-1}, ku_G] \to [S^{V-1}, KU_G]$$

which are isomorphisms since ku_G is the (-2)-connected cover of KU_G. Finally, suppose that $m = 1$. Then we have by assumption $|G| \geq 2j - 1$, which implies that $j - 1 \leq (|G| - 1)/2$ and hence $\rho_G \supseteq \mathbb{R} + \sigma + ... + \sigma^{j-1}$. Thus there is a representation V such that the maps in question are isomorphic to the maps shown below.

$$[S^{V-1}, ku_G] \to [S^{V-1}, KU_G]$$
$$[S^{V-2}, ku_G] \to [S^{V-2}, KU_G]$$

The first map above is an isomorphism since ku_G is the (-2)-connected cover of KU_G, while the second map is injective for the same reason. □

Remark: Since the restriction maps of $\underline{RU}$ are not injective, the irregular and regular 0-slices are not equal. Thus the SSS is NOT the same as the RSSS for KU_G.

We next obtain the slices.

Corollary 5.2. *If $G = C_n$ then the odd slices of KU_G are zero. Also, for any $j \in \mathbb{Z}$ we have the following.*

$$P_{2j}^{2j} KU_G \cong S^{\mathbb{C}+\sigma+...+\sigma^{j-1}} \wedge H(\underline{RU}/(1 - \sigma^j))$$

Proof. The first statement is immediate, since $P_{2n}KU_G \xrightarrow{\cong} P_{2n-1}KU_G$. For the second statement, by the periodicity of the slice tower, we may assume that $0 \leq j < |G|$. Then Theorem 5.1 gives the $2j$-slice as

$$S^{\mathbb{C}+\sigma+...+\sigma^{j-1}} \wedge Cofib(S^{\sigma^j} \wedge ku_G \to ku_G),$$

so we must show that

$$Cofib(S^{\sigma^j} \wedge ku_G \to ku_G) \cong H(\underline{RU}/(1-\sigma^j)).$$

For this we utilize the long exact sequence of homotopy groups and the commutative diagram below.

$$\begin{tikzcd} S^{\sigma^j} \wedge ku_G \arrow[r] \arrow[d] & ku_G \arrow[d] \\ S^{\sigma^j} \wedge KU_G \arrow[r, "\cong"] & KU_G \end{tikzcd}$$

Since ku_G is the (-2)-connected cover of KU_G, we obtain an isomorphism on the left and right sides of the above square after applying the functor $[X, -]$ when X is 0-connected. If X is (-1)-connected we obtain an isomorphism on the right side and a monomorphism on the left side. Applying this to sphere spectra, we see that the map $S^{\sigma^j} \wedge ku_G \to ku_G$ is an isomorphism on $\underline{\pi}_k$ for $k > 0$ and a monomorphism on $\underline{\pi}_0$. It follows that

$$Cofib(S^{\sigma^j} \wedge ku_G \to ku_G) \cong H\big(coker\big(\underline{\pi}_0(S^{\sigma^j} \wedge ku_G) \to \underline{\pi}_0(ku_G)\big)\big).$$

Now the map $S^0 \wedge ku_G \to S^{\sigma^j} \wedge ku_G$ is surjective on $\underline{\pi}_0$, and the composite

$$ku_G \cong S^0 \wedge ku_G \to S^{\sigma^j} \wedge ku_G \to ku_G$$

is multiplication by the composite

$$S^0 \to S^{\sigma^j} \xrightarrow{\beta_j} ku_G.$$

This element of $RU(G)$ is $1 - \sigma^j$, so the result follows. □

Next we let $G = C_p$ with p prime, and describe the first quadrant in the RSSS. Most of these groups coincide with the corresponding groups in the HFPSS. Hence we must compute only the groups near the vanishing line. First we consider odd p.

The only groups that can be on the vanishing line occur when the number of the slice is divisible by $|G| = p$. Since the odd slices are zero, we consider even multiples of p. Then the group on the vanishing line coming from the $2jp$-slice is

$$\pi_{2j}^G(S^{j\mathbb{C}[G]} \wedge H\underline{RU}) \cong \pi_0^G(S^{j(\mathbb{C}[G]-\mathbb{C})} \wedge H\underline{RU})$$

for all $j > 0$. Since $\mathbb{C}[G] - \mathbb{C} \cong \sigma + ... + \sigma^{p-1}$ has zero fixed points, it is easy to see that this is isomorphic to

$$\begin{aligned} coker\big(t_e^G : RU(e) \to RU(G)\big) &\cong coker\big(\mathbb{Z} \xrightarrow{1+\sigma+...+\sigma^{p-1}} \mathbb{Z}[\sigma]/(1-\sigma^p)\big) \\ &\cong \mathbb{Z}^{p-1}. \end{aligned}$$

We also know that π_{2j+1}^G is zero, since it is contained in an odd-degree cohomology group of G with coefficients in $\mathbb{Z}$. If the number of the slice is $2j$ with j not divisible by p, then σ^j generates the character group so we have

$$\underline{RU}/(1-\sigma^j) \cong \underline{\mathbb{Z}}$$

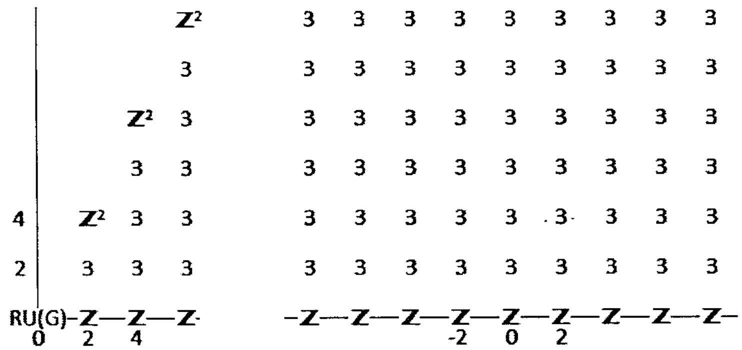
and the first nonzero homotopy group reduces as before to the cokernel of the transfer, which is the cokernel of

$$\mathbb{Z} \xrightarrow{p} \mathbb{Z},$$

and hence is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, when $j > 1$. When $j = 1$ we have

$$\pi_2^G(S^{\mathbb{C}} \wedge H\underline{\mathbb{Z}}) \cong \mathbb{Z}.$$

The first quadrant of the RSSS and the entire HFPSS for $p = 3$ are illustrated below. (In the figure, '3' represents a copy of $\mathbb{Z}/3\mathbb{Z}$.)



There can be no nonzero differentials in either spectral sequence (at least in the first quadrant, in the case of the RSSS) because of the way the groups are spaced. The result for $p = 2$ is the same, though the argument is slightly different. When $p = 2$ we have the following,

$$P^{4j}_{4j} KU_G \cong S^{j\mathbb{C}[G]} \wedge H\underline{RU}$$
$$P^{4j+2}_{4j+2} KU_G \cong S^{j\mathbb{C}[G]+\mathbb{C}} \wedge H\underline{\mathbb{Z}}$$

so that $\pi^G_{2j+1}(P^{4j+2}_{4j+2} KU_G) = 0$. The other calculations are the same as in the odd case. Finally, we give the slice filtration for the positive homotopy groups of KU_G when $G = C_p$.

Proposition 5.3. *If $G = C_p$ and $m > 0$ then for $0 < s \leq m(p-1)$ we have*

$$F^{2s}\pi^G_{2m} KU_G = F^{2s-1}\pi^G_{2m} KU_G = \beta^m I^s,$$

where $I \subseteq RU(G)$ is the augmentation ideal.

Proof. For $s = m(p-1)$ this is easy to deduce from the explicit form of the slice tower and the fact that, if j is not divisible by p, then $1 - \sigma^j$ generates I. Now by Corollary I.8.8 we know that $F^1\pi^G_{2m} KU_G$ is the kernel of the restriction map $r^G_e : RU(G) \to \mathbb{Z}$, which is I. We also know from the form of the slice tower that $F^{2s}\pi^G_{2m} KU_G = F^{2s-1}\pi^G_{2m} KU_G$ for any s. Next, there is a Bott element $\beta \in E^{0,2}_2$ such that multiplication by β induces an isomorphism

$$\beta\cdot : E_2^{a,b} \xrightarrow{\cong} E_2^{a,b+2}$$

when $b > a$ and $a < (p-1)(b-a)$. It follows that, if $q > 0$ and $a \leq 2q(p-1)$ then

$$\beta^r(F^a\pi^G_{2q}KU_G) = F^a\pi^G_{2(q+r)}KU_G$$

for any $r > 0$. Now we have $F^2\pi^G_2 KU_G = \beta I$, so we obtain

$$\beta^s I^s \subseteq F^{2s}\pi_{2s}KU_G.$$

Choosing $N \geq s, m$ and combining the above facts we have

$$\beta^N I^s \subseteq \beta^{N-s}F^{2s}\pi^G_{2s}KU_G = F^{2s}\pi^G_{2N}KU_G = \beta^{N-m}F^{2s}\pi^G_{2m}KU_G,$$

which immediately implies that

$$\beta^m I^s \subseteq F^{2s}\pi^G_{2m}KU_G.$$

Now we have equality when $s = 1$, so the result follows by induction on s since we have $I^2 = pI$ and thus the filtration quotients at positive s are isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for both filtrations. □

The deduction of the Atiyah-Segal completion theorem for C_p is now quite simple. Since KU_G is *split* (see [May1]), we have the following isomorphism.

$$KU^*(BG) \cong \pi_{-*}(KU_G)^{hG}$$

Now there are no differentials in the homotopy fixed point spectral sequence for KU_G, so it converges strongly to these groups. Since we have a Bott element in $E_2^{0,2}$ we can identify all of the even homotopy groups of $(KU_G)^{hG}$, together with their filtrations coming from the HFPSS, which we denote by F^s. We have the following.

$$\begin{aligned} KU^{2m+1}(BG) &= 0 \\ KU^{2m}(BG) &\cong \varprojlim_s \pi_{-2m}(KU_G)^{hG}/F^s\pi_{-2m}(KU_G)^{hG} \end{aligned}$$

Now we know by the results of this section that the maps

$$\pi^G_{2m}KU_G/F^{2m(p-1)}\pi^G_{2m}KU_G \to \pi_{2m}(KU_G)^{hG}/F^{2m(p-1)}\pi_{2m}(KU_G)^{hG}$$

are isomorphisms when $m > 0$. Thus we have commutative diagrams

$$\begin{tikzcd} \beta^{m+1}RU(G)/\beta^{m+1}I^{(m+1)(p-1)} \arrow[r, "\cong"] \arrow[d, "\beta^{-1}\cdot"'] & \pi_{2(m+1)}(KU_G)^{hG}/F^{2(m+1)(p-1)}\pi_{2(m+1)}(KU_G)^{hG} \arrow[d, "\beta^{-1}\cdot"] \\ \beta^m RU(G)/\beta^m I^{m(p-1)} \arrow[r, "\cong"] & \pi_{2m}(KU_G)^{hG}/F^{2m(p-1)}\pi_{2m}(KU_G)^{hG} \end{tikzcd}$$

and hence an isomorphism as below.

$$\varprojlim_s RU(G)/I^s \xrightarrow{\cong} \pi_0(KU_G)^{hG}$$

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Appendix A

Miscellany

1 Transfinite Filtrations

In many places in the present work we consider transfinite sequences of maps. We always assume that a transfinite sequence $\{X_\alpha\}$ satisfies the condition

$$X_\alpha = \varinjlim_{\beta<\alpha} X_\beta$$

when α is a limit element. We prefer to think of arbitrary well-ordered indexing sets, instead of just ordinals, since one can then freely expand a single map into another well-ordered sequence, as in the following. There are many situations where one has pushout diagrams of the form shown below.

$$\begin{tikzcd} \coprod_\gamma A_\gamma \arrow[r, "\coprod_\gamma i_\gamma"] \arrow[d] & \coprod_\gamma B_\gamma \arrow[d] \\ X_\alpha \arrow[r] & X_{\alpha+1} \end{tikzcd}$$

This pushout may be hopelessly complicated to analyze, so we perform the following, purely categorical trick. Well-order the indexing set of the coproduct (the γ's). Then the pushout $X_{\alpha+1}$ can be expressed as a well-ordered colimit of a sequence of pushouts of the individual maps i_γ. This trick is alluded to in the proof of Proposition IV.4.7 and used implicity thereafter.

2 A Cube Lemma

There are several places in Chapter IV where we show that a certain map, obtained as a colimit of a map of transfinite sequences, is an h-cofibration. Our method is as follows. Suppose we are given a map of transfinite sequences, as below.

$$\begin{tikzcd}
\cdots \arrow[r] & X_\alpha \arrow[r] \arrow[d] & X_{\alpha+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & Y_\alpha \arrow[r] & Y_{\alpha+1} \arrow[r] & \cdots
\end{tikzcd}$$

Now being an h-cofibration is equivalent to having the LLP with respect to certain maps, so to show that the colimit map is an h-cofibration it suffices to show that

- $X_0 \to Y_0$ is an h-cofibration, and
- the pushout product maps $X_{\alpha+1} \cup_{X_\alpha} Y_\alpha \to Y_{\alpha+1}$ are h-cofibrations.

Now the maps $X_\alpha \to X_{\alpha+1}$ and $Y_\alpha \to Y_{\alpha+1}$ are often pushouts themselves, so we utilize the following lemma, which is valid in all cocomplete categories.

Lemma 2.1. *Suppose given a commutative diagram as below.*

$$\begin{tikzcd}
A \arrow[rr, "i"] \arrow[dr, "j"'] \arrow[dd] & & B \arrow[dr] \arrow[dd] & \\
& A' \arrow[rr] \arrow[dd] & & B' \arrow[dd] \\
C \arrow[rr, "k"] \arrow[dr, "l"'] & & D \arrow[dr] & \\
& C' \arrow[rr] & & D'
\end{tikzcd}$$

If the front and back faces are pushouts, then so is the diagram below.

$$\begin{tikzcd}
A' \cup_A B \arrow[r, "i \square j"] \arrow[d] & B' \arrow[d] \\
C' \cup_C D \arrow[r, "k \square l"] & D'
\end{tikzcd}$$

3 Change of Ring Functors

In this section we prove the change of ring isomorphism cited in Section III.3. Let $f : k \to k'$ be a weak equivalence of commutative ring S_G-modules. We have a pair of adjoint functors, called *restriction* and *extension of scalars*, as shown below.

$$assoc_k \underset{k' \wedge_k (\,-\,)}{\overset{forget}{\rightleftarrows}} assoc_{k'} \tag{3.1}$$

Theorem 3.2. *The change of ring functors in 3.1 form a Quillen equivalence.*

Proof. It is trivial that these functors form a Quillen pair, and that the right adjoint creates the weak equivalences on $assoc_{k'}$. Thus, let X be a cofibrant object of $assoc_k$; it suffices to prove that the unit of the adjunction

$$X \to k' \wedge_k X$$

is a weak equivalence (see, for example, the Appendix of [MMSS]). Now the initial k-algebra (that is, k) is not a cofibrant k-module, but it clearly suffices to prove that the map $k \to X$ is a q-cofibration of k-modules. For this we may assume that X is a $k\mathbb{A}(I)$-cell. Thus, X can be given a transfinite filtration $\{X_\alpha\}$ such that $X_0 = k$ and each $X_{\alpha+1}$ can be written as the colimit of a sequence of k-modules Y_i, with $Y_0 = X_\alpha$ and pushout diagrams as below,

$$\begin{tikzcd} X_\alpha^{\wedge i+1} \wedge (\partial_A B^{\wedge i}) \arrow[r] \arrow[d] & X_\alpha^{\wedge i+1} \wedge B^{\wedge i} \arrow[d] \\ Y_{i-1} \arrow[r] & Y_i \end{tikzcd}$$

where $A \to B$ is some generating q-cofibration of k-modules (and the smash products are taken over k). It is now easy to prove by transfinite induction, using Lemma 2.1, that smashing with X_α preserves q-cofibrations of k-modules. Applying transfinite induction again, we see that all of the above pushouts are q-cofibrations of k-modules. $\square$

Remark: The fact that the unit map is a cofibration of k modules is a slight strengthening of Theorem 4.1(3) of [SS].

Remark: Similar theorems for modules and commutative algebras also hold. Using Lemma IV.3.3, they are actually easier to prove than the above theorem; however, we do not need them in the present work.

4 The S Model Structures

At some key points in Chapters I and IV we require some facts about an alternative model structure on equivariant orthogonal spectra introduced in [Sto]. These are called $\mathbb{S}$ *model structures.* They are defined by pulling back the classical model structures from all subgroups. Hence, if we let $I(G)$ and $J(G)$ denote the classical generating cofibrations and acyclic cofibrations, respectively (and $I(G)^+$, $J(G)^+$ the positive generators), then the sets

$$\{G_+ \wedge_H I(H) : H \subseteq G\}$$
$$\{G_+ \wedge_H J(H) : H \subseteq G\}$$

are the generating cofibrations and acyclic cofibrations, respectively, for the $\mathbb{S}$ model structure for G (similarly for the positive versions). In this section we record some of the basic facts about these model structures that are not apparent to the author from [Sto] or [MM].

First of all, we must prove that these model structures actually exist. This is taken care of by the following two lemmas.

Lemma 4.1. *If X is a retract of a $\{G_+ \wedge_H I(H) : H \subseteq G\}$-cell, then the underlying prespectrum of X is cofibrant. That is, $X(V)$ is G-cofibrant for each representation V and the structure maps of X are G-cofibrations.*

We omit the proof, which easily reduces by Lemma 2.1 to the case $X = S^{-V}$.

Lemma 4.2. *If $X \to Y$ is a weak equivalence of H-spectra, with $H \subseteq G$ and X and Y retracts of $\{H_+ \wedge_K I(K) : K \subseteq H\}$-cells, then the induced map*

$$G_+ \wedge_H X \to G_+ \wedge_H Y$$

is a weak equivalence.

Proof. Take cofibrant replacements $X' \to X$ and $Y' \to Y$ in the classical level model structure corresponding to the levels that are restrictions of G-representations, and complete the diagram below.

$$\begin{tikzcd} X' \arrow[r, dotted] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Since the vertical maps are weak equivalences in each level which is a restriction of a G-representation, and all of these spectra are levelwise cofibrant by Lemma 4.1, the vertical maps are homotopy equivalences in these levels. Hence, after applying induction they are level homotopy equivalences. Furthermore, after applying induction the top horizontal map is a weak equivalence since induction is a left Quillen functor on the model structure corresponding to the levels that are restrictions of G-representations. Hence, after induction the bottom horizontal map is also a weak equivalence. $\square$

Of course, once the existence of the model structures is established, it follows immediately that induction and restriction functors form Quillen pairs, and it is also clear that restriction preserves (acyclic) cofibrations. There is no longer any need to be careful about which representations one is indexing on. Of course, the identity functor (with itself) forms a Quillen equivalence relating the classical model structure to the $\mathbb{S}$ model structure. Similar remarks apply for the positive version.

To prove that the $\mathbb{S}$ model structure is monoidal, and pull it back to a model structure on k-algebras (k a commutative ring spectrum), we require the following lemma.

Lemma 4.3. *If X is $\mathbb{S}$-cofibrant then X is flat.*

Proof. This is Proposition 2.3.29 of [Sto]; however, the author had difficulty following the argument there. We proceed as in the proof of Lemma IV.3.3(i), using Spanier Whitehead duality. First let Y be an arbitrary orthogonal G-spectrum, and let $\mathcal{U}$ be a complete G-universe. Then since $\mathcal{U}$ is also a complete H-universe for any subgroup H of G, it is clear that the map

$$Y \to \operatorname*{hocolim}_{V \in \mathcal{U}} F(S^{-V} \wedge S^V, Y)$$

is a fibrant replacement in the $\mathbb{S}$ model structure. Thus, if X is compact and $\mathbb{S}$-cofibrant, then maps from X to Y in the homotopy category can be computed as

$$\varinjlim_{V} [S^{-V} \wedge S^V \wedge X, Y].$$

With this complication, the rest of the proof follows the pattern of the proof of Lemma IV.3.3(i). $\square$

Next, recall the Quillen pair from [MM], shown below.

$$Sp_G^{\mathcal{O}} \underset{\mathbb{N}}{\overset{\mathbb{N}^{\#}}{\rightleftarrows}} \mathcal{M}_G$$

It is easy to see that the right adjoint $\mathbb{N}^{\#}$ commutes with restriction; hence, the left adjoint $\mathbb{N}$ commutes with induction. Since induction preserves cofibrations of S_G-modules, and a weak equivalence of cofibrant S_G-modules is a homotopy equivalence, we obtain the following.

Lemma 4.4. *The pair of functors*

$$Sp_G^{\mathcal{O}} \underset{\mathbb{N}}{\overset{\mathbb{N}^{\#}}{\rightleftarrows}} \mathcal{M}_G$$

is a Quillen equivalence relating the positive $\mathbb{S}$ model structure to the q model structure on S_G-modules.

Corollary 4.5. *If X is positive $\mathbb{S}$-cofibrant then the unit map $X \to \mathbb{N}^{\#}\mathbb{N}X$ is a weak equivalence.*

Finally, we seek to pull back the positive $\mathbb{S}$ model structure to commutative algebra categories. For this we introduce the following considerations. First note that in fact, evaluating an orthogonal G-spectrum X in level V defines a functor

$$Sp_G^{\mathcal{O}} \to (O(V) \rtimes G)Top_*$$

with a left adjoint that we denote by $\mathcal{G}_V$; a spectrum of the form $\mathcal{G}_V X$ is called *semifree*. In [Sto], the author shows that the $\mathbb{S}$-cofibrations can also be generated by maps of the form

$$\mathcal{G}_{\mathbb{R}^n}\big((O(\mathbb{R}^n) \rtimes G)/H \times (S^{d-1} \to D^d)\big)_+ \tag{4.6}$$

for subgroups H of $O(\mathbb{R}^n) \rtimes G$ such that $H \cap O(\mathbb{R}^n) = \{1\}$. Now let $\mathcal{F}_G[i]$ denote the family of subgroups of $G \times \Sigma_i$ that have trivial intersection with Σ_i, and let $E_G\Sigma_i$ be the universal space for this family. We have the following.

Lemma 4.7. *[Lemma 2.3.34 of [Sto]] Let Y be an orthogonal G-spectrum and let $X = \mathcal{G}_{\mathbb{R}^n}\big((O(\mathbb{R}^n) \rtimes G/H)_+ \wedge K\big)$ for some based CW complex K, $n > 0$ and some subgroup H of $O(\mathbb{R}^n) \rtimes G$ such that $H \cap O(\mathbb{R}^n) = \{1\}$ and the H-representation defined by the projection $H \to O(\mathbb{R}^n)$ has a nonzero fixed vector. Then the quotient map*

$$(E_G\Sigma_{i+} \wedge_{\Sigma_i} X^i) \wedge Y \to (X^i/\Sigma_i) \wedge Y$$

is a weak equivalence.

Corollary 4.8. *If X is positive $\mathbb{S}$-cofibrant, then the quotient map*

$$E_G\Sigma_{i+} \wedge_{\Sigma_i} X^i \to X^i/\Sigma_i$$

is a weak equivalence.

Proof. By using the generating cofibrations 4.6, and using the fact that the restriction of $E_G\Sigma_i$ to $G \times \Sigma_{i_1} \times ... \times \Sigma_{i_k}$ is homotopy equivalent to $E_G\Sigma_{i_1} \times ... \times E_G\Sigma_{i_k}$, this easily reduces to proving that a smash product of these maps is a weak equivalence when the X's are of the form $\mathcal{G}_{\mathbb{R}^n}\big((O(\mathbb{R}^n) \rtimes G/H)_+ \wedge K\big)$ as in Lemma 4.7. The result follows by writing such a smash product as a composite of maps involving one smash factor at a time and applying Lemma 4.7 to each. $\square$

Corollary 4.9. *If X is positive $\mathbb{S}$-cofibrant and weakly contractible then any symmetric power of X is weakly contractible.*

Proof. By Corollary 4.8 it suffices to show that $E_G\Sigma_{i+} \wedge_{\Sigma_i} X^i$ is weakly contractible. It is not hard to see that this spectrum is positive $\mathbb{S}$-cofibrant, and thus by Corollary 4.5 it suffices to show that

$$\mathbb{N}(E_G\Sigma_{i+} \wedge_{\Sigma_i} X^i) \cong E_G\Sigma_{i+} \wedge_{\Sigma_i} \mathbb{N}(X)^i$$

is contractible. This is immediate, since $\mathbb{N}(X)$ is contractible by Corollary 4.5. $\square$

The remaining fact one needs to prove the model structures on commutative algebras is the following.

Corollary 4.10. *If B is positive $\mathbb{S}$-cofibrant then symmetric powers of B are flat.*

Proof. As before, by using the generators 4.6 this reduces to the case where B is of the form $\mathcal{G}_{\mathbb{R}^n}\big((O(\mathbb{R}^n) \rtimes G/H)_+ \wedge K\big)$ as in Lemma 4.7, which further reduces us to showing that $E_G\Sigma_{i+} \wedge_{\Sigma_i} B^i$ is flat in this case. However, this last spectrum is positive $\mathbb{S}$-cofibrant, so the result follows by Lemma 4.3. $\square$

Of course, once the model structures on commutative algebras have been constructed, one can show that they are related by Quillen equivalences to the classical model structures (which can be proven to exist as a corollary of the above). Furthermore, it is immediate that the pair of functors

$$comm_H \underset{N_H^G}{\overset{\mathrm{Res}_H^G}{\rightleftarrows}} comm_G$$

is a Quillen pair relating these $\mathbb{S}$ model structures, where N_H^G is the norm functor of [HHR]. Classically, this pair is only a Quillen pair when one uses the model structure on $comm_H$ derived from the levels which are restrictions of G-representations. Finally, we remark that the above results can be used to prove all of the standard change of ring isomorphisms relative to the $\mathbb{S}$ model structures, and that $(\mathbb{N}, \mathbb{N}^{\#})$ is a Quillen equivalence relating the $\mathbb{S}$ model structure on commutative rings to the model category of commutative ring S_G-modules.

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