Enumerative Algebraic Geometry via Techniques of Symplectic Topology and Analysis of Local Obstructions

by

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Abstract

Enumerative geometry of algebraic varieties is a fascinating field of mathematics that dates back to the nineteenth century. We introduce new computational tools into this field that are motivated by recent progress in symplectic topology and its influence on enumerative geometry. The most straightforward applications of the methods developed are to enumeration of rational curves with a cusp of specified nature in projective spaces. A general approach for counting positive-genus curves with a fixed complex structure is also presented. The applications described include enumeration of rational curves with a (3,4)-cusp, genus-two and genus-three curves with a fixed complex structure in the two-dimensional complex projective space, and genus-two curves with a fixed complex structure in the three-dimensional complex projective space. Our constructions may be applicable to problems in symplectic topology as well.

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Contents

Chapter 1

Introduction

For more than a hundred years, tools of algebraic geometry had been the dominant force behind progress in enumerative geometry of algebraic varieties. More recently, developments in symplectic topology have lead to solutions of many fundamental problems in this field and to the introduction of new concepts into algebraic geometry itself. In this chapter, we give examples of classical problems in enumerative geometry, describe relations of recent advances in symplectic topology to enumerative geometry, outline our approach to a large class of enumerative questions, and present solutions to some of them.

1.1 Background

Enumerative algebraic geometry is an area of mathematics that attempts to determine the number of geometric objects that satisfy pre-specified geometric conditions. Typically, the objects are (complex) curves in a smooth algebraic manifold. The curves to be counted are usually required to determine a given homology class and pass through a collection of subvarieties. Often one also specifies the singularities each curve is to have and some data, such as the complex structure, about the normalization of each curve. Perhaps, the most classical example of an enumerative problem is

Question 1 *If d is a positive integer, what is the number n_d of degree-d rational curves that pass through $3d-1$ points in general position in the complex projective plane $\mathbb{P}^2$?*

Since the number of (complex) lines through any two distinct points is one, $n_1=1$. A little bit of algebraic geometry and topology gives $n_2=1$ and $n_3=12$. It is far harder to find that $n_4=620$, but this number was computed as early as the middle of the nineteenth century. The higher-degree numbers remained unknown until the early 1990s, when Kontsevich announced a recursive formula for the numbers n_d. A version of this formula can be found at the end of Section 11.3.

The recursive relation announced by Kontsevich is deduced in [KM] from the associativity property of multiplication in the quantum cohomology of $\mathbb{P}^2$. Quantum cohomology of symplectic manifolds originated in two fields very different from algebraic geometry. In his seminal paper [G], Gromov initiated the study of pseudoholomorphic curves in symplectic manifolds and demonstrated their usefulness by proving the Symplectic Non-Squeezing Theorem and obtaining a number of other important results. On the other hand, physicists had long worked with correlation functions of topological field theories, and Gromov's work suggested a natural setting for such functions. They predicted the associativity property for quantum multiplication, which was then used in [KM] to derive the recursive formula.

Around the same time, pseudoholomorphic maps were used to rigorously define new invariants of symplectic manifolds, including quantum cohomology. The crucial fact needed to show the associativity of quantum multiplication is a composition law for the behavior of pseudoholomorphic maps. In [RT], two composition laws are proved for maps from arbitrary-genus complex curves into positive symplectic manifolds. These two composition laws insure that all such invariants, with fixed complex structure, can be computed from the ordinary cohomology of the space and the simplest possible genus-zero invariants. For example, in projective spaces, these invariants are determined by the numbers of lines passing through various collections of linear subspaces (i.e. points, lines, etc.) in general position. The latter can be computed using Schubert calculus. It is also shown in [RT] that the genus-zero symplectic invariants in $\mathbb{P}^n$ agree with the enumerative invariants such as those of Question 1. Thus, [RT] gives a complete solution to Question 1 and its generalization to higher-dimensional projective spaces:

Question 2 *Suppose $n \geq 2$, d, and N are positive integers, and $\mu = (\mu_1, \ldots, \mu_N)$ is an N-tuple of proper subvarieties of $\mathbb{P}^n$ in general position such that*

$$\mathit{codim}_{\mathbb{C}}\mu \equiv \sum_{l=1}^{l=N} \mu_l = d(n+1) + n - 3 + N. \tag{1.1}$$

What is the number $n_d(\mu)$ of degree-d rational curves that pass through the subvarieties $\mu_1, \ldots, \mu_N$?

Condition (1.1) is necessary to insure that the expected answer is finite and not clearly zero. For easy geometric reasons, it is sufficient to solve Question 2, as well as other similar questions, for tuples μ of linear subspaces of $\mathbb{P}^n$ of codimension at least two. Thus, in our computations in later chapters we assume that the constraints μ for $\mathbb{P}^2$ are points and for $\mathbb{P}^3$ are points and lines.

Following the work of [G] and [K], moduli spaces of stable maps into algebraic manifolds became objects of much research in algebraic geometry. Algebraic geometers usually denote by $\overline{\mathfrak{M}}_{g,N}(\mathbb{P}^n, d)$ the stable-map compactification of the space of equivalence class of degree-d holomorphic maps from genus-g Riemann surfaces with N marked points. These spaces are described as algebraic stacks in [FP]. While their cohomology is not entirely understood, it is shown in [P2] that the intersections of certain tautological cohomology classes in $\overline{\mathfrak{M}}_{0,N}(\mathbb{P}^n, d)$ can be computed. The computability of some of these intersections is essential for applicability of the methods described in this thesis to enumerative geometry.

1.2 Counting Rational Curves with Singularities

A slight variation on Question 1 is

Question 3 *If d is a positive integer, what is the number $|S_1(\mu)|$ of degree-d rational curves that have a cusp and pass through a tuple μ of $3d-2$ points in general position in $\mathbb{P}^2$?*

Because of the constraints imposed, the cusp of each such curve will necessarily be a simple, or $(2,3)$, cusp. In general, if $1<k<m$, point p of curve $\mathcal{C}$ in $\mathbb{P}^2$ is called a (k,m)-cusp of $\mathcal{C}$ if there exists a parameterization of $\mathcal{C}$ near p of the form

$$t \longrightarrow \big(t^k, t^m + o(t^m)\big), \qquad 0 \longrightarrow p,$$

for some choice of local coordinates near p.

It is easy to see that the first three numbers of Question 3 are 0, 0, and 24. Algebraic geometers have also obtained a general formula for these numbers; see [P2] or [V1]. However, their methods are not easily generalizable to curves with higher-order cusps or to higher-dimensional projective spaces. In this thesis, we prove

Proposition 1.1 *If d is a positive integer, the number of rational degree-d curves that pass through a tuple μ of $3d-4$ points in general position in $\mathbb{P}^2$ and have a $(3,4)$-cusp is given by*

$$\begin{aligned}\big|\mathcal{S}_{1;2}(\mu)\big| =& \big\langle 33a^2c_1^2(\mathcal{L}^*) + 18ac_1^3(\mathcal{L}^*) + 4c_1^4(\mathcal{L}^*), [\bar{\mathcal{V}}_1(\mu)]\big\rangle + 3\big|\mathcal{V}_3(\mu)\big| \\ &- \big\langle 21a^2 + 9a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big) + 2\big(c_1^2(\mathcal{L}_1^*)+c_1^2(\mathcal{L}_2^*)\big) + c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*), [\bar{\mathcal{V}}_2(\mu)]\big\rangle\end{aligned}$$

d	2	3	4	5	6	7
$\lvert\mathcal{S}_{1;2}(\mu)\rvert$	0	0	147	54,612	23,177,124	14,617,373,280

It is easy to see that the degree-two and -three numbers should be zero. The degree-four number given above agrees with the one computed by P. Aluffi based on [AF2]. The spaces $\mathcal{V}_k(\mu)$ of Proposition 1.1 consist of equivalence classes of rational maps of total degree-d that have pass through the $3d-4$ points. The top cohomology intersections of Proposition 1.1 are closely related to intersections of tautological classes in $\overline{\mathfrak{M}}_{0,N}(\mathbb{P}^2,d)$. In particular, they are computable. An algorithm for computing these numbers is described in Section 6.10. The spaces $\mathcal{V}_k(\mu)$ and the relevant cohomology classes are defined in Section 6.3, using the notation described in Chapter 2.

Proposition 1.1 is a fairly straightforward application of the methods developed in this thesis. These methods can be used to enumerate rational curves with specified singularities of "local" nature, i.e. every component of the normalization of each curve contains at most one point of the preimage of the singular points. In this definition, we do not consider simple nodes of a plane curve to be singularities. Singularities of local nature include cusps of any type. On the other hand, these methods cannot be used to enumerate irreducible curves that have a tacnode or two cusps. However, we can enumerate rational two-component curves that meet at a tacnode. Lemma 6.4 counts such curves in $\mathbb{P}^3$. The dimension of the projective space is not an obstruction to applying these methods. In fact, they should be adaptable to some other algebraic varieties, such the complex homogeneous manifolds and del Pezzo surfaces. However, intersections of tautological classes on moduli of stable maps into such manifolds are not understood yet.

We now outline our approach to counting rational curves with singularities based on the example of Question 3. Denote by $\mathcal{V}_1(\mu)$ the space of equivalence classes of degree-d maps from S^2 with $3d-1$ marked points such that $3d-2$ of the marked points are mapped to the designated points in $\mathbb{P}^2$. Let $\hat{0}$ be the remaining point. The number of Question 3 is the cardinality of the set $\mathcal{S}_1(\mu)$ consisting of the maps in $\mathcal{V}_1(\mu)$ with the differential vanishing at $\hat{0}$. This is the zero set of section $\mathcal{D}$ of bundle $L^*\otimes \mathrm{ev}^*T\mathbb{P}^2$. Here L^* is the universal cotangent bundle at $\hat{0}$, whose chern class is closely related to $c_1(\mathcal{L}^*)$. The evaluation map $\mathrm{ev}\colon \mathcal{V}_1(\mu)\longrightarrow\mathbb{P}^2$ sends each element of $\mathcal{V}_1(\mu)$ to its value at $\hat{0}$. The section $\mathcal{D}$ extends over $\bar{\mathcal{V}}_1(\mu)$ and is transversal to zero over $\mathcal{V}_1(\mu)$. Thus, the cardinality of $\mathcal{S}_1(\mu)=\mathcal{D}^{-1}\cap\mathcal{V}_1(\mu)$ is euler class of $L^*\otimes\mathrm{ev}^*T\mathbb{P}^2$ minus the contribution to the euler class from the boundary of $\mathcal{V}_1(\mu)$, i.e. $\bar{\mathcal{V}}_1(\mu)-\mathcal{V}_1(\mu)$. What this means and how this contribution can be computed in good cases is discussed in Chapter 5.

If we are to have any hope of computing the $\mathcal{D}$-contribution of $\partial\bar{\mathcal{V}}_1(\mu)$ to the euler class of $L^*\otimes\mathrm{ev}^*T\mathbb{P}^2$, we need to understand the behavior of $\mathcal{D}$ near $\partial\bar{\mathcal{V}}_1(\mu)$. This behavior is described by the analytic estimate of Theorem 6.2, which is a consequence of Theorem 3.33. Theorem 3.33 is obtained by gluing holomorphic maps into $\mathbb{P}^n$, or an arbitrary Kahler manifold, at the singular points of the domains. Unlike standard gluing constructions, our pregluing step involves only the domains of the maps, and not the target. This pregluing step is one of the key ingredients in deriving the analytic estimate of Theorem 6.2. About half of the derivation argument is actually contained in Section 7.3. The other key ingredient is the construction of $\bar{\mathcal{V}}_1(\mu)$ as an S^1-quotient with associated line bundle L. Once the total space of the S^1-bundle is constructed, it is easy to define $\mathcal{D}$; see Sections 2.6 and 6.2.

Proposition 1.1 is proved by viewing $\mathcal{S}_{1;2}(\mu)$ as the set of zeros of a bundle section $\mathcal{D}^{(2)}$ over $\bar{\mathcal{S}}_1(\mu)$ that lie in $\mathcal{S}_1(\mu)$. In the given case, $\mathcal{S}_1(\mu)$ is two-dimensional over $\mathbb{C}$. The behavior of $\mathcal{D}^{(2)}$ near $\partial\bar{\mathcal{S}}_1(\mu)$ can be deduced from the formulas that appear in the proof of Theorem 6.2. The argument can be extended to higher-order cusps and higher-dimensional projective spaces. The applications collected in Chapter 6 are the ones needed in Chapters 9 and 10.

1.3 Counting Curves with Complex Structure Fixed

There are two natural versions of Question 1 for curves of positive genus in projective spaces:

Question 4 *Suppose g, $n\geq 2$, d, and N are positive numbers and μ is an N-tuple of subvarieties of $\mathbb{P}^n$ in general position such that*

$$\mathrm{codim}_{\mathbb{C}}\mu = d(n+1)-(n-3)(g-1)+N.$$

What is the number $\tilde{n}_{g,d}(\mu)$ of genus-g degree-d curves in $\mathbb{P}^n$ that pass through the constraints μ.

Question 5 *Suppose g, $n\geq 2$, d, and N are positive numbers and μ is an N-tuple of subvarieties of $\mathbb{P}^n$ in general position such that*

$$\mathrm{codim}_{\mathbb{C}}\mu = \begin{cases} d(n+1)-n(g-1)+N, & \text{if } g\geq 2\\ d(n+1)-n(g-1)+N-1, & \text{if } g=1.\end{cases}$$

What is the number $n_{g,d}(\mu)$ of genus-g degree-d curves in $\mathbb{P}^n$ that have a given generic complex structure on the normalization and pass through the constraints μ.

The $n=2$ case of Question 5 has been worked out via algebraic methods in [CH]. A beautifully short solution to the $n=2$, $g=1$ case of Question 4 via a degeneration argument is given in [P1].

However, an approach to Question 4 that lies much closer to symplectic topology seems to hold a great promise of wide applications. As mentioned in Section 1.1, the symplectic invariant $\mathrm{RT}_{g,d}(\cdot;\cdot)$ is computable for all projective spaces and

$$n_d(\mu) = \mathrm{RT}_{0,d}(\mu_1, \mu_2, \mu_3; \mu_4, \ldots, \mu_N).$$

It is shown in [I] that the difference

$$\mathrm{RT}_{1,d}(\mu_1; \mu_2, \ldots, \mu_{3d-1}) - 2n_{1,d}(\mu)$$

can be expressed in terms of intersections of tautological classes on the spaces $\bar{\mathcal{V}}_k(\mu)$, or equivalently in terms of the numbers $n_{d'}(\mu')$ with $d' \leq d$ and constraints μ' related to μ. In this thesis, we prove

Theorem 1.2 *If* $n_{2,d} = n_{2,d}\big((3d-2)\ pts\big)$ *in* $\mathbb{P}^2$,

$$n_{2,d} = 3(d^2-1)n_d + \frac{1}{2}\sum_{d_1+d_2=d}\Big(d_1^2d_2^2 + 28 - 16\frac{9d_1d_2-1}{3d-2}\Big)\binom{3d-2}{3d_1-1}d_1d_2n_{d_1}n_{d_2}.$$

d	1	2	3	4	5	6	7
$n_{2,d}$	0	0	0	14,400	6,350,400	3,931,128,000	3,718,909,209,600

Theorem 1.3 *If d is a positive integer and μ is a tuple of p points and q lines in general position in $\mathbb{P}^3$ with $2p+q=4d-3$,*

$$2n_{2,d}(\mu) = RT_{2,d}(\cdot;\mu) - CR_2(\mu), \qquad \textit{where}$$

$$\frac{1}{2}CR_2(\mu) = \big\langle 480a^3c_1(\mathcal{L}^*) + 476a^2c_1^2(\mathcal{L}^*) + 240ac_1^3(\mathcal{L}^*) + 49c_1^4(\mathcal{L}^*), \bar{\mathcal{V}}_1(\mu)\big\rangle + 36\big|\mathcal{V}_3(\mu)\big|$$
$$- \big\langle 324a^2 + 144a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big) + 27\big(c_1^2(\mathcal{L}_1^*)+c_1^2(\mathcal{L}_2^*)\big) + 25c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*), \bar{\mathcal{V}}_2(\mu)\big\rangle.$$

degree	4			5		6
(p,q)	(3,7)	(2,9)	(1,11)	(8,1)	(0,17)	(10,1)
$n_{2,d}(\mu)$	14,400	307,200	4,748,160	9,600	7,494,574,433,280	1,301,760

Theorem 1.4 *If d is a positive integer and μ is a tuple of $3d-4$ points in general position in $\mathbb{P}^2$,*

$$n_{3,d} = RT_{3,d}(\cdot;\mu) - CR_3(\mu), \qquad \textit{where}$$

$$\frac{1}{12}CR_3(\mu) = \big\langle 413a^2c_1^2(\mathcal{L}^*) + 210ac_1^3(\mathcal{L}^*) + 44c_1^4(\mathcal{L}^*), [\bar{\mathcal{V}}_1(\mu)]\big\rangle + 18|\bar{\mathcal{V}}_3(\mu)|$$
$$-\big\langle 217a^2 + 84a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big) + 16\big(c_1^2(\mathcal{L}_1^*)+c_1^2(\mathcal{L}_2^*)\big) + 10c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*), [\bar{\mathcal{V}}_2(\mu)]\big\rangle.$$

d	2	3	4	5	6	7
$n_{3,d}$	0	0	14,280	9,469,152	6,573,686,112	6,289,178,278,656

The formulas of Theorems 1.2 and 1.3 produce numbers that satisfy all the classical checks the author is aware of. They are described in detail at the end of Section 9.6. We mention here only the most interesting one. For fairly simple reasons, the number $n_{2,4}(\mu)$ for 3 points and 7 lines in $\mathbb{P}^3$ must be the same as the number $n_{2,4}$ in $\mathbb{P}^2$. Our numbers satisfy this property. The formula of Theorem 1.2 contradicts that of [KQR], which was obtained by a degeneration argument similar to that of [P1] in the genus-one case. In Chapter 11, we correct the errors in [KQR] and recover the formula of Theorem 1.2 via the same degeneration argument. As for the genus-three numbers, $n_{3,2}$ and $n_{3,3}$ must be zero. Our number $n_{3,4}$ agrees with that of [AF1]. Theorem 1.4 is stated for a generic complex structure on the normalization Σ, i.e. Σ is not hyperelleptic, has no hyperflexes under the canonical embedding into $\mathbb{P}^2$, and its automorphism group is trivial. If Σ is not hyperelleptic, but has n hyperflexes, the corresponding fixed-complex-structure number is $\big(n_{3,d}-2n|\mathcal{S}_{1;2}(\mu)|\big)/\big|\text{Aut}(\Sigma)\big|$; see Chapter 10 for more details.

We now outline the proof of these theorems. Let (Σ, j_Σ) be a nonsingular generic Riemann surface of genus $g\geq 2$. Denote by $\mathcal{H}_{\Sigma,d}(\mathbb{P}^n)$ the set of simple holomorphic maps from Σ to $\mathbb{P}^n$ of degree d. With notation as above, let

$$\mathcal{H}_{\Sigma,d}(\mu) = \big\{(y_1,\ldots,y_N;u)\colon u\in\mathcal{H}_{\Sigma,d}(\mathbb{P}^n);\ y_l\in\Sigma,\ u(y_l)\in\mu_l\ \forall l=1,\ldots,N\big\}.$$

The cardinality of this set is precisely $\big|\text{Aut}(\Sigma)\big|n_{g,d}(\mu)$. If

$$\nu \in \Gamma\big(\Sigma\times\mathbb{P}^n;\Lambda^{0,1}\pi_\Sigma^*T^*\Sigma\otimes\pi_{\mathbb{P}^n}^*T\mathbb{P}^n\big),$$

let $\mathcal{M}_{\Sigma,\nu,d}$ denote the set of all smooth maps u from Σ to $\mathbb{P}^n$ of degree d such that $\bar{\partial}u|_z=\nu|_{(z,u(z))}$ for all $z\in\Sigma$. If μ is as above, put

$$\mathcal{M}_{\Sigma,\nu,d}(\mu) = \big\{(y_1,\ldots,y_N;u)\colon u\in\mathcal{M}_{\Sigma,\nu,d};\ y_l\in\Sigma,\ u(y_l)\in\mu_l\ \forall l=1,\ldots,N\big\}.$$

For a generic ν, $\mathcal{M}_{\Sigma,\nu,d}$ is a smooth finite-dimensional oriented manifold, and $\mathcal{M}_{\Sigma,\nu,d}(\mu)$ is a zero-dimensional finite submanifold of $\mathcal{M}_{\Sigma,\nu,d}\times\Sigma^N$, whose cardinality (with sign) depends only the homology classes of $\mu_1,\ldots,\mu_N$; see [RT]. The symplectic invariant $\text{RT}_{g,d}(;\mu)$ is the signed cardinality of the set $\mathcal{M}_{\Sigma,\nu,d}(\mu)$.

If $t>0$ is very small, all elements of $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$ lie either near $\mathcal{H}_{\Sigma,d}(\mu)$, the space of holomorphic maps from Σ with a tree of spheres attached, or the space of multiply-covered maps from Σ. The maps in these spaces must have degree d and must pass through the constraints μ. The space of such multiply-covered maps from Σ is empty in the relevant cases. There is a natural bijection between the elements of $\mathcal{H}_{\Sigma,d}(\mu)$ and the nearby elements of $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$. We denote the number of elements of $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$ near the space of maps with singular domains by $CR_g(\mu)$. Since the signed cardinality of $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$ is always $RT_{g,d}(;\mu)$, the three theorems are really statements about the number $CR_g(\mu)$, the contribution of spaces of holomorphic maps with singular domains to $RT_{g,d}(;\mu)$; see Section 9.6 regarding the first theorem.

In order to compute $CR_g(\mu)$, we partition such spaces of maps into smooth manifolds $\mathcal{M}_{\mathcal{T}}(\mu)$. For each $b \in \mathcal{M}_{\mathcal{T}}$ and every sufficiently small element v of the bundle of gluing parameters $F^{\emptyset}\mathcal{T}$, we first construct a nearly holomorphic map $u_v : \Sigma \longrightarrow \mathbb{P}^n$. We then attempt to solve the equation

$$\bar{\partial} \exp_{u_v} \xi = t\nu \Longleftrightarrow \bar{\partial} u_v + D_v \xi + N_v \xi = t\nu \in \Gamma^{0,1}(u_v) \tag{1.2}$$

for a small vector field $\xi \in \Gamma(u_v)$ along u_v. Since we need to count the number of elements of $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$ that lie nearly $\mathcal{M}_{\mathcal{T}}(\mu)$, we require that ξ lie in a subspace $\tilde{\Gamma}_+(v)$ complementary to the "tangent bundle" $\Gamma_-(v)$ of the space $\{u_v : v \in F^{\emptyset}\mathcal{T}_\delta\}$. The cokernel of the operator D_b induces an orthogonal splitting

$$\Gamma^{0,1}(v) = \Gamma_-^{0,1}(v) \oplus \Gamma_+^{0,1}(v)$$

such that $\pi_+^{0,1} \circ D_v : \tilde{\Gamma}_+(v) \longrightarrow \Gamma_+^{0,1}(v)$ is an isomorphism and $\Gamma_-^{0,1}(v)$ is isomorphic to the cokernel of the operator D_b. If v and t are sufficiently small, depending on b, the $\pi_+^{0,1}$-part of equation (1.2) has a unique solution ξ_v, which also solves the entire equation (1.2) if and only if

$$\pi_-^{0,1}\big(t\nu - \bar{\partial} u_v - D_v \xi_v - N_v \xi_v\big) = 0 \in \Gamma_-^{0,1}(u_v) \approx \text{coker } D_b. \tag{1.3}$$

The bundle coker D_b is then the obstruction bundle for solving equation (1.2) in the sense of [T], and equation (1.3) describes the obstruction. It remains to adjust for the constraints and extract the leading-order term from (1.3). The latter part depends on the choices of the above splittings of $\Gamma(v)$ and $\Gamma^{0,1}(v)$. In order to extract the leading-order term from (1.3), we need the composite $\pi_-^{0,1} \circ D_v$ to be sufficiently small on $\tilde{\Gamma}_+(v)$. Allowable choices are discussed in Chapter 3, and choices that are good for our purposes are described in Sections 7.2, 8.1, and 10.1. Using Proposition 7.6 and Section 5.1, we then reduce the problem of counting of the zeros (1.2) to the problem of counting the zeros of affine maps between finite-rank vector bundles over fairly simple compact spaces. The latter problem is addressed in Section 5.3.

The purely analytic parts of this thesis, i.e. Chapters 3 and 4, are stated for arbitrary Kahler manifolds. This slight generalization causes no additional difficulties. In fact, the general arguments should extend to arbitrary symplectic manifolds with fairly minor additional complications. Thus, it is hoped that some parts of this thesis will find applications to problems in symplectic topology itself.

Chapter 2

Spaces of Stable Maps

While there is a very good understanding of what constitutes a stable map, there is little in a way of commonly accepted notation for stable maps and various spaces of stable maps. In this chapter, we recall the definition of bubble and stable maps as well as set up analytically and computationally convenient notation. Our notation for bubble maps evolved from that of [Mc]. We also define various spaces of bubble maps and restate the definition of the Gromov topology on the set of all bubble maps in our notation.

Section 2.1 describes the most frequently used abbreviations and identifications. Sections 2.2-2.4 set up notation for prestable curves with a fixed complex structure on the principal components. The fundamental part of our modification to the pregluing step of standard gluing constructions for pseudoholomorphic maps is introduced in Section 2.3. Its description involves only prestable curves. Section 2.5 describes bubble maps into a manifold V and defines a topology on the space of such tuples. In Section 2.6, we define various spaces of (pseudo)holomorphic bubble and stable maps into an almost complex manifold (V,J). Some of the definitions in that section require assumptions about the energy of holomorphic maps $S^2 \longrightarrow V$. These assumptions are satisfied if (V,ω,J) is a symplectic manifold with a compatible almost complex structure; see the end of Section 2.1 for more details.

2.1 Notation

Definition 2.1 *A finite partially ordered set I is a* <u>*linearly ordered set*</u> *if for all $i_1,i_2,h\in I$ such that $i_1,i_2<h$, either $i_1\le i_2$ or $i_2\le i_1$.*
A linearly ordered set I is a <u>*rooted tree*</u> *if I has a unique minimal element, i.e. there exists $\hat{0}\in I$ such that $\hat{0}\le h$ for all $h\in I$.*
If I and I' are linearly ordered sets, bijection $\phi: I \longrightarrow I'$ is an <u>*isomorphism of linearly ordered sets*</u> *if for all $h,i\in I$, $i<h$ if and only if $\phi(i)<\phi(h)$.*

Let I be a linearly ordered set. Denote the subset of the non-minimal elements of I by $\hat{I}$, i.e.

$$\hat{I} = \{h\in I: i<h \text{ for some } i\in I\}.$$

For every $h\in\hat{I}$, the set $\{i\in I: i<h\}$ has a unique maximal element ι_h, i.e.

$$\iota_h<h \quad\text{and}\quad i\le\iota_h \text{ for all } i\in I \text{ s.t. } i<h.$$

For reasons clarified in Section 2.2, $\iota\colon \hat{I} \longrightarrow I$ will be called the attaching map of I. It is clear from Definition 2.1 that I has a unique splitting $I = \bigsqcup_{k\in K} I_k$ such that $I_k \subset I$ is a rooted tree. The attaching map of I restricts to the attaching map of each I_k, which will still be denoted by ι.

Let I be a rooted tree. We denote the unique minimal element of I by $\hat{0}_I$, or simply by $\hat{0}$ if there is no ambiguity. If I^*, I_*, and $I*$ are rooted trees, we will write $\hat{I}^*$, $\hat{I}_*$, and $\hat{I}*$ for $\widehat{I^*}$, $\widehat{I_*}$, and $\widehat{I*}$, respectively; here $*$ denotes any string of symbols. If $i\in I$, let

$$D_iI = \{h\in I\colon h>i\}, \qquad \bar{D}_iI = D_iI \cup \{i\}.$$

Every rooted tree I has a number of subsets that are rooted trees; the subsets $\bar{D}_iI$ are one example. If H is a subset of I, the set

$$I^{(H)} \equiv \{i\in I\colon i\not> h \ \forall h\in H\}$$

is also a rooted tree. If $i\in I$, denote $I^{(\{i\})}$ by $I^{(i)}$. If H is a subset of $\hat{I}$, let

$$I^H = \{i\in I\colon i\not\geq h \ \forall h\in H\}, \quad I(H) = H\cup\{\hat{0}_I\}.$$

If $h\in\hat{I}$, denote $I^{\{h\}}$ by I^h.

If M_1 and M_2 are two sets, let M_1+M_2 be the disjoint union of M_1 and M_2. Finally, if N is a nonnegative integer, let $[N]=\{1,\ldots,N\}$.

We now introduce some analytic notation. Let $\beta\colon \mathbb{R}\longrightarrow[0,1]$ be a smooth function such that

$$\beta(t) = \begin{cases} 0, & \text{if } t\leq 1; \\ 1, & \text{if } t\geq 2, \end{cases} \qquad \text{and} \qquad \beta'(t)>0 \quad \text{if } t\in(1,2). \tag{2.1}$$

If $r>0$, let $\beta_r\in C^\infty(\mathbb{R};\mathbb{R})$ be given by $\beta_r(t)=\beta(r^{-\frac{1}{2}}t)$. Note that

$$\operatorname{supp}(\beta_r) = [r^{\frac{1}{2}}, 2r^{\frac{1}{2}}], \quad \|\beta_r'\|_{C^0} \leq C_\beta r^{-\frac{1}{2}}, \quad \text{and} \quad \|\beta_r''\|_{C^0} \leq C_\beta r^{-1}. \tag{2.2}$$

Throughout this thesis, β and β_r will refer to these smooth bump functions.

Let $q_N, q_S\colon \mathbb{C}\longrightarrow S^2\subset\mathbb{R}^3$ be the stereographic projections mapping the origin in $\mathbb{C}$ to the north and south poles, respectively. Explicitly,

$$q_N(z) = \Big(\frac{2z}{1+|z|^2}, \frac{1-|z|^2}{1+|z|^2}\Big) \in \mathbb{C}\times\mathbb{R}, \quad q_S(z) = \Big(\frac{2z}{1+|z|^2}, \frac{-1+|z|^2}{1+|z|^2}\Big). \tag{2.3}$$

We denote the south pole of S^2, i.e. the point $(0,0,-1)\in\mathbb{R}^3$, by ∞. We identify $\mathbb{C}$ with $S^2-\{\infty\}$ via the map q_N. If $x, z\in S^2-\{\infty\}$, define $\phi_x z\in\mathbb{C}$ by

$$\phi_x z = z - x \equiv q_N^{-1}(z) - q_N^{-1}(x). \tag{2.4}$$

Note that the map $\phi_x\colon S^2-\{\infty\}\longrightarrow\mathbb{C}$ is biholomorphic. If g is a Riemannian metric on a Riemann surface (Σ, j) of positive genus, $x\in\Sigma$ and $v\in T_x\Sigma$, we write $\exp_{g,x} v\in\Sigma$ for the exponential of v defined with respect to the Levi-Civita connection of g. Let $\mathrm{inj}_g x$ denote the corresponding injectivity radius at x and d_g the distance function. If $x, z \in \Sigma$ are such that $d_g(x,z) < \mathrm{inj}_g x$, define $\phi_{g,x} z \in T_x\Sigma$ by

$$\exp_{g,x} \phi_{g,x} z = z, \quad |\phi_{g,x} z|_{g,x} < \mathrm{inj}_g x. \tag{2.5}$$

Note that if g is flat on a neighborhood U of x in Σ, then $\phi_{g,x}|U$ is holomorphic.

Let (V, ω, J, g_V) be a Kahler manifold. Denote the corresponding Levi-Civita connection, exponential map, and distance function by ∇^V, $\exp_V$ and d_V, respectively. For every $\lambda \in H_2(V;\mathbb{Z})$, let $|\lambda| = \langle \omega, \lambda \rangle$. The number $|\lambda|$ is the g_V-energy of any holomorphic map $u: \Sigma \longrightarrow V$ such that $u_*[\Sigma] = \lambda$; see [MS]. By rescaling ω, it can be assumed that $|\lambda| \geq 1$, whenever there exists a nonconstant holomorphic map $S^2 \longrightarrow V$ in the λ-homology class. If g is any Kahler metric on (V, J), denote the corresponding Levi-Civita connection, exponential map, distance function, injectivity radius, and the parallel transport along the geodesic for $X \in TV$ by ∇^g, $\exp_g$, d_g, inj_g, and $\Pi_{g,X}$, respectively. If $q \in V$ and $\delta \in \mathbb{R}$, let

$$B_g(q, \delta) = \{q' \in V : d_g(q, q') \leq \delta\}.$$

In our construction, we allow g vary in a smooth family. Without causing any additional difficulty in the gluing construction, consideration of such families simplifies computations in specific cases; see Sections 6.2 and 7.3. If (S, j) is a smooth Riemann surface and $u \in C^\infty(S; V)$, put

$$\Gamma(u) = \Gamma(S; u^*TV), \quad \Gamma^1(u) = \Gamma(S; T^*S \otimes u^*TV);$$
$$\Gamma^{0,1}(u) = \Gamma(S; \Lambda^{0,1}T^*S \otimes u^*TV), \quad \bar{\partial} u = \frac{1}{2}\big(du + J \circ du \circ j\big) \in \Gamma^{0,1}(u).$$

We denote by D_V and D_g the linearization of $\bar{\partial}$-operator with respect to the metrics g_V and g on V, respectively. Since both metrics are Kahler, D_V and D_g commute with J and have no zeroth-order term; see Appendix A.

It should be mentioned that it is not essential for the gluing construction of Chapter 3 that (V, J, g) is Kahler or even symplectic. If (V, J, g) is not Kahler, we would need to choose an orientation on certain spaces of holomorphic maps and take the induced orientation on the cokernel bundle; see Section 3.2. Dropping the Kahler assumption would slightly complicate the notation, but not the analysis. Furthermore, definitions in Section 2.6 make use of the function $|\cdot|$ of the previous paragraph.

2.2 Bubble Trees

Let S be either the Riemann sphere S^2 or a smooth Riemann surface Σ of genus at least 2. Allowing the genus-one case would lead to somewhat more complicated notation, but would have no effect on the analysis done in Chapter 3. Denote by S^* the open subset $S - \{\infty\}$ if $S = S^2$ and S itself if $S = \Sigma$.

Definition 2.2 *A* *bubble tree based on* S *is a tuple* $\mathcal{T} = \big(S, I; x\big)$, *where*
(1) I *is a rooted tree and* $x: \hat{I} \longrightarrow S \cup S^2$ *is a map;*

(2) $x_h \in S^*$ *if* $\iota_h = \hat{0}$ *and* $x_h \in S^2 - \{\infty\}$ *otherwise;*
(3) if $h_1 \neq h_2$ *and* $\iota_{h_1} = \iota_{h_2}$, $x_{h_1} \neq x_{h_2}$.

Given a bubble tree T as above, let Σ_{T} be the complex curve

$$\Sigma_{\mathsf{T}} = \Big((\{\hat{0}\} \times S) \sqcup \bigsqcup_{h \in \hat{I}} (\{h\} \times S^2) \Big) \Big/ \sim,$$

where $(h, \infty) \sim (\iota_h, x_h)$ for $h \in \hat{I}$. The subset $\Sigma_{\mathsf{T},\hat{0}} \equiv \{\hat{0}\} \times S$ of Σ_{T} is the *principal component* of T or Σ_{T}. If $h \in \hat{I}$, let $\Sigma_{\mathsf{T},h} = \{h\} \times S^2$. For every $i \in I$, $\Sigma_{\mathsf{T},i}$ will be called the *ith bubble component* of T or Σ_{T} or simply a *bubble component.* Let $\Sigma^*_{\mathsf{T},i}$ and Σ^*_{T} denote the open subsets of smooth points of $\Sigma_{\mathsf{T},i}$ and Σ_{T}, respectively, i.e.

$$\Sigma^*_{\mathsf{T},i} = \begin{cases} \Sigma_{\mathsf{T},i} - \{(i,\infty)\} - \{(i, x_h) : \iota_h = i\}, & \text{if } i \in \hat{I}; \\ S - \{(\hat{0}, x_h) : \iota_h = \hat{0}\}, & \text{if } i = \hat{0}; \end{cases} \qquad \Sigma^*_{\mathsf{T}} = \bigcup_{i \in I} \Sigma^*_{\mathsf{T},i}.$$

The complement of Σ^*_{T} in Σ_{T} are the *singular points* of Σ_{T}.

For every $i \in I$, let $\mathsf{T}^{(i)} = \big(S, I^{(i)}; x|\hat{I}^{(i)}\big)$. Similarly, for each $h \in \hat{I}$, let $\mathsf{T}^h = \big(S, I^h; x|\hat{I}^h\big)$. These tuples are again bubble trees based on S. The complex curve $\Sigma_{\mathsf{T}^{(i)}}$ is obtained from Σ_{T} by dropping all bubble components descendent from the ith bubble component. The curve Σ_{T^h} is obtained by dropping the hth bubble component along with all bubble components descendent from it.

If $S = S^2$ and $h \in \hat{I}$, we denote the map ϕ_{x_h} defined in (2.4) by $\phi_{\mathsf{T},h}$. If $z \in \Sigma_{\mathsf{T},i}$, put

$$r_{\mathsf{T},h}(z) = \begin{cases} |\phi_{\mathsf{T},h} z|, & \text{if } i = \iota_h \text{ and } z \neq \infty; \\ 100, & \text{otherwise.} \end{cases} \tag{2.6}$$

If $\delta > 0$, let $B_{\mathsf{T},h}(\delta) = \{z \in \Sigma_{\mathsf{T}} : r_{\mathsf{T},h}(z) < \delta\}$. Put

$$r_{\mathsf{T}} = \min_{h \in \hat{I}} \Big(|q_S^{-1}(x_h)|, \min\{r_{\mathsf{T},h}(\iota_l, x_l) : l \neq h\} \Big). \tag{2.7}$$

If $S = \Sigma$ and $h \in \hat{I}$ is such that $\iota_h \in \hat{I}$, we again let $\phi_{\mathsf{T},h}$ denote the function ϕ_{x_h} of (2.4) and define $r_{\mathsf{T},h}$ and $B_{\mathsf{T},h}(\delta)$ as above. If g is a Riemannian metric on Σ, $\iota_h = \hat{0}$, and $z \in \Sigma_{\mathsf{T},i}$, put

$$r_{\mathsf{T},g,h}(z) = \begin{cases} d_g(x_h, z), & \text{if } i = \hat{0}; \\ 100, & \text{otherwise.} \end{cases} \tag{2.8}$$

We denote by $\phi_{\mathsf{T},g,h}$ the function ϕ_{g,x_h} of (2.5) and by $B_{\mathsf{T},g,h}(\delta)$ the ball $B_g(x_h, \delta)$. Put

$$r_{\mathsf{T}} g = \min\Big(\min_{\iota_h = \hat{0}} \{r_{\mathsf{T},g,h}(\iota_l, x_l) : l \neq h\}, \min_{\iota_h \neq \hat{0}} \big(|q_S^{-1}(x_h)|, \min\{r_{\mathsf{T},h}(\iota_l, x_l) : l \neq h\} \big) \Big). \tag{2.9}$$

We say g is a T*-admissible Riemannian metric* on Σ if there exists $\delta > 0$ such that for all $h \in \hat{I}$ with $\iota_h = \hat{0}$ the metric g is flat on $B_{\mathsf{T},g,h}(\delta)$.

2.3 The Basic Gluing Construction

In this section, we describe a gluing construction on bubble trees, which is the basis of all the other gluing constructions in this paper. Lemma 2.3 plays a very important role for computational aspects of this thesis.

Let $\mathcal{T}=(S,I;x)$ be a bubble tree. If $h\in\hat{I}$, put

$$F^{(0)}_{h,\mathcal{T}}=\begin{cases}\mathbb{C}, & \text{if } x_h\in S^2;\\ T_{x_h}\Sigma, & \text{if } x_h\in\Sigma,\end{cases}\qquad F^{(0)}_{\mathcal{T}}=\bigoplus_{h\in\hat{I}}F^{(0)}_{h,\mathcal{T}}. \tag{2.10}$$

If $S=S^2$, for any $\delta>0$, put

$$F^{(0)}_{\mathcal{T},\delta}=\Big\{v=(\mathcal{T},v_{\hat{I}})\colon v_{\hat{I}}\in F^{(0)}_{\mathcal{T}},\ |v|\equiv\sum_{h\in\hat{I}}|v_h|<\delta\Big\}.$$

Let $\delta_{\mathcal{T}}\in(0,1)$ be such that $8\delta_{\mathcal{T}}^{\frac{1}{2}}<r_{\mathcal{T}}$. If $S=\Sigma$ and g is an admissible metric on Σ, put

$$F^{(0)}_{\mathcal{T},g,\delta}=\Big\{v=(\mathcal{T},v_{\hat{I}})\colon v_{\hat{I}}\in F^{(0)}_{\mathcal{T}},\ |v|_g\equiv\sum_{\iota_h=\hat{0}}|v_h|_g+\sum_{\iota_h\neq\hat{0}}|v_h|<\delta\Big\},$$

where $|v_h|_g=|v_h|_{g,x_h}$. Let $\delta_{\mathcal{T}}g\in(0,1)$ be such that $8(\delta_{\mathcal{T}}g)^{\frac{1}{2}}<r_{\mathcal{T}}g$ and the metric g is flat on $B_g\big(x_h,4(\delta_{\mathcal{T}}g)^{\frac{1}{2}}\big)$ for all $h\in\hat{I}$ with $\iota_h=\hat{0}$. We now construct a family of bubble trees parameterized by $F^{(0)}_{\mathcal{T},\delta}$ with $\delta=\delta_{\mathcal{T}}$ if $S=S^2$ and by $F^{(0)}_{\mathcal{T},g,\delta}$ with $\delta=\delta_{\mathcal{T}}g$ if $S=\Sigma$.

First, for every $h\in\hat{I}$ and $v_h\in F^{(0)}_{\mathcal{T},h}$ with

$$|v_h|\in(0,\delta)\ \text{ if } x_h\in S^2\quad\text{and}\quad |v_h|_g\in(0,\delta)\ \text{ if } x_h\in\Sigma,$$

we define local stretching maps

$$q_{h,(x_h,v_h)}\colon\Sigma_{\mathcal{T}(\iota_h)}\longrightarrow\Sigma_{\mathcal{T}(h)}\ \text{ if } x_h\in S^2\quad\text{and}\quad q_{g,h,(x_h,v_h)}\colon\Sigma_{\mathcal{T}(\iota_h)}\longrightarrow\Sigma_{\mathcal{T}(h)}\ \text{ if } x_h\in\Sigma$$

as follows. If $x_h\in S^2$, let $p_{h,(x_h,v_h)}\colon B_{\mathcal{T},h}(\delta^{\frac{1}{2}})\longrightarrow\mathbb{C}\cup\infty$ be given by

$$p_{h,(x_h,v_h)}(z)=\big(1-\beta_{|v_h|}(2|\phi_{\mathcal{T},h}z|)\big)\overline{\Big(\frac{v_h}{\phi_{\mathcal{T},h}z}\Big)}. \tag{2.11}$$

Define $q_{h,(x_h,v_h)}\colon\Sigma_{\mathcal{T}h}\longrightarrow\Sigma_{\mathcal{T}(h)}$ by

$$q_{h,(x_h,v_h)}(z)=\begin{cases}\big(h,q_S(p_{h,(x_h,v_h)}(z))\big), & \text{if } |v_h|^{-\frac{1}{2}}r_{\mathcal{T},h}(z)\le 1;\\ \Big(\iota_h,\phi^{-1}_{\mathcal{T},h}\big(\beta_{|v_h|}(|\phi_{\mathcal{T},h}z|)(\phi_{\mathcal{T},h}z)\big)\Big), & \text{if } 1\le|v_h|^{-\frac{1}{2}}r_{\mathcal{T},h}(z)\le 2;\\ z, & \text{otherwise.}\end{cases} \tag{2.12}$$

Note that $q_{h,(x_h,v_h)}$ is smooth and is a diffeomorphism, except on the circle $r_{\mathcal{T},h}(z)=|v_h|^{\frac{1}{2}}$ in $\Sigma_{\mathcal{T},\iota_h}$. The map $q_{h,(x_h,v_h)}$ stretches $B_{\mathcal{T},h}(|v_h|^{\frac{1}{2}})$ around the sphere $\Sigma_{\mathcal{T},h}$. If $x_h\in\Sigma$, similarly to the above, let $p_{g,h,(x_h,v_h)}: B_{\top,g,h}(\delta^{\frac{1}{2}}) \longrightarrow \mathbb{C}\cup\infty$ be given by

$$p_{g,h,(x_h,v_h)}(z) = \big(1-\beta_{|v_h|_g}(2|\phi_{\top,g,h}z|_g)\big)\overline{\Big(\frac{v_h}{\phi_{\top,g,h}z}\Big)}. \tag{2.13}$$

Note that the ratio $v_h/\phi_{\top,g,h}z$ is well-defined as an extended complex number, since $T_{x_h}\Sigma$ is one-dimensional and $v_h\neq 0$. We then define $q_{g,h,(x_h,v_h)}: \Sigma_{\top^h} \longrightarrow \Sigma_{\top(h)}$ as in (2.12), but replacing $\phi_{\top,h}z$ and $p_{h,(x_h,v_h)}$ by $\phi_{\top,g,h}$ and $p_{g,h,(x_h,v_h)}$, respectively.

If $S=S^2$, for every $h\in I$ and $v\in F^{(0)}_{\top,\delta}$, we now define a bubble tree $\top_h(v)$ and a smooth map $q_{v,h}: \Sigma_{\top_h(v)} \longrightarrow \Sigma_{\top(h)}$. Choose an ordering of I consistent with its partial ordering. If $h=\hat{0}$, we take $I_h(v)=\{\hat{0}\}$, $\top_h(v)=(S,I_h(v);)$, and $q_{v,h}=Id_S$. Suppose $h\neq\hat{0}$ and

$$\top_{h-1}(v) = \big(S,\hat{I}_{h-1}(v);x_h(v)\big)$$

with $I_{h-1}(v)\subset I$. If $v_h=0$, put

$$I_h(v)=I_{h-1}(v)\cup\{h\}, \quad \big(\iota_{h,l}(v),x_{h,l}(v)\big) = \begin{cases} \big(\iota_{h-1,l}(v),x_{h-1,l}(v)\big), & \text{if } l\in I_{h-1}(v);\\ q^{-1}_{v,\iota_h}(\iota_h,x_h), & \text{otherwise.}\end{cases}$$

Let $q_{v,h}|\Sigma_{\top_{h-1}(v)}=q_{v,h-1}$ and $q_{v,h}(h,z)=(h,z)$. If $v_h\neq 0$, let

$$I_h(v)=I_{h-1}(v), \quad \big(\iota_{h,l}(v),x_{h,l}(v)\big) = \big(\iota_{h-1,l}(v),x_{h-1,l}(v)\big).$$

Take $q_{v,h}=q_{h,(x_h,v_h)}\circ q_{v,h-1}$. Inductively this procedure defines a bubble tree $\top(v)=\top_{h^*}(v)$ based on S and a smooth map $q_v=q_{v,h^*}: \Sigma_{\top(v)} \longrightarrow \Sigma_\top$, which is a diffeomorphism outside of $|I-I(v)|$ disjoint circles, where h^* is the largest element of I. The resulting bubble tree and map are independent of the choice of the extension of the partial ordering. While the domains of the maps $q_{v,h}$ do depend on such a choice, whenever we make use of the maps $q_{v,h}$ below, the result will also be independent of the choice. If $S=\Sigma$, for every $h\in I$ and $v\in F^{(0)}_{\top,g,\delta}$, we define bubble tree $\top_{g,h}(v)$ and maps $q_{g,v,h}: \Sigma_{\top_{g,h}(v)} \longrightarrow \Sigma_{\top(h)}$ similarly to the above, but replacing $q_{h,(x_h,v_h)}$ by $q_{g,h,(x_h,v_h)}$ whenever $\iota_h=\hat{0}$. We let $\top_g(v)=\top_{g,h^*}(v)$ and $q_{g,v}=q_{g,v,h^*}$. As before, $q_{g,v}$ is smooth and a diffeomorphism outside of $|I-I(v)|$ disjoint circles.

If $S=S^2$ and $v_h\neq 0$, put

$$\begin{aligned} A^+_{v,h} &= q^{-1}_{v,\iota_h}\Big(\big\{z\in\Sigma_{\top,\iota_h}: |v_h|^{\frac{1}{2}} \le r_{\top,h}(z) \le 2|v_h|^{\frac{1}{2}}\big\}\Big);\\ A^-_{v,h} &= q^{-1}_{v,\iota_h}\Big(\big\{z\in\Sigma_{\top,\iota_h}: \frac{1}{2}|v_h|^{\frac{1}{2}} \le r_{\top,h}(z) \le |v_h|^{\frac{1}{2}}\big\}\Big). \end{aligned} \tag{2.14}$$

Note that $A^\pm_{v,h}\subset\Sigma_{\top(v),i^*_h(v)}$, where

$$i^*_h(v) = \min\big\{i\in I: i<h \text{ and } v_{h'}\neq 0 \text{ if } i<h'<h\big\} = \max\big\{i\in I(v): i<h\big\}.$$

If $S=\Sigma$ and $v_h\neq 0$, we similarly define

$$\begin{aligned} A^+_{g,v,h} &= q^{-1}_{g,v,\iota_h}\Big(\big\{z\in\Sigma_{\top,\iota_h}\colon |v_h|_g^{\frac{1}{2}}\leq r_{\top,g,h}(z)\leq 2|v_h|_g^{\frac{1}{2}}\big\}\Big);\\ A^-_{g,v,h} &= q^{-1}_{g,v,\iota_h}\Big(\big\{z\in\Sigma_{\top,\iota_h}\colon \frac{1}{2}|v_h|_g^{\frac{1}{2}}\leq r_{\top,g,h}(z)\leq |v_h|_g^{\frac{1}{2}}\big\}\Big), \end{aligned} \tag{2.15}$$

where $|v_h|_g$ and $r_{\top,g,h}$ denote $|v_h|$ and $r_{\top,h}$ if $\iota_h\in\hat{I}$.

Lemma 2.3 *If $S=S^2$, the map q_v is holomorphic outside of the annuli $A^\pm_{v,h}$ with $v_h\neq 0$. For such h,*

$$\|dq_{h,(x_h,v_h)}\|_{C^0(q_{v,\iota_h}(A^\pm_{v,h}))}\leq C;$$
$$\bar\partial\big(q_v\circ q^{-1}_{v,\iota_h}\big)\big|_z = -2|v_h|^{-\frac{1}{2}}\Big(\frac{v_h}{\phi_{\top,h}z}\Big)dq_S\big|_{p_{h,(x_h,v_h)}z}\circ\partial\beta\big|_{2|v_h|^{-\frac{1}{2}}\phi_{\top,h}z}\circ d\phi_{\top,h}\big|_z\quad \forall z\in q_{v,\iota_h}(A^-_{v,h}),$$

where the norm is computed with respect to the standard metric on S^2, and β is a viewed as a function on $\mathbb{C}$ via the standard norm on $\mathbb{C}$. If $S=\Sigma$, the map $q_{g,v}$ is holomorphic outside of the annuli $A^\pm_{g,v,h}$ with $v_h\neq 0$. For such h,

$$\|dq_{g,h,(x_h,v_h)}\|_{C^0(q_{g,v,\iota_h}(A^\pm_{g,v,h}))}\leq C_g \quad \textit{if } \iota_h=\hat{0};\qquad \|dq_{h,(x_h,v_h)}\|_{C^0(q_{g,v,\iota_h}(A^\pm_{g,v,h}))}\leq C \quad \textit{if } \iota_h\neq\hat{0};$$
$$\bar\partial\big(q_{g,v}\circ q^{-1}_{g,v,\iota_h}\big)\big|_z = -2|v_h|^{-\frac{1}{2}}\Big(\frac{v_h}{\phi_{\top,h}z}\Big)dq_S\big|_{p_{h,(x_h,v_h)}z}\circ\partial\beta\big|_{2|v_h|^{-\frac{1}{2}}\phi_{\top,h}z}\circ d\phi_{\top,h}\big|_z\quad \forall z\in q_{g,v,\iota_h}(A^-_{v,h}),$$

where we regard β as a function on $T_{x_h}\Sigma$ via the metric g and denote $\phi_{g,\top,h}$ by $\phi_{\top,h}$ if $\iota_h=\hat{0}$.

Proof: The first statement in each of the two cases is immediate from the construction. The estimates on the differential of $q_{h,(x_h,v_h)}$ and $q_{g,h,(x_h,v_h)}$ follow from (2.2). Suppose $S=\Sigma$, $\iota_h=\hat{0}$, $v_h\neq 0$ and $z\in A^-_{g,v,h}$. Since $q_{g,v}=q_{g,h,(x_h,v_h)}$ on $A^-_{g,v,h}$ and q_S is anti-holomorphic, from (2.13) we obtain

$$\begin{aligned} \bar\partial q_{g,v}\big|_z &= dq_S\big|_{p_{g,h,(x_h,v_h)}z}\circ\partial p_{g,h,(x_h,v_h)}\big|_z\\ &= -2|v_h|_g^{-\frac{1}{2}}\Big(\frac{v_h}{\phi_{\top,g,h}z}\Big)dq_S\big|_{p_{g,h,(x_h,v_h)}z}\circ\partial\beta\big|_{2|v_h|_g^{-\frac{1}{2}}\phi_{\top,g,h}z}\circ d\phi_{\top,g,h}\big|_z. \end{aligned}$$

The other cases are proved similarly, since $q_{g,v}\circ q^{-1}_{g,v,\iota_h}=q_{h,(x_h,v_h)}$ on $q_{g,v,\iota_h}(A^-_{g,v,h})$ and a similar statement holds in the case $S=S^2$.

2.4 Curves with Marked Points

Definition 2.4 *If M is a finite set, a* <u>*curve with M-marked points based on S*</u> *is a tuple*

$$\mathcal{C}=\big(S,M,I;x,(j,y)\big),\qquad \textit{where}$$

(1) $\top_{\mathcal{C}}\equiv(S,I;x)$ is a bubble tree based on S, and $j\colon M\longrightarrow I$ and $y\colon M\longrightarrow S\cup S^2$ are maps;
(2) $j_l\in I$, $(j_l,y_l)\in\Sigma^*_{\top_{\mathcal{C}},j_l}$, and $y_l\neq\infty$ for all $l\in M$;
(3) for any $l_1,l_2\in M$ with $l_1\neq l_2$ and $j_{l_1}=j_{l_2}$, $y_{l_1}\neq y_{l_2}$.
The curve $\mathcal{C}$ is <u>*stable*</u> *if $\big|\{h:\iota_h=i\}\big|+\big|\{l:j_l=i\}\big|\geq 2$ for all $i\in\hat{I}$ if $S=\Sigma$ and all $i\in I$ if $S=S^2$.*

Via the construction in Section 2.2, such a tuple $\mathcal{C}$ corresponds to a complex curve $\Sigma_{\mathcal{C}} \equiv \Sigma_{\top_{\mathcal{C}}}$ with marked points $\{(j_l, y_l)\}_{l\in M}$. For each $i\in I$, we denote by $\Sigma_{\mathcal{C},i}$ and $\Sigma^*_{\mathcal{C},i}$ the surfaces $\Sigma_{\top_{\mathcal{C}},i}$ and $\Sigma^*_{\top_{\mathcal{C}},i}$, respectively.

With notation as above, for every $h\in\hat{I}$, let $F^{(0)}_{h,\mathcal{C}}$ and $F^{(0)}_{\mathcal{C}}$ denote the spaces $F^{(0)}_{h,\top_{\mathcal{C}}}$ and $F^{(0)}_{\top_{\mathcal{C}}}$, respectively. If $S=S^2$, put

$$r_{\mathcal{C}} = \min\Big(r_{\top_{\mathcal{C}}}, \min_{l\in M}\big(|q_S^{-1}y_l|, \min\{r_{\top_{\mathcal{C}},h}(j_l,y_l)\colon h\in\hat{I}\},\\ \min\{|\phi_{y_l}^{-1}y_h|\colon h\neq l, j_h=j_l\}\big)\Big). \tag{2.16}$$

Let $\delta_{\mathcal{C}}\in(0,1)$ be such that $16(|I|+|M|)\delta_{\mathcal{C}}^{\frac{1}{2}}<r_{\mathcal{C}}$. If $\upsilon=(\mathcal{C},\upsilon_{\hat{I}})$ with $\upsilon_{\hat{I}}\in F^{(0)}_{\mathcal{C}}$ and $|\upsilon|<\delta_{\mathcal{C}}$, we construct a curve $\mathcal{C}(\upsilon)$ with M-marked points as follows. Let

$$\top(\upsilon) = \big(S, I(\upsilon); x(\upsilon)\big) \quad \text{and} \quad q_\upsilon\colon \Sigma_{\top(\upsilon)} \longrightarrow \Sigma_{\mathcal{C}}$$

be the bubble tree and the smooth map defined in Section 2.3. Then we take

$$\mathcal{C}(\upsilon) = \big(S, M, I(\upsilon); x(\upsilon), (j(\upsilon), y(\upsilon))\big),$$

where $(j_l(\upsilon), y_l(\upsilon))\in\Sigma_{\top(\upsilon),j_l(\upsilon)}$ is defined by

$$q_\upsilon(j_l(\upsilon), y_l(\upsilon)) = (j_l, y_l).$$

Similarly, if $S=\Sigma$ and g is an admissible Riemannian metric on Σ, put

$$r_{\mathcal{C}}g = \min\Big(r_{\top_{\mathcal{C}}}g, \min_{j_l=\hat{0}}\big(\min_{\iota_h=\hat{0}}\{r_{\top_{\mathcal{C}},g,h}(j_l,y_l)\}, \min\{|\phi_{g,y_l}^{-1}y_h|_g\colon h\neq l, j_h=\hat{0}\}\big),\\ \min_{j_l\neq\hat{0}}\big(|q_S^{-1}y_l|, \min_{\iota_h\neq 0}\{r_{\top_{\mathcal{C}},h}(j_l,y_l)\}, \min\{|\phi_{y_l}^{-1}y_h|\colon h\neq l, j_h=j_l\}\big)\Big). \tag{2.17}$$

Let $\delta_{\mathcal{C}}g\in(0,1)$ be such that $16(|I|+|M|)(\delta_{\mathcal{C}}g)^{\frac{1}{2}}<r_{\mathcal{C}}g$ and g is flat in $B_g\big(x_h, 8(\delta_{\mathcal{C}}g)^{\frac{1}{2}}\big)$ for all $h\in\hat{I}$ with $\iota_h=\hat{0}$. If $\upsilon\in F^{(0)}_{\mathcal{C}}$ and $|\upsilon|_g<\delta_{\mathcal{C}}g$, we construct a curve $\mathcal{C}_g(\upsilon)$ with M-marked points in the same way as above, but replacing q_υ and $\top(\upsilon)$ by $q_{g,\upsilon}$ and $\top_g(\upsilon)$.

Definition 2.5 *An* <u>*isomorphism of curves with M-marked points*</u> $\mathcal{C}=\big(S,M,I;x,(j,y)\big)$ *and* $\mathcal{C}'=\big(S,M,I';x',(j',y')\big)$, *is a tuple of maps,*

$$\phi_0\colon I\longrightarrow I', \quad \phi_{1,\hat{0}}\colon S\longrightarrow S, \quad \phi_{1,h}\colon S^2\longrightarrow S^2 \;\; \textit{for } h\in I, \qquad \textit{where}$$

(1) ϕ_0 is an isomorphism of the linearly ordered sets I and I' and $\phi_0(j_l)=j'_l$ for all $l\in M$;
(2) $\phi_{1,i}$ is a biholomorphic map for all $i\in I$ and $\phi_{1,\hat{0}}$ is the identity map if $S=\Sigma$;
(3) $\phi_{1,i}(\infty)=\infty$ for all $i\in I$ if $S=S^2$ and for all $i\in\hat{I}$ if $S=\Sigma$;
(4) $\phi_{1,\iota_h}(x_h)=x'_{\phi_0(h)}$ for $h\in\hat{I}$ and $\phi_{1,j_l}(y_l)=y'_l$ for all $l\in M$.

Such a set of maps corresponds to a continuous map $\Sigma_{\mathcal{C}}\longrightarrow\Sigma_{\mathcal{C}'}$ that maps the lth marked point (j_l,y_l) on $\Sigma_{\mathcal{C}}$ to the lth marked point (j'_l,y'_l) on $\Sigma_{\mathcal{C}'}$ and is biholomorphic on each component of $\Sigma_{\mathcal{C}}$. Note that if $\mathcal{C}$ is stable, $\mathcal{C}$ has no nontrivial automorphisms. Let $[\mathcal{C}]$ denote the equivalence class of $\mathcal{C}$ in the set of all curves based on S with marked points.

Denote by $\bar{\mathcal{M}}_{S,M}$ the set of all equivalence classes of stable curves based on S with M-marked points. If $S=S^2$, $\bar{\mathcal{M}}_{S,M}$ can be identified with the moduli space $\bar{\mathcal{M}}_{0,|M|+1}$ of all stable rational curves with $|M|+1$ marked points, or more canonically with the space $\bar{\mathcal{M}}_{0,M}$ of all stable rational curves with the marked points labeled by the set $M+\{\hat{0}\}$. If $S=\Sigma$ has genus bigger than two and is generic, $\bar{\mathcal{M}}_{S,M}$ is the closed subset of $\bar{\mathcal{M}}_{g,M}$ consisting of all stable curves of genus g with M-marked points that have a fixed complex structure on the principal component. If S has genus two, $\bar{\mathcal{M}}_{S,M}$ is a double cover of the corresponding set for $g=2$, since any smooth genus-two curve has a holomorphic automorphism of order two; see [GH, p254]. The reason we require $\phi_{1,\hat{0}}=Id_\Sigma$ is that the symplectic invariant of [RT] disregards the automorphisms of Σ.

2.5 Bubble Maps

Definition 2.6 *A V-valued bubble map is a tuple* $b=(S,M,I;x,(j,y),u)$, *where*
(1) I is a linearly ordered set, which is a rooted tree if $S=\Sigma$;
(2) $u\colon I\longrightarrow C^\infty(S;V)\cup C^\infty(S^2;V)$ is a map;
(3) if $I=\bigsqcup_{k\in K} I_k$ is the splitting of I into rooted trees, then $M=\bigsqcup_{k\in K} M_k$ for some subsets M_k of M such that $\mathcal{C}_k=(S,M_k,I_k;x|\hat{I}_k,(j,y)|M_k)$ is an M_k-marked curve based on S;
(4) $u_h\in C^\infty(S;V)$ if $h\in I-\hat{I}$, $u_h\in C^\infty(S^2;V)$ if $h\in\hat{I}$, and such that $u_h(\infty)=u_{\iota_h}(x_h)$ for all $h\in\hat{I}$;
(5) for all $i\in\hat{I}$ if $S=\Sigma$ and $i\in I$ if $S=S^2$,

$$\big|\{h\in\hat{I}\colon \iota_h=i\}\big|+\big|\{l\in M\colon j_l=i\}\big|<2\Longrightarrow u_{i*}[S^2]\neq 0\in H_2(V;\mathbb{Z}).$$

The bubble map b is simple if I is a rooted tree; b is holomorphic if $\bar{\partial}u_i=0$ for all $i\in I$.

With notation as in Definition 2.6, every bubble map b corresponds to a continuous map

$$u_b\colon \Sigma_b\equiv\bigsqcup_{k\in K}\Sigma_{\mathcal{C}_k}\longrightarrow V,$$

which is smooth on the components of $\Sigma_{\mathcal{C}_k}$. If $h\in\hat{I}_k$, let $F^{(0)}_{h,b}=F^{(0)}_{h,\mathcal{C}_k}$. Similarly, let

$$F^{(0)}_b=\bigoplus_{k\in K}F^{(0)}_{\mathcal{C}_k},\quad \Sigma^*_b=\bigcup_{k\in K}\Sigma^*_{\mathcal{C}_k}\subset\Sigma_b,\quad \Sigma^*_{b,i}=\Sigma^*_{\mathcal{C}_k,i}\subset\Sigma_{b,i}\equiv\Sigma_{\mathcal{C}_k,i},$$

whenever $i\in I_k\subset I$. If b is simple, denote by T_b the bubble tree $\mathsf{T}_{\mathcal{C}_k}$ for the unique element $k\in K$.

Definition 2.7 *An isomorphism of V-valued bubble maps* $b=(S,M,I;x,(j,y),u)$ *and* $b'=(S,M,I';x',(j',y'),u')$ *is a tuple of maps*

$$\phi_0\colon I\longrightarrow I',\quad \phi_{1,i}\colon S\longrightarrow S\ \ \textit{for } i\in I-\hat{I},\quad \phi_{1,i}\colon S^2\longrightarrow S^2\ \ \textit{for } i\in\hat{I},\qquad \textit{where}$$

(1) ϕ_0 is an isomorphism of the linearly ordered sets I and I' with $\phi_0(j_l)=j'_l$ for all $l\in M$;
(2) $\phi_{1,i}$ is a biholomorphic map for all $i\in I$ and is the identity map if $S=\Sigma$ and $i\notin\hat{I}$;
(3) $\phi_{1,i}(\infty)=\infty$ for all $i\in I$ if $S=S^2$ and for all $i\in\hat{I}$ if $S=\Sigma$;
(4) $\phi_{1,\iota_h}(x_h)=x'_{\phi_0(h)}$ for all $h\in\hat{I}$ and $\phi_{1,j_l}(y_l)=y'_l$ for all $l\in M$;
(5) $u'_{\phi_0(i)}\circ\phi_{1,i}=u_i$ for all $i\in I$.

Such a set of maps corresponds to a continuous map $\Sigma_b \longrightarrow \Sigma_{b'}$ that maps the marked points of b to the marked points of b', intertwines the maps $u_b\colon \Sigma_b \longrightarrow V$ and $u_{b'}\colon \Sigma_{b'} \longrightarrow V$, and is biholomorphic on each component $\Sigma_{b,i}$ of Σ_b. Let G_b denote the group of automorphisms of the bubble map b. This group is necessarily finite by the stability condition (4) of Definition 2.6. If $\lambda \in H_2(V;\mathbb{Z})$, let

$$\bar{\mathcal{C}}^{\infty}_{(\lambda;M)}(S;V) = \big\{b = \big(S,M,I;x,(j,y),u\big) \text{ is } V\text{-valued bubble map}\colon \sum_{i\in I} u_{i*}[\Sigma_{b,i}] = \lambda\big\}/\sim;$$

$$\mathcal{C}^{\infty}_{(\lambda;M)}(S;V) = \big\{b = \big(S,\{\hat{0}\};,(\hat{0},y),u_{\hat{0}}\big) \text{ is } V\text{-valued bubble map}\colon u_{\hat{0}*}[S]=\lambda\big\}/\sim,$$

where the equivalence relation is given by isomorphisms of V-valued bubble maps. If $\mu = \mu_M$ is an M-tuple of submanifolds of V, let

$$\bar{\mathcal{C}}^{\infty}_{(\lambda;M)}(S;\mu) = \big\{b = \big[S,M,I;x,(j,y),u\big] \in \bar{\mathcal{C}}^{\infty}_{(\lambda;M)}(S;V)\colon u_{j_l}(y_l)\in \mu_l \ \forall l\in M\big\},$$

$$\mathcal{C}^{\infty}_{(\lambda;M)}(S;\mu) = \big\{b = \big[S,M,\{\hat{0}\};,(\hat{0},y),u_{\hat{0}}\big] \in \mathcal{C}^{\infty}_{(\lambda;M)}(S;V)\colon u_{\hat{0}}(y_l)\in \mu_l \ \forall l\in M\big\}.$$

A topology on $\bar{\mathcal{C}}^{\infty}_{(\lambda;M)}(S;V)$ and its subsets $\mathcal{C}^{\infty}_{(\lambda;M)}(S;V)$, $\bar{\mathcal{C}}^{\infty}_{(\lambda;M)}(S;\mu)$, $\mathcal{C}^{\infty}_{(\lambda;M)}(S;\mu)$ is defined below.

Definition 2.8 *Let $b^* = \big(S,M,I^*;x^*,(j^*,y^*),u^*\big)$ and $b_k = \big(S,M,I_k;x_k,(j_k,y_k),u_k\big)$ be simple bubble maps. If $S = S^2$, the sequence $\{b_k\}$ converges to b^* if for all k sufficiently large one can choose*
(a) M-marked curves $\mathcal{C}_k = \big(S,M,I^*;x'_k,(j^*,y^*)\big)$, and
(b) vectors $v_k \in F^{(0)}_{\mathcal{C}_k}$ with $16|v_k| < r^2_{\mathcal{C}_k}$,
such that with $\upsilon_k = (\mathcal{C}_k, v_k)$,
(1) $\lim_{k\longrightarrow\infty} x'_{k,h} = x^*_h$ for all $h \in \hat{I}$, and $\lim_{k\longrightarrow\infty} |v_k| = 0$;
2) $\mathcal{C}(\upsilon_k) = \big(S,M,I_k;x_k,(j_k,y(\upsilon_k))\big)$,

$$\lim_{k\longrightarrow\infty} q_{\upsilon_k}(j_{k,l},y_{k,l}) = (j^*_l,y^*_l) \quad \forall l\in M, \quad \textit{and} \quad \lim_{k\longrightarrow\infty} \sup_{z\in\Sigma_{\mathcal{C}(\upsilon_k)}} d_V\big(u_{b^*}(q_{\upsilon_k}(z)),u_{b_k}(z)\big) = 0.$$

If $S = \Sigma$, convergence is defined in the same way, but $|v_k|$ and $\mathcal{C}(\upsilon_k)$ are replaced by $|v_k|_g$ and $\mathcal{C}_g(\upsilon_k)$, respectively, for a T_{b^*}-admissible metric g on Σ.

This notion of convergence is independent of the choice of an admissible metric on Σ. Definition 2.8 induces a topology on the space $\bar{\mathcal{C}}^{\infty}_{(\lambda;M)}(S;V)$, which will be referred to as the Gromov topology.

2.6 Stratums of Bubble Maps

We now introduce the notion of a bubble type. We then define various spaces of holomorphic bubble maps indexed by bubble types and vector bundles over them.

Definition 2.9 *A bubble type is a tuple $\mathcal{T} = \big(S,M,I;j,\underline{\lambda}\big)$, such that*
(1) I is a linearly ordered set, and $j\colon M \longrightarrow I$ and $\underline{\lambda}\colon I \longrightarrow H_2(S;\mathbb{Z})$ are maps;
(2) for all $i\in\hat{I}$ if $S=\Sigma$ and all $i\in I$ if $S=S^2$, $\lambda_i \neq 0$ if $|\{h\colon \iota_h = i\}| + |\{l\colon j_l = i\}| < 2$.
Bubble type $\mathcal{T}$ is simple if I is a rooted tree; $\mathcal{T}$ is basic if $\hat{I} = \emptyset$.
Two bubble types $\mathcal{T} = \big(S,M,I;j,\underline{\lambda}\big)$ and $\mathcal{T}' = \big(S,M,I';j',\underline{\lambda}'\big)$ are equivalent if there exists

an isomorphism of linearly ordered sets $\phi_0: I \longrightarrow I'$ *such that* $\phi_0(j_l) = j'_l$ *for all* $l \in M$ *and* $\lambda'_{\phi_0(i)} = \lambda_i$ *for all* $i \in I$.
If $\mathcal{T}^* = (S, M, I^*; j^*, \underline{\lambda}^*)$ *and* $\mathcal{T} = (S, M, I; j, \underline{\lambda})$ *are two bubble types,* $\underline{\mathcal{T}^* < \mathcal{T}}$ *if* $I \subset I^*$,

$$j_l = \max\{i \in I : i \leq j^*_l\} \quad \forall l \in M \quad \text{and} \quad \lambda_i = \sum_{i = \max\{i' \leq h : i' \in I\}} \lambda^*_h \quad \forall i \in I.$$

If $\mathcal{T} = (S, M, I; j, \underline{\lambda})$ *is a bubble type, a* $\underline{\mathcal{T}\text{-bubble map}}$ *is a bubble map* $b = (S, M, I; x, (j, y), u)$ *such that* $u_{i*}[\Sigma_{b,i}] = \lambda_i \in H_2(V; \mathbb{Z})$ *for all* $i \in I$.

The splitting of I into rooted trees I_k induces a splitting of $\mathcal{T}$ into simple bubble types

$$\mathcal{T}_k = (S, M_k, I_k; j_k, \underline{\lambda}_k),$$

where j_k and $\underline{\lambda}_k$ are the restrictions of j and $\underline{\lambda}$ to M_k and I_k, respectively. Similarly, each $\mathcal{T}$-bubble map b corresponds to a K-tuple of bubble maps $b_K = (b_k)_{k \in K}$, where b_k is a $\mathcal{T}_k$-bubble map.

We denote the equivalence class of the bubble type $\mathcal{T}$ by $[\mathcal{T}]$ and the group of automorphisms of $\mathcal{T}$ that fix all minimal elements of I by $\mathcal{A}(\mathcal{T})$. This group acts naturally on the set of all $\mathcal{T}$-bubble maps. The partial ordering on the set of bubble types induces a partial ordering on the set of their equivalence classes. If b and b' are $\mathcal{T}$- and $\mathcal{T}'$-bubble maps, respectively, such that $[b] = [b']$, then $[\mathcal{T}] = [\mathcal{T}']$. Furthermore, if $\{b_k\}$ is a sequence of $\mathcal{T}$-bubble maps, b^* is $\mathcal{T}^*$-bubble map, and $[b_k]$ converges to $[b^*]$ with respect to the Gromov topology, then $[\mathcal{T}^*] \leq [\mathcal{T}]$.

Let $\mathcal{T} = (S, M, I; j, \underline{\lambda})$ be a bubble type. We denote by $\langle \mathcal{T} \rangle$ the basic bubble type such that $\langle \mathcal{T} \rangle \geq \mathcal{T}$. It can be described explicitly as follows. Let $I = \bigsqcup_{k \in K} I_k$ be the splitting of I into rooted trees and $M = \bigsqcup_{k \in K} M_k$ the corresponding splitting of M; see Definition 2.6. It can be assumed that $K = I - \hat{I}$ and k is the unique minimum element of I_k. For every $k \in K$ and $l \in M_k$, let

$$\lambda'_k = \sum_{i \in I_k} \lambda_i, \qquad j'_l = k.$$

Then $\langle \mathcal{T} \rangle = (S, M, K; j', \underline{\lambda}')$.

If $\mathcal{T} = (S, M, I; j, \underline{\lambda})$ is a simple bubble type and $i \in I$, let

$$D_i\mathcal{T} = D_iI, \quad \bar{D}_i\mathcal{T} = \bar{D}_iI, \quad H_i\mathcal{T} = \{h \in \hat{I} : \iota_h = i\}, \quad M_i\mathcal{T} = \{l \in M : j_l = i\}.$$

If H is a subset of $\hat{I}$, let $\mathcal{T}(H) = (S, M, H \cup \hat{0}; j', \underline{\lambda}')$, where

$$j'_l = \max\{i \in H \cup \hat{0} : i \leq j_l\} \quad \text{and} \quad \lambda'_i = \sum_{i^*_H \leq h \leq i} \lambda_h \text{ with } i^*_H = \max\{i^* \in H \cup \hat{0} : i^* \leq i\}.$$

Then $\mathcal{T}(H)$ is again a bubble type. The bubble type $\mathcal{T}(H)$ is the bubble type obtained by gluing $\mathcal{T}$-bubble maps with the parameter $v_{\hat{I}}$ such that $v_h = 0$ if and only if $h \in H$; see Chapter 3.

Given a bubble type $\mathcal{T}=(S,M,I;j,\underline{\lambda})$, let $d(\mathcal{T})\colon I\longrightarrow\mathbb{R}$ be given by

$$d_i(\mathcal{T}) = |\lambda_i| + \big|\{l\in M : j_l=i\}\big| + \sum_{\iota_h=i} d_h(\mathcal{T}) \quad \forall i\in I. \tag{2.18}$$

Since I is a linearly ordered set, the numbers $d_i(\mathcal{T})$ are uniquely defined by (2.18). If

$$b = \big(S,M,I;x,(j,y),u\big)$$

is a $\mathcal{T}$-bubble map, b is *$\mathcal{T}$-balanced* if for all $i\in\hat{I}$

(B1) $\int_{\mathbb{C}} |du_i\circ q_N|^2 z + \sum_{\iota_h=i} d_h(\mathcal{T})x_h + \sum_{j_l=i} y_l = 0$;

(B2) $\int_{\mathbb{C}} |du_i\circ q_N|^2 \beta(|z|) + \sum_{\iota_h=i} d_h(\mathcal{T})\beta(|x_h|) + \sum_{j_l=i} \beta(|y_l|) = \frac{1}{2}$.

The integrals above are computed with respect to the metric g_V on V. Recall that we consider $\mathbb{C}$ to be a subset of S^2 via the map q_N. Thus, x_h and y_l can be viewed as complex numbers, as done above. If $S=S^2$ and b is as above, b is *completely $\mathcal{T}$-balanced* (or *cb*) if (B1) and (B2) hold for all $i\in I$.

Denote by $\mathcal{H}_{\mathcal{T}}$ the set of all holomorphic $\mathcal{T}$-bubble maps. Let

$$PSL_2^{(0)} = \{g\in PSL_2 : g(\infty)=\infty\}, \qquad \mathcal{G}_{\mathcal{T}} = \prod_{h\in\hat{I}} PSL_2^{(0)}.$$

The group $\mathcal{G}_{\mathcal{T}}$ acts on $\mathcal{H}_{\mathcal{T}}$ as follows. If

$$b = \big(S,M,I;x,(j,y),u\big) \in \mathcal{H}_{\mathcal{T}} \quad \text{and} \quad g = g_{\hat{I}} \in \mathcal{G}_{\mathcal{T}},$$

define $gb = \big(S,M,I;gx,(j,gy),(gu)\big)$ by

$$(gx)_h = \begin{cases} g_{\iota_h}x_h, & \text{if } \iota_h\in\hat{I};\\ x_h, & \text{if } \iota_h\notin\hat{I};\end{cases} \quad (gy)_l = \begin{cases} g_{j_l}y_l, & \text{if } j_l\in\hat{I};\\ y_l, & \text{if } j_l\notin\hat{I};\end{cases} \quad (gu)_i = \begin{cases} g_i\cdot u_i, & \text{if } i\in\hat{I};\\ u_i, & \text{if } i\notin\hat{I},\end{cases}$$

where for any map $f\colon S^2\longrightarrow V$ and $g\in PSL_2$, we define

$$g\cdot f\colon S^2\longrightarrow V \quad \text{by} \quad \{g\cdot f\}(z) = f(g^{-1}z).$$

Let $\mathcal{M}_{\mathcal{T}}^{(0)}\subset\mathcal{H}_{\mathcal{T}}$ denote the subset of $\mathcal{T}$-balanced holomorphic maps and

$$G_{\mathcal{T}} \equiv \prod_{h\in\hat{I}} S^1 \subset \mathcal{G}_{\mathcal{T}}.$$

Since every element of $G_{\mathcal{T}}$ is a map on I, $\mathcal{A}(\mathcal{T})$ acts naturally on $G_{\mathcal{T}}$. The semi-direct product $\mathcal{A}(\mathcal{T})\ltimes G_{\mathcal{T}}$ acts on $\mathcal{M}_{\mathcal{T}}^{(0)}$ and all the stabilizers are finite. Denote the quotient by $\mathcal{M}_{\mathcal{T}}$, and let

$$\bar{\mathcal{M}}_{\mathcal{T}} = \bigcup_{\mathcal{T}'\le\mathcal{T}} \mathcal{M}_{\mathcal{T}'}.$$

If $\mathcal{A}(\mathcal{T})=\{1\}$, corresponding to the quotient $\mathcal{M}_{\mathcal{T}}=\mathcal{M}_{\mathcal{T}}^{(0)}/G_{\mathcal{T}}$, we obtain $|\hat{I}|$ line (orbi)-bundles

$$\{L_h\mathcal{T}\longrightarrow\mathcal{M}_{\mathcal{T}}: h\in\hat{I}\},$$

that carry natural norms:

$$|[b,c_h]|=|c_h| \qquad \text{if} \quad b\in\mathcal{M}_{\mathcal{T}}^{(0)} \quad \text{and} \quad c_h\in\mathbb{C}.$$

If $\mathcal{A}(\mathcal{T})\neq\{1\}$, the fiber products and connect sums of the above line bundles taken over each orbit of $\mathcal{A}(\mathcal{T})$ are well-defined. Let $F_h^{(0)}\mathcal{T}\longrightarrow\mathcal{M}_{\mathcal{T}}^{(0)}$ be the bundle with the fiber $F_{h,b}^{(0)}$ at $b\in\mathcal{M}_{\mathcal{T}}^{(0)}$, i.e.

$$F_h^{(0)}\mathcal{T}=\begin{cases}\mathcal{M}_{\mathcal{T}}^{(0)}\times\mathbb{C}, & \text{if } x_h\in S^2;\\ \pi_h^*T\Sigma, & \text{if } x_h\in\Sigma,\end{cases} \qquad \text{where} \quad \pi_h(b)=x_h,$$

with notation as above. The action of $G_{\mathcal{T}}$ on $\mathcal{M}_{\mathcal{T}}^{(0)}$ lifts to an action on each bundle $F_h^{(0)}\mathcal{T}$ by

$$g\cdot(b,v_h)=\begin{cases}\big(g\cdot b, g_{\iota_h}g_h^{-1}v_h\big), & \text{if } \iota_h\in\hat{I};\\ \big(g\cdot b, g_h^{-1}v_h\big), & \text{if } \iota_h\notin\hat{I}.\end{cases}$$

Here and in the rest of the paper, we identify S^1 with the unit complex numbers in the usual way. Let $F_h\mathcal{T}$ be the line orbi-bundle over $\mathcal{M}_{\mathcal{T}}$ given by

$$F_h\mathcal{T}=F_h^{(0)}\mathcal{T}/G_{\mathcal{T}}.$$

This bundle has a natural norm unless $\iota_h=\hat{0}$ and $S=\Sigma$. In such a case, any metric g on Σ induces a norm on $F_h\mathcal{T}$. Let

$$F^{(0)}\mathcal{T}=\bigoplus_{h\in\hat{I}}F_h^{(0)}\mathcal{T}, \quad F_b^{(0)}\mathcal{T}=F^{(0)}\mathcal{T}|b; \quad F\mathcal{T}=\bigoplus_{h\in\hat{I}}F_h\mathcal{T}, \quad F_{[b]}^{(0)}\mathcal{T}=F^{(0)}\mathcal{T}|[b].$$

Note that if $\mathcal{T}^*<\mathcal{T}$, there is a natural splitting

$$\big(\mathcal{A}(\mathcal{T}^*)\ltimes G_{\mathcal{T}^*}\big)=\big(\mathcal{A}(\mathcal{T})\ltimes G_{\mathcal{T}}\big)\times G,$$

with G determined by $\mathcal{T}$ and $\mathcal{T}^*$. Thus, $G_{\mathcal{T}}$ acts on $\mathcal{M}_{\mathcal{T}^*}^{(0)}$ and the line bundles $F_h^{(0)}\mathcal{T}^*$, while $G_{\mathcal{T}^*}$ acts on $\mathcal{M}_{\mathcal{T}}^{(0)}$ and $F_h^{(0)}\mathcal{T}$.

If $S=S^2$, let

$$\mathcal{B}_{\mathcal{T}}=\big\{b=\big(S,M,I;x,(j,y),u\big)\in\mathcal{H}_{\mathcal{T}}: b \text{ is cb};\ u_{i_1}(\infty)=u_{i_2}(\infty)\ \forall i_1,i_2\in I-\hat{I}\big\}.$$

Denote by $\mathcal{U}_{\mathcal{T}}^{(0)}\subset\mathcal{M}_{\mathcal{T}}$ the quotient $\mathcal{B}_{\mathcal{T}}/\big(\mathcal{A}(\mathcal{T})\ltimes G_{\mathcal{T}}\big)$. The group

$$G_{\mathcal{T}}^*\equiv\prod_{i\in I-\hat{I}}S^1$$

acts on $\mathcal{U}_{\mathcal{T}}^{(0)}$ and $\mathcal{M}_{\mathcal{T}}$ as follows. If

$$[b] = \big[(S^2, M, I; x, (j,y), u)\big] \in \mathcal{M}_{\mathcal{T}} \quad \text{and} \quad g = (g_i)_{i\in I-\hat{I}} \in G^*_{\mathcal{T}},$$

define $g[b] = \big[(S^2, M, I; gx, (j, gy), gu)\big]$ by

$$(gx)_h = \begin{cases} x_h, & \text{if } \iota_h \in \hat{I}; \\ g_{\iota_h} x_h, & \text{if } \iota_h \notin \hat{I}; \end{cases} \quad (gy)_l = \begin{cases} y_l, & \text{if } j_l \in I; \\ g_{j_l} y_l, & \text{if } j_l \notin \hat{I}; \end{cases} \quad (gu)_i = \begin{cases} u_i, & \text{if } i \in \hat{I}; \\ g_i \cdot u_i, & \text{if } i \notin \hat{I}. \end{cases}$$

As in the previous paragraph, all the stabilizers are finite. Furthermore, this $G_{\mathcal{T}^*}$-action on $\mathcal{M}_{\mathcal{T}}$ naturally lifts to an action on $\mathcal{M}_{\mathcal{T}}^{(0)}$ and along with the $G_{\mathcal{T}}$-action on $\mathcal{M}_{\mathcal{T}}^{(0)}$ induces an action of $\tilde{G}_{\mathcal{T}} \equiv G^*_{\mathcal{T}} \times G_{\mathcal{T}}$ on $\mathcal{M}_{\mathcal{T}}^{(0)}$ as well as on $F_h^{(0)}\mathcal{T}$ by

$$(g^*, g) \cdot (b, v_h) = \begin{cases} \big((g^*, g) \cdot b, g_{\iota_h} g_h^{-1} v_h\big), & \text{if } \iota_h \in \hat{I}; \\ \big((g^*, g) \cdot b, g^*_{\iota_h} g_h^{-1} v_h\big), & \text{if } \iota_h \notin \hat{I}. \end{cases}$$

Note that $G^*_{\mathcal{T}'} = G^*_{\mathcal{T}}$ whenever $\mathcal{T}' \leq \mathcal{T}$. Let

$$\mathcal{U}_{\mathcal{T}} = \mathcal{U}_{\mathcal{T}}^{(0)}/G^*_{\mathcal{T}}, \qquad \bar{\mathcal{U}}_{\mathcal{T}}^{(0)} = \bigcup_{\mathcal{T}' \leq \mathcal{T}} \mathcal{U}_{\mathcal{T}'}^{(0)}, \qquad \bar{\mathcal{U}}_{\mathcal{T}} = \bigcup_{\mathcal{T}' \leq \mathcal{T}} \mathcal{U}_{\mathcal{T}'}.$$

With respect to the Gromov topology, the space $\bar{\mathcal{U}}_{\mathcal{T}}^{(0)}$ is Hausdorff and compact if (V, ω, J) is semipositive; see [RT]. Furthermore, $G^*_{\mathcal{T}}$ acts continuously on $\mathcal{U}_{\mathcal{T}'}^{(0)}$ as can be easily seen from Definition 2.8. If follows that $\bar{\mathcal{U}}_{\mathcal{T}}$ is also Hausdorff and compact in the quotient topology if (V, ω, J) is semipositive. Denote by $\{L_i\mathcal{T} \longrightarrow \bar{\mathcal{U}}_{\mathcal{T}} : i \in I - \hat{I}\}$ the line orbi-bundles corresponding to the quotient $\bar{\mathcal{U}}_{\mathcal{T}} = \bar{\mathcal{U}}_{\mathcal{T}}^{(0)}/G^*_{\mathcal{T}}$. Let

$$\mathcal{F}_h\mathcal{T} = \Big(F_h^{(0)}\mathcal{T}\big|\mathcal{B}_{\mathcal{T}}\Big)/\tilde{G}_{\mathcal{T}} \longrightarrow \mathcal{U}_{\mathcal{T}}, \quad \mathcal{F}_{h,[b]}\mathcal{T} = \mathcal{F}_h\mathcal{T}\big|[b]; \quad \mathcal{F}\mathcal{T} = \bigoplus_{h\in\hat{I}} \mathcal{F}_h\mathcal{T}, \quad \mathcal{F}_{[b]}\mathcal{T} = \mathcal{F}\mathcal{T}\big|[b].$$

The line bundles $\mathcal{F}_h\mathcal{T}$ have natural norms, defined as in the previous paragraph.

If $\mathcal{T} = (S, M, I; j, \underline{\lambda})$ is a bubble type and $b = \big(S, M, I; x, (j,y), u\big)$ is a $\mathcal{T}$-bubble map, for any $l \in M$, let $\text{ev}_l : \mathcal{H}_{\mathcal{T}} \longrightarrow V$ be the map given by

$$\text{ev}_l\big((S, M, I; x, (j,y), u)\big) = u_{j_l}(y_l).$$

This map descends to the quotients defined above and induces continuous maps on the spaces $\bar{\mathcal{M}}_{\mathcal{T}}$, $\bar{\mathcal{U}}_{\mathcal{T}}^{(0)}$, and $\bar{\mathcal{U}}_{\mathcal{T}}$. If $\mu = \mu_M$ is an M-tuple of submanifolds in V, put

$$\mathcal{H}_{\mathcal{T}}(\mu) = \big\{b \in \mathcal{H}_{\mathcal{T}} : \text{ev}_l(b) \in \mu_l \ \forall l \in M\big\}.$$

Define spaces $\mathcal{M}_{\mathcal{T}}^{(0)}(\mu)$, $\mathcal{M}_{\mathcal{T}}(\mu)$, $\bar{\mathcal{M}}_{\mathcal{T}}(\mu)$, etc. similarly. If $S = S^2$, we define another evaluation map,

$$\text{ev} : \mathcal{B}_{\mathcal{T}} \longrightarrow V \quad \text{by} \quad \text{ev}\big((S^2, M, I; x, (j,y), u)\big) = u_{\hat{0}}(\infty),$$

where $\hat{0}$ is any minimal element of I. This map induces continuous maps on the spaces $\bar{\mathcal{U}}_{\mathcal{T}}^{(0)}$ and $\bar{\mathcal{U}}_{\mathcal{T}}$. If $\mu = \mu_{\tilde{M}}$ is an $\tilde{M}$-tuple of constraints, let

$$\mathcal{U}_{\mathcal{T}}(\mu) = \left\{ b \in \mathcal{U}_{\mathcal{T}} : \mathrm{ev}_l(b) \in \mu_l \ \forall l \in \tilde{M} \cap M, \ \mathrm{ev}(b) \in \mu_l \ \forall l \in \tilde{M} - M \right\}$$

and define $\mathcal{U}_{\mathcal{T}}^{(0)}(\mu)$, etc. similarly.

Chapter 3

The Gluing Construction

We now present a gluing construction on the spaces $\mathcal{M}_{\mathcal{T}}(\mu)$ such that $\mathcal{H}_{\mathcal{T}}$ is a smooth manifold with the tangent bundle isomorphic to the kernel of the linearization of the $\bar{\partial}$-operator, as defined below. The space $\mathcal{H}_{\mathcal{T}}$ is well-known to be smooth if the linearization of the $\bar{\partial}$-operator is surjective; see [MS]. However, surjectivity of the linearization is not a necessary condition; see Chapter 7 for examples. In fact, there are two main cases of primarily interest to us. The first is when $\mathcal{T}=(S^2,M,I;j,\underline{\lambda})$ and the linearization of the $\bar{\partial}$-operator is indeed surjective. In this case, we give an analytic description of a neighborhood of $\mathcal{U}_{\mathcal{T}}(\mu)$ in $\bar{\mathcal{U}}_{\langle\mathcal{T}\rangle}(\mu)$ for a generic set of constraints μ. The second case is when $\mathcal{T}=(\Sigma,M,I;j,\underline{\lambda})$ and the cokernels of the linearization of the $\bar{\partial}$-operator form a vector bundle over $\mathcal{H}_{\mathcal{T}}$, which will be the analogue of Taubes's obstruction bundle of [T] in the gluing construction below. Using the same analysis as in the first case, we describe any sufficiently nice element of $C^{\infty}_{(\lambda;M)}(\Sigma;\mu)$ lying near $\mathcal{M}_{\mathcal{T}}(\mu)$, where $\lambda=\sum\lambda_i$. If

$$\nu\in\Gamma\big(\Sigma\times V;\Lambda^{0,1}\pi_{\Sigma}^*T^*\Sigma\otimes\pi_V^*TV\big)$$

and $t\in\mathbb{R}^+$, the elements of the space

$$\mathcal{M}_{\Sigma,t\nu,\lambda}(\mu)\equiv\big\{(y_1,\ldots,y_N;u)\in C^{\infty}_{\lambda,N}(\Sigma;\mu):\bar{\partial}u|_z=t\nu|_{(z,u(z))}\big\}$$

lying near $\mathcal{M}_{\mathcal{T}}(\mu)$ will correspond to the zero set of a section of the obstruction bundle.

3.1 Notation

For our gluing construction, we fix a smooth family $\{g_{V,b}:b\in\mathcal{M}_{\mathcal{T}}\}$ of Kahler metrics on (V,J). We assume that this family is $\big(\mathcal{A}(\mathcal{T})\ltimes G_{\mathcal{T}}\big)$-invariant if $S=\Sigma$ and $\big(\mathcal{A}(\mathcal{T})\ltimes\tilde{G}_{\mathcal{T}}\big)$-invariant if $S=S^2$. If $b\in\mathcal{M}_{\mathcal{T}}$, $X,Y\in T_qV$, and $u:(\mathcal{D},j)\longrightarrow V$ is a smooth map from a one-dimensional complex manifold, let

$$\exp_{b,q}X=\exp_{g_b,q}X,\quad \nabla^b=\nabla^{g_b},\quad \Pi_{b,X}Y=\Pi_{g_b,X}Y,\quad D_{b,u}=D_{g_b,u};$$

see Section 2.1 for more details. If $S=\Sigma$, we also choose a smooth family

$$\{g_{\mathcal{T},x}:x=(x)_{\{h:\iota_h=\hat{0}\}};\ x_h\in\Sigma;\ x_{h_1}\neq x_{h_2}\text{ if }h_1\neq h_2\}$$

of Riemannian metrics on Σ such that each metric $g_{\mathcal{T},x}$ is flat on a neighborhood of x_h in Σ for all $h\in\hat{I}$ with $\iota_h=\hat{0}$. Existence of such a family of metrics is shown in [FO]. If

$$b=\big(\Sigma,M,I;x,(j,y),u\big)\in\mathcal{H}_{\mathcal{T}},$$

let $g_{b,\hat{0}}$ denote the metric $g_{\mathcal{T},(x)_{\{h:\iota_h=\hat{0}\}}}$ on Σ. If $i\in\hat{I}$, we write $g_{b,i}$ for the standard metric on S^2. Similarly, if $S=S^2$, for all $i\in I$, we write $g_{b,i}$ for the standard metric on S^2.

If $b=\big(S,M,I;x,(j,y),u\big)\in\mathcal{H}_{\mathcal{T}}$, let

$$\Gamma'(b)=\bigoplus_{i\in I}\Gamma(u_i);\quad \Gamma(b)=\Gamma(u_b)=\big\{\xi_I\in\Gamma'(b):\xi_h(\infty)=\xi_{\iota_h}(x_h)\ \forall h\in\hat{I}\big\};$$
$$\Gamma^1(b)=\Gamma^1(u_b)=\bigoplus_{i\in I}\Gamma^1(u_i);\qquad \Gamma^{0,1}(b)=\Gamma^{0,1}(u_b)=\bigoplus_{i\in I}\Gamma^{0,1}(u_i).$$

Define $D_b:\Gamma(b)\longrightarrow\Gamma^{0,1}(b)$ by

$$\big(D_b\xi_I\big)_i=D_{b,u_i}\xi_i\qquad\forall i\in I.$$

We denote the kernel of operator D_b on $\Gamma(b)$ by $\Gamma_-(b)$. If $\xi\in\Gamma(u_i)$ or $\xi\in\Gamma^1(u_i)$, let $\|\xi\|_{b,C^k}$ and $\|\xi\|_{b,2}$ denote the C^k- and L^2-norms of ξ computed with respect to the metrics $g_{V,b}$ on V and $g_{b,i}$ on $\Sigma_{b,i}$. If $\xi=\xi_I\in\Gamma'(b)$ or $\xi\in\Gamma^1(b)$, put

$$\|\xi\|_{b,C^k}=\sum_{i\in I}\|\xi_i\|_{b,C^k},\quad \|\xi\|_{b,2}=\sum_{i\in I}\|\xi_i\|_{b,2}.$$

Let $\pi_{b,-}:\Gamma(b)\longrightarrow\Gamma_-(b)$ be the (L^2,b)-orthogonal projection map.

The space $\mathcal{P}_b\mathcal{T}$ of perturbations of bubble map b is the collection of tuples $\sigma=(\xi_{\hat{I}};w_{\hat{I}+M})$, where

$$\xi_i\in\Gamma(u_i)\ \forall i\in I,\quad w_h\in F^{(0)}_{h,b}\ \forall h\in\hat{I},\quad w_l\in\begin{cases}\mathbb{C}, & \text{if } l\in M\ \&\ \Sigma_{b,j_l}=S^2;\\ T_{y_l}\Sigma, & \text{if } l\in M\ \&\ \Sigma_{b,j_l}=\Sigma.\end{cases}$$

If σ is sufficiently small, we define $\exp_b\sigma=\big(S,M,I;x(\sigma),(j,y(\sigma)),u_\sigma\big)$ by

$$x_h(\sigma)=\begin{cases}x_h+w_h, & \text{if } \Sigma_{b,i_h}=S^2;\\ \exp_{g_{b,\hat{0}},x_h}w_h, & \text{if } \Sigma_{b,i_h}=\Sigma;\end{cases}\qquad y_l(\sigma)=\begin{cases}y_l+w_l, & \text{if } \Sigma_{b,j_l}=S^2;\\ \exp_{g_{b,\hat{0}},y_l}w_l, & \text{if } \Sigma_{b,j_l}=\Sigma;\end{cases}$$

and $u_{\sigma,i}=\exp_{b,u_i}\xi_i$. If $z\in\Sigma$, let $|v|_b=|v|_{g_{b,\hat{0}},x}$. For consistency, if $v\in\mathbb{C}$, let $|v|_b=|v|$. Along with the (L^2,b)-norm on the vector fields defined above, we obtain an inner-product on the space of tuples σ as above.

In order to get a good description of the spaces $\mathcal{M}^{(0)}_{\mathcal{T}}$ as submanifolds of $\mathcal{H}_{\mathcal{T}}$, we describe an action of an open subset of 0 in $\big(\mathbb{C}\oplus\mathbb{R}\oplus\mathbb{R}\big)^{\hat{I}}$ on bubble maps and distinguished elements $\sigma^{(k)}_{(b,i)}\in\mathcal{P}_b\mathcal{T}$ that correspond to this action. If $(c,r,\theta)=(c,r,\theta)_{\hat{I}}\in\big(\mathbb{C}\times\mathbb{R}\times\mathbb{R}\big)^{\hat{I}}$ and b is a bubble map as above, we define

$$(c,r,\theta)\cdot b = \big(S, M, I; (c,r,\theta)x, (j,(c,r,\theta)y), (c,r,\theta)u\big)$$

by setting

$$\big((c,r,\theta)x\big)_h = e^{i\theta_{\iota_h}}(1+r_{\iota_h})(x_h + c_{\iota_h}), \qquad \big((c,r,\theta)y\big)_l = e^{i\theta_{j_l}}(1+r_{j_l})(y_l + c_{j_l}),$$
$$\big((c,r,\theta)u\big)_i\big(q_N(z)\big) = u_i\big(q_N\big((1+r_i)^{-1}e^{-i\theta_i}z - c_i\big)\big).$$

If (c,r,θ) is sufficiently small, $(c,r,\theta)\cdot b$ is again a bubble map, i.e. the maps into V still agree at the nodes, and the nodes and the marked points are still all distinct. In fact, the values of the maps at the nodes or the marked points do not change, i.e.

$$\big((c,r,\theta)u\big)_{\iota_h}\big(((c,r,\theta)x)_h\big) = u_{\iota_h}(x_h), \quad \big((c,r,\theta)u\big)_h(\infty) = u_h(\infty),$$
$$\big((c,r,\theta)u\big)_{j_l}\big(((c,r,\theta)y)_l\big) = u_{j_l}(y_l).$$

Furthermore, if $b \in \mathcal{H}_{\mathcal{T}}$, $(c,r,\theta)\cdot b \in \mathcal{H}_{\mathcal{T}}$. If b is of type $\mathcal{T}$, the above describes the action of a neighborhood of the identity in $\mathcal{G}_{\mathcal{T}}$ on the space of stable maps of type $\mathcal{T}$. The action by $\mathbb{C}$ corresponds to the translations of $\mathbb{C}$, by the first $\mathbb{R}$-component to dilations about the origin, and by the last $\mathbb{R}$-component to rotations about the origin. If $S = S^2$ and $(c,r,\theta) \in \big(\mathbb{C}\times\mathbb{R}\times\mathbb{R}\big)^I$ is sufficiently small, we define $(c,r,\theta)\cdot b$ similarly.

If $u \in C^\infty(S^2;V)$, define $\xi_u^{(1)},\ldots,\xi_u^{(4)} \in \Gamma(u)$ by:

$$\xi_u^{(1)}(q_N(z)) = -d(u\circ q_N)\Big|_z\frac{\partial}{\partial s}, \qquad \xi_u^{(2)}(q_N(z)) = -d(u\circ q_N)\Big|_z\frac{\partial}{\partial t}$$
$$\xi_u^{(3)}(q_N(z)) = -d(u\circ q_N)\Big|_z\Big(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t}\Big) = -rd(u\circ q_N)\Big|_z\frac{\partial}{\partial r},$$
$$\xi_u^{(3)}(q_N(z)) = d(u\circ q_N)\Big|_z\Big(t\frac{\partial}{\partial s} - s\frac{\partial}{\partial t}\Big) = -d(u\circ q_N)\Big|_z\frac{\partial}{\partial \theta}.$$

where we write $z = s+it \in \mathbb{C}$ and $r = \sqrt{s^2+t^2}$. These vector fields extend smoothly by zero over the south pole. For any $x \in S^2 - \{\infty\}$, let $w_x^{(1)},\ldots,w_x^{(4)} \in \mathbb{C}$ be given by

$$w_x^{(1)} = 1, \quad w_x^{(2)} = i, \quad w_x^{(3)} = x, \quad w_x^{(4)} = ix.$$

If b is a bubble map as above, $k = 1,\ldots,4$, $i^* \in \hat{I}$ if $S = \Sigma$ and $i^* \in I$ if $S = S^2$, let

$$\sigma_{(b,i^*)}^{(k)} = \big((\xi_{(b,i^*)}^{(k)})_I, (w_{(b,i^*)}^{(k)})_{\hat{I}+M}\big)$$

be given by

$$\xi_{(b,i^*),i}^{(k)} = \begin{cases} \xi_{u_i}^{(k)}, & \text{if } i = i^*; \\ 0, & \text{if } i \neq i^*; \end{cases} \quad w_{(b,i^*),h}^{(k)} = \begin{cases} w_{x_h}^{(k)}, & \iota_h = i^*; \\ 0, & \iota_h \neq i^*; \end{cases} \quad w_{(b,i^*),l}^{(k)} = \begin{cases} w_{y_l}^{(k)}, & j_l = i^*; \\ 0, & j_l \neq i^*. \end{cases}$$

The tuples $\sigma_{(b,i^*)}^{(k)}$ correspond to the infinitesimal action of $\mathcal{G}_{\mathcal{T}}$ on the space of stable maps of type $\mathcal{T}$.

Finally, if X is any space, $F \longrightarrow X$ a normed vector bundle, and $\delta\colon X \longrightarrow \mathbb{R}$ is any function, let

$$F_\delta = \big\{(b,v)\in F\colon |v|_b < \delta(b)\big\}.$$

Similarly, if Ω is a subset of F, let $\Omega_\delta = F_\delta \cap \Omega$. If $\upsilon=(b,v)\in F$, denote by b_υ the image of υ under the bundle projection map, i.e. b in this case.

3.2 The Basic Setup

In this section, we describe our assumptions on the smooth structure of $\mathcal{H}_\mathcal{T}$ and state some of their implications.

Definition 3.1 *Bubble type $\mathcal{T}=(S^2,M,I;j,\underline{\lambda})$ is* <u>*(V,J)-regular*</u> *if for all*

$$b = \big(S,M,I;x,(j,y),u\big) \in \mathcal{H}_\mathcal{T}$$

(1) $D_{b,u_i}\colon \Gamma(u_i)\longrightarrow\Gamma^{0,1}(u_i)$ is onto for all $i\in I$;
(2) $\ker D_{b,u_i}\longrightarrow T_{u_i(\infty)}V$, $\xi\longrightarrow\xi(\infty)$, is onto for all $i\in I$.

Definition 3.2 *Simple bubble type $\mathcal{T}=(S,M,I;j,\underline{\lambda})$ is* <u>*(V,J)-semiregular*</u> *if*
(1) the space $\mathcal{H}_{(S,\emptyset,\{\hat{0}\};,\lambda_{\hat{0}})}$ is a complex manifold of the same dimension as $\ker D_{g_V,b}$ for all $b\in\mathcal{H}_{(S,\emptyset,\{\hat{0}\};,\lambda_{\hat{0}})}$, and there exist $\delta, C\in C^\infty(\mathcal{H}_{(S,\emptyset,\{\hat{0}\};,\lambda_{\hat{0}})};\mathbb{R}^+)$ and for each $b=(S,\emptyset,\{\hat{0}\};,u_{\hat{0}})$ in $\mathcal{H}_{(S,\emptyset,\{\hat{0}\};,\lambda_{\hat{0}})}$ a map

$$h_{\mathcal{T},\hat{0};b}\colon \big\{\xi\in\ker D_{g_V,b}\colon \|\xi\|_{g_V,C^0}<\delta(b)\big\}\longrightarrow\Gamma(u_{\hat{0}}) \qquad \textit{such that}$$
$$\big\|h_{\mathcal{T},\hat{0};b}(\xi)\big\|_{g_V,b}\le C(b)\|\xi\|^2_{g_V,b},\quad \big\|h_{\mathcal{T},\hat{0};b}(\xi)-h_{\mathcal{T},\hat{0};b}(\xi')\big\|_{g_V,C^0}\le C(b)\big\|\xi-\xi'\big\|_{g_V,C^0},$$

for all $\xi,\xi'\in\ker D_{g_V,b}$ with $\|\xi\|_{g_V,C^0},\|\xi'\|_{g_V,C^0}<\delta(b)$, and the map

$$H_{(S,\emptyset,\{\hat{0}\};,\lambda_{\hat{0}}),b}\colon \big\{\xi\in\ker D_{g_V,b}\colon \|\xi\|_{g_V,C^0}<\delta(b)\big\}\longrightarrow\mathcal{H}_{(S,\emptyset,\{\hat{0}\};,\lambda_{\hat{0}})},$$
$$H_{(S,\emptyset,\{\hat{0}\};,\lambda_{\hat{0}}),b}(\xi)=\exp_{g_V,u_{\hat{0}}}\big(\xi+h_{\mathcal{T},\hat{0};b}(\xi)\big),$$

is an orientation-preserving diffeomorphism onto an open neighborhood of b in $\mathcal{H}_{(S,\emptyset,\{\hat{0}\};,\lambda_{\hat{0}})}$. Furthermore, the family of maps $\big\{H_{(S,\emptyset,\{\hat{0}\};,\lambda_{\hat{0}}),b}\colon b\in\mathcal{H}_{(S,\emptyset,\{\hat{0}\};,\lambda_{\hat{0}})}\big\}$ is smooth.
(2) for all $b=\big(S,M,I;x,(j,y),u\big)\in\mathcal{H}_\mathcal{T}$
(2a) $D_{b,u_h}\colon\Gamma(u_h)\longrightarrow\Gamma^{0,1}(u_h)$ is onto for all $h\in\hat{I}$;
(2b) $\ker D_{b,u_h}\longrightarrow T_{u_h(\infty)}V$, $\xi\longrightarrow\xi(\infty)$, is onto for all $h\in\hat{I}$.

Remarks: (1) All conditions in both definitions above are independent of the choice of metric on V.
(2) Condition (1) of Definition 3.2 says that $\mathcal{H}_{(S,\emptyset,\{\hat{0}\};,\lambda_{\hat{0}})}$ is a smooth manifold modeled on $\ker D_b$ for $b\in\mathcal{H}_{(S,\emptyset,\{\hat{0}\};,\lambda_{\hat{0}})}$, as would be the case if $D_b\colon\Gamma(u_b)\longrightarrow\Gamma^{0,1}(u_b)$ were surjective.
(3) The conditions of Definitions 3.1 and 3.2 insure that $\mathcal{H}_\mathcal{T}$ is a smooth manifold; see Proposition 3.3 below. However, (2) of Definition 3.1 and (2b) of Definition 3.2 are somewhat stronger than necessary to show that $\mathcal{H}_\mathcal{T}$ is smooth. They allow us to obtain the second part of (1) in Proposition 3.3, which is used in the proof of surjectivity of the gluing map; see Section 4.3. These two conditions hold for all complex homogeneous manifolds; see [RT].

Note that if $\mathcal{T}$ is semiregular, the homotopy invariance of the index implies that the vector spaces

$$\Gamma_-(b) \equiv \text{coker } D_b \approx \ker D_b^* \subset \Gamma^{0,1}(b), \quad b \in \mathcal{H}_\mathcal{T},$$

form a vector bundle over $\mathcal{H}_\mathcal{T}$. Here D_b^* denotes the formal adjoint of D_b with respect to a metric g on S; it is a J-linear operator. The space $\ker D_b^*$ is independent of a conformal choice of the metric g. The bundle $\Gamma_- \longrightarrow \mathcal{H}_\mathcal{T}$ will be called the *$\mathcal{T}$-cokernel bundle.* It is $(\mathcal{A}(\mathcal{T}) \ltimes G_\mathcal{T})$-equivariant, and thus descends to a bundle $\Gamma_- \longrightarrow \mathcal{M}_\mathcal{T}$, which will be the analogue of Taubes's obstruction in our gluing setting.

Let $\mathcal{T} = (S, M, I; j, \underline{\lambda})$ be a bubble type. If $b = (S, M, I; x, (j, y), u) \in \mathcal{H}_\mathcal{T}$, put

$$\mathcal{K}_b\mathcal{T} = \Big\{\sigma = (\xi, w_{\hat{I}+M}) \in \mathcal{P}_b\mathcal{T} : \xi_i \in \ker(D_{b,u_i})\ \forall i \in I;\ \langle \sigma, \sigma^{(k)}_{(b,h)} \rangle = 0\ \forall h \in \hat{I}, k \in [4];$$
$$\xi_h(\infty) = \xi_{\iota_h}(x_h) + du_{\iota_h}\big|_{x_h} w_h\ \forall h \in \hat{I}\Big\}.$$

If $\sigma = (\xi, w_{\hat{I}+M}) \in \mathcal{K}_b\mathcal{T}$, let

$$\|\sigma\|_{b,C^k} = \|\xi\|_{b,C^k} + \sum_{h \in \hat{I}} |w_h|_b + \sum_{l \in M} |w_l|_b.$$

We take the default norm on $\mathcal{K}_b\mathcal{T}$ be given by $\|\cdot\|_{b,C^0}$. If $b' = (S, M, I; x', (j, y'), u)$ and $\delta > 0$, we say $d(b, b') < \delta$ if there exists $\sigma \in \mathcal{P}_b\mathcal{T}$ such that $\exp_b \sigma = b'$ and $\|\sigma\|_{b,C^0} \leq \delta$.

Proposition 3.3 *(1) If $\mathcal{T} = (S, M, I; j, \underline{\lambda})$ is a regular or semiregular bubble type, $\mathcal{H}_\mathcal{T}$ is a complex manifold and there exist $\epsilon_\mathcal{T}, C_\mathcal{T} \in C^\infty(\mathcal{M}^{(0)}_\mathcal{T}; \mathbb{R}^+)$ with the following property. If $b^* \in \mathcal{H}_\mathcal{T}$ and*

$$b = (S, M, I; x, (j, y), u) \quad \text{is s.t.} \quad d(b^*, b) < \epsilon_\mathcal{T}(b^*) \text{ and } \bar{\partial} u_i = 0\ \forall i \in I,$$

there exist $\xi_i \in \Gamma(u_i)$ for $i \in \hat{I}$ such

$$\|\xi_i\|_{g_V, C^0} \leq C_\mathcal{T}(b^*) \sum_{h \in \hat{I}} d_V\big(u_{\iota_h}(x_h), u_h(\infty)\big) \quad \text{and} \quad b' = (S, M, I; x, (j, y), u') \in \mathcal{H}_\mathcal{T},$$

where $u'_{\hat{0}} = u_{\hat{0}}$ and $u'_i = \exp_{g_V, u_i} \xi_i$ if $i \in \hat{I}$.
(2) The space $\mathcal{M}^{(0)}_\mathcal{T}$ is a smooth oriented manifold on which the group $G_\mathcal{T}$ acts smoothly. The maps

$$\begin{aligned}
ev &: \mathcal{M}^{(0)}_\mathcal{T} \longrightarrow V, & ev(S, M, I; x, (j, y), u) &= u_{\hat{0}}(\infty), \\
ev_l &: \mathcal{M}^{(0)}_\mathcal{T} \longrightarrow V, & ev_l(S, M, I; x, (j, y), u) &= u_{j_l}(y_l), \\
du_i|_z &: \mathcal{M}^{(0)}_\mathcal{T} \longrightarrow T^*\Sigma_{\mathcal{T},i} \otimes u_i^* TV, & du_i|_z(S, M, I; x, (j, y), u) &= du_i|_z,
\end{aligned}$$

are smooth. In particular, $u_i \longrightarrow \|du_i\|_{b,C^0}$ defines a continuous function on $\mathcal{M}^{(0)}_\mathcal{T}$.
(3) There exist $\delta_\mathcal{T}, C_\mathcal{T} \in C^\infty(\mathcal{M}^{(0)}_\mathcal{T}; \mathbb{R}^+)$ and smooth maps

$$h_{\mathcal{T},b} = h^{(1)}_{\mathcal{T},b} \oplus h^{(2)}_{\mathcal{T},b} : \mathcal{K}_b\mathcal{T}_{\delta_\mathcal{T}(b)} \longrightarrow \Gamma'(b) \oplus (\mathbb{C} \oplus \mathbb{R})^{\hat{I}},$$

such that $\left\|h_{\mathcal{T},b}(\sigma)\right\|_{b,C^0} \le C_{\mathcal{T}}(b)\|\sigma\|^2_{b,C^0}$,

$$\left\|h_{\mathcal{T},b}(\sigma) - h_{\mathcal{T},b}(\sigma')\right\|_{b,C^0} \le C_{\mathcal{T}}(b)\left(\|\sigma\|_{b,C^0} + \|\sigma'\|_{b,C^0}\right)\|\sigma - \sigma'\|_{b,C^0},$$

and each map

$$H^{(0)}_{\mathcal{T},b}\colon \left\{(\sigma,\theta)\in\mathcal{K}_b\mathcal{T}_{\delta_{\mathcal{T}}(b)}\times\mathbb{R}^{\hat{I}} : |\theta|<\pi\right\} \longrightarrow \mathcal{M}^{(0)}_{\mathcal{T}},$$
$$H^{(0)}_{\mathcal{T},b}(b,\sigma,\theta) = \left(h^{(2)}_{\mathcal{T},b}(\sigma),\theta\right)\cdot \exp_b\left(\sigma + h^{(1)}_{\mathcal{T},b}(\sigma)\right),$$

is orientation-preserving diffeomorphism onto an open neighborhood of b in $\mathcal{M}^{(0)}_{\mathcal{T}}$.

Proof: (1) Let $\mathcal{T}_i = (\Sigma_{\mathcal{T},i}, \{l\colon j_l=i\}+\{h\colon \iota_h=i\}, \{\hat{0}\}; \hat{0}, \lambda_i)$. By (1) of Definition 3.1, and (1) and (2a) of Definition 3.2, $\mathcal{H}_{\mathcal{T}_i}$ is a complex manifold for all $i\in I$. Let

$$\triangle^{\hat{I}}_V = \left\{(q,q)_{\hat{I}} \in \prod_{\hat{I}}(V\times V) : q_h\in V\right\}.$$

The submanifold $\triangle^{\hat{I}}_V$ is the $\hat{I}$-product of the diagonal in $V\times V$. Since V is oriented, so is the normal bundle of $\triangle^{\hat{I}}_V$. Claim (1) of the proposition follows by applying the Implicit Function Theorem, (2) of Definition 3.1 and (2b) of Definition 3.2 to the smooth map

$$\mathrm{ev}_{\hat{I}}\colon \prod_{i\in I}\mathcal{H}_{\mathcal{T}_i} \longrightarrow \prod_{\hat{I}}(V\times V), \quad \mathrm{ev}_h\big((S,M,I;x,(j,y),u)\big) = \big(u_h(\infty), u_{\iota_h}(x_h)\big).$$

Note that $\mathcal{H}_{\mathcal{T}} = \mathrm{ev}^{-1}_{\hat{I}}\left(\triangle^{\hat{I}}_V\right)$.
(2) For any $u\in C^\infty(S^2;V)$, define $\tilde{\Psi}u\in\mathbb{C}$, $\Psi^{(3)}u\in\mathbb{R}$, and $\Psi u\in\mathbb{C}\times\mathbb{R}$ by

$$\Psi u = \left(\tilde{\Psi}u, \Psi^{(3)}u\right) = \left(\int_{\mathbb{C}} |du\circ q_N|^2 z, \int_{\mathbb{C}} |du\circ q_N|^2\beta(|z|) - \frac{1}{2}\right),$$

where the integrals are computed using the metric g_V. For $i^*\in\hat{I}$ if $S=\Sigma$ and $i^*\in I$ if $S=S^2$, we define maps

$$\Psi_{\mathcal{T},i^*}\colon \prod_{i\in I}\mathcal{H}_{\mathcal{T}_i} \longrightarrow \mathbb{C}\times\mathbb{R} \qquad \text{by}$$
$$\Psi_{\mathcal{T},i^*}\big(S,M,I;x,(j,y),u\big) = \Big(\tilde{\Psi}u_{i^*} + \sum_{\iota_h=i^*} d_h(\mathcal{T})x_h + \sum_{j_l=i^*} y_l,$$
$$\Psi^{(3)}u_{i^*} + \sum_{\iota_h=i^*} d_h(\mathcal{T})\beta(|x_h|) + \sum_{j_l=i^*}\beta(|y_l|)\Big).$$

These maps $\Psi_{\mathcal{T},i^*}$ are smooth, since the smooth structure on all $\mathcal{H}_{\mathcal{T}_i}$ is described similarly to (1) of Definition 3.2. Furthermore, if $b\in\mathcal{M}^{(0)}_{\mathcal{T}}$, $i^*\in\hat{I}$, and $k^*=1,2,3$, since $\Psi_{\mathcal{T},i}(b)=0$ for all i and β' does not change sign, by Lemma 3.4,

$$d\Psi^{(k^*)}_{\mathcal{T},i^*}\Big|_b \sigma^{(k)}_{(b,i)} \begin{cases} = 0, & \text{if } i\neq i^*;\\ \neq 0, & \text{if } i=i^*, k=k^*;\\ = 0, & \text{if } k\neq k^*\neq 3,\end{cases}$$

where $k=1,2,3$. By (2) of Definition 3.1 and (2b) of Definition 3.2, it follows that the map

$$\prod_{i\in I}\mathcal{H}_{\mathcal{T}_i}\longrightarrow(\mathbb{C}\times\mathbb{R})^{\hat{I}}\times\prod_{\hat{I}}(V\times V),\quad b\longrightarrow\Big(\big(\Psi_{\mathcal{T},i}(b)\big)_{i\in\hat{I}},\mathrm{ev}_{\hat{I}}(b)\Big),$$

is transversal to the submanifold $\{0\}\times\Delta_V^{\hat{I}}$. The preimage of this submanifold is precisely the space $\mathcal{M}_{\mathcal{T}}^{(0)}$. Thus, $\mathcal{M}_{\mathcal{T}}^{(0)}$ is a smooth oriented manifold by the Implicit Function Theorem.

Lemma 3.4 *For any $k\in[4]$ and $u\in C^\infty(S^2;V)$, $\xi^{(k)}(\infty)=0$. Furthermore,*

$$\tilde{\Psi}\big((c,r,\theta)\cdot u\big)=(1+r)\big(\tilde{\Psi}u+c\|du\|_2^2\big)\quad\forall(c,r)\in\mathbb{C}\times\mathbb{R};\tag{3.1}$$

$$\frac{d}{dr}\Psi^{(3)}\big((0,r,\theta)\cdot u\big)\Big|_{r=0}=\int_{\mathbb{C}}|d(u\circ q_N)|^2\beta'(|z|)|z|,\tag{3.2}$$

where $(c,r)\cdot u$ is defined as in Section 3.1. Finally, $D_u\xi_u^{(k)}=0$ if $\bar{\partial}u=0$.

Proof: The first and last statements are immediate. To prove (3.1), we use the change of $z\longrightarrow(1+r)^{-1}z-c$.

$$\begin{aligned}\int_{\mathbb{C}}\big|d\big(((c,r)\cdot u)\circ q_N\big)\big|^2z&=\int_{\mathbb{C}}(1+r)^{-2}\big|d(u\circ q_N)\big|^2_{(1+r)^{-1}z-c}z\\&=(1+r)\int_{\mathbb{C}}\big|d\big(u\circ q_N\big)\big|^2(z+c)=(1+r)\big(\tilde{\Psi}u+c\|du\|_2^2\big),\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{d}{dr}\int_{\mathbb{C}}\big|d\big((r\cdot u)\circ q_N\big)\big|^2\beta(|z|)\Big|_{r=0}&=\frac{d}{dr}\int_{\mathbb{C}}\big|d(u\circ q_N)\big|^2\beta\big((1+r)|z|\big)\Big|_{r=0}\\&=\int_{\mathbb{C}}\big|d(u\circ q_N)\big|^2\beta'\big(|z|\big)|z|.\end{aligned}$$

The lemma is now proved, since the action by the θ-component does not change $\tilde{\Psi}$.

If $\mathcal{T}=(S^2,M,I;j,\underline{\lambda})$ is a regular bubble type, with notation as above, let

$$\tilde{\mathcal{K}}_b\mathcal{T}=\big\{\sigma=(\xi_I,w_{M+\hat{I}})\in\mathcal{K}_b\mathcal{T}\colon\langle\sigma,\sigma^k_{(b,\hat{0})}\rangle=0\ \forall k\in[4],\ \xi_{i_1}(\infty)=\xi_{i_1}(\infty)\ \forall i_1,i_2\in I-\hat{I}\big\}.$$

By (3) of Definition 3.1 and the same argument as in the proof of Proposition 3.3, we can construct smooth maps $h^{(1)}_{\mathcal{T},b}\times h^{(2)}_{\mathcal{T},b}:\tilde{\mathcal{K}}_b\mathcal{T}_{\delta(b)}\longrightarrow\Gamma'(b)\times(\mathbb{C}\times\mathbb{R})^I$ such that each map

$$\begin{aligned}H^{(0)}_{\mathcal{T},b}&:\big\{(b,\sigma,\theta)\in\mathcal{K}_b\tilde{\mathcal{T}}_{\delta(b)}\times\mathbb{R}^I:|\theta|<\pi\big\}\longrightarrow\mathcal{B}_{\mathcal{T}},\\H^{(0)}_{\mathcal{T},b}(\sigma,\theta)&=\big(h^{(2)}_{\mathcal{T},b}(\sigma),\theta\big)\cdot\exp_b\big(\sigma+h^{(1)}_{\mathcal{T},b}(\sigma)\big),\end{aligned}$$

is orientation-preserving diffeomorphism onto an open neighborhood of b in $\mathcal{B}_{\mathcal{T}}$.

3.3 Construction of Nearly Holomorphic Bubble Maps

Let $\mathcal{T}=(S,M,I;j,\underline{\lambda})$ be a simple bubble type. In this section, for all $b\in\mathcal{M}_{\mathcal{T}}^{(0)}$ and $\upsilon=(b,v_{\hat{I}})$ with $v_{\hat{I}}\in F_b^{(0)}\mathcal{T}$ sufficiently small, we construct a bubble map $b(\upsilon)$ with domain Σ_υ, where Σ_υ is as in Section 2.3. The map $u_{b(\upsilon)}$ will be just the composite $u_b\circ q_\upsilon$. We then define a Riemannian metric $g_{\upsilon,i}$ and a nonnegative function $\rho_{\upsilon,i}$ on each component $\Sigma_{\upsilon,i}$ of Σ_υ. The metrics will be such that the C^0-norm of the differential of q_υ is bounded independently of $v_{\hat{I}}$. The nonnegative functions are used to modify the Sobolev norms, in such a way that the norm of the inverse of the operator $D_{b(\upsilon)}$ on certain subspaces of $\Gamma(b(\upsilon))$ is bounded independently of $v_{\hat{I}}$.

By Proposition 3.3, $\mathcal{M}_{\mathcal{T}}^{(0)}$ is a smooth manifold. If $S=S^2$, let $\delta_{\mathcal{T}}\in C^\infty(\mathcal{M}_{\mathcal{T}}^{(0)};\mathbb{R}^+)$ be a $(\mathcal{A}(\mathcal{T})\ltimes\tilde{G}_{\mathcal{T}})$-invariant function such that $\delta_{\mathcal{T}}(b)<r_{\mathrm{T}_b}$ for all $b\in\mathcal{M}_{\mathcal{T}}^{(0)}$. If $S=\Sigma$, let $\delta_{\mathcal{T}}\in C^\infty(\mathcal{M}_{\mathcal{T}}^{(0)};\mathbb{R}^+)$ be a $(\mathcal{A}(\mathcal{T})\ltimes G_{\mathcal{T}})$-invariant function such that for all

$$b=\big(\Sigma,M,I;x,(j,y),u\big)\in\mathcal{M}_{\mathcal{T}}^{(0)},$$

(A1) $4\delta_{\mathcal{T}}$ is smaller than the function δ of Lemma B.1;
(A2) $4\delta_{\mathcal{T}}(b)<r_{\mathcal{C}_b}g_{b,\hat{0}}$.
In both cases, it can be assumed that $\delta_{\mathcal{T}}$ does not exceed $\frac{1}{4}$.

If H is a subset of $\hat{I}$, put

$$F^{(H)}\mathcal{T}=\big\{\upsilon=(b,v_{\hat{I}})\in F^{(0)}\mathcal{T}\colon v_h=0 \text{ if and if } h\in H\big\},$$
$$F^{H}\mathcal{T}=\big\{\upsilon=[b,v_{\hat{I}}]\in F\mathcal{T}\colon v_h=0 \text{ if and if } h\in H\big\}.$$

For any $\upsilon=(b,v_{\hat{I}})\in F^{(0)}\mathcal{T}$, let $|\upsilon|$ denote $|\upsilon|_{g_b}$ if $S=\Sigma$. From now on, we assume that $\delta\in C^\infty(\mathcal{M}_{\mathcal{T}}^{(0)};\mathbb{R}^+)$ is a $(\mathcal{A}(\mathcal{T})\ltimes G_{\mathcal{T}})$-invariant function if $S=\Sigma$ and a $(\mathcal{A}(\mathcal{T})\ltimes\tilde{G}_{\mathcal{T}})$-invariant function if $S=S^2$ such that $8\delta^{\frac{1}{2}}\le\delta_{\mathcal{T}}$. If

$$\upsilon=\big(b_\upsilon,v_{\hat{I}}\big)=\big((S,M,I;x,(j,y),u),v_{\hat{I}}\big)\in F^{(0)}\mathcal{T}_\delta,$$

let $q_\upsilon\colon\Sigma_\upsilon\longrightarrow\Sigma_{b_\upsilon}$ be the smooth map defined in Section 2.3 for

$$\upsilon=\big(\mathcal{C},v_{\hat{I}}\big)=\big((S,M,I;x,(j,y)),v_{\hat{I}}\big),$$

using the metric $g_{b_\upsilon,\hat{0}}$ on Σ if $S=\Sigma$. Let $u_\upsilon=u_{b_\upsilon}\circ q_\upsilon$ and $b(\upsilon)=\big(\mathcal{C}(\upsilon),u_\upsilon\big)$.

We now define a Riemannian metric $g_{\upsilon,i}$ on $\Sigma_{\upsilon,i}$ for each $i\in I(\upsilon)\subset I$. Along the way, we construct a metric $g_{\upsilon,i}$ on $\Sigma_{b_\upsilon,i}$ for each $i\in I$. Suppose $i\in I$ and for all $h\in\hat{I}$ such that $\iota_h=i$, we have constructed a metric $g_{\upsilon,h}$ on $\Sigma_{b_\upsilon,h}$. For each $h\in\hat{I}$ such that $\iota_h=i$ and $v_h\neq0$, let $\tilde{g}_{\upsilon,i,h}$ denote the metric on $B_{b_\upsilon,h}\big(2\delta(b_\upsilon)^{\frac{1}{2}}\big)$ which is the pullback of the metric $g_{\upsilon,h}$ by the map

$$z\longrightarrow q_N\Big(\frac{\phi_{b_\upsilon,h}}{v_h}\Big),\qquad\text{where}\quad \phi_{b_\upsilon,h}=\begin{cases}\phi_{\mathrm{T}_{b_\upsilon},h}, & \text{if } x_h\in S^2;\\ \phi_{\mathrm{T}_{b_\upsilon},g_b,h}, & \text{if } x_h\in\Sigma.\end{cases}$$

This metric is conformal with the original metric $g_{b_\upsilon,i}$ on $\Sigma_{b_\upsilon,i}$, because the maps $\phi_{b,h}$ are holomorphic on the set $\{r_{b,h} \le \delta_{\mathcal{T}}(b)\}$ and the metric $g_{v,h}$ is conformal with the standard metric on $\mathbb{C}$. Thus, there exists a smooth positive function $\lambda_{v,i,h}$ such that $\tilde{g}_{v,i,h} = \lambda^2_{v,i,h} g_{b_v,i}$. Let $\lambda_{v,i} \in C^\infty(\Sigma_{b_v,i};\mathbb{R}^+)$ be given by

$$\lambda_{v,i}(z) = \begin{cases} \lambda_{v,i,h}(z) + \beta_{|v_h|}\big(r_{b_v,h}(z)\big)\big(1-\lambda_{v,i,h}(z)\big), & \text{if } \iota_h = i \text{ and } r_{b_v,h}(z) \le 2|v_h|^{\frac{1}{2}};\\ 1, & \text{if } r_{b_v,h}(z) \ge 2|v_h|^{\frac{1}{2}}\ \forall h \in \hat{I}. \end{cases}$$

Since I is a rooted tree, this procedure defines metrics $g_{v,i}$ for each $i \in I(v)$.

In addition, we define a smooth nonnegative function $\rho_{v,i}$ on $\Sigma_{v,i}$ for each $i \in I(v)$. As in the previous paragraph, along the way we define a function $\rho_{v,i}$ for each $i \in I$. Suppose $i \in I$ and for all $h \in \hat{I}$ such that $\iota_h = i$, we have constructed a smooth function $\rho_{v,h}$ on $\Sigma_{b_v,h}$. For $h \in \hat{I}$ with $\iota_h = i$ and $z \in \Sigma_{b_v,i}$ with $|z|_h \equiv r_{b_v,h}(z) \le 2\delta_{\mathcal{T}}(b_v)$, put

$$\rho_{v,i}(z) = \begin{cases} \rho_{v,h}(q_{h,v_h}z) + \beta\Big(\frac{\delta_{\mathcal{T}}(b_v)|z|_h}{|v_h|}\Big)\Big\{\Big(|z|_h^2 + \frac{|v_h|^2}{|z|_h^2}\Big) - \tilde{\rho}_{v,h}(q_{h,v_h}z)\Big\}, & \text{if } |z|_h \le \delta_{\mathcal{T}}(b_v);\\ \Big(|z|_h^2 + \frac{|v_h|^2}{|z|_h^2}\Big) + \beta\Big(\frac{|z|_h}{\delta_{\mathcal{T}}(b_v)}\Big)\Big\{1 - \Big(|z|_h^2 + \frac{|v_h|^2}{|z|_h^2}\Big)\Big\}, & \text{if } |z|_h \ge \delta_{\mathcal{T}}(b_v), \end{cases}$$

if $v_h \neq 0$, where q_{h,v_h} is defined as in Section 2.4, using the metric $g_{b_v,\hat{0}}$ on Σ if $S = \Sigma$. If $v_h = 0$ and z is as above, let

$$\rho_{v,i}(z) = |z|_h^2 + \beta\Big(\frac{|z|_h}{\delta_{\mathcal{T}}(b_v)}\Big)\big\{1 - |z|_h^2\big\}.$$

If $|z|_h \ge 2\delta_{\mathcal{T}}(b_v)$ for all $h \in \hat{I}$ with $\iota_h = i$ and $v_i \neq 0$ if $i > 0$, set $\rho_{v,i}(z) = 1$. Otherwise, let

$$\rho_{v,i}(z) = |q_S^{-1}(z)|^2 + \beta\big(\delta_{\mathcal{T}}(b_v)|q_S^{-1}(z)|\big)\big\{1 - |q_S^{-1}(z)|^2\big\}.$$

This construction defines nonnegative functions $\rho_{v,i}$ on $\Sigma_{v,i}$ for all $i \in I(v)$.

We finally define norms on the spaces $\Gamma(u_v)$ and $\Gamma^1(u_v)$. If $\eta_i \in \Gamma^1(u_{v,i})$, put

$$2\|\eta_i\|_{v,p;i} = \Big(\int_{\Sigma_{v,i}} |\eta_i|^p\Big)^{\frac{1}{p}} + \Big(\int_{\Sigma_{v,i}} \rho_{v,i}^{-\frac{p-2}{p}} |\eta_i|^2\Big)^{\frac{1}{2}}, \tag{3.3}$$

where $|\eta_i|$ and the integrals are computed with respect to the metric $g_{v,i}$ on $\Sigma_{v,i}$ and g_{V,b_v} on V. Denote by $\|\eta_i\|_{v,C^0;i}$ the C^0-norm of η_i with respect to these metrics. If $\eta = \eta_{I(v)} \in \Gamma^1(u_v)$, let

$$\|\eta\|_{v,p} = \sum_{i\in I(v)} \|\eta_i\|_{v,p;i}, \qquad \|\eta\|_{v,C^0} = \sum_{i \in I(v)} \|\eta_i\|_{v,C^0;i}.$$

Similarly, for any $\xi_i \in \Gamma(u_{v,i})$, put

$$2\|\xi_i\|_{v,p;i} = \Big(\int_{\Sigma_{v,i}} |\xi_i|^p\Big)^{\frac{1}{p}} + \Big(\int_{\Sigma_{v,i}} \rho_{v,i}^{-\frac{p-2}{p}} |\xi_i|^2\Big)^{\frac{1}{2}}; \quad \|\xi\|_{v,p,1;i} = \|\xi_i\|_{v,p;i} + \|\nabla\xi_i\|_{v,p;i}, \tag{3.4}$$

where we again use the metrics $g_{v,i}$ on $\Sigma_{v,i}$ and g_{V,b_v} on V as in (3.3). Denote by $\|\xi_i\|_{v,C^0;i}$

the C^0-norm of ξ_i with respect to the metric g_{V,b_v} on V. If $\xi = \xi_{I(v)} \in \Gamma(u_v)$, let

$$\|\xi\|_{v,p} = \sum_{i\in I(v)} \|\xi_i\|_{v,p;i}, \qquad \|\xi\|_{v,p,1} = \sum_{i\in I(v)} \|\xi_i\|_{v,p,1;i}, \qquad \|\xi\|_{v,C^0} = \sum_{i\in I(v)} \|\xi_i\|_{v,C^0;i}.$$

Note that even though the functions $\rho_{v,i}^{-\frac{p-2}{p}}$ have poles at the singular points of Σ_v, all smooth one-forms and vector fields have finite norms defined by (3.3) and (3.4), respectively, since $\frac{p-2}{p} < 1$. We denote by $L^p_1(v)$ the completion of $\Gamma(u_v)$ with respect to the $(v,p,1)$-norm and by $L^p(v)$ the completion of $\Gamma^{0,1}(u_v)$ with respect to the (v,p)-norm. Finally, let

$$D_v : \Gamma(u_v) \longrightarrow \Gamma^{0,1}(u_v)$$

denote the linearization of the $\bar{\partial}$-operator at u_v with respect to the metric g_{V,b_v} on V.

Lemma 3.5 *If $\mathcal{T}$ is a simple bubble type and $p>2$, there exist $\delta, C \in C^\infty(\mathcal{M}^{(0)}_{\mathcal{T}}; \mathbb{R}^+)$ such that for all $v \in F^{(0)}\mathcal{T}_\delta$,*
(1) $\|du_v\|_{v,C^0} \le C(b_v)$ and $\|\bar{\partial} u_v\|_{v,p} \le C(b_v)|v|^{\frac{1}{p}}$;
(2) $\|D_v\xi\|_{v,p} \le C(b_v)\|\xi\|_{v,p,1}$ for all $\xi \in \Gamma(u_v)$;
(3) $\|\xi\|_{v,C^0} \le C(b_v)\|\xi\|_{v,p,1}$ for all $\xi \in \Gamma(u_v)$;
(4) $\|\xi\|_{v,p,1} \le C(b_v)\big(\|D_v\xi\|_{v,p} + \|\xi\|_{v,p}\big)$ for all $\xi \in \Gamma(u_v)$.

Proof: If $h \in I - I(v)$ and $S = S^2$, let $A^\pm_{v,h}$ be the annulus as in Section 2.3. If $S = \Sigma$, let $A^\pm_{v,h}$ denote $A^\pm_{g_{b_v},v,h}$. By definition of the norms, q_v is an isometry outside of such annuli, and by Lemma 2.3 the C^0-norm of dq_v is bounded on such annuli independently of $v_{\hat{I}}$. Thus, the first part of (1) follows from (2) of Proposition 3.3. Since $\rho_v \ge |v_h|$ on $A_{v,h}$, the second part of (1) follows from Lemma 2.3. Statement (2) of the lemma is immediate from the definition of the norms. The last two claims are proved in Appendix B; see Propositions B.7 and B.11. In fact, the C^0-norm of ξ is bounded by the usual L^p_1-norm of ξ.

3.4 Scale of Variations

In Section 3.6, we consider perturbations of the bubble maps $\{b(v)\}$ in directions "away" from the space of such bubble maps. More precisely, we look at replacing u_v by $\exp_{b_v,u_v}\xi$ with ξ lying in a certain subspace of $L^p_1(v)$ complementary to "the tangent space" of the space of maps $\{b(v)\}$. If $\mathcal{T}$ is regular, one obvious candidate for such a subspace is the (L^2, v)-orthogonal complement of the kernel of D_v. While the construction in Section 3.6 would go through, we would run into significant difficulty showing injectivity and surjectivity of the gluing map; see Sections 4.2 and 4.5. In this section, we start by describing a choice of the complementary subspace which will work for the purposes of Sections 3.6, 4.2, and 4.5. We then describe norms on the tangent spaces to $F\mathcal{T}$ and the properties of our setup that are sufficient to show injectivity and surjectivity of the gluing map.

Suppose $v = \big((S,M,I;x,(j,y),u),v\big) \in F^{(0)}\mathcal{T}_\delta$, where $\mathcal{T}$ is a simple bubble type as before. For any $\xi \in \Gamma(b_v)$, define $R_v\xi \in L^p_1(v)$ by

$$\{R_v\xi\}(z) = \xi(q_v(z)).$$

Note that $R_v\xi$ is smooth outside of the $|I-I(v)|$ circles mapped by q_v to the nodes of Σ_{v_b} and is continuous everywhere, since $\Gamma(b_v)$ is the set of smooth vector fields on the components of Σ_{b_v} that agree at the nodes. It follows that $R_v\xi$ is indeed of class L^p_1. Let $\Gamma_-(v)$ be the image of $\ker(D_{b_v})$ under the map R_v. This space models the "tangent bundle" to the space of maps $\{b(v)\}$. Denote by $\Gamma_+(v)$ its (L^2,g_v)-orthogonal complement in $L^p_1(v)$. Let $\pi_{v,-}$ and $\pi_{v,+}$ be the (L^2,g_v)-orthogonal projections onto $\Gamma_-(v)$ and $\Gamma_+(v)$, respectively.

With $H\subset\hat{I}$ and $v\in F^{(H)}\mathcal{T}_\delta$, let

$$T_vF^H\mathcal{T}=\{\varpi=(\xi,w_{\hat{I}+M},\theta_{\hat{I}},r_{\hat{I}-H})\colon(\xi,w_{\hat{I}+M})\in\mathcal{K}_{b_v}\mathcal{T};\ \theta_h,r_h\in\mathbb{R}\};$$
$$\tilde{T}_vF^H\mathcal{T}=\{(\xi,w_{\hat{I}+M},\theta_{\hat{I}},r_{\hat{I}-H})\in T_vF^H\mathcal{T}\colon w_h=0\ \forall h\in H\}.$$

Given ϖ as above, put

$$\|\varpi\|=\|\xi\|_{b_v,C^0}+\sum_{h\in\hat{I}}|w_h|_{b_v}+\sum_{l\in M}|w_l|_{b_v}+\sum_{h\in\hat{I}}|\theta_h|+\sum_{h\in\hat{I}-H}|r_h|.$$

If $\delta_\mathcal{T}$ and $H^{(0)}_{\mathcal{T},b_v}$ are as in Proposition 3.3 and $\|\varpi\|<\delta_\mathcal{T}(b_v)$, put

$$b_\varpi\equiv\big(S,M,I;x(\varpi),(j,y(\varpi)),u(\varpi)\big)=H^{(0)}_{\mathcal{T},b_v}\big(\xi,w_{\hat{I}+M};\theta_{\hat{I}}\big)\in\mathcal{M}^{(0)}_\mathcal{T},$$
$$v_h(\varpi)=\begin{cases}(1+r_h)\begin{cases}v_h, & \text{if } x_h\in S^2;\\ d\phi^{-1}_{b_v,h}|_{\phi_{b_v,h}x_h(\varpi)}v_h, & \text{if } x_h\in\Sigma;\end{cases} & \text{if } h\notin H;\\ 0, & \text{if } h\in H;\end{cases}\qquad v(\varpi)\equiv\big(b_\varpi,(v(\varpi))_{\hat{I}}\big).$$

Then $v(\varpi)\in F^{(H)}\mathcal{T}_{2\delta}$ if $\|\varpi\|<\delta(b_v)$ for some $\delta\in C^\infty(\mathcal{M}^{(0)}_\mathcal{T};\mathbb{R}^+)$ sufficiently small. If $H=\emptyset$, $T_vF^H\mathcal{T}=\tilde{T}_vF^H\mathcal{T}$ models the tangent space of $[v]$ in $F^H\mathcal{T}$. If $H\neq\emptyset$, the bundle $F^H\mathcal{T}$ and the construction of the previous section lift to a bundle ${}^HF\mathcal{T}$ over

$$\mathcal{M}^H_\mathcal{T}\equiv\mathcal{M}^{(0)}_\mathcal{T}/\{g_{\hat{I}}\in G_\mathcal{T}\colon g_h=1\ \forall g\in H\}.$$

Then $T_vF^H\mathcal{T}$ models the tangent space of $[v]$ in ${}^HF\mathcal{T}$. On the other hand, $\tilde{T}_vF^H\mathcal{T}$ models the tangent space of $[v]$ in the restriction of ${}^HF\mathcal{T}$ to the subspace

$$\{[b'=(S,M,I;x',(j,y'),u)]\in\mathcal{M}^H_\mathcal{T}\colon x'_h=x_h\ \forall h\in H\}.$$

The reason for defining subspaces $\tilde{T}_vF^H\mathcal{T}$ is that if $x'_h\neq x_h$ for some $h'\in H$, $b(v)$ and $b(v')$ do not have the same singular points for all $v\in F^{(H)}_b\mathcal{T}$ and $v\in F^{(H)}_{b'}\mathcal{T}$. Since the perturbation construction of Section 3.6 does not change the singular points of $b(v)$ and $b(v')$, the resulting bubble maps $\tilde{b}(v)$ and $\tilde{b}(v')$ will necessarily be different.

We now define norms on $T_vF^H\mathcal{T}$, which make the estimates in Lemma 3.6 dependent only on b_v. If $h\in\hat{I}-H$, let

$$w'_h=\phi_{b_v,h}q_{v(\varpi),\iota_h}\big(q^{-1}_{v,\iota_h}(\iota_h,x_h)\big)\in F^{(0)}_{h,b_v}\qquad\text{if } q_{v(\varpi),\iota_h}\big(q^{-1}_{v,\iota_h}(\iota_h,x_h)\big)\in\Sigma_{b_v,\iota_h}.$$

In such a case, let $\|\varpi\|_{\upsilon,h}=\left|\frac{w'_h+w_h}{v_h}\right|$. Otherwise, put $\|\varpi\|_{\upsilon,h}=1$. Let

$$\|\varpi\|_\upsilon = \|\varpi\| + \sum_{h\in\hat{I}-H} \|\varpi\|_{\upsilon,h}.$$

In order to simplify notation, we replace $\upsilon(\varpi)$ by ϖ whenever there is no ambiguity. If $\|\varpi\|_\upsilon$ is sufficiently small, define $\zeta_\varpi\in\Gamma'(u_\upsilon)$ by

$$\exp_{b_\upsilon,u_\upsilon}\zeta_\varpi = u_\varpi, \quad \|\zeta_\varpi\|_{b_\upsilon,C^0} < \text{inj }g_{V,b_\upsilon}.$$

Similarly, for $l\in M$, define $w_l(\varpi)\in T_{y_l(\upsilon)}\Sigma_{\upsilon,j_l(\upsilon)}$ by

$$\exp_{g_\upsilon,y_l(\upsilon)} w_l(\varpi) = y_l\left(\upsilon(\varpi)\right), \quad |w_l(\varpi)| \equiv |w_l(\varpi)|_{g_\upsilon} < \text{inj}_{y_l(\upsilon)} g_\upsilon.$$

If $\varpi\in\tilde{T}F_\upsilon^H\mathcal{T}$ and $\xi\in\Gamma(u_\upsilon)$, let $R_\varpi\xi\in\Gamma(u_\varpi)$ be the vector field given by

$$R_\varpi\xi(z) = \Pi_{b_\upsilon,\zeta_\varpi(z)}\xi(z).$$

Note that since $b(\upsilon)$ and $b(\varpi)$ have the same singular points whenever $\varpi\in\tilde{T}F_\upsilon^H\mathcal{T}$, $\Pi_{b_\upsilon,\zeta_\varpi}$ does indeed map $\Gamma(u_\upsilon)$ to $\Gamma(u_\varpi)$. If $\eta\in\Gamma^1(u_\upsilon)$, we define $R_\varpi\eta\in\Gamma^1(u_\varpi)$ similarly. Let S_ϖ denote the inverse of R_ϖ.

Lemma 3.6 *There exist $\delta, C\in C^\infty(\mathcal{M}_\mathcal{T}^{(0)};\mathbb{R}^+)$ such that for all $\upsilon\in F^{(H)}\mathcal{T}_\delta$ and $\varpi\in\tilde{T}_\upsilon F^H\mathcal{T}_\delta$,*
(1) $C(b_\upsilon)^{-1}\|\varpi\|_\upsilon \le \|\zeta_\varpi\|_{\upsilon,p,1} + \sum_{l\in M}|w_l(\varpi)|_{g_\upsilon} \le C(b_\upsilon)\|\varpi\|_\upsilon$;
(2) $\left\|\frac{g_{V,b_\varpi}}{g_{V,b_\upsilon}}-1\right\|_{C^3} \le C(b_\upsilon)\|\varpi\|$, $\left\|\frac{g_\varpi}{g_\upsilon}-1\right\|_{C^0} \le C(b_\upsilon)\|\varpi\|_\upsilon$ *and* $\left\|\frac{\rho_\varpi}{\rho_\upsilon}-1\right\|_{C^0} \le C(b_\upsilon)\|\varpi\|_\upsilon$;
(3) $\left\|S_\varpi du_\varpi - du_\upsilon\right\|_{\upsilon,p} \le C(b_\upsilon)\|\varpi\|_\upsilon$ *and* $\left\|S_\varpi\bar{\partial}u_\varpi - \bar{\partial}u_\upsilon\right\|_{\upsilon,p} \le C(b_\upsilon)|\upsilon|^{\frac{1}{p}}\|\varpi\|_\upsilon$;
(4) $\left\|S_\varpi\nu - \nu\right\|_{\upsilon,p} \le C(b_\upsilon)\|\varpi\|_\upsilon$;
(5) $\left\|S_\varpi D_\varpi R_\varpi\xi - D_\upsilon\xi\right\|_{\upsilon,p} \le C(b_\upsilon)\|\varpi\|_\upsilon\|\xi\|_{\upsilon,p,1}$ *for all* $\xi\in\Gamma(u_\upsilon)$;
(6) $\left\|S_\varpi\pi_{\varpi,\pm}R_\varpi\xi - \pi_{\upsilon,\pm}\xi\right\|_{\upsilon,p,1} \le C(b_\upsilon)\|\varpi\|_\upsilon\|\xi\|_{\upsilon,p,1}$ *for all* $\xi\in\Gamma(u_\upsilon)$,

Proof: The first statement of (2) is clear. Proofs of (1), the last two claims of (2), (3), and (6) are direct, though lengthy, computations, all of the same nature. The statement of (4) is immediate from (1). Estimate (5) follows from (2) and basic Riemannian geometry estimates as in Appendix A.1.

Remark: Claim (6) above is proved by choosing an orthonormal basis $\{\xi_{b,i}\}$ for the kernel of D_b for b lying near b_υ in $\mathcal{M}_\mathcal{T}^{(0)}$, so that each $\xi_{b,i}$ varies smoothly with b. Then the claim follows immediately from an estimate on $S_\varpi R_{\upsilon(\varpi)}\xi_{b_\varpi,i} - R_\upsilon\xi_{b,i}$, since the projection maps can be expressed in terms of inner-products with $\xi_{b,i}$. Note that if we had defined $\Gamma_-(\upsilon)$ to be the kernel of D_υ in the case $\mathcal{T}$ is regular, this claim, if true, would have been much harder to prove because of the presence of small eigenvalues of $D_\upsilon^*D_\upsilon$; see Section 3.6 for more details.

If $\upsilon\in F^{(0)}\mathcal{T}_\delta$, $(\Sigma_\upsilon, g_\upsilon)$ can be viewed as a connected sum of the surfaces $\{(\Sigma_{\mathcal{T},i}, g_{b_\upsilon,i})\}$ with very thin necks. If $\varpi\in\mathcal{K}_{b_\upsilon}\mathcal{T}\subset T_\upsilon F^\emptyset\mathcal{T}$ is as above and $|w_h|\ge 2|v_h|^{\frac{1}{2}}$, the maps $u_\upsilon:\Sigma\longrightarrow V$ and $u_\varpi:\Sigma\longrightarrow V$ are very far apart in the C^0-norm even if $\|\varpi\|$ is small. However, we can still compare the two maps and the various objects of Lemma 3.6, appropriately defined, on the corresponding direct summands. If the gluing map of Section 3.6 is defined only on $F^{(\emptyset)}\mathcal{T}_\delta$, and not on $F\mathcal{T}_\delta$, we need to be able to do such comparisons in order to adjust the gluing map in the presence of constraints μ; see Section 3.8.

In order to state an analogue of Lemma 3.6 with $\|\varpi\|_v$ for $\varpi\in\tilde{T}_\varpi F^H\mathcal{T}$ replaced by $\|\varpi\|$ for $\varpi\in\mathcal{K}_{b_v}\mathcal{T}\subset\tilde{T}_\varpi F^\emptyset\mathcal{T}$, for each $\varpi\in\mathcal{K}_{b_v}\mathcal{T}_{\delta(b)}$, with δ sufficiently small, we construct a smooth map

$$\tilde{q}_\varpi\colon(\Sigma_v,g_v)\longrightarrow(\Sigma_\varpi,g_\varpi),$$

which is almost an isometry. The map will depend only on the elements $w_h\in F^{(0)}_{b,h}$. The structure of the construction is similar to the construction of the map q_v in Section 2.3. For each $h\in\hat{I}$ with $\iota_h=\hat{0}$, let $\tilde{p}_{h,\varpi}:B_{b_v,h}(4\delta_{\mathcal{T}}(b_v))\longrightarrow\Sigma$ be the (holomorphic) $(g_{b_v,\hat{0}},g_{b_\varpi,\hat{0}})$-isometry provided by Lemma B.1. If $r_{b_v,h}(z)\le 2\delta_{\mathcal{T}}(b_v)$, put

$$\tilde{q}_{h,\varpi}(z)=\phi^{-1}_{b_v,h}\Big\{\phi_{b_v,h}\tilde{p}_{h,\varpi}(z)+\beta_{\delta^2_{\mathcal{T}}(b_v)}\big(r_{b_v,h}(z)\big)\big(\phi_{b_v,h}(z)-\phi_{b_v,h}\tilde{p}_{h,\varpi}(z)\big)\Big\}.$$

The $\tilde{q}_{h,\varpi}$ extends to rest of Σ by identity. If $h\in\hat{I}$ and $\iota_h\neq\hat{0}$, we similarly define $\tilde{q}_{h,(x_h,w_h)}:\Sigma_{b,\iota_h}\longrightarrow\Sigma_{b,\iota_h}$ by

$$\tilde{q}_{h,\varpi}(z)=\begin{cases}\phi^{-1}_{b_v,h}\Big\{\phi_{b_v,h}(z)+w_h-\beta_{\delta^2_{\mathcal{T}}(b_v)}\big(r_{b_v,h}(z)\big)w_h\Big\}, & \text{if } r_{b_v,h}(z)\le 2\delta_{\mathcal{T}}(b_v);\\ z, & \text{if } r_{b_v,h}(z)\ge 2\delta_{\mathcal{T}}(b_v).\end{cases}$$

Let $\tilde{q}_{\varpi,\hat{0}}=Id_\Sigma$. If $h\in\hat{I}$ and $\tilde{q}_{\varpi,\iota_h}:\Sigma\longrightarrow\Sigma$ has been constructed, let

$$\tilde{q}_{\varpi,h}(z)=\begin{cases}q^{-1}_{\varpi,\iota_h}\big(\tilde{p}_{h,\varpi}(z)\big(q_{\varpi,\iota_h}(\tilde{q}_{\varpi,\iota_h}(z))\big)\big), & \text{if } r_{b_v,h,h}\big(q_{\varpi,\iota_h}(z)\big)\le 2\delta_{\mathcal{T}}(b_v);\\ \tilde{q}_{\varpi,\iota_h}(z), & \text{if } r_{b_v,h,h}\big(q_{\varpi,\iota_h}(z)\big)\ge 2\delta_{\mathcal{T}}(b_v).\end{cases}$$

Going through all of I, we obtain a map $\tilde{q}_\varpi:\Sigma\longrightarrow\Sigma$, which shifts the connect-summands of (Σ,g_v) to the connect-summands of (Σ,g_ϖ). The important properties of such maps $\tilde{q}_\varpi$ as summarized below.

Lemma 3.7 *There exist $\delta,C\in C^\infty(\mathcal{M}^{(0)}_{\mathcal{T}};\mathbb{R})$ and a smooth family of maps*

$$\big\{\tilde{q}_\varpi:\Sigma\longrightarrow\Sigma\mid\varpi\in\mathcal{K}_{b_v}\mathcal{T}_{\delta(b_v)}\subset T_vF^{(\emptyset)}_v\mathcal{T},\ v\in F^{(\emptyset)}_{b_v}\mathcal{T}_{\delta(b_v)}\big\},\qquad\textit{such that}$$

(1) $\tilde{q}_0=Id_\Sigma$ and $q_v=q_\varpi\circ\tilde{q}_\varpi$ on $\Sigma^*_{b_v,i}=\Sigma^*_{b_\varpi,i}$ outside of the annuli

$$A_\varpi=\tilde{q}^{-1}_{\varpi,\iota_h}q^{-1}_{\varpi,\iota_h}\Big(\big\{z\in\Sigma_{b_v,\iota_h}\colon\delta_{\mathcal{T}}(b_v)\le r_{b,h}(z)\le 2\delta_{\mathcal{T}}(b_v)\big\}\Big),$$

which contain no marked points of $b(v)$ or $b(\varpi)$.
(2) $\left|\frac{\tilde{q}^*_\varpi g_\varpi}{g_v}-\frac{\tilde{q}^*_{\varpi'}g_{\varpi'}}{g_v}\right|\le C(b_v)\|\varpi-\varpi'\|$ for all $\varpi,\varpi'\in\mathcal{K}_{b_v}\mathcal{T}_{\delta(b_v)}$.

These maps q_ϖ allow us to compare operators on vector fields and one-forms on (Σ_v,u_v) and (Σ_ϖ,u_ϖ) whenever $\|\varpi\|$ is sufficiently small. Define $\zeta'_\varpi\in\Gamma(u_v)$ by

$$\exp_{b_v,u_v}\zeta'_\varpi=u_\varpi\circ\tilde{q}_\varpi,\qquad\|\zeta'_\varpi\|_{b_v,C^0}\le\text{inj }g_{b_v}.$$

For $\xi\in\Gamma(u_v)$, let $R'_\varpi\xi\in\Gamma(u_\varpi)$ be given by

$$\{R'_\varpi\xi\}(z)=\Pi_{b_v,\zeta'_\varpi(\tilde{q}_\varpi^{-1}(z))}\xi\big(\tilde{q}_\varpi^{-1}(z)\big).$$

Similarly, for any $\eta\in\Gamma^{0,1}(u_v)$, let $R'_\varpi\eta\in\Gamma^{0,1}(u_\varpi)$ be given by

$$\{R'_\varpi\eta\}\big|_z=\Pi_{b_v,\zeta'_\varpi(\tilde{q}_\varpi^{-1}(z))}\circ\eta\big|_{\tilde{q}_\varpi^{-1}(z)}\circ\partial\tilde{q}_\varpi^{-1}\big|_z.$$

Denote by S'_ϖ the inverse of R'_ϖ. Similarly to Lemma 3.6, we have

Lemma 3.8 *There exist $\delta, C\in C^\infty(\mathcal{M}_{\mathcal{T}}^{(0)};\mathbb{R}^+)$ such that for all $\upsilon\in F^{(\emptyset)}\mathcal{T}_\delta$ and $\varpi\in\mathcal{K}_{b_v}\mathcal{T}$,*
(1) $C(b_v)^{-1}\|\varpi\|\le\|\zeta'_\varpi\|_{v,p,1}+\sum_{l\in M}|w_l(\varpi)|_{g_v}\le C(b_v)\|\varpi\|$;
(2) $\big\|S'_\varpi du_\varpi-du_v\big\|_{v,p}\le C(b_v)\|\varpi\|$ and $\big\|S'_\varpi\bar\partial u_\varpi-\bar\partial u_v\big\|_{v,p}\le C(b_v)|v|^{\frac{1}{p}}\|\varpi\|$;
(3) $\big\|S'_\varpi\nu-\nu\big\|_{v,p}\le C(b_v)\|\varpi\|$;
(4) $\big\|S'_\varpi D_\varpi R'_\varpi\xi-D_v\xi\big\|_{v,p}\le C(b_v)\|\varpi\|\|\xi\|_{v,p,1}$ for all $\xi\in\Gamma(u_v)$;
(5) $\big\|S'_\varpi\pi_{\varpi,\pm}R'_\varpi\xi-\pi_{v,\pm}\xi\big\|_{v,p,1}\le C(b_v)\|\varpi\|\|\xi\|_{v,p,1}$ for all $\xi\in\Gamma(u_v)$.

3.5 Obstruction Bundle Setup

In the next section, we look for solution of the equation $\bar\partial\exp_{b_v,u_v}\xi=t\nu$ with ξ lying in a fixed complement of $\Gamma_-(v)$. If t is sufficiently small, we are able to solve this equation up to an element of a vector bundle of the same rank as the dimension of $\Gamma_-(b_v)$, called obstruction bundle. This element is the obstruction to solving the equation. There are choices to be made for this obstruction bundle as well as for the subspace complementary to $\Gamma_-(v)$. We describe in this section what conditions these choices must satisfy for the gluing construction to work properly. For the rest of this thesis, p will denote denote a real number number strictly greater than 2.

If $b^*=\big(S,M,I;x^*,(j,y^*),u^*\big)\in\mathcal{M}_{\mathcal{T}}^{(0)}$ and $b=\big(S,M,I;x,(j,y),u\big)=H_{\mathcal{T},b^*}(\sigma,\theta)$ for some $\sigma\in\mathcal{K}_{b^*}\mathcal{T}$ and $\theta\in\mathbb{R}^{\hat{I}}$, let $\xi_{b^*,b}=\xi_{b^*,b,I}\in\Gamma'(b)$ be given by

$$\exp_{b^*,u_i^*}\xi_{b^*,b,i}=u_i,\quad\big\|\xi_{b^*,b,i}\big\|_{C^0}<\operatorname{inj}g_{V,b^*}.$$

Let $\Pi_{b^*,b}=\Pi_{b^*,\xi_{b^*,b}}$.

Definition 3.9 *Suppose $b^*=\big(S,M,I;x^*,(j,y^*),u^*\big)$, $b_k=\big(S,M,I;x_k,(j,y_k),u_k\big)\in\mathcal{M}_{\mathcal{T}}^{(0)}$, and $\upsilon_k=(b_k,v_k)\in F^{(0)}\mathcal{T}$ are such that the sequences $\{b_k\}$ and $\{|v_k|_{b_k}\}$ converge to $b^*\in\mathcal{M}_{\mathcal{T}}^{(0)}$ and $0\in\mathbb{R}$, respectively.*
(1) The sequence $\{\xi_k\in L_1^p(u_{v_k})\}$ C^0-converges to $\xi^*\in\Gamma'(b^*)$ if
(1a) the sequence $\{\Pi_{b^*,b_k}^{-1}(\xi_k\circ q_{v_k}^{-1})\}$ C^0-converges to ξ^* on compact subsets of $\Sigma_{b^*}^*$;
(1b) there exists $C>0$ such that $\|\xi_k\|_{v_k,p,1}<C$ for all k.
(2) The sequence of subspaces $\{V_k\subset\Gamma(u_{v_k})\}$ C^0-converges to subspace $V^*\subset\Gamma(b^*)$ if there exists a sequence of bases $\big\{\{\xi_{k,i}\}_{i=1}^{i=N}\subset V_k\big\}$ such that
(2a) for each i fixed, the sequence $\{\xi_{k,i}\}$ C^0-converges to some $\xi_i^*\in V^*$;
(2b) the set $\{\xi_i^*\}$ has cardinality N and is basis for V^*.

Lemma 3.10 *If the sequence* $\{v_k\} \subset F^{(0)}\mathcal{T}$ *converges to* $b^* \in \mathcal{M}_{\mathcal{T}}^{(0)}$ *and the sequences* $\{\xi_k \in L_1^p(v_k)\}$ *and* $\{\tilde{\xi}_k \in L_1^p(v_k)\}$ *converge to* $\xi^* \in \Gamma'(b^*)$ *and* $\tilde{\xi}^* \in \Gamma'(b^*)$*, respectively,*

$$\lim_{k\longrightarrow\infty} \langle\langle \xi_k, \tilde{\xi}_k \rangle\rangle_{v_k,2} = \langle\langle \xi^*, \tilde{\xi}^* \rangle\rangle_{v^*,2}.$$

Proof: If $v_k \longrightarrow b^*$, the metrics $g_{V,b_{v_k}}$ on V and $g_{b_{v_k},i}$ on $\Sigma_{\mathcal{T},i}$ C^0-converge to g_{V,b^*} and $g_{b^*,i}$, respectively. On the other hand, by (1b) of Definition 3.9 and (2) of Lemma 3.5, there exists $C > 0$ such that

$$\|\xi_k\|_{v_k,C^0}, \|\tilde{\xi}_k\|_{v_k,C^0} < C \qquad \forall k.$$

Thus, the claim follows from (1a) of Definition 3.9.

Definition 3.11 *Suppose* Ω *is an open subset of* $F^{(\emptyset)}\mathcal{T}$ *such that* $b(v)$ *is defined for all* $v \in \Omega$*. An* $(\mathcal{A}(\mathcal{T}) \ltimes G_{\mathcal{T}})$*-invariant smooth complex subbundle* $\tilde{\Gamma}_- \longrightarrow \Omega$ *of the Banach bundle* $L_1^p \longrightarrow \Omega$ *is a* <u>*tangent-space model*</u> *over* Ω *if*

(1) for every sequence $\{v_k\} \subset \Omega$ *converging to* $b^* \in \mathcal{M}_{\mathcal{T}}^{(0)}$*, a subsequence of* $\{\tilde{\Gamma}_-(v_k)\}$ C^0*-converges to a subspace* $V^* \subset \Gamma(b)$ *such that* $\pi_{b,-} : V^* \longrightarrow \Gamma_-(b^*)$ *is an isomorphism;*

(2) if $\tilde{\pi}_{v,-} : L_1^p(v) \longrightarrow \tilde{\Gamma}_-(v)$ *is the* (L^2, v)*-orthogonal projection onto* $\tilde{\Gamma}_-(v)$*, there exist* $\delta, C \in C^\infty(\mathcal{M}_{\mathcal{T}}^{(0)}; \mathbb{R}^+)$ *such that for all* $v \in \Omega_\delta$ *and all* $\xi \in \Gamma(u_v)$*,*

(2a) $\left\| S_\varpi \tilde{\pi}_{\varpi,-} R_\varpi \xi - \tilde{\pi}_{v,-}\xi \right\|_{v,2} \le C(b_v)\|\varpi\|_v \|\xi\|_{v,p,1}$ *for all* $\varpi \in T_v F^{(\emptyset)}\mathcal{T}_{\delta(b_v)}$*;*

(2b) $\left\| S'_\varpi \tilde{\pi}_{\varpi,-} R'_\varpi \xi - \tilde{\pi}_{v,-}\xi \right\|_{v,2} \le C(b_v)\|\varpi\| \|\xi\|_{v,p,1}$ *for all* $\varpi \in \mathcal{K}_{b_v}\mathcal{T} \subset T_v F^{(\emptyset)}\mathcal{T}_{\delta(b_v)}$*.*

One example of a tangent-space model is $\{\Gamma_-(v) : v \in F^{(\emptyset)}\mathcal{T}_\delta\}$. In such a case, the limit V^* in (1) of Definition 3.11 is $\Gamma_-(b^*)$ and thus depends only on b^*, and not on the sequence $\{v_k\}$. However, for computational reasons, it is sometimes advantageous to work with other choices. With the choices of Section 8.1, the limit V^* in (1) of Definition 3.11 in fact often depends on the sequence. The following lemma collects some of the implications of (1) of Definition 3.11. Condition (2) is needed in Sections 4.2 and 4.5. For any tangent space model over Ω and $v \in \Omega$, we denote the (L^2, v)-orthogonal complement of $\tilde{\Gamma}_-(v)$ by $\tilde{\Gamma}_+(v)$. Write $\tilde{\Gamma}_+^{0,1}(v)$ for the image of $\tilde{\Gamma}_+(v)$ under the operator $\tilde{D}_v$.

Lemma 3.12 *Let* $\tilde{\Gamma}_- \longrightarrow \Omega$ *be a tangent-space model. Then there exist* $C, \delta \in C^\infty(\mathcal{M}_{\mathcal{T}}^{(0)}; \mathbb{R})$ *such that for all* $v \in \Omega_\delta$

(1a) $\|\xi\|_{v,p,1} \le C(b_v)\|\xi\|_{v,2}$ *for all* $\xi \in \tilde{\Gamma}_-(v)$*;*

(1b) $\|\tilde{\pi}_{v,-}\xi\|_{v,p,1} \le C(b_v)\|\xi\|_{v,p,1}$ *for all* $\xi \in \Gamma(u_v)$*;*

(2a) $L_1^p(v) = \Gamma_-(v) \oplus \tilde{\Gamma}_+(v)$*;*

(2b) if $\tilde{\pi}_-$ *and* $\tilde{\pi}_+$ *are the projection maps corresponding to the above decomposition,*

$$\|\tilde{\pi}_{v,\pm}\xi\|_{v,p,1} \le C(b_v)\|\xi\|_{v,p,1} \qquad \forall \xi \in \Gamma(u_v).$$

Proof: (1) Suppose there exists a sequence $\{v_k \in \Omega\}$ converging to $b^* \in \mathcal{M}_{\mathcal{T}}^{(0)}$ and a sequence $\{\xi_k \in \tilde{\Gamma}_-(v_k)\}$ such that $\|\xi_k\|_{v_k,p,1} = 1$, while $\|\xi\|_{v_k,2} \longrightarrow 0$. Since $\|\xi_k\|_{v_k,p,1} = 1$, by (2) of Lemma 3.16 and (1) of Definition 3.11, a subsequence of $\{\xi_k\}$ C^0-converges to some nonzero $\xi^* \in \Gamma(b^*)$. However, since $\|\xi_k\|_{v_k,2} \longrightarrow 0$, $\|\xi^*\|_{b^*,2} = 0$ by Lemma 3.10. This is a contradiction, and thus (1a) holds. Claim (1b) is an immediate consequence of (1a) and (2) of Lemma 3.16.

(2) Claim (2a) is equivalent to saying that no nonzero element of $\tilde{\Gamma}_-(v)$ is orthogonal to $\Gamma_-(v)$. So, suppose $v_k \longrightarrow b^* \in \mathcal{M}_{\mathcal{T}}^{(0)}$ and $\{\xi_k \in \tilde{\Gamma}_-(v_k)\}$ is such that ξ_k is orthogonal

to $\Gamma_-(v)$ and $\|\xi_k\|_{v_k,p,1}=1$. Since $\xi_k\in\tilde\Gamma_-(v_k)$ and $\|\xi_k\|_{v_k,p,1}=1$, by (1) of Definition 3.11, a subsequence of $\{\xi_k\}$ converges to some nonzero $\xi^*\in\Gamma(b^*)$. By Lemma 3.10, ξ^* is orthogonal to $\Gamma_-(v)$. However, this contradicts the second part of (1) of Definition 3.11.
(3) Due to (1b), Claim (2b) is equivalent to saying that there exist $C,\delta\in C^\infty(\mathcal{M}_\mathcal{T};\mathbb{R})$ such that

$$\|\xi\|_{v,p,1}\le C(b_v)\|\bar\pi_{v,-}\xi\|_{v,p,1}\qquad\forall v\in\Omega_\delta \text{ and } \xi\in\Gamma_-(v).$$

Suppose there exists a sequence $\{v_k\}\subset\Omega$ converging to some $b^*\in\mathcal{M}_\mathcal{T}^{(0)}$ and a sequence $\{\xi_k\in\tilde\Gamma_-(v_k)\}$ such that $\|\bar\pi_{v_k,-}\xi_k\|_{v_k,2}\longrightarrow 0$, while $\|\xi_k\|_{v_k,p,1}=1$. By Definition 3.11, a subsequence of $\{\tilde\Gamma_-(v_k)\}$ converges to a subspace $V\subset\Gamma(b)$. On the other hand, a subsequence of $\{\xi_k\}$ C^0-converges to a nonzero element $\xi^*\in\Gamma_-(b^*)$, which must be orthogonal to V by Lemma 3.10. This contradicts the second part of (1) of Definition 3.11.

Definition 3.13 *Suppose Ω is an open subset of $F^{(\emptyset)}\mathcal{T}$ such that $b(v)$ is defined for all $v\in\Omega$. An $(\mathcal{A}(\mathcal{T})\ltimes G_\mathcal{T})$-invariant smooth complex subbundle $\Gamma_-^{0,1}(v)\longrightarrow\Omega$ of the Banach bundle $L^p\longrightarrow\Omega$ with the same rank as $\Gamma_-^{0,1}\longrightarrow\mathcal{M}_\mathcal{T}^{(0)}$ is an obstruction bundle if*
(1) there exists $C\in C^\infty(\mathcal{M}_\mathcal{T}^{(0)};\mathbb{R})$ such that

$$\|\eta\|_{v,p}\le C(b_v)\|\eta\|_2 \quad\textit{and}\quad \|D_v^*\eta\|_{v,1}\le C(b_v)|v|^{\frac{1}{p}}\qquad\forall v\in\Omega,\ \eta\in\Gamma_-^{0,1}(v);$$

(2) if $\pi_{v,-}^{0,1}: L^p(v)\longrightarrow\Gamma_-^{0,1}(v)$ is the (L^2,v)-orthogonal projection onto $\Gamma_-^{0,1}(v)$, there exists $\delta\in C^\infty(\mathcal{M}_\mathcal{T}^{(0)};\mathbb{R}^+)$ such that for all $v\in\Omega_\delta$ and all $\eta\in\Gamma^{0,1}(u_v)$,
(2a) $\big\|S_\varpi\pi_{\varpi,-}^{0,1}R_\varpi\eta-\pi_{v,-}^{0,1}\eta\big\|_{v,2}\le C(b_v)\|\varpi\|_v\|\xi\|_{v,p}$ for all $\varpi\in T_vF^{(\emptyset)}\mathcal{T}_{\delta(b_v)}$;
(2b) $\big\|S'_\varpi\pi_{\varpi,-}^{0,1}R'_\varpi\eta-\pi_{v,-}^{0,1}\eta\big\|_{v,2}\le C(b_v)\|\varpi\|\|\xi\|_{v,p}$ for all $\varpi\in\mathcal{K}_{b_v}\mathcal{T}\subset T_vF^{(\emptyset)}\mathcal{T}_{\delta(b_v)}$.

Such an obstruction bundle is related to the cokernel bundle $\Gamma_-^{0,1}\longrightarrow\mathcal{M}_\mathcal{T}^{(0)}$. However, if $\hat I\neq\emptyset$, the low-eigenspaces of $D_vD_v^*$ are too large to form an obstruction bundle; see the remark below. Examples of bundles that satisfy Definition 3.13 can be found in Section 7.2. Given such an obstruction bundle, we denote by $\pi_{v,+}^{0,1}$ the (L^2,v)-orthogonal projection onto $\Gamma_+^{0,1}(v)$, the (L^2,v)-orthogonal complement of $\Gamma_-^{0,1}(v)$. The following lemma is clear from (1) of Definition 3.13.

Lemma 3.14 *If $\Gamma_-^{0,1}\longrightarrow\Omega$ is an obstruction bundle, there exists $C\in C^\infty(\mathcal{M}_\mathcal{T};\mathbb{R})$ such that*

$$\|\pi_{v,\pm}^{0,1}\eta\|_{v,p}\le C(b_v)\|\eta\|_{v,p}\qquad\forall v\in\Omega,\ \eta\in\Gamma^{0,1}(u_v).$$

Definition 3.15 *If $\mathcal{T}$ is a semiregular bubble type, an obstruction bundle setup for $(V,J,\mathcal{T})$ is a tuple $(\delta,\tilde\Gamma_-,\Gamma_-^{0,1},R)$, where*
(1) $\delta\in C^\infty(\mathcal{M}_\mathcal{T}^{(0)};\mathbb{R}^+)$ is $(\mathcal{A}(\mathcal{T})\ltimes G_\mathcal{T})$-invariant and $b(v)$ is defined for all $v\in F^{(0)}\mathcal{T}_\delta$;
(2) $\tilde\Gamma_-\longrightarrow F^{(\emptyset)}\mathcal{T}_\delta$ and $\Gamma_-^{0,1}\longrightarrow F^{(\emptyset)}\mathcal{T}_\delta$ are a tangent-space model and an obstruction bundle, respectively;
(3) $R:\pi^*\Gamma_-^{0,1}\longrightarrow\Gamma_-^{0,1}$ is a smooth oriented $(\mathcal{A}(\mathcal{T})\ltimes G_\mathcal{T})$-equivariant bundle isomorphism over $F^{(0)}\mathcal{T}_\delta$, where $\pi: F^{(0)}\mathcal{T}_\delta\longrightarrow\mathcal{M}_\mathcal{T}^{(0)}$ is the bundle projection map.

For the rest of this chapter, we fix such an obstruction bundle setup. However, whenever we refer to $\delta\in C^\infty(\mathcal{M}_\mathcal{T};\mathbb{R}^+)$, we will mean any function smaller than the function δ in Definition 3.15. The following lemma states some of the consequences of our setup that are crucial for the construction of the next section. If $\mathcal{T}$ is a regular bubble type, we take $\tilde{\Gamma}_-(\upsilon)$ and $\Gamma_-^{0,1}(\upsilon)$ to be $\Gamma_-(\upsilon)$ and $\{0\}$, respectively, and define the other bundles and the projection maps in the same way.

Lemma 3.16 *If $\mathcal{T}$ is a simple bubble type, there exist $\delta, C \in C^\infty(\mathcal{M}_{\mathcal{T}}^{(0)};\mathbb{R}^+)$ such that for any $\upsilon \in F^{(0)}\mathcal{T}_\delta$ if $\mathcal{T}$ is regular and any $\upsilon \in F^{(\emptyset)}\mathcal{T}_\delta$ if $\mathcal{T}$ is semiregular,*
(1) $\|\xi\|_{\upsilon,p,1} \le C(b_\upsilon)\|D_\upsilon\xi\|_{\upsilon,p}$ for all $\xi \in \Gamma_+(\upsilon)$ and all $\xi \in \tilde{\Gamma}_+(\upsilon)$;
(2) $\|\pi_{\upsilon,-}^{0,1}\eta\|_{\upsilon,p} \le C(b_\upsilon)|\upsilon|^{\frac{1}{p}}\|\eta\|_{\upsilon,p}$ for all $\eta \in \tilde{\Gamma}_+^{0,1}(\upsilon)$;
(3) $\pi_{\upsilon,+}^{0,1} : \tilde{\Gamma}_+^{0,1}(\upsilon) \longrightarrow \Gamma_+^{0,1}(\upsilon)$ is an isomorphism with the norm of the inverse bounded by $C(b_\upsilon)$.

Proof: (1) The first statement of the lemma is proved in Appendix B; see Proposition B.13. It is consequence of (2) and (4) of Lemma 3.5 and of (1) of Definition 3.11. The second claim is immediate from (1) of Definition 3.13 and the first claim.
(2) Let W be the (L^2, g_υ)-orthogonal complement of $\pi_{\upsilon,+}^{0,1}(\tilde{\Gamma}_+^{0,1}(\upsilon))$ in $\Gamma_+^{0,1}(\upsilon)$. The second claim implies that

$$L^p(\upsilon) = \big(\Gamma_-^{0,1}(\upsilon) \oplus W\big) \oplus \tilde{\Gamma}_+^{0,1}(\upsilon). \tag{3.5}$$

Since $\tilde{\Gamma}_+^{0,1}(\upsilon)$ is the image of $\tilde{\Gamma}_+(\upsilon)$ under D_υ, with respect to the decompositions (3.5) and $L_1^p(\upsilon) = \Gamma_-(\upsilon) \oplus \tilde{\Gamma}_+(\upsilon)$,

$$D_\upsilon = \begin{vmatrix} D_\upsilon^{(--)} & 0 \\ D_\upsilon^{(+-)} & D_\upsilon^{(++)} \end{vmatrix}.$$

Since $D_\upsilon^{(++)}$ is an isomorphism by (1) of the lemma,

$$\begin{aligned} \text{ind } D_\upsilon &= \text{ind } D_\upsilon^{(--)} = \dim\Gamma_-(\upsilon) - \big(\dim\Gamma_-^{0,1}(\upsilon) + \dim W\big) \\ &= \big(\dim\Gamma_-(b_\upsilon) - \dim\Gamma_-^{0,1}(b_\upsilon)\big) - \dim W = \text{ind } D_{b_\upsilon} - \dim W. \end{aligned} \tag{3.6}$$

On the other hand, by the Index Theorem, with $n = \dim_{\mathbb{C}} V$,

$$\begin{aligned} \text{ind } D_\upsilon &= 2\Big(\sum_{h \in \hat{I}(\upsilon)} \big(\langle c_1(V,J), \lambda_i(\upsilon)\rangle - n(g(\Sigma_{\mathcal{T},i}) - 1)\big) - n\big(|\hat{I}(\upsilon)| - 1\big)\Big) \\ &= 2\Big(\sum_{h \in \hat{I}(\upsilon)} \langle c_1(V,J), \lambda_i\rangle - n(g(S) - 1)\Big) = \text{ind } D_{b_\upsilon}. \end{aligned} \tag{3.7}$$

By equations (3.6) and (3.7), $W = \{0\}$, and the last claim of the lemma follows from the second one.

Remark: It is essential for claim (1) of Lemma 3.16 that $p > 2$. The operator $D_\upsilon^* D_\upsilon$ has at least $|\hat{I}|(\dim V)$ eigenvalues that tend to 0 as $|\upsilon| \longrightarrow 0$. The corresponding eigenfunctions converge to vector fields on the components of Σ_b that do not agree at the nodes. If $\mathcal{T}$ is semiregular, the operator D_{b_υ} has cokernel $\Gamma_-^{0,1}(b)$. In such a case, the number of low eigenvalues of $D_\upsilon^* D_\upsilon$, including 0, is $(\dim\Gamma_-^{0,1}(b)) + |\hat{I}|(\dim V)$.

Let $\tilde{\pi}_{\upsilon,+}^{0,1} : \Gamma_+^{0,1}(\upsilon) \longrightarrow \tilde{\Gamma}_+^{0,1}(\upsilon)$ denote the inverse of $\pi_{\upsilon,+}^{0,1} : \tilde{\Gamma}_+^{0,1}(\upsilon) \longrightarrow \Gamma_+^{0,1}(\upsilon)$. We extend $\tilde{\pi}_{\upsilon,+}^{0,1}$ to all of $L^p(\upsilon)$ by taking it to be $\tilde{\pi}_{\upsilon,+}^{0,1} \circ \pi_{\upsilon,+}^{0,1}$. If $\eta \in \tilde{\Gamma}_+^{0,1}(\upsilon)$, let $P_\upsilon\eta \in \tilde{\Gamma}_+(\upsilon)$ be the unique element such that $D_\upsilon P_\upsilon \eta = \eta$. We extend P_υ to all of $L^p(\upsilon)$ by taking it to be $P_\upsilon \circ \tilde{\pi}_{\upsilon,+}^{0,1}$. From Lemma 3.16, we immediately obtain

Corollary 3.17 *If $\mathcal{T}$ is a simple bubble type, there exist $\delta, C \in C^\infty(\mathcal{M}_{\mathcal{T}}^{(0)}; \mathbb{R}^+)$ such that for all $\upsilon \in F^{(0)}\mathcal{T}_\delta$ if $\mathcal{T}$ is regular and $\upsilon \in F^{(\emptyset)}\mathcal{T}_\delta$ if $\mathcal{T}$ is semiregular,*
(1) $\|\tilde{\pi}^{0,1}_{\upsilon,+}\eta\|_{\upsilon,p} \le C(b_\upsilon)\|\eta\|_{\upsilon,p}$ for all $\eta \in \Gamma^{0,1}(\upsilon)$;
(2) $\|P_\upsilon\eta\|_{\upsilon,p,1} \le C(b_\upsilon)\|\eta\|_{\upsilon,p}$ for all $\eta \in \Gamma^{0,1}(\upsilon)$.

3.6 The Gluing Map

In this section, we look for small vector fields $\xi \in \tilde{\Gamma}_+(\upsilon)$ such that $\exp_{b_\upsilon,u_\upsilon}\xi$ is holomorphic if $\mathcal{T}$ is regular and lies in $\mathcal{M}_{\Sigma,t\nu,\lambda}$ if $\mathcal{T}$ is semiregular. In Section 4.5, we show that all holomorphic maps if $\mathcal{T}$ is regular and all maps in $\mathcal{M}_{\Sigma,t\nu,\lambda} \times \Sigma^M$ if $\mathcal{T}$ is semiregular that lie near $\mathcal{M}_{\mathcal{T}}$ with respect to the Gromov topology can be obtained in this way.

If $\xi \in \Gamma(u_\upsilon)$, define $\exp_\upsilon \xi \colon \Sigma_\upsilon \longrightarrow V$ and $\bar{\partial}_\upsilon \xi \in \Gamma^{0,1}(u_\upsilon)$ by

$$\{\exp_\upsilon \xi\}(z) = \exp_{b_\upsilon,u_\upsilon(z)} \xi(z), \quad \{\bar{\partial}_\upsilon \xi\}|_z = \Pi^{-1}_{b_\upsilon,\xi(z)} \circ \bar{\partial}\{\exp_\upsilon \xi\}\big|_z.$$

If $S = \Sigma$ and $\nu \in \Gamma(\Sigma; \Lambda^{0,1}\pi_\Sigma^* T^*\Sigma \otimes \pi_V^* TV)$, let $\nu_{\upsilon,\xi} \in \Gamma^{0,1}(u_\upsilon)$ be given by

$$\nu_{\upsilon,\xi}|_z = \Pi^{-1}_{b_\upsilon,\xi(z)} \circ \nu|_{(z,\{\exp_\upsilon \xi\}z)}.$$

Then,

$$\bar{\partial}\{\exp_\upsilon \xi\}(\cdot) = t\nu|_{(\cdot,\{\exp_\upsilon \xi\}(\cdot))} \Longleftrightarrow \bar{\partial}_\upsilon \xi = t\nu_{\upsilon,\xi}. \tag{3.8}$$

Write

$$\bar{\partial}_\upsilon \xi = \bar{\partial} u_\upsilon + D_\upsilon \xi + N_\upsilon \xi \quad \text{and} \quad \nu_{\upsilon,\xi}|_z = \nu|_{(z,u_\upsilon(z))} + L_{\nu,\upsilon}\xi|_z. \tag{3.9}$$

Then the second equation in (3.8) is equivalent to

$$D_\upsilon \xi + N_{\upsilon,t\nu}\xi = t\nu - \bar{\partial} u_\upsilon, \tag{3.10}$$

and by Proposition A.11 and (1) of Lemma 3.5, there exist $C_{\bar{\partial}}, \delta \in C^\infty(\mathcal{M}_{\mathcal{T}}^{(0)}; \mathbb{R}^+)$ such that for any $\upsilon \in F^{(0)}\mathcal{T}_\delta$ and $\xi_1, \xi_2 \in \Gamma(u_\upsilon)$,

$$\|N_{\upsilon,t\nu}\xi_1 - N_{\upsilon,t\nu}\xi_2\|_{\upsilon,p} \le C_{\bar{\partial}}(b_\upsilon)\left(\|\xi_1\|_{\upsilon,p,1} + \|\xi_2\|_{\upsilon,p,1} + t\right)\|\xi_1 - \xi_2\|_{\upsilon,p,1}. \tag{3.11}$$

If $\mathcal{T}$ is semiregular, the term ν will be fixed, and we will be looking for solutions of (3.10) with $t > 0$ very small for $\upsilon \in F^{(\emptyset)}\mathcal{T}_\delta$. If $\mathcal{T}$ is regular, we will consider (3.10) with $t = 0$ and $\upsilon \in F^{(0)}\mathcal{T}_\delta$. In both cases, we will consider only solutions ξ of (3.10) that lie in the subspace $\tilde{\Gamma}_+(\upsilon)$ of $L^p_1(\upsilon)$, since the subspace $\Gamma_-(\upsilon)$ corresponds to moving along the image of the pregluing map $\upsilon \longrightarrow b(\upsilon)$.

Vector field $\xi = P_\upsilon\eta$ with $\eta \in \Gamma^{0,1}_+(\upsilon)$ solves equation (3.10) if and only if

$$\eta + \pi^{0,1}_{\upsilon,+} N_{\upsilon,t\nu} P_\upsilon \eta = \pi^{0,1}_{\upsilon,+}\left(t\nu - \bar{\partial} u_\upsilon\right) \tag{3.12}$$

$$\text{and} \qquad \pi^{0,1}_{\upsilon,-}\left(t\nu - \bar{\partial} u_\upsilon - \tilde{\pi}^{0,1}_{\upsilon,+}\eta - N_{\upsilon,t\nu}P_\upsilon\eta\right) = 0. \tag{3.13}$$

Denote the map $\eta \longrightarrow \pi^{0,1}_{\upsilon,+} N_{\upsilon,t\nu} P_\upsilon \eta$ by $N^+_{\upsilon,t\nu}$. By Corollary 3.17 and equation (3.11), there exist $\tilde{C}_{\bar{\partial}}, \delta \in C^\infty(\mathcal{M}_{\mathcal{T}}^{(0)}; \mathbb{R}^+)$ such that for any $\upsilon \in F^{(0)}\mathcal{T}_\delta$ if $\mathcal{T}$ is regular and $\upsilon \in F^{(\emptyset)}\mathcal{T}_\delta$ if $\mathcal{T}$

is semiregular,

$$\big\|N^+_{v,t\nu}\eta_1 - N^+_{v,t\nu}\eta_2\big\|_{v,p} \le \tilde{C}_{\bar{\delta}}(b_v)\big(\|\eta_1\|_{v,p} + \|\eta_2\|_{v,p} + t\big)\|\eta_1 - \eta_2\|_{v,p} \tag{3.14}$$

for all $\eta_1, \eta_2 \in \Gamma^{0,1}_+(v)$ such that $\|\eta_1\|_{v,p}, \|\eta_2\|_{v,p} \le \delta(b)$.

Lemma 3.18 *There exist $\epsilon, \delta \in C^\infty(\mathcal{M}^{(0)}_{\mathcal{T}}; \mathbb{R}^+)$ such that for all $v \in F^{(0)}\mathcal{T}_\delta$ and $t=0$ if $\mathcal{T}$ is regular, $v \in F^{(\emptyset)}\mathcal{T}_\delta$ and $t \in [0; \delta(b_v)]$ if $\mathcal{T}$ is semiregular, and $\alpha \in \Gamma^{0,1}_+(v)$ with $\|\alpha\|_{v,p} < \epsilon(b_v)$, the equation*

$$\eta + N^+_{v,t\nu}\eta = \alpha$$

has a unique solution η_α in $\Gamma^{0,1}_+(v)$ such that $\|\eta_\alpha\|_{v,p} \le 2\epsilon(b_v)$. Furthermore, such a solution satisfies $\|\eta_\alpha\|_{v,p} \le 2\|\alpha\|_{v,p}$.

Proof: Put $\epsilon(b) = (6\tilde{C}_{\bar{\delta}}(b))^{-1}$, where $\tilde{C}_{\bar{\delta}}$ is as in (3.14). Define

$$\Psi_\alpha \colon \big\{\eta \in \Gamma^{0,1}_+(v) \colon \|\eta\|_{v,p} \le \ 2\|\alpha\|_{v,p}\big\} \longrightarrow \Gamma^{0,1}_+(v)$$

by $\Psi_\alpha(\eta) = \alpha - N^+_{v,t\nu}\eta$. By equation (3.14),

$$\|\Psi_\alpha(\eta)\|_{v,p} \le \|\alpha\|_{v,p} + \tilde{C}_{\bar{\delta}}(b_v)\big(\|\eta\|_{v,p} + t\big)\|\eta\|_{v,p} \le 2\|\alpha\|_{v,p};$$
$$\|\Psi_\alpha(\eta_1) - \Psi_\alpha(\eta_2)\|_{v,p} \le \tilde{C}_{\bar{\delta}}(b_v)(\|\eta_1\|_{v,p} + \|\eta_2\|_{v,p} + t)\|\eta_1 - \eta_2\|_{v,p} \le \frac{5}{6}\|\eta_1 - \eta_2\|_{v,p}.$$

It follows that Ψ_α is a contracting operator, and thus has a unique fixed point η_α, i.e.

$$\eta_\alpha + N^+_{v,t\nu}\eta_\alpha = \alpha, \quad \text{and} \quad \|\eta_\alpha\|_{v,p} \le 2\|\alpha\|_{v,p}.$$

The uniqueness claim follows immediately by taking the difference of the corresponding equations.

Corollary 3.19 *If $\mathcal{T}$ is a simple bubble type, there exist $\delta, \epsilon, C \in C^\infty(\mathcal{M}^{(0)}_{\mathcal{T}}; \mathbb{R}^+)$ such that for all $v \in F^{(0)}\mathcal{T}_\delta$ and $t = 0$ if $\mathcal{T}$ is regular and $v \in F^{(\emptyset)}\mathcal{T}_\delta$ and $t \in [0; \delta(b_v)]$ if $\mathcal{T}$ is semiregular, there exists a unique $\eta_{v,t\nu} \in \Gamma^{0,1}(v)$ such that $\eta_{v,t\nu}$ satisfies equation (3.12) and $\|\eta_{v,t\nu}\|_{v,p} \le \epsilon(b_v)$. Furthermore,*

$$\|\eta_{v,t\nu}\|_{v,p} \le C(b_v)\big(t + |v|^{\frac{1}{p}}\big).$$

Proof: This corollary follows from Lemmas 3.18 and 3.5.

We now put $\xi_{v,t\nu} = P_v\eta_{v,t\nu}$ and $\tilde{u}_{v,t\nu} = \exp_v \xi_{v,t\nu}$. Replacing u_v in $b(v)$ by $\tilde{u}_{v,t\nu}$, we obtain a new bubble map that will be called $\tilde{b}_{t\nu}(v)$. If $\mathcal{T}$ is regular (and thus $t=0$), we will write $\tilde{u}_v$ and $\tilde{b}(v)$ for $\tilde{u}_{v,0}$ and $\tilde{b}_0(v)$, respectively. We can assume that the functions δ, ϵ and C of Corollary 3.19 are $\big(\mathcal{A}(\mathcal{T}) \ltimes \tilde{G}_{\mathcal{T}}\big)$-invariant if $S = S^2$ and $\big(\mathcal{A}(\mathcal{T}) \ltimes G_{\mathcal{T}}\big)$-invariant if $S = \Sigma$. For $\mathcal{T}$ regular, we have thus constructed a *gluing* map

$$\tilde{\gamma}^{(0)}_{\mathcal{T}} \colon F^{(0)}\mathcal{T}_\delta \longrightarrow \bar{\mathcal{M}}_{\langle\mathcal{T}\rangle}, \quad v \longrightarrow \tilde{b}(v).$$

Since this map is $\big(\mathcal{A}(\mathcal{T}) \ltimes G_{\mathcal{T}}\big)$-invariant, as can be seen from the construction, $\tilde{\gamma}^{(0)}_{\mathcal{T}}$ induces a map on the quotient

$$\tilde{\gamma}_{\mathcal{T}} \colon F\mathcal{T}_\delta \longrightarrow \bar{\mathcal{M}}_{\langle\mathcal{T}\rangle}. \tag{3.15}$$

By the smooth dependence of solutions of (3.12), the restrictions

$$\tilde{\gamma}_{\mathcal{T}}^{(0)}: F^{(H)}\mathcal{T}_\delta \longrightarrow \mathcal{M}_{\mathcal{T}(H)}^{(0)}$$

are smooth. However, continuity of $\tilde{\gamma}_{\mathcal{T}}$ on all of $F\mathcal{T}_\delta$ is not immediate. In the next section, we show the map $\tilde{\gamma}_{\mathcal{T}}$ is a homeomorphism onto a neighborhood of $\mathcal{M}_{\mathcal{T}}$ in $\bar{\mathcal{M}}_{\langle\mathcal{T}\rangle}$. If $\mathcal{T}$ is semiregular and $t>0$, we have constructed a map

$$\tilde{\gamma}_{\mathcal{T},t\nu}^{(0)}: F^{(\emptyset)}\mathcal{T}_\delta\Big|\epsilon^{-1}(-t,t) \longrightarrow C^\infty_{(\lambda;M)}(\Sigma;V),$$

which again is $\big(\mathcal{A}(\mathcal{T})\ltimes G_{\mathcal{T}}\big)$-invariant and thus descends to a map

$$\tilde{\gamma}_{\mathcal{T},t\nu}: F^{\emptyset}\mathcal{T}_\delta\Big|\big(\epsilon^{-1}(-t,t)/\mathcal{A}(\mathcal{T})\ltimes G_{\mathcal{T}}\big) \longrightarrow C^\infty_{(\lambda;M)}(\Sigma;V). \tag{3.16}$$

The map $u_{\tilde{b}_{t\nu}(v)}$ lies in $\mathcal{M}_{\Sigma,t\nu,\lambda}$ if and only if equation (3.13) is satisfied, i.e.

$$R_v\psi_{\mathcal{T},t\nu}(v) \equiv t\nu - \bar{\partial}u_v - \tilde{\pi}^{0,1}_{v,+}\eta_{v,t\nu} - N_{v,t\nu}P_v\eta_{v,t\nu} = 0 \in \Gamma^{0,1}_{-}(v), \tag{3.17}$$

since $\eta_{v,t\nu}$ satisfies equation (3.12).

3.7 An Implicit Function Theorem

In this section, we prove a refined version of the Implicit Function Theorem. It will be used in the rest of this chapter to modify the gluing maps of Section 3.6 for the spaces $\mathcal{M}_{\mathcal{T}}(\mu)$, $\mathcal{U}_{\mathcal{T}}(\mu)$, etc.

Let $\mathcal{S}$ be a smooth oriented manifold, and $\mathcal{NS}$, $\mathcal{N}^\mu$, and F oriented Riemannian vector bundles over $\mathcal{S}$. We denote by b, $(b,\vec{n})$, (b,σ), and (b,v) general elements of $\mathcal{S}$, $\mathcal{NS}$, $\mathcal{N}^\mu$, and F, respectively. If Ω is any subset of F and $\delta>0$, let

$$\Omega(\delta) = \big\{(b,\vec{n},v)\in\mathcal{NS}\oplus F: (b,v)\in\Omega;\ |\vec{n}|,|v|<\delta\big\}.$$

Let U be an open neighborhood of $\mathcal{S}$ in $\mathcal{NS}\oplus\mathcal{N}^\mu\oplus F$ and $h: U\Diamond a\mathbb{R}^k$ a smooth map such that

$$h(b,\vec{n},\sigma,v) = h(b,\vec{n},\sigma,0), \quad h|\mathcal{S}=0, \quad \text{and} \quad d(h:\mathcal{N}^\mu_b\longrightarrow\mathbb{R}^n)_{(b,0)}:\mathcal{N}^\mu_b\longrightarrow\mathbb{R}^k$$

is an orientation-preserving isomorphism for all $b\in\mathcal{S}$. Let $\tilde{U}$ be a subset of U such that $\tilde{U}$ is the fiber product along $\mathcal{S}$ of an open neighborhood of $\mathcal{S}$ in $\mathcal{NS}\oplus\mathcal{N}^\mu$ and an open subset Ω of F. Suppose $\delta_{\mathcal{S}}>0$, $C\in C^\infty(\mathcal{S};\mathbb{R}^+)$, and $\tilde{h}_t:\tilde{U}\longrightarrow\mathbb{R}^k$ is a family of smooth functions with $t\in[0,\delta_{\mathcal{S}}]$ such that

$$\big|\tilde{h}_t-h\big|_{(b,\vec{n},\sigma,v)}, \Big|\frac{\partial\tilde{h}_t}{\partial\sigma}-\frac{\partial h}{\partial\sigma}\Big|_{(b,\vec{n},\sigma,v)} \le C(b)\big(|v|^{\frac{1}{p}}+t\big) \quad \forall t\in(0,\delta_{\mathcal{S}}),\ (b,\vec{n},\sigma,v)\in\tilde{U},$$

where $\frac{\partial h}{\partial\sigma}$ denotes the differential of h along the fibers of $\mathcal{N}^\mu$.

Lemma 3.20 *Let B be an open ball about $0\in\mathbb{R}^k$. If $f: B\longrightarrow\mathbb{R}^k$ is a smooth function and*

$$k\big|Df|_z - Df|_0\big| < \big|(Df|_0)^{-1}\big|^{-1} \quad \forall z\in B,$$

then f is injective on B.

Proof: Let f_i denote the ith component of f. By the Mean Value Theorem, for all $x,y\in B$, there exists $z_i(x,y)\in B$ such that

$$\big|f_i(x)-f_i(y)\big| = \big|Df_i|_{z_i(x,y)}\big||x-y|.$$

Adding up these equations over all i, we obtain

$$\begin{aligned}\sum_{i=1}^{i=k}\big|f_i(x)-f_i(y)\big| &\ge \sum_{i=1}^{i=k}\big|Df_i|_0\big||x-y| - k\sup_{z\in B}\big|Df|_z - Df|_0\big||x-y|\\ &\ge \Big(\big|(Df|_0)^{-1}\big|^{-1} - k\sup_{z\in B}\big|Df|_z-Df|_0\big|\Big)|x-y|.\end{aligned}$$

Lemma 3.21 *For every precompact subset K of $\mathcal{S}$, there exists $\epsilon>0$ such that for all $t\in(0,\epsilon)$ and $(b,\vec{n},v)\in\Omega(\epsilon)|K$, the map*

$$\big\{(b,\sigma)\in\mathcal{N}^{\mu}\colon |\sigma|<\epsilon\big\} \longrightarrow \tilde{h}_t(b,\vec{n},\sigma,v)$$

is defined and injective, and its differential defines an orientation-preserving isomorphism between $\mathcal{N}_b^{\mu}$ and $\mathbb{R}^k$.

Proof: The map above is defined as long as

$$\big\{(b,\vec{n},\sigma,v)\in\mathcal{NS}\oplus\mathcal{N}_b^{\mu}\oplus F\colon b\in K, (b,\vec{n},v)\in\Omega(\epsilon), |\sigma|<\epsilon\big\}\subset\tilde{U}.$$

Since K is precompact, existence of $\delta>0$ such that the last inclusion holds is trivial. The other two statements follow from the third property of h and the second property of $\tilde{h}_t$ (see above); Lemma 3.20 is needed to prove the injectivity. Note that the variation of $\frac{\partial\tilde{h}_t}{\partial\sigma}$ over K can be bounded from the variation $\frac{\partial h}{\partial\sigma}$ and the second property of $\tilde{h}_t$.

Lemma 3.22 *For every precompact subset K of $\mathcal{S}$ and $\epsilon>0$ sufficiently small, there exists $\delta>0$ such that for all $t\in(0,\delta)$ and $(b,\vec{n},v)\in\Omega(\delta)|K$, the image of the map*

$$\{(b,\sigma)\in\mathcal{N}^{\mu}\colon |\sigma|<\epsilon\} \longrightarrow \tilde{h}_t(b,\vec{n},\sigma,v)$$

contains $0\in\mathbb{R}^k$.

Proof: We assume $\epsilon>0$ does not exceed the number provided by Lemma 3.21. Then by precompactness of K and the proof of Lemma 3.21,

$$\varepsilon\equiv\min\big\{|h(b,\vec{n},\sigma,v)|\colon (b,\vec{n},v)\in\Omega_\epsilon|K,\ (b,\sigma)\in\mathcal{N}^{\mu},\ |\sigma|=\frac{1}{2}\epsilon\big\}>0. \tag{3.18}$$

Since for each $(b,\vec{n},v)\in\Omega(\epsilon)|K$, the image of the map

$$\{(b,\sigma)\in\mathcal{N}^{\mu}\colon |\sigma|<\epsilon\} \longrightarrow h(b,\vec{n},\sigma,v)$$

contains a neighborhood of 0 in $\mathbb{R}^k$ and $\tilde{h}_t$ is continuous, the claim follows from the first property of $\tilde{h}_t$ along with equation (3.18).

Corollary 3.23 *For every precompact open subset K of $\mathcal{S}$, there exist $\delta, C>0$ with the following property. For all $t\in(0,\delta)$, there exists a smooth section*

$$\varphi_t \in \Gamma\big(\Omega(\delta)|K; \pi^*\mathcal{N}^\mu\big),$$

where $\pi : \Omega(\delta)|K \longrightarrow K$ is the bundle projection map, such that

$$\Omega(\delta)|K \longrightarrow \tilde{h}_t^{-1}(0), \quad (b,\vec{n},v) \longrightarrow \big(b,\vec{n},\varphi_t(b,\vec{n},v),v\big),$$

is an orientation-preserving diffeomorphism. Furthermore,

$$\big|\varphi_t(b,\vec{n},v)\big| \le C\big(|v|^{\frac{1}{p}} + t + |\vec{n}|\big) \quad \forall (b,\vec{n},v)\in\Omega(\delta)|K.$$

Finally, if G is a group that acts on the space $\mathcal{S}$ and bundles $\mathcal{NS}$, $\mathcal{N}^\mu$, and F, and preserves h, $\tilde{h}_t$, Ω, and K, then φ_t is G-equivariant.

Proof: With ϵ as provided by Lemma 3.21, let $\delta>0$ be as provided by Lemma 3.22. Then

$$F_t : \big\{(b,\vec{n},\sigma,v) : (b,\vec{n},v)\in\Omega(\delta)|K, |\sigma|<\epsilon\big\} \longrightarrow \Omega(\delta)\times\mathbb{R}^k, \;\; F_t(b,\vec{n},\sigma,v) = \big(b,\vec{n},v,\tilde{h}_t(b,\vec{n},v)\big)$$

is a diffeomorphism onto an open subset W of the target space. The inverse of F_t must have the form

$$F_t^{-1}(b,\vec{n},v,\tilde{\sigma}) = \big(b,\vec{n},\phi_t(b,\vec{n},v,\tilde{\sigma}),v\big)$$

for some smooth function ϕ_t. By Lemma 3.22, $(\Omega(\delta)|K)\times\{0\}\subset W$. Thus,

$$\varphi_t\in\Gamma\big(\Omega(\delta)|K;\pi^*\mathcal{N}^\mu\big), \quad \varphi_t(b,\vec{n},v) = \phi_t(b,\vec{n},v,0),$$

is a well-defined section, and by definition of ϕ_t,

$$\Omega(\delta)|K \longrightarrow \tilde{h}_t^{-1}(0), \quad (b,\vec{n},v) \longrightarrow \big(b,\vec{n},\varphi_t(b,\vec{n},v),v\big),$$

is a diffeomorphism. It is orientation-preserving by Lemma 3.21. The estimate on φ_t follows from the three properties of h, the first property of $\tilde{h}_t$, and the proof of Lemma 3.20. The final statement of the lemma is clear, since our construction commutes with the G-action.

3.8 The Orientation of $\mathcal{M}_{\Sigma,t\nu,\lambda}(\mu)$ and the Gluing Map

At this point, our treatments of regular and semiregular cases diverge. In this section, we assume that $\mathcal{T} = (\Sigma,[N],I;j,\underline{\lambda})$ is a semiregular bubble type and μ is an N-tuple of constraints in general position as defined below. Let $\lambda = \sum\lambda_i$ as before. We recall how each element of $\mathcal{M}_{\Sigma,t\nu,\lambda}(\mu)$ is assigned a sign and then specialize to the elements $\tilde{b}_{t\nu}(v)\in\mathcal{M}_{\Sigma,t\nu,\lambda}(\mu)$. We conclude this section with Theorem 3.29 that describes the elements of $\mathcal{M}_{\Sigma,t\nu,\lambda}(\mu)$ lying near the space $\mathcal{M}_{\mathcal{T}}(\mu)$.

Definition 3.24 *(1) Section $\nu\in\Gamma^{0,1}(\Sigma\times V;\Lambda^{0,1}_{J,j}\pi_\Sigma^*T^*\Sigma\otimes\pi_V^*TV)$ is $\underline{\lambda\text{-regular}}$ if for all $t\in(0,1)$ and $u\in\mathcal{M}_{\Sigma,t\nu,\lambda}$, the operator $(D_{V,u}-\nabla^V_{\cdot}\nu) : \Gamma(u) \longrightarrow \Gamma^{0,1}(u)$ is surjective.*
(2) If ν is λ-regular, N-tuple μ of oriented submanifolds of V is $\underline{\nu\text{-regular}}$ if for all $t\in(0,1)$

and $b=(\Sigma,[N],\{\hat{0}\};,(\hat{0},y),u_{\hat{0}})\in\mathcal{M}_{\Sigma,t\nu,\lambda}(\mu)$

$$\bigoplus_{l\in[N]} T_{u_{\hat{0}}(y_l)}V = Im\ dev_{[N]}\big|_b + \bigoplus_{l\in[N]} T_{u_{\hat{0}}(y_l)}\mu_l \qquad where$$

$$dev_{[N]}\big|_b : \ker\big(D_{V,u_{\hat{0}}} - \nabla^V_{\cdot}\nu\big) \oplus \bigoplus_{l\in[N]} T_{y_l}\Sigma \longrightarrow \bigoplus_{l\in[N]} T_{u_{\hat{0}}(y_l)}V, \quad dev_l\big|_b(\xi,w_{[N]}) = \xi(y_l) + du_{\hat{0}}\big|_{y_l} w_l.$$

(3) If $\mathcal{T}$ is a (V,J)-semiregular bubble type, tuple μ of oriented submanifolds of V is <u>*$\mathcal{T}$-regular*</u> *if*

$$\bigoplus_{l\in[N]} T_{u_{j_l}(y_l)}V = Im\ dev_{[N]}\big|_b + \bigoplus_{l\in[N]} T_{u_{j_l}(y_l)}\mu_l \qquad \forall\ b=(\Sigma,[N],I;x,(j,y),u)\in\mathcal{H}_{\mathcal{T}}(\mu);$$

$$where \qquad dev_{[N]}\big|_b : \mathcal{K}_b\mathcal{T} \longrightarrow \bigoplus_{l\in[N]} T_{u_{j_l}(y_l)}V, \quad dev_l\big|_b(\xi_I, w_{\hat{I}+[N]}) = \xi_{j_l}(y_l) + du_{j_l}\big|_{y_l} w_l.$$

(4) If $\mathcal{T}$ is a (V,J)-semiregular bubble type, $\mathcal{S}\subset\mathcal{M}_{\mathcal{T}}$ is a smooth submanifold, and $\tilde{\mathcal{S}}\subset\mathcal{M}^{(0)}_{\mathcal{T}}$ is the preimage of $\mathcal{S}$ under the quotient projection map, N-tuple μ of oriented submanifolds of V is <u>*$\mathcal{S}$-regular*</u> *if*

$$\bigoplus_{l\in[N]} T_{u_{j_l}(y_l)}V = dev_{[N]}\big|_b\big(\mathcal{K}_b\mathcal{T}\cap T_b\tilde{\mathcal{S}}\big) + \bigoplus_{l\in[N]} T_{u_{j_l}(y_l)}\mu_l \qquad \forall\ b\in\tilde{\mathcal{S}}(\mu)\equiv\tilde{\mathcal{S}}\cap\mathcal{M}^{(0)}_{\mathcal{T}}(\mu).$$

Note that all four definitions above are independent of the choice of metrics on V. Throughout this section, we assume that ν is λ-regular, $\mathcal{T}$ is semiregular, and μ is ν- and $\mathcal{T}$-regular.

The space $\mathcal{M}_{\Sigma,t\nu,\lambda}$ consists of the maps $u\colon\Sigma\longrightarrow V$ such that $\bar{\partial}u|_z = t\nu(z,u(z))$ for all $z\in\Sigma$. Thus, the tangent space at u can be described as

$$T_u\mathcal{M}_{\Sigma,t\nu,\lambda} = \big\{\xi\in\Gamma(\Sigma;u^*TV)\colon D_{V,u}\xi - tL_{\nu,u}\xi = 0\big\},$$

where $L_{\nu,u}\xi$ is defined by

$$\{L_{\nu,u}\xi\}(z) = \nabla^V_{\xi(z)}\nu\big|_{(z,u(z))}.$$

The operator $D_{V,u}-tL_{\nu,u}$ is independent of the choice of the connection along $\mathcal{M}_{\Sigma,t\nu,\lambda}$ and by assumption has no cokernel if $t\in(0,1)$. An orientation on $\mathcal{M}_{\Sigma,t\nu,\lambda}$ is determined by an orientation of the bundle $\Lambda^{top}_{\mathbb{R}}T\mathcal{M}_{\Sigma,t\nu,\lambda}$ over $\mathcal{M}_{\Sigma,t\nu,\lambda}$, which is the determinant line bundle of the elliptic operator $D_{V,u}-tL_{\nu,u}$. Since $L_{\nu,u}$ has order zero, the operator $D_{V,u}-tL_{\nu,u}$ is homotopic through elliptic operators to the operator $D_{V,u}$. Thus, $\Lambda^{top}_{\mathbb{R}}T\mathcal{M}_{\Sigma,t\nu,\lambda}$ is homotopic to

$$\det(D_{V,u}) = \big(\Lambda^{top}_{\mathbb{R}}(\ker\ D_{V,u})\big)\otimes\big(\Lambda^{top}_{\mathbb{R}}(\text{coker}\ D_{V,u})\big);$$

see [LM]. Since $D_{V,u}$ commutes with J, $\ker D_{V,u}$ and coker $D_{V,u}$ are both complex vector spaces and thus have natural orientations, which induce an orientation on the determinant line bundle of $D_{V,u}$ and via a homotopy of operators on the determinant bundle of $D_{V,u}-tL_{\nu,u}$. It follows that $\mathcal{M}_{\Sigma,t\nu,\lambda}\times\Sigma^N$ is naturally oriented. If μ is a ν-regular tuple of submanifold of V of total codimension

$$\text{codim}\ \mu = \dim\mathcal{M}_{\Sigma,t\nu,\lambda}\times\Sigma^N = \text{ind}\ D_{V,u} + 2|N| = 2\Big(\langle c_1(V,J),\lambda\rangle + (\dim V)\big(1-g(\Sigma)\big) + |N|\Big),$$

the differential of the map

$$\mathrm{ev}_{[N]}\colon \mathcal{M}_{\Sigma,t\nu,\lambda}\times\Sigma^N \longrightarrow \prod_{l\in[N]} V, \quad (\Sigma,[N],\{\hat{0}\};,(\hat{0},y),u_{\hat{0}}) \longrightarrow \big(u_{\hat{0}}(y_l)\big)_{l\in[N]},$$

induces an isomorphism between $T\mathcal{M}_{\Sigma,t\nu,\lambda}\oplus T\Sigma^N$ and the normal bundle of μ in V^N at each point of $\mathcal{M}_{\Sigma,t\nu,\lambda}(\mu)$. Here we identify the N-tuple μ with the submanifold

$$\prod_{l\in[N]}\mu_l\subset\prod_{l\in[N]}V\equiv V^N.$$

Since the normal bundle of μ is oriented, the evaluation map also induces an orientation on $T\mathcal{M}_{\Sigma,t\nu,\lambda}\oplus T\Sigma^N$ along $\mathcal{M}_{\Sigma,t\nu,\lambda}(\mu)$. Each element $b\in\mathcal{M}_{\Sigma,t\nu,\lambda}(\mu)$ is assigned a plus sign or is positively oriented if the two orientations agree, and a minus sign otherwise.

For any $v\in F^{(\emptyset)}\mathcal{T}$ such that q_v is defined, let $L_{\nu,v}\colon\Gamma(u_v)\longrightarrow\Gamma^{0,1}(u_v)$ be given by

$$\{L_{\nu,v}\xi\}(z)=\nabla^{b_v}_{\xi(z)}\nu\big|_{(z,f(z))}.$$

Denote by $\Gamma^{0,1}_{t,+}(v)$ the image of $\Gamma_+(v)$ under the map $D_v-tL_{\nu,v}$.

Lemma 3.25 *For any compact subset K of $\mathcal{M}^{(0)}_{\mathcal{T}}$, there exist $\delta,C>0$ such that for all $v\in F^{(\emptyset)}\mathcal{T}_\delta|K$ and $t\in(0,\delta)$,*
(1) $\|\xi\|_{v,p,1}\le C\|D_v\xi-tL_{\nu,v}\xi\|_{v,p}$ for all $\xi\in\Gamma_+(v)$;
(2) $L^p(v)=\Gamma^{0,1}_{t,+}(v)\oplus\Gamma^{0,1}_-(v)$;
(3) if $D^{--}_{v,t}$ and $L^{--}_{\nu,v,t}$ are the $(-,-)$-components of D_v and $L_{\nu,v}$ with respect to the decompositions $L^p_1(v)=\Gamma_+(v)\oplus\Gamma_-(v)$ and $L^p(v)=\Gamma^{0,1}_{t,+}(v)\oplus\Gamma^{0,1}_-(v)$, then

$$\pi_{v,-}\colon\ker\big\{D_v-tL_{\nu,v}\colon L^p_1(v)\longrightarrow L^p(v)\big\}\longrightarrow\ker\big\{D^{--}_{v,t}-tL^{--}_{\nu,v,t}\colon\Gamma_-(v)\longrightarrow\Gamma^{0,1}_-(v)\big\}$$

is an orientation-preserving isomorphism, provided one of the two operators is surjective.

Proof: (1) The first claim is immediate from (1) of Lemma 3.16 and (2) of Lemma 3.5. The second is obtained by the same argument as in the proof of (3) of Lemma 3.16.
(2) By construction, $\pi_{v,-}$ is an isomorphism of the two kernels of the lemma. In particular, $D_v-tL_{\nu,v}$ is surjective if and only if $D^{--}_{v,t}-tL^{--}_{\nu,v,t}$ is. Define

$$\Phi_\tau\colon L^p_1(v)\oplus\Gamma^{0,1}_-(v)\longrightarrow L^p(v)\quad\text{and}\quad\Psi_\tau\colon\Gamma_-(v)\oplus\Gamma^{0,1}_-(v)\longrightarrow\Gamma^{0,1}_-(v)\qquad\text{by}$$
$$\Phi_\tau(\xi,\eta)=D_v\xi+\tau tL_{\nu,v}\xi+\eta\quad\text{and}\quad\Psi_\tau(\xi,\eta)=\tau\big(D^{--}_{v,t}+tL^{--}_{\nu,v,t}\big)\xi+\eta.$$

The first map is surjective for all $\tau\in[0,1]$ by (2) of the lemma, while the surjectivity of the second map is immediate from the definition. Furthermore, the maps

$$\phi_\tau\colon\ker\Phi_\tau\longrightarrow\Gamma_-(v),\quad\phi_\tau(\xi,\eta)=\pi_{v,-}\xi,\qquad\text{and}\quad\psi_\tau\colon\ker\Psi_\tau\longrightarrow\Gamma_-(v),\quad\psi_\tau(\xi,\eta)=\xi,$$

are isomorphisms such that

$$\psi_1^{-1}\phi_1(\xi,0)=\pi_{v,-}\xi;\quad\text{if }\phi_1(\xi,\eta)=\psi_1(\xi',\eta'),\ \eta=\eta';\quad\psi_0^{-1}\phi_0J=J\psi_0^{-1}\phi_0.$$

It follows that $\pi_{v,-}$ is an orientation-preserving map between the two kernels of the lemma.

If K is a precompact open subset of $\mathcal{M}_{\mathcal{T}}$ and $\delta>0$ is such that $\tilde{b}_{t\nu}(v)$ is defined for all $v\in F^{\emptyset}\mathcal{T}_{\delta}|K$ and $t\in(0,\delta)$, let $\mathcal{M}(K,\delta)$ and $\tilde{\mathcal{M}}_{t\nu}(K,\delta)$ denote the images of $F^{\emptyset}\mathcal{T}_{\delta}|K$ under the maps $\gamma_{\mathcal{T}}$ and $\tilde{\gamma}_{\mathcal{T},t\nu}$, respectively. Both maps are continuous and injective; see Section 4.2. The smooth structure of $F\mathcal{T}$ induces smooth structures on $\mathcal{M}(K,\delta)$ and $\tilde{\mathcal{M}}_{t\nu}(K,\delta)$, with the tangent bundles described by

$$T_{b(v)}\mathcal{M}(K,\delta)=\big\{\zeta'_{\varpi}=\frac{d}{d\tau}\zeta_{\tau\varpi}\big|_{\tau=0}:\varpi\in T_vF\mathcal{T}\big\}\oplus\bigoplus_{l\in[N]}T_{y_l(v)}\Sigma;$$

$$T_{\tilde{b}_{t\nu}(v)}\tilde{\mathcal{M}}_{t\nu}(K,\delta)=\big\{\tilde{\zeta}'_{\varpi}=\frac{d}{d\tau}\tilde{\zeta}_{\tau\varpi}\big|_{\tau=0}:\varpi\in T_vF\mathcal{T}\big\}\oplus\bigoplus_{l\in[N]}T_{y_l(v)}\Sigma,$$

where $T_vF\mathcal{T}$ denotes $T_vF^{\emptyset}\mathcal{T}$; see Section 3.4. It is easy to see that $\varpi\longrightarrow\zeta'_{\varpi}$ is nearly complex linear and $\pi_{v,-}$ is almost the identity on the first component of $T_{b(v)}\mathcal{M}(K,\delta)$; both error terms are bounded by $C_K|v|$. Furthermore, by (1) of Lemma 3.6 and Corollary 4.7, $\varpi\longrightarrow\tilde{\zeta}'_{\varpi}$ also nearly computes with the complex structures and $\Pi_{b_v,\xi_{v,t\nu}}\pi_{v,-}\Pi^{-1}_{b_v,\xi_{v,t\nu}}$ is almost the identity on the first component of $T_{\tilde{b}_{t\nu}(v)}\tilde{\mathcal{M}}_{t\nu}(K,\delta)$; in the given case, the error terms are bounded by $C_K\big(t+|v|^{\frac{1}{p}}\big)$. Thus, the orientations of $\mathcal{M}(K,\delta)$ and $\tilde{\mathcal{M}}_{t\nu}(K,\delta)$ induced by the natural orientation of $F\mathcal{T}$ agree with the orientations induced from the natural orientation on $\Gamma_-(v)\oplus\bigoplus_{l\in[N]}T_{y_l(v)}\Sigma$ via the maps $\pi_{v,-}\oplus id$ and $\pi_{v,-}\Pi^{-1}_{b_v,\xi_{v,t\nu}}\oplus id$, respectively.

By construction in Section 3.6,

$$\tilde{\psi}_{t\nu}:\tilde{\mathcal{M}}_{t\nu}(K,\delta)\longrightarrow\Gamma^{0,1},\quad v\longrightarrow t\nu|_{\tilde{u}_{v,t\nu}}-\bar{\partial}\tilde{u}_{v,t\nu}\in\Gamma^{0,1}(\tilde{u}_{v,t\nu}),$$

determines a section of the bundle $\Pi\Gamma^{0,1}_-$ over $\tilde{\mathcal{M}}_{t\nu}(K,\delta)$, given by

$$\Pi\Gamma^{0,1}_-(\tilde{b}_{t\nu}(v))=\Pi_{b_v,\xi_{v,t\nu}}\Gamma^{0,1}_-(v).$$

Note that the zero set of this section is precisely the space $(\mathcal{M}_{\Sigma,t\nu,\lambda}\times\Sigma^N)\cap\tilde{\mathcal{M}}_{t\nu}(K,\delta)$. A linearization of this section is given by

$$\begin{aligned}\nabla_{\tilde{\zeta}'_{\varpi}}(t\nu-\bar{\partial}\tilde{u}_{v,t\nu})&\equiv\Pi_{b_v,\xi_{v,t\nu}}\pi^{0,1}_{v,t,-}\nabla^{v}_{\pi_{v,-}\Pi^{-1}_{b_v,\xi_{v,t\nu}}\tilde{\zeta}'_{\varpi}}\Pi^{-1}_{b_v,\xi_{v,t\nu}}(t\nu-\bar{\partial}\tilde{u}_{v,t\nu})\\&=-\Pi_{b_v,\xi_{v,t\nu}}\big(D^{--}_{v,t}-tL^{--}_{v,\nu,t}\big)\,\pi_{v,-}\Pi^{-1}_{b_v,\xi_{v,t\nu}}\tilde{\zeta}'_{\varpi},\end{aligned}$$

where $\pi^{0,1}_{v,t,-}:L^p(v)=\Gamma^{0,1}_{+,t}(v)\oplus\Gamma^{0,1}_-(v)\longrightarrow\Gamma^{0,1}_-(v)$ is the projection map.

Corollary 3.26 *For any compact subset K of $\mathcal{M}^{(0)}_{\mathcal{T}}$, there exists $\delta>0$ such that for all $t\in(0,\delta)$, the orientation of $(\mathcal{M}_{\Sigma,t\nu,\lambda}\times\Sigma^N)\cap\tilde{\mathcal{M}}_{t\nu}(K,\delta)$ as the zero set of the section $\tilde{\psi}_{t\nu}$ agrees with its natural orientation.*

Proof: Suppose $\tilde{b}_{t\nu}(v)\in(\mathcal{M}_{\Sigma,t\nu,\lambda}\times\Sigma^N)\cap\tilde{\mathcal{M}}_{t\nu}(K,\delta)$. Since we can use any connection in $\tilde{u}^*_{v,t\nu}TV$ to define the natural orientation on $T_{\tilde{u}_{v,t\nu}}\mathcal{M}_{\Sigma,t\nu,\lambda}$, we can write

$$\big\{D_{\tilde{u}_{v,t\nu}}-tL_{\nu,\tilde{u}_{v,t\nu}}\big\}\xi=\Pi_{b_v,\xi_{v,t\nu}}\big\{D_v-tL_{\nu,v}\big\}\Pi^{-1}_{b_v,\xi_{v,t\nu}}\xi\quad\forall\xi\in\Gamma(\tilde{u}_{v,t}).$$

Thus, by Lemma 3.25, $\pi_{v,-}\circ\Pi^{-1}_{b_v,\xi_{v,t\nu}}\oplus id$ induces an orientation-preserving isomorphism between $T_{\tilde{b}_{t\nu}(v)}\mathcal{M}_{\Sigma,t\nu,\lambda}\oplus\bigoplus_{l\in[N]}T_{y_l(v)}\Sigma$ and $\ker(D^{--}_{v,t}-L^{--}_{\nu,v,t})\oplus\bigoplus_{l\in[N]}T_{y_l(v)}\Sigma$ with their natural orientations. By the preceding paragraph, the same is true for the zero set of $\tilde{\psi}_{t\nu}$.

If μ is an N-tuple of constraints as above, let

$$\mathcal{M}(K,\delta;\mu)=\{b(v)\in\mathcal{M}(K,\delta):\mathrm{ev}_{[N]}(b(v))\in\mu\},$$
$$\tilde{\mathcal{M}}_{t\nu}(K,\delta;\mu)=\{\tilde{b}_{t\nu}(v)\in\tilde{\mathcal{M}}_{t\nu}(K,\delta):\mathrm{ev}_{[N]}(\tilde{b}_{t\nu}(v))\in\mu\}.$$

Then $\mathcal{M}(K,\delta;\mu)$ and $\tilde{\mathcal{M}}_{t\nu}(K,\delta;\mu)$ are smooth manifolds. In fact, the smoothness of $\mathcal{M}(K,\delta;\mu)$ is immediate from the smoothness of $F\mathcal{T}|\mathcal{M}_{\mathcal{T}}(\mu)$, which is a consequence of $\mathcal{T}$-regularity of μ, while the smoothness of $\tilde{\mathcal{M}}_{t\nu}(K,\delta;\mu)$ follows from Lemma 3.28 below. Furthermore, since μ is ν-regular, the section $\tilde{\psi}_{t\nu}$ is transversal to zero in $\Pi\Gamma^{0,1}_{-}$ over $\tilde{\mathcal{M}}_{t\nu}(K,\delta;\mu)$. By Corollary 3.26, the sign of $\tilde{b}_{t\nu}(v)\in\mathcal{M}_{\Sigma,t\nu,\lambda}(\mu)$ defined at the beginning of this section is its sign as an element of the zero set of the section $\tilde{\psi}_{t\nu}$ of $\Pi\Gamma^{0,1}_{-}$ over $\tilde{\mathcal{M}}_{t\nu}(K,\delta;\mu)$.

If $b=(\Sigma,[N],I;x,(j,y),u)\in\mathcal{M}^{(0)}_{\mathcal{T}}(\mu)$, let

$$\mathcal{K}^{\mu}_b\mathcal{T}=\{(\xi,w_{\hat{I}+[N]})\in\mathcal{K}_b\mathcal{T}:\ \xi_{j_l}(y_l)+du_{j_l}|_{y_l}w_l\in T_{u_{j_l}(y_l)}\mu_l\ \forall l\in[N]\}.$$

Denote by $\mathcal{N}^{\mu}_b\mathcal{T}$ the (L^2,b)-orthogonal complement of $\mathcal{K}^{\mu}_b\mathcal{T}$ in $\mathcal{K}_b\mathcal{T}$. Note that by (3) of Definition 3.24,

$$\bigoplus_{l\in[N]}T_{u_{j_l}(y_l)}V=d\mathrm{ev}_{[N]}\big|_b(\mathcal{N}^{\mu}_b\mathcal{T})\oplus\bigoplus_{l\in[N]}T_{u_{j_l}(y_l)}\mu_l.$$

We denote by $\tilde{\mathcal{N}}^{\mu}\mathcal{T}$ the bundle over $\mathcal{M}^{(0)}_{\mathcal{T}}(\mu)$ with fibers $\mathcal{N}^{\mu}_b\mathcal{T}$ and by $\mathcal{N}^{\mu}\mathcal{T}\longrightarrow\mathcal{M}_{\mathcal{T}}(\mu)$ its quotient by the natural $G_{\mathcal{T}}$-action.

Suppose $\mathcal{S}\subset\mathcal{M}_{\mathcal{T}}$ is a smooth oriented submanifold such that μ is $\mathcal{S}$-regular. Denote by $\tilde{\mathcal{S}}\subset\mathcal{M}^{(0)}_{\mathcal{T}}$ the preimage of $\mathcal{S}$ under the quotient projection map. Let $\mathcal{N}\mathcal{S}\longrightarrow\mathcal{S}$ and $\mathcal{N}\tilde{\mathcal{S}}\longrightarrow\tilde{\mathcal{S}}$ be the normal bundles. Choose a $(\mathcal{A}(\mathcal{T})\ltimes G_{\mathcal{T}})$-equivariant orientation-preserving identification $\tilde{\phi}_{\mathcal{S}}:\mathcal{N}\tilde{\mathcal{S}}_{\delta}\longrightarrow\mathcal{M}^{(0)}_{\mathcal{T}}$ of neighborhoods of $\tilde{\mathcal{S}}$ in $\mathcal{N}\tilde{\mathcal{S}}$ and $\mathcal{M}^{(0)}_{\mathcal{T}}$. Let $\tilde{\Phi}_{\mathcal{S}}:\pi^*_{\mathcal{N}\tilde{\mathcal{S}}}F^{(0)}\mathcal{T}\longrightarrow F^{(0)}\mathcal{T}$ be a $(\mathcal{A}(\mathcal{T})\ltimes G_{\mathcal{T}})$-equivariant vector-bundle isomorphism covering $\tilde{\phi}_{\mathcal{S}}$ such that $\tilde{\Phi}_{\mathcal{S}}$ is the identity on $\tilde{\mathcal{S}}$. Let

$$\phi_{\mathcal{S}}:\mathcal{N}\mathcal{S}_{\delta}\longrightarrow\mathcal{M}_{\mathcal{T}}\quad\text{and}\quad\Phi_{\mathcal{S}}:\pi^*_{\mathcal{N}\mathcal{S}}F\mathcal{T}\longrightarrow F\mathcal{T}$$

be the maps induced by $\tilde{\phi}_{\mathcal{S}}$ and $\tilde{\Phi}_{\mathcal{S}}$, respectively. Put

$$\mathcal{S}(\mu)=\mathcal{S}\cap\mathcal{M}_{\mathcal{T}}(\mu),\quad\tilde{\mathcal{S}}(\mu)=\tilde{\mathcal{S}}\cap\mathcal{M}^{(0)}_{\mathcal{T}}(\mu).$$

Since μ is $\mathcal{S}$-regular, we can choose a $(\mathcal{A}(\mathcal{T})\ltimes G_{\mathcal{T}})$-equivariant orientation-preserving identification $\tilde{\phi}^{\mu}_{\mathcal{S}}:\tilde{\mathcal{N}}^{\mu}\mathcal{T}_{\delta}|\tilde{\mathcal{S}}(\mu)\longrightarrow\tilde{\mathcal{S}}$. Let

$$\tilde{\Phi}^{\mu}_{\mathcal{S}}:\pi^*_{\tilde{\mathcal{N}}^{\mu}\mathcal{T}}(\mathcal{N}\tilde{\mathcal{S}}\oplus F^{(0)}\mathcal{T})\longrightarrow\mathcal{N}\tilde{\mathcal{S}}\oplus F^{(0)}\mathcal{T}$$

be a $(\mathcal{A}(\mathcal{T})\ltimes G_{\mathcal{T}})$-equivariant splitting-preserving vector-bundle isomorphism covering $\tilde{\phi}_{\mathcal{S}}^{\mu}$ such that $\tilde{\Phi}_{\mathcal{S}}^{\mu}$ is the identity on $\tilde{\mathcal{S}}(\mu)$. Denote by

$$\phi_{\mathcal{S}}^{\mu}\colon \mathcal{N}^{\mu}\mathcal{T}_{\delta}|\mathcal{S}(\mu)\longrightarrow\tilde{\mathcal{S}}\quad\text{and}\quad \Phi_{\mathcal{S}}^{\mu}\colon \pi_{\mathcal{N}^{\mu}\mathcal{T}}^{*}(\mathcal{N}\mathcal{S}\oplus F\mathcal{T})\longrightarrow\mathcal{N}\mathcal{S}\oplus F\mathcal{T}$$

the maps induced by $\tilde{\phi}_{\mathcal{S}}^{\mu}$ and $\tilde{\Phi}_{\mathcal{S}}^{\mu}$, respectively.

Definition 3.27 *With notation as above and in Section 3.4, tuple $(\Phi_{\mathcal{S}},\Phi_{\mathcal{S}}^{\mu})$ is a regularization of $\mathcal{S}(\mu)$ if for all $b\in\tilde{\mathcal{S}}(\mu)$, $\vec{n}\in\mathcal{N}_b\tilde{\mathcal{S}}_{\delta(b)}$, and $\sigma\in\tilde{\mathcal{N}}_b^{\mu}\mathcal{T}_{\delta(b)}$, there exists $\varpi(\vec{n},\sigma)\in\mathcal{K}_{\tilde{\phi}_{\mathcal{S}}(b,\vec{n})}\mathcal{T}$ such that*

$$\tilde{\Phi}_{\mathcal{S}}\tilde{\Phi}_{\mathcal{S}}^{\mu}(b,\sigma;\vec{n},v)=\big\{\tilde{\Phi}_{\mathcal{S}}(b,\vec{n};v)\big\}\big(\varpi(\vec{n},\sigma)\big)\qquad\forall v\in F_b^{(0)}\mathcal{T}.$$

Note that if μ is $\mathcal{S}$-regular, $\mathcal{S}(\mu)$ admits a normalization. In fact, we can start with any choice of $\Phi_{\mathcal{S}}$ and $\Phi_{\mathcal{S}}^{\mu}|\pi_{\mathcal{N}^{\mu}\mathcal{T}}^{*}\mathcal{N}\mathcal{S}$ as in the preceding paragraph, and then choose $\Phi_{\mathcal{S}}^{\mu}|\pi_{\mathcal{N}^{\mu}\mathcal{T}}^{*}F\mathcal{T}$ so that the triple satisfies the requirements of the definition. In the applications of Theorem 3.29 in Chapters 8 and 10, the exact choice of $\Phi_{\mathcal{S}}^{\mu}$ does not matter, but that of $\Phi_{\mathcal{S}}$ does play a role.

For the purposes of Theorem 3.29, we assume that $\Phi_{\mathcal{S}}$ and $\Phi_{\mathcal{S}}^{\mu}$ also encode the lifts of $\phi_{\mathcal{S}}$ and $\phi_{\mathcal{S}}^{\mu}$ to the bundles $\pi_{\mathcal{N}\mathcal{S}}^{*}\Gamma_{-}^{0,1}\longrightarrow\mathcal{S}$ and $\pi_{\mathcal{N}^{\mu}\mathcal{T}}^{*}\Gamma_{-}^{0,1}\longrightarrow\mathcal{S}(\mu)$, respectively. Put

$$\begin{aligned} F^{(0)}\mathcal{S}&=\mathcal{N}\tilde{\mathcal{S}}\oplus F^{(0)}\mathcal{T}, &\quad F^{(\emptyset)}\mathcal{S}&=\big\{(b,\vec{n},v_{\hat{I}})\in F^{(0)}\mathcal{S}\colon v_{\hat{I}}\in F_b^{(\emptyset)}\mathcal{T}\big\};\\ F\mathcal{S}&=\mathcal{N}\mathcal{S}\oplus F\mathcal{T}, &\quad F^{\emptyset}\mathcal{S}&=\big\{[b,\vec{n},v_{\hat{I}}]\in F^{(0)}\mathcal{S}\colon [b,v_{\hat{I}}]\in F_b^{\emptyset}\mathcal{T}\big\}.\end{aligned}$$

Lemma 3.28 *For any $(\mathcal{A}(\mathcal{T})\ltimes G_{\mathcal{T}})$-invariant precompact open subset K of $\tilde{\mathcal{S}}(\mu)$, there exist an open neighborhood U_K of K in $\mathcal{M}_{\mathcal{T}}^{(0)}$ and $\delta,C>0$ with the following property. If $t\in(0,\delta)$, there exists a smooth $(\mathcal{A}(\mathcal{T})\ltimes G_{\mathcal{T}})$-equivariant section*

$$\tilde{\varphi}_{\mathcal{S},t\nu}^{\mu}\in\Gamma\big(F^{(\emptyset)}\mathcal{S}_{\delta}|K;\pi_{F^{(0)}\mathcal{S}}^{*}\tilde{\mathcal{N}}^{\mu}\mathcal{T}\big),$$

such that $\|\tilde{\varphi}_{\mathcal{S},t\nu}^{\mu}(v)\|_{b_v,C^0}\le C\big(t+|v|^{\frac{1}{p}}\big)$ for all $v\in F^{(\emptyset)}\mathcal{S}_{\delta}|K$ and

$$F^{\emptyset}\mathcal{S}_{\delta}|K\longrightarrow\tilde{\mathcal{M}}_{t\nu}(U_K,\delta;\mu),\quad [b,\vec{n},v]\longrightarrow\tilde{\gamma}_{\mathcal{T},t\nu}\Big(\tilde{\Phi}_{\mathcal{S}}\big(\tilde{\Phi}_{\mathcal{S}}^{\mu}\tilde{\varphi}_{\mathcal{S},t\nu}^{\mu}(b,\vec{n},v)\big)\Big),$$

is an orientation-preserving diffeomorphism.

Proof: Since μ is a regular value of $\mathrm{ev}_{[N]}|\mathcal{S}$ and K is precompact, there exists $\delta>0$ such that the map

$$\big\{(b,\vec{n},v,\sigma)\in F^{(0)}\mathcal{S}\oplus\tilde{\mathcal{N}}^{\mu}\mathcal{T}\big|K\colon |\vec{n}|,\|\sigma\|_{b,C^0}<\delta\big\}\longrightarrow F^{(0)}\mathcal{T},\quad (b,\vec{n},v,\sigma)\longrightarrow\tilde{\Phi}_{\mathcal{S}}\tilde{\Phi}_{\mathcal{S}}^{\mu}(b,\sigma;\vec{n},v),$$

is a $G_{\mathcal{T}}$-equivariant orientation-preserving diffeomorphism onto its image. Thus, if $\delta>0$ is sufficiently small, there exists $C>0$ such that, with notation as in Definition 3.27,

$$C^{-1}\|\sigma-\sigma'\|_{b,C^0}\le\big\|\varpi(\vec{n},\sigma)-\varpi(\vec{n},\sigma')\big\|_{b,C^0}\le C\|\sigma-\sigma'\|_{b,C^0}\quad\forall\, b\in\tilde{\mathcal{S}}(\mu),\ \vec{n}\in\mathcal{N}_b\tilde{\mathcal{S}}_{\delta},\ \sigma,\sigma'\in\tilde{\mathcal{N}}_b^{\mu}\mathcal{T}.$$

Then by Corollary 4.11 and definition of S'_{ϖ} in Section 3.4,

$$\Big|d_V\big(\tilde\phi_S\tilde\Phi^\mu_S(b,\sigma;\vec n),\tilde\gamma_{\mathcal T,t\nu}\big(\tilde\Phi_S\tilde\Phi^\mu_S(b,\sigma;\vec n,v)\big)\big)-d_V\big(\tilde\phi_S\tilde\Phi^\mu_S(b,\sigma';\vec n),\tilde\gamma_{\mathcal T,t\nu}\big(\tilde\Phi_S\tilde\Phi^\mu_S(b,\sigma';\vec n,v)\big)\big)\Big|$$
$$\le C\big(t+|v|^{\frac1p}\big)\|\sigma-\sigma'\|_{b,C^0}\qquad\forall\ t\in(0,\delta),\ b\in\tilde{\mathcal S}(\mu),\ \vec n\in\mathcal N_b\tilde{\mathcal S}_\delta,\ \sigma,\sigma'\in\tilde{\mathcal N}^\mu_b\mathcal T, v\in F^{(\emptyset)}_b\mathcal T_\delta.$$

On a neighborhood of $\mathrm{ev}_{[N]}(b)\in\mu$, we can identify the normal bundle of μ in V^N g_V-isometrically with the trivial hermitian bundle of the same rank. Let π denote the projection onto the fiber. Since μ is $\mathcal S$-regular,

$$\|\sigma-\sigma'\|_{b,C^0}\le C\big|\pi\mathrm{ev}_{[N]}\big(\tilde\phi_S\tilde\Phi^\mu_S(b,\sigma;\vec n)\big)-\pi\mathrm{ev}_{[N]}\big(\tilde\phi_S\tilde\Phi^\mu_S(b,\sigma';\vec n)\big)\big|$$

for all $b\in\tilde{\mathcal S}(\mu)$, $\vec n\in\mathcal N_b\tilde{\mathcal S}_\delta$, and $\sigma,\sigma'\in\tilde{\mathcal N}^\mu_b\mathcal T$. Thus, we can apply Corollary 3.23 to

$$h=\pi\circ\mathrm{ev}_{[N]}\circ\tilde\phi_S\circ\tilde\Phi^\mu_S\quad\text{and}\quad\tilde h_t=\pi\circ\mathrm{ev}_{[N]}\circ\tilde\gamma_{\mathcal T,t\nu}\circ\tilde\Phi_S\circ\tilde\Phi^\mu_S.$$

We obtain $\delta,\epsilon>0$ and for each $t\in(0,\delta)$ a section $\tilde\varphi^\mu_{S,t\nu}$ with the claimed bound such that the map

$$F^{(\emptyset)}\mathcal S_\delta|K\longrightarrow\Big\{(b,\vec n,v_{\hat I},\sigma)\colon\|\sigma\|_{b,C^0}<\epsilon,\ \mathrm{ev}_{[N]}\tilde\gamma_{\mathcal T,t\nu}\big(\tilde\Phi_S\big(\tilde\Phi^\mu_S(b,\vec n;\sigma,v)\big)\big)\in\mu\Big\},$$
$$(b,\vec n,v)\longrightarrow\big(b,\vec n,v,\tilde\varphi^\mu_{S,t\nu}(b,\vec n,v)\big),$$

is an orientation-preserving diffeomorphism. Since

$$\big\{[b,\vec n,v,\sigma]\colon[b,\vec n,v]\in F^{\emptyset}\mathcal S_\delta,\|\sigma\|_{b,C^0}<\epsilon\big\}\longrightarrow\tilde{\mathcal M}_{t\nu}(U_K,\delta),$$
$$[b,\vec n,\sigma,v]\longrightarrow\tilde\gamma_{\mathcal T,t\nu}\big(\tilde\Phi_S\big(\tilde\Phi^\mu_S(b,\vec n;\sigma,v)\big)\big),$$

is orientation-preserving by the discussion above and our assumptions on $\tilde\phi_S$, the claim follows. Above

$$U_K=\tilde\phi_S\Big(\tilde\Phi^\mu_S\big(\{(b,\vec n,\sigma)\in\mathcal N\tilde{\mathcal S}\oplus\tilde{\mathcal N}^\mu\mathcal T|K\colon\|\vec n\|_{b,C^0}<\epsilon,\|\sigma\|_{b,C^0}<\epsilon\}\big)\Big).$$

Theorem 3.29 *Suppose $\lambda\in H_2(V;\mathbb Z)$, $\mathcal T=\big(\Sigma,[N],I;j,\underline\lambda\big)$ is a (V,J)-semiregular bubble type, with $\sum_{i\in I}\lambda_i=\lambda$ and cokernel bundle $\Gamma^{0,1}_-\longrightarrow\mathcal M_{\mathcal T}$, and $(\tilde\Gamma_-,\Gamma^{0,1}_-,R)$ is an obstruction bundle setup. Let $\mathcal S\subset\mathcal M_{\mathcal T}$ be a smooth oriented submanifold,*

$$\nu\in\Gamma^{0,1}(\Sigma\times V;\Lambda^{0,1}_{J,j}\pi^*_\Sigma T^*\Sigma\otimes\pi^*_VTV)$$

a λ-regular section, μ a ν-, $\mathcal T$-, and $\mathcal S$-regular N-tuple of submanifolds of V of total codimension

$$\mathrm{codim}\ \mu=2\big(\langle c_1(V,J),\lambda\rangle+(\dim_{\mathbb C}V)\big(1-g(S)\big)+|N|\big),$$

and (Φ_S,Φ^μ_S) is regularization of $\mathcal S(\mu)$. Then for every precompact open subset K of $\mathcal S(\mu)$, there exist a neighborhood U_K of K in $\bar C^\infty_{(\lambda;N)}(\Sigma;\mu)$ and $\delta,\epsilon,C>0$ with the following property. For every $t\in(0,\epsilon)$, there exist a section

$$\varphi^\mu_{S,t\nu}\in\Gamma\big(F^{\emptyset}\mathcal S_\delta|K;\pi^*_{F\mathcal S}\mathcal N^\mu\mathcal T\big),\quad\textit{with}\quad\big\|\varphi^\mu_{S,t\nu}(v)\big\|_{b_v,C^0}\le C\big(t+|v|^{\frac1p}\big),$$

and a sign-preserving bijection between $\mathcal M_{\Sigma,t\nu,\lambda}(\mu)\cap U_K$ and the zero set of the section $\psi^\mu_{S,t\nu}$

defined by

$$\psi^{\mu}_{\mathcal{S},t\nu} \in \Gamma\big(F^{\emptyset}\mathcal{S}_{\delta}|K; \pi^*_{F\mathcal{S}}\Gamma^{0,1}_{-}\big), \quad \Phi^{\mu}_{\mathcal{S}}\big(\varphi^{\mu}_{\mathcal{S},t\nu}(v); \psi^{\mu}_{\mathcal{S},t\nu}(v)\big) = \psi_{\mathcal{S},t\nu}\big(\Phi^{\mu}_{\mathcal{S}}(\varphi^{\mu}_{\mathcal{S},t\nu}(v))\big);$$
$$\psi_{\mathcal{S},t\nu} \in \Gamma\big(F^{\emptyset}\mathcal{S}_{\delta}|(\mathcal{S}\cap U_K); \pi^*_{F\mathcal{S}}\Gamma^{0,1}_{-}\big), \quad \Phi_{\mathcal{S}}\big(v; \psi_{\mathcal{S},t\nu}(v)\big) = \psi_{\mathcal{T},t\nu}\big(\Phi_{\mathcal{S}}(v)\big);$$
$$\psi_{\mathcal{T},t\nu} \in \Gamma\big(F^{\emptyset}\mathcal{T}_{\delta}|(\mathcal{M}_{\mathcal{T}}\cap U_K); \pi^*_{F\mathcal{T}}\Gamma^{0,1}_{-}\big), \quad R_v\psi_{\mathcal{T},t\nu}(v) = \pi^{0,1}_{v,-}\big(t\nu_{v,t} - \bar{\partial}u_v - D_v\xi_{v,t\nu} - \tilde{\eta}_{v,t\nu}\big),$$

for some $\nu_{v,t}, \tilde{\eta}_{v,t\nu} \in \Gamma^{0,1}(u_v)$ *and* $\xi_{v,t\nu} \in \tilde{\Gamma}_+(v)$, *dependent smoothly on* v, *such that*

$$\big\|\nu_{v,t} - \nu\big\|_{v,2} \le C\big(t + |v|^{\frac{1}{p}}\big), \quad \big\|\xi_{v,t\nu}\big\|_{v,p,1} \le C\big(t + |v|^{\frac{1}{p}}\big), \quad \big\|\tilde{\eta}_{v,t\nu}\big\|_{v,p} \le C\big(t + |v|^{\frac{1}{p}}\big)^2.$$

Furthermore, if $z \in \Sigma$ *and* $(B_{b_v}(u_v(z), C\delta), J, g_{V,b_v})$ *is isometric to a ball in* $\mathbb{C}^n$*, then* $\tilde{\eta}_{v,t\nu}(z) = 0$.

Remark: In specific applications, the main goal would be to express the number of zeros of $\psi^{\mu}_{\mathcal{S},t\nu}$ in terms of the cohomology ring of a closure of $\mathcal{S}_{\mathcal{T}}(\mu)$. One of the significant intermediate steps is to extract the leading-order terms from the section $\psi^{\mu}_{\mathcal{S},t\nu}$. If $\lambda_{\hat{0}} = 0$, the estimate on $\nu_{v,t}$ given above easily leads to a sufficiently good estimate on $\pi^{0,1}_{v,-}\nu_{v,t}$; see Section 7.3. In such a case, one can also extract the first-order term from $\pi^{0,1}_{v,-}\bar{\partial}u_v$, which suffices for the computation in [I]. A power-series expansion for $\pi^{0,1}_{v,-}\bar{\partial}u_v$ is given in Proposition 7.6, and terms of up to third degree are needed in Chapter 8. With the choice of metrics of Chapter 7, the term $\pi^{0,1}_{v,-}\tilde{\eta}_{v,t\nu}$ vanishes. The remaining term is shown to be secondary for a good choice of the obstruction bundle setup.

Proof of Theorem 3.29: Let $\delta, \epsilon > 0$ be as in Lemma 3.28 and its proof. We take $\varphi^{\mu}_{\mathcal{S},t\nu}$ to be the section descendent from the $(\mathcal{A}(\mathcal{T}) \ltimes G_{\mathcal{T}})$-equivariant section $\tilde{\varphi}^{\mu}_{\mathcal{S},t\nu}$. Denote by U'_K the open set U_K of Lemma 3.28. By Corollary 4.22, there exists a neighborhood U_K of K in $\bar{C}^{\infty}_{(\lambda;N)}(\Sigma;\mu)$ such that $\mathcal{M}_{\Sigma,t\nu,\lambda}(\mu) \cap U_K$ is contained in $\tilde{\mathcal{M}}_{t\nu}(U'_K,\delta;\mu)$. The neighborhood U_K can always be chosen to contain all the zeros of the section $\tilde{\psi}_{t\nu}$ of the bundle $\Pi\Gamma^{0,1}_{-}$ over $\mathcal{M}_{\Sigma,t\nu,\lambda}(\mu) \cap U_K$. By Corollary 3.26, $\mathcal{M}_{\Sigma,t\nu,\lambda}(\mu) \cap U_K$ is precisely the oriented zero set of the section $\tilde{\psi}_{t\nu}$. Since the map

$$F^{\emptyset}\mathcal{S}_{\delta}|K \longrightarrow \tilde{\mathcal{M}}_{t\nu}(U'_K,\delta;\mu), \quad v \longrightarrow \tilde{\gamma}_{\mathcal{T},t\nu}\big(\Phi_{\mathcal{S}}\Phi^{\mu}_{\mathcal{S}}(\varphi^{\mu}_{\mathcal{S},t\nu}(v))\big),$$

is an orientation-preserving diffeomorphism by Lemma 3.28, it induces a sign-preserving bijection between the zero set of $\tilde{\psi}_{t\nu}$ on $\tilde{\mathcal{M}}_{t\nu}(U'_K,\delta;\mu)$, and the zero set of the section

$$(\tilde{\gamma}_{\mathcal{T},t\nu}\Phi_{\mathcal{S}}\Phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu})^*\tilde{\psi}_{t\nu} \in \Gamma\big(F^{\emptyset}\mathcal{S}_{\delta}|K; (\tilde{\gamma}_{\mathcal{T},t\nu}\Phi_{\mathcal{S}}\Phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu})^*\Pi\Gamma^{0,1}_{-}\big).$$

By equation (3.17), under the canonical identification

$$(\tilde{\gamma}_{\mathcal{T},t\nu}\gamma_{\mathcal{T}}^{-1})^*\Pi\Gamma^{0,1}_{-} = \Gamma^{0,1}_{-} \longrightarrow \mathcal{M}(U'_K,\delta),$$

the section $(\tilde{\gamma}_{\mathcal{T},t\nu}\gamma_{\mathcal{T}}^{-1})^*\tilde{\psi}_{t\nu}$ corresponds to the section $\psi_{t\nu}$ given by

$$\begin{aligned} \psi_{t\nu}\big(b(v)\big) &= t\nu|_{u_v} - \bar{\partial}u_v - \tilde{\pi}^{0,1}_{v,+}\eta_{v,t\nu} - N_{v,t\nu}P_v\eta_{v,t\nu} \\ &= \pi^{0,1}_{v,-}\big(t\nu|_{u_v} - \bar{\partial}u_v - \tilde{\pi}^{0,1}_{v,+}\eta_{v,t\nu} - N_{v,t\nu}P_v\eta_{v,t\nu}\big) \\ &= \pi^{0,1}_{v,-}\big(t\nu_{v,\xi_{v,t\nu}} - \bar{\partial}u_v - \tilde{\pi}^{0,1}_{v,+}\eta_{v,t\nu} - N_vP_v\eta_{v,t\nu}\big). \end{aligned} \tag{3.19}$$

The second equality above is automatic, since $\psi_{t\nu}(b(v)) \in \Gamma_-^{0,1}(v)$; the third follows from the definition of $N_{v,t\nu}$ in Section 3.6. The bounds on the terms $\nu_{v,\xi_v,t\nu}$, $\eta_{v,t\nu}$, $N_v P_v \eta_{v,t\nu}$ also follow from Section 3.6. By definition of $\Gamma_-^{0,1}$ in Section 3.4 and equation (3.19), under the canonical identification

$$\gamma_{\mathcal{T}}^* \Gamma_-^{0,1} = \pi_{F\mathcal{T}}^* \Gamma_-^{0,1} \longrightarrow F^{\emptyset}\mathcal{T}_\delta \big| U_K',$$

the section $\gamma_{\mathcal{T}}^* \psi_{t\nu}$ corresponds to the section $\psi_{\mathcal{T},t\nu}$, described in the statement of the theorem.

The next proposition describes a special case of the above theorem. It is obtained by fixing a metric g on Σ and going through the construction analogous to that in Section 3.6 and then modification for constraints as above. The sign statement below follows from the fact that the (L^2, g, g_V)-projection $\ker(D_{V,u_b} - \tau t L_{\nu,u_b}) \longrightarrow \ker D_{V,u_b}$ is an isomorphism for all $\tau \in [0,1]$, t sufficiently small, and $b \in \mathcal{M}_{\Sigma,t\nu,\lambda^*}(\mu)$ sufficiently close to K.

Proposition 3.30 *Suppose $\lambda \in H_2(V;\mathbb{Z})$, $\mathcal{T} = (\Sigma, [N], \{\hat{0}\}; \hat{0}, \lambda)$ is a (V,J)-regular bubble type,*

$$\nu \in \Gamma^{0,1}(\Sigma \times V; \Lambda_{J,j}^{0,1} \pi_\Sigma^* T^*\Sigma \otimes \pi_V^* TV)$$

is any section, and μ is a $\mathcal{T}$-regular N-tuple of submanifolds of V of total codimension

$$codim\ \mu = 2\big(\langle c_1(V,J), \lambda\rangle + (\dim_{\mathbb{C}} V)(1 - g(S)) + |N|\big).$$

Then $\mathcal{M}_{\mathcal{T}}(\mu)$ is a discrete set and for every finite subset K of $\mathcal{S}(\mu)$, there exist a neighborhood U_K of K in $\bar{C}^\infty_{(\lambda;N)}(\Sigma;\mu)$, $\epsilon > 0$, and for each $t \in (0,\epsilon)$ a sign-preserving bijection between K and $\mathcal{M}_{\Sigma,t\nu,\lambda}(\mu) \cap U_K$.

3.9 Gluing Maps for Spaces $\bar{\mathcal{U}}_{\mathcal{T}}^{(0)}(\mu)$ and Orientations

We now consider the case $\mathcal{T} = (S^2, M, I; j, \underline{\lambda})$ is a regular bubble type. However, most of the analysis in this section applies to any regular bubble type $\mathcal{T}$. Let μ be a generic $\tilde{M}$-tuple of submanifolds in V, as defined below. If $I = \bigsqcup_{k\in K} I_k$ is the decomposition of I into rooted trees and $\{\mathcal{T}_k\}$ are the corresponding simple types derived from $\mathcal{T}$, the product gluing map,

$$(\tilde{\gamma}_{\mathcal{T}_k})_{k\in K} \colon \prod_{k\in K} F\mathcal{T}_{k,\delta_k} \longrightarrow \prod_{k\in K} \bar{\mathcal{M}}_{\langle \mathcal{T}_k\rangle},$$

may not map the total space of the bundle over $\mathcal{U}_{\mathcal{T}}^{(0)}(\mu)$ into $\bar{\mathcal{U}}_{\langle\mathcal{T}\rangle}^{(0)}(\mu)$. In this section, we remedy this deficiency of the product gluing map. We also show that all the spaces $\bar{\mathcal{U}}_{\langle\mathcal{T}\rangle}^{(0)}(\mu)$ and $\bar{\mathcal{U}}_{\langle\mathcal{T}\rangle}(\mu)$ are naturally oriented topological orbifolds and the gluing maps defined below preserve orientations.

Definition 3.31 *If $\mathcal{T}$ is a (V,J)-regular bubble type, $\tilde{M}$-tuple μ of oriented submanifolds of V is <u>$\mathcal{T}$-regular</u> if the manifolds $\{\mu_l : l \in \tilde{M} - M\}$ intersect transversally in V and*

$$T_{ev(b)}V \oplus \bigoplus_{l\in\tilde{M}\cap M} T_{u_{j_l}(y_l)}V = Im\ dev\big|_b + Im\ dev_{\tilde{M}\cap M}\big|_b + T_{ev(b)} \bigcap_{l\in\tilde{M}-M} \mu_l + \bigoplus_{l\in\tilde{M}\cap M} T_{u_{j_l}(y_l)}\mu_l$$

for all $b=(S^2,M,I;x,(j,y),u)\in\mathcal{B}_{\mathcal{T}}(\mu)$.

Let $\mathcal{T}$, $\mathcal{T}_k$, K, μ, and b be as above. Denote by $b_k=\big(S^2,M_k,I_k;x|\hat{I}_k,(j,y)|M_k,u|I_k\big)$ the corresponding $\mathcal{T}_k$-bubble map; see Section 2.6. Let $\mathcal{N}_b^{\mu}\mathcal{T}$ be the (L^2,b)-orthogonal complement of

$$\tilde{\mathcal{K}}_b^{\mu}\mathcal{T}=\big\{(\xi,w_{\hat{I}+M})\in\tilde{\mathcal{K}}_b\mathcal{T}\colon \xi_{j_l}(y_l)+du_{j_l}|_{y_l}w_l\in T_{u_{j_l}(y_l)}\mu_l\ \forall l\in\tilde{M}\cap M,\\ \xi_i(\infty)\in T_{\mathrm{ev}(b)}\bigcap_{l\in\tilde{M}-M}\mu_l\ \forall i\in I-\hat{I}\big\}$$

in $\bigoplus_{k\in K}\mathcal{K}_{b_k}\mathcal{T}_k$. Denote by $\tilde{\mathcal{N}}^{\mu}\mathcal{T}\longrightarrow\mathcal{B}_{\mathcal{T}}(\mu)$ and $\mathcal{N}^{\mu}\mathcal{T}\longrightarrow\mathcal{U}_{\mathcal{T}}^{(0)}(\mu)$ the corresponding vector bundles. Let

$$\mathcal{N}^{\mu}\mathcal{B}_{\mathcal{T}}=\tilde{\mathcal{N}}^{\mu}\mathcal{T}\oplus(\mathbb{C}\oplus\mathbb{R})^K\longrightarrow\mathcal{B}_{\mathcal{T}}(\mu),\quad \mathcal{N}^{\mu}\mathcal{U}_{\mathcal{T}}^{(0)}=\mathcal{N}^{\mu}\mathcal{T}\oplus(\mathbb{C}\oplus\mathbb{R})^K\longrightarrow\mathcal{U}_{\mathcal{T}}^{(0)}(\mu);\\ F^{(0)}\mathcal{T}=\bigoplus_{k\in K}F^{(0)}\mathcal{T}_k\longrightarrow\mathcal{B}_{\mathcal{T}},\quad F\mathcal{T}=\bigoplus_{k\in K}F\mathcal{T}_k\longrightarrow\mathcal{U}_{\mathcal{T}}^{(0)}.$$

The last two vector bundles carry norms induced from the norms on $F^{(0)}\mathcal{T}_k$, while we define norms on the first two by

$$\big|(b,\sigma,(c,r))\big|=\|\sigma\|_{b,C^0}+|(c,r)|,$$

if $\sigma\in\mathcal{N}_b^{\mu}\mathcal{T}\subset\bigoplus_{k\in K}\tilde{\mathcal{K}}_{b_k}\mathcal{T}_k$ and $(c,r)\in(\mathbb{C}\oplus\mathbb{R})^K$. If δ is sufficiently small, define

$$\tilde{\phi}_{\mathcal{T}}^{\mu}\colon\mathcal{N}^{\mu}\mathcal{B}_{\mathcal{T},\delta}\longrightarrow\prod_{k\in K}\mathcal{M}_{\mathcal{T}_k}^{(0)}\quad\text{by}\quad\tilde{\phi}_{\mathcal{T}}^{\mu}(\sigma,(c,r))=\big((c_k,r_k)\cdot H_{\mathcal{T}_k}^{(0)}(\sigma_k)\big)_{k\in K}\in\prod_{k\in K}\mathcal{M}_{\mathcal{T}_k}^{(0)},$$

where $H_{\mathcal{T}_k}^{(0)}$ is as at the end of Section 3.2 and $(c_k,r_k)\cdot$ denotes the action of a neighborhood of

$$0\in\mathbb{C}\times\mathbb{R}=\mathbb{C}\times\mathbb{R}\times\{0\}\subset\mathbb{C}\times\mathbb{R}\times\mathbb{R}$$

described in Section 3.1. Since $\tilde{\phi}_{\mathcal{T}}^{\mu}$ is $\big(\mathcal{A}(\mathcal{T})\ltimes\tilde{G}_{\mathcal{T}}\big)$-equivariant, it descends to a $G_{\mathcal{T}}^*$-equivariant map

$$\phi_{\mathcal{T}}^{\mu}\colon\mathcal{N}^{\mu}\mathcal{U}_{\mathcal{T},\delta}^{(0)}\longrightarrow\prod_{k\in K}\mathcal{M}_{\mathcal{T}_k}.$$

Let $\Phi_{\mathcal{T}}^{\mu}\colon\pi_{\mathcal{N}^{\mu}\mathcal{U}_{\mathcal{T}}}^*F\mathcal{T}\longrightarrow F\mathcal{T}$ be a $G_{\mathcal{T}}^*$-equivariant vector-bundle map covering the map $\phi_{\mathcal{T}}^{\mu}$ such that $\Phi_{\mathcal{T}}^{\mu}$ is the identity over $\mathcal{U}_{\mathcal{T}}^{(0)}(\mu)$. Denote by $\tilde{\Phi}_{\mathcal{T}}^{\mu}$ the lift of $\Phi_{\mathcal{T}}^{\mu}$ to $\mathcal{N}^{\mu}\mathcal{B}_{\mathcal{T},\delta}$. Let $\Phi_{\mathcal{T},k}^{\mu}$ and $\tilde{\Phi}_{\mathcal{T},k}^{\mu}$ be kth components of $\Phi_{\mathcal{T}}^{\mu}$ and $\tilde{\Phi}_{\mathcal{T}}^{\mu}$, respectively.

Lemma 3.32 *There exist* $\big(\mathcal{A}(\mathcal{T})\ltimes\tilde{G}_{\mathcal{T}}\big)$*-invariant functions* $\delta,C\in C^{\infty}(\mathcal{B}_{\mathcal{T}}(\mu);\mathbb{R}^+)$ *and a* $\big(\mathcal{A}(\mathcal{T})\ltimes\tilde{G}_{\mathcal{T}}\big)$*-equivariant section*

$$\tilde{\varphi}_{\mathcal{T}}^{\mu}\in\Gamma\big(F^{(0)}\mathcal{T}_\delta;\pi_{F^{(0)}\mathcal{T}}^*\mathcal{N}^{\mu}\mathcal{B}_{\mathcal{T},\delta}\big),$$

such that $\left|\tilde{\varphi}^{\mu}_{\mathcal{T}}(\upsilon)\right| \le C(b_\upsilon)|\upsilon|^{\frac{1}{p}}$ *and*

$$F\mathcal{T}_\delta \longrightarrow \bar{\mathcal{U}}^{(0)}_{\langle\mathcal{T}\rangle}(\mu), \quad \upsilon \longrightarrow \left(\tilde{\gamma}_{\mathcal{T}_k}\left(\Phi^{\mu}_{\mathcal{T},k}\varphi^{\mu}_{\mathcal{T}}(\upsilon)\right)\right)_{k\in K},$$

is a homeomorphism onto an open neighborhood of $\mathcal{U}^{(0)}_{\mathcal{T}}(\mu)$ *in* $\bar{\mathcal{U}}^{(0)}_{\langle\mathcal{T}\rangle}(\mu)$. *Furthermore, the restriction of this map to* $F^{\emptyset}\mathcal{T}_\delta$ *is an orientation-preserving diffeomorphism onto an open subset of* $\mathcal{U}^{(0)}_{\langle\mathcal{T}\rangle}(\mu)$.

Proof: Denote by $\mathcal{N}_{\mathcal{T}}\mu$ the normal bundle of

$$X_{\mathcal{T}}(\mu) \equiv \left\{x_K \in V^K\colon x_{k_1} = x_{k_2} \in \mu_l \ \forall k_1, k_2 \in K,\ l \in \tilde{M} - M\right\} \times \prod_{l\in\tilde{M}\cap M} \mu_l \subset V^K \times V^{\tilde{M}\cap M}.$$

Let $\tilde{\mathcal{N}}_{\mathcal{T}}\mu = \mathcal{N}X_{\mathcal{T}}\mu \oplus (\mathbb{C}\oplus\mathbb{R})^K$. Since the $(\mathcal{A}(\mathcal{T})\ltimes\tilde{G}_{\mathcal{T}}$-action does not change any evaluation maps and the constraints are in general position, the differential of the map

$$\Psi_{\mathcal{T},\tilde{M}}\colon \prod_{k\in K} \mathcal{M}^{(0)}_{\mathcal{T}_k} \longrightarrow V^K \times V^{\tilde{M}\cap M} \times (\mathbb{C}\times\mathbb{R})^K,$$
$$\Psi_{\mathcal{T},\tilde{M}} = \left((\mathrm{ev}_{\mathcal{T}_k}(b_k))_{k\in K}; (\mathrm{ev}_l(b))_{l\in M\cap\tilde{M}}; (\Psi_{\mathcal{T}_k,\hat{0}_{I_k}}(b_k))_{k\in K}\right),$$

where $\Psi_{\mathcal{T}_k,\hat{0}_{I_k}}(b_k) \in \mathbb{C}\times\mathbb{R}$ is as in Section 3.2, induces an isomorphism between $\mathcal{N}^{\mu}_b\mathcal{B}_{\mathcal{T}}$ and $\tilde{\mathcal{N}}_{\mathcal{T}}\mu$. This isomorphism is orientation-preserving by definition of orientations. Thus,

$$\tilde{\phi}^{\mu}_{\mathcal{T}}\colon \mathcal{N}^{\mu}\mathcal{B}_{\mathcal{T},\delta} \longrightarrow \prod_{k\in K} \mathcal{M}^{(0)}_{\mathcal{T}_k}$$

is an orientation-preserving diffeomorphism onto an open neighborhood of $\mathcal{B}_{\mathcal{T}}(\mu)$ in $\prod_{k\in K}\mathcal{M}^{(0)}_{\mathcal{T}_k}$, provided $\delta \in C^\infty(\mathcal{B}_{\mathcal{T}}(\mu);\mathbb{R}^+)$ is sufficiently small. By the same argument as in Section 3.8, for any simple bubble type $\mathcal{T}'$, the map

$$\tilde{\gamma}_{\mathcal{T}'}\colon F^{\emptyset}\mathcal{T}'_\delta \longrightarrow \mathcal{M}_{\langle\mathcal{T}'\rangle} = \mathcal{H}_{\langle\mathcal{T}'\rangle}$$

is an orientation-preserving diffeomorphism onto an open subset of $\mathcal{M}_{\langle\mathcal{T}'\rangle}$ as long as $\delta \in C^\infty(\mathcal{M}_{\mathcal{T}'};\mathbb{R}^+)$ is sufficiently small. Along with Corollary 4.23, this implies that the product map

$$\prod_{k\in K}\tilde{\gamma}_{\mathcal{T}_k}\colon \prod_{k\in K} F\mathcal{T}_{k,\delta} \longrightarrow \prod_{k\in K}\bar{\mathcal{M}}_{\langle\mathcal{T}_k\rangle}$$

is a homeomorphism onto an open neighborhood of $\prod_{k\in K}\mathcal{M}_{\mathcal{T}_k}$ in $\prod_{k\in K}\bar{\mathcal{M}}_{\langle\mathcal{T}_k\rangle}$ and its restriction to the preimage of $\prod_{k\in K}\mathcal{M}_{\langle\mathcal{T}_k\rangle}$ is an orientation-preserving diffeomorphism. The lemma now follows by applying an argument similar to the proof of Lemma 3.28 to the functions

$$h(\upsilon) = \Psi_{\mathcal{T},\tilde{M}}\left(\tilde{\phi}^{\mu}_{\mathcal{T}}(\sigma,(c,r))\right), \quad \tilde{h}(\upsilon) = \tilde{h}_0(\upsilon) = \Psi_{\mathcal{T},\tilde{M}}\left(\left(\tilde{\gamma}_k\Phi^{\mu}_{\mathcal{T},k}(\upsilon)\right)_{k\in K}\right),$$

where we write $\upsilon = \left(\sigma,(c,r),v\right)$, with $\left(\sigma,(c,r)\right) \in \mathcal{N}^{\mu}\mathcal{B}_{\mathcal{T}}$ and $v \in F^{(0)}\mathcal{T}$. Since $\mathcal{B}_{\mathcal{T}}(\mu)$ is generally not precompact in $\prod_{k\in K}\mathcal{M}^{(0)}_{\mathcal{T}_k}$, we end up with $\delta, C \in C^\infty(\mathcal{B}_{\mathcal{T}}(\mu);\mathbb{R}^+)$, instead of $\delta, C \in \mathbb{R}^+$. Another difference is that $\tilde{h}$ is not necessarily smooth with respect to the standard smooth structure on $\mathcal{N}^\mu\mathcal{B}_\mathcal{T} \oplus F^{(0)}\mathcal{T}$. However, we can put a smooth structure on the total space such that the composite maps

$$\mathcal{N}^\mu\mathcal{B}_{\mathcal{T},\delta} \oplus F^{(0)}\mathcal{T} \longrightarrow F^{(0)}\mathcal{T} \longrightarrow \mathbb{R}, \quad v \longrightarrow \tilde{\Phi}^\mu_\mathcal{T}(v), \; v_h \longrightarrow |v_h|^{\frac{1}{3p}}, \quad h \in \hat{I}_k, \; k \in K,$$

are smooth, whenever $\delta \in C^\infty(\mathcal{B}_\mathcal{T}(\mu); \mathbb{R}^+)$ is sufficiently small. Then by Corollary 4.5, $\tilde{h}$ is C^2, which is sufficient for the arguments of Section 3.7. Finally, in the given case $\tilde{h}$ is defined on all of $(\mathcal{N}^\mu\mathcal{B}_\mathcal{T} \oplus F^{(0)}\mathcal{T})_\delta$ and thus the second condition on $\tilde{h}_t$ in Section 3.7 is redundant.

Suppose $\mathcal{T}$ is a bubble type and μ is an $\tilde{M}$-tuple of constraints in general position. By Lemma 3.32, there exist $G^*_\mathcal{T}$-invariant functions $\delta, C \in C^\infty(\mathcal{U}^{(0)}_\mathcal{T}(\mu); \mathbb{R}^+)$ and a $G^*_\mathcal{T}$-equivariant section

$$\varphi^\mu_\mathcal{T} \in \big(F\mathcal{T}_\mu; \pi^*_{F\mathcal{T}}\mathcal{N}^\mu\mathcal{U}^{(0)}_{\mathcal{T},\delta}\big)$$

such that $|\varphi^\mu_\mathcal{T}(v)| \le C(b_v)|v|^{\frac{1}{p}}$ and

$$\tilde{\gamma}^\mu_\mathcal{T} \colon F\mathcal{T}_\delta \longrightarrow \bar{\mathcal{U}}^{(0)}_{\langle\mathcal{T}\rangle}(\mu), \quad \tilde{\gamma}^\mu_\mathcal{T}(v) = \big(\tilde{\gamma}_{\mathcal{T}_k}\big(\Phi^\mu_{\mathcal{T},k}\varphi^\mu_\mathcal{T}(v)\big)\big)_{k\in K},$$

is a homeomorphism onto a neighborhood of $\mathcal{U}^{(0)}_\mathcal{T}(\mu)$ in $\bar{\mathcal{U}}^{(0)}_{\langle\mathcal{T}\rangle}(\mu)$, which is an orientation-preserving diffeomorphism on a dense open subset of the domain. If $\mathcal{T}'$ is another regular bubble type such that $\langle\mathcal{T}\rangle = \langle\mathcal{T}'\rangle$ and μ is $\mathcal{T}'$-regular, it follows that

$$\tilde{\gamma}^{\mu\;-1}_\mathcal{T}\tilde{\gamma}^\mu_{\mathcal{T}'} \colon \tilde{\gamma}^{\mu\;-1}_{\mathcal{T}'}\big(\tilde{\gamma}^\mu_\mathcal{T}(F\mathcal{T}_\delta)\big) \longrightarrow \tilde{\gamma}^{\mu\;-1}_\mathcal{T}\big(\tilde{\gamma}^\mu_{\mathcal{T}'}(F\mathcal{T}'_{\delta'})\big)$$

is a homeomorphism provided $\delta' \in C^\infty(\mathcal{U}^{(0)}_{\mathcal{T}'}(\mu); \mathbb{R}^+)$ is sufficiently small. Furthermore, by the above it is orientation-preserving on a dense open subset of its domain. It follows that $\tilde{\gamma}^{\mu-1}_\mathcal{T}\tilde{\gamma}^\mu_{\mathcal{T}'}$ is an orientation-preserving homeomorphism everywhere. We thus obtain

Theorem 3.33 *Let $\mathcal{T}^* = (S^2, M, I^*; j, \underline{\lambda}^*)$ be a basic bubble type and μ an $\tilde{M}$-tuple of constraints such that μ is $\mathcal{T}$-regular for every bubble type $\mathcal{T} \le \mathcal{T}^*$.*
(1) The spaces $\bar{\mathcal{U}}^{(0)}_{\mathcal{T}^*}(\mu)$ and $\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$ are oriented topological orbifolds.
(2) Suppose $\mathcal{T} < \mathcal{T}^*$, $\phi^\mu_\mathcal{T} \colon \mathcal{N}^\mu\mathcal{T}_{\tilde{\delta}} \longrightarrow \mathcal{U}^{(0)}_\mathcal{T}$ is a $G_{\mathcal{T}^*}$-equivariant identification of neighborhoods of $\mathcal{U}^{(0)}_\mathcal{T}(\mu)$ in $\mathcal{N}^\mu\mathcal{T}$ and in $\mathcal{U}^{(0)}_\mathcal{T}$, and $\Phi^\mu_\mathcal{T} : \pi^*_{\mathcal{N}^\mu\mathcal{T}}F\mathcal{T} \longrightarrow F\mathcal{T}$ is a lift of $\phi^\mu_\mathcal{T}$ such that $\Phi^\mu_\mathcal{T}|\mathcal{U}^{(0)}_\mathcal{T}(\mu) = 1$. Then there exist $G_{\mathcal{T}^*}$-invariant functions $\delta, C \in C^\infty(\mathcal{U}^{(0)}_\mathcal{T}(\mu); \mathbb{R}^+)$ and a $G_{\mathcal{T}^*}$-equivariant continuous orientation-preserving identification,

$$\tilde{\gamma}^\mu_\mathcal{T} : F\mathcal{T}_\delta \longrightarrow \bar{\mathcal{U}}^{(0)}_{\mathcal{T}^*}(\mu),$$

of neighborhoods of $\mathcal{U}^{(0)}_\mathcal{T}(\mu)$ in $F\mathcal{T}$ and in $\bar{\mathcal{U}}^{(0)}_{\mathcal{T}^*}(\mu)$, which is smooth on $F^\emptyset\mathcal{T}_\delta \longrightarrow \mathcal{U}_{\mathcal{T}^*}(\mu)$. Furthermore, for every $v \in F\mathcal{T}_\delta$, there exists $\sigma(v) \in \mathcal{N}^\mu\mathcal{T}$ such that

$$\|\sigma(v)\|_{b^*} \le C(b^*)|v|^{\frac{1}{p}} \quad \textit{and} \quad u_{\tilde{\gamma}^\mu_\mathcal{T}(v)} = \exp_{V, u_{b'} \circ q_{\Phi^\mu_\mathcal{T}(\sigma(v))}} \xi_v, \quad \textit{where} \quad \Phi^\mu_\mathcal{T}(\sigma(v)) \equiv [b', v'],$$

for some $\xi_v \in \Gamma(u_{b'} \circ q_{\Phi^\mu_\mathcal{T}(\sigma(v))})$ with $\|\xi_v\|_{C^0} \le C(b_v)|v|^{\frac{1}{p}}$.

Remark: The descriptive statement (2) of Theorem 3.33 leads to the analytic estimate of Theorem 6.2, which is one of the crucial ingredients in our computations.

Chapter 4

Technical Issues

In this chapter, we prove continuity, injectivity, and surjective of the gluing map of Section 3.6. Proofs of continuity and injectivity are standard arguments. The goal is to use a standard approach for proving surjectivity as well, but before we can do so, certain issues specific to the construction of Chapter 3 have to be addressed. The trickiest one is perhaps that our gluing construction is done using a family of metrics on Riemann surface Σ parameterized by bubble maps that are flat near the singular points of the domain of each map. If the genus of Σ is one, this issue can be avoided, since we can use a metric on Σ which is flat everywhere. Of course, if the genus of Σ is not one, such a metric does not exist.

4.1 Continuity of the Gluing Map

Let $\mathcal{T} = (S, M, I; j, \underline{\lambda})$ be a simple regular bubble type and H a nonempty subset of $\hat{I}$. Suppose $\upsilon_k \in F^{(\emptyset)}\mathcal{T}_\delta$ and the sequence υ_k converges to $\upsilon^* \in F^{(H)}\mathcal{T}_\delta$. In this section, we show that $\tilde{\gamma}_{\mathcal{T}}(\upsilon_k)$ converges to $\tilde{\gamma}_{\mathcal{T}}(\upsilon^*)$ in the Gromov topology. Our main interest is the case $S = S^2$.

It is sufficient to assume that $\pi_h(\upsilon_k) = \pi_h(\upsilon^*)$ if $h \notin H$. In particular, $b \equiv b_{\upsilon^*} = b_{\upsilon_k}$. Denote by $\tilde{\mathcal{T}} = \mathcal{T}(H)$ the bubble type of $b(\upsilon^*)$. For each k, define

$$\tilde{\upsilon}_k = \big(\tilde{b}(\upsilon^*), (\tilde{\upsilon}_k)_H\big) \in F^{(0)}_{b(\upsilon^*)}\tilde{\mathcal{T}}$$

as follows. If $h \in H$, put

$$i_H h = \min\big\{i < h \colon \text{if } h' \in I \ \&\ i < h' < h,\ h' \notin H\big\}.$$

Since I is a rooted tree, $i_H h$ is well-defined. Let

$$\tilde{\upsilon}_{k,h} = \prod_{i_H h < h' \leq h} \upsilon_{k,h'}.$$

Since $\upsilon_k \longrightarrow \upsilon^*$, $\tilde{\upsilon}_{k,h} \longrightarrow 0$ for all $h \in H$. Furthermore, by construction $\Sigma_{\upsilon_k} = \Sigma_{\tilde{\upsilon}_k}$ and $q_{\upsilon_k} = q_{\upsilon^*} \circ q_{\tilde{\upsilon}_k}$. In particular, $u_{\upsilon_k} = u_{\upsilon^*} \circ q_{\tilde{\upsilon}_k}$.

For any $h \in H$ and $\delta > 0$, let

$$A_{h,\delta,k} = q_{\upsilon_k}^{-1}\big(\{(\iota_h, z): r_{b,h}(z) \le \delta\} \cup \{(h,z): |q_S^{-1}(z)| \le \delta\}\big) \subset \Sigma_{\upsilon_k},$$
$$A^*_{h,\delta} = q_{\upsilon^*}^{-1}\big(\{(\iota_h, z): r_{b,h}(z) \le \delta\} \cup \{(h,z): |q_S^{-1}(z)| \le \delta\}\big) \subset \Sigma_{\upsilon^*}, \quad \Sigma^*_\delta = \Sigma_{\upsilon^*} - \bigcup_{h \in H} A^*_{h,\delta}.$$

It is convenient to make the following definitions. If $\eta_k \in L^p(\upsilon_k)$, the sequence $\{\eta_k\}$ *converges to* $\eta^* \in L^p(\upsilon^*)$ if $q_{\tilde{\upsilon}_k}^{-1*}\eta_k$ converges to η^* in the L^p-norm on all precompact open subsets of $\Sigma^*_{\upsilon^*}$ and

$$\lim_{\delta \longrightarrow 0} \lim_{k \longrightarrow 0} \|\eta_k\|_{\upsilon_k, L^p(A_{h,\delta,k})} = 0 \qquad \forall h \in H. \tag{4.1}$$

If $\xi_k \in L^p_1(\upsilon_k)$, the sequence $\{\xi_k\}$ *converges to* $\xi^* \in L^p_1(\upsilon^*)$ if $\xi_k \circ q_{\tilde{\upsilon}_k}^{-1}$ converges to ξ^* in the L^p_1-norm on all precompact open subsets of $\Sigma^*_{\upsilon^*}$ and

$$\lim_{\delta \longrightarrow 0} \lim_{k \longrightarrow 0} \|\xi_k\|_{\upsilon_k, L^p_1(A_{h,\delta,k})} = 0 \qquad \forall h \in H. \tag{4.2}$$

In (4.1) and (4.2), we use the modified Sobolev norms.

Lemma 4.1 *There exist $C, \delta \in C^\infty\big(\mathcal{M}^{(0)}_{\mathcal{T}}; \mathbb{R}^+\big)$ such that for any sequence $\{\upsilon_k \in F^{(\emptyset)}\mathcal{T}_\delta\}$ converging to υ^* as above and $\xi \in \Gamma_+(\upsilon^*)$*

$$\big\|\pi_{\upsilon_k,-}(\xi \circ q_{\tilde{\upsilon}_k})\big\|_{\upsilon_k,p,1} \le C(b)\big|\upsilon_k - \upsilon^*\big|\|\xi\|_{\upsilon^*,p,1}.$$

Proof: Note that $\Gamma_-(\upsilon_k) = \{\xi^- \circ q_{\tilde{\upsilon}_k} : \xi^- \in \Gamma(\upsilon^*)\}$. Thus, the difference $q^*_{\tilde{\upsilon}_k}\pi_{\upsilon^*,-} - \pi_{\upsilon_k,-}q^*_{\tilde{\upsilon}_k}$ arises entirely from the difference between the metrics $q^*_{\tilde{\upsilon}_k}g_{\upsilon^*}$ and g_{υ_k}. By construction, the two metrics differ only on the annuli $A_{h,2|\upsilon_{k,h}|^{\frac{1}{2}},k}$ for $h \in H$. Thus, the claim follows from (2) of Lemma 3.5.

Lemma 4.2 *If η_k converges to η^*, then $P_{\upsilon_k}\eta_k$ converges to $P_{\upsilon^*}\eta^*$.*

Proof: (1) Let $\{\epsilon_k\}, \{\delta_k\} \subset \mathbb{R}^+$ be sequences converging to zero such that for all $h \in H$,

$$\|\eta^*\|_{\upsilon^*, L^p(A^*_{h,\delta_k})} \le \epsilon_k; \quad \big\|P_{\upsilon^*}\eta^*\big\|_{\upsilon^*, L^p_1(A^*_{h,\delta_k})} \le \epsilon_k, \quad \lim_{k \longrightarrow \infty} \|\eta_k\|_{\upsilon_k, L^p(A_{h,\delta_{k^*},k})} < \epsilon_k. \tag{4.3}$$

For every $k^* > 0$, choose N_{k^*} such that for all $k > N_{k^*}$

$$\big\|q_{\tilde{\upsilon}_k}^{-1*}\eta_k - \eta^*\big\|_{\upsilon^*, L^p(\Sigma^*_{\delta_{k^*}})} \le \epsilon_{k^*} \quad \text{and} \quad \|\eta_k\|_{\upsilon_k, L^p(A_{h,\delta_{k^*},k})} \le \epsilon_{k^*} \ \ \forall h \in H. \tag{4.4}$$

It can be assumed that $2|\upsilon_k - \upsilon^*|^{\frac{1}{2}} \le \delta_{k^*}, \epsilon_{k^*}$ whenever $k > N_{k^*}$. For any $k > N_{k^*}$, let $\tilde{\eta}_{k^*,k} \in L^p(\upsilon^*)$ be given by

$$\tilde{\eta}_{k^*,k} = \begin{cases} q_{\tilde{\upsilon}_k}^{-1*}\tilde{\eta}_k, & \text{on } \Sigma^*_{\delta_{k^*}}; \\ 0, & \text{outside of } \Sigma^*_{\delta_{k^*}}. \end{cases}$$

Then $\|\tilde{\eta}_{k^*,k}\|_{\upsilon^*,p} \le \|\eta_k\|_{\upsilon_k,p}$. Let

$$\tilde{P}_{k^*,k}\eta_k = q^*_{\tilde{\upsilon}_k}P_{\upsilon^*}\tilde{\eta}_{k^*,k} \in L^p_1(\upsilon_k).$$

Then by Lemma 3.16 and the first assumptions of (4.3) and (4.4),

$$\begin{aligned} \big\|q_{\tilde{v}_k}^{-1*}\tilde{P}_{k^*,k}\eta_k - P_{v^*}\eta^*\big\|_{v^*,L_1^p(\Sigma^*_{\delta_{k^*}})} &\le \big\|P_{v^*}\tilde{\eta}_{k^*,k} - P_{v^*}\eta^*\big\|_{v^*,p,1} \\ &\le C(b)\big\|\tilde{\eta}_{k^*,k} - \eta^*\big\|_{v^*,p} \le 2C(b)\epsilon_{k^*}. \end{aligned} \tag{4.5}$$

Since $\|dq_{\tilde{v}_k}\|_{C^0} \le C(b)$, by (4.3) and the first assumption of (4.4) for all $h \in H$,

$$\begin{aligned} \big\|\tilde{P}_{k^*,k}\eta_k\big\|_{v_k,L_1^p(A_{h,\delta_{k^*},k})} &\le C(b)\big\|P_{v^*}\tilde{\eta}_{k^*,k}\big\|_{v^*,L_1^p(A^*_{h,\delta_{k^*}})} \\ &\le C(b)\Big(\big\|P_{v^*}\eta^*\big\|_{v^*,L_1^p(A^*_{h,\delta_{k^*}})} + \big\|P_{v^*}\tilde{\eta}_{k^*,k} - P_{v^*}\eta^*\big\|_{v^*,p,1}\Big) \le C'(b)\epsilon_{k^*}. \end{aligned} \tag{4.6}$$

(2) We now show that $\tilde{P}_{k^*,k}\eta_k$ is close to $P_{v_k}\eta_k$. By Lemmas 3.16 and 4.1,

$$\begin{aligned} \big\|\tilde{P}_{k^*,k}\eta_k - P_{v_k}\eta_k\big\|_{v_k,p,1} &\le C(b)\Big(\big\|D_{v_k}\tilde{P}_{k^*,k}\eta_k - \eta_k\big\|_{v_k,p} + \big\|\pi_{v_k,-}\tilde{P}_{k^*,k}\eta_k\big\|_{v_k,p,1}\Big) \\ &\le C(b)\Big(\big\|D_{v_k}\tilde{P}_{k^*,k}\eta_k - \eta_k\big\|_{v_k,p} + |v_k - v^*|\|\eta_k\|_{v_k,p}\Big). \end{aligned} \tag{4.7}$$

Since $D_{v^*}P_{v^*}\tilde{\eta}_{k^*,k} = \tilde{\eta}_{k^*,k}$ and $q_{\tilde{v}_k}$ is holomorphic outside of the annuli $A_{h,\delta_{k^*},k}$,

$$D_{v_k}\tilde{P}_{k^*,k}\eta_k = \eta_k \quad \text{on } \Sigma_{v_k} - \bigcup_{h\in H} A_{h,\delta_{k^*},k}; \tag{4.8}$$

By equation (4.6),

$$\big\|D_{v_k}\tilde{P}_{k^*,k}\eta_k\big\|_{v_k,L^p(A_{h,\delta_{k^*},k})} \le C(b)\big\|\tilde{P}_{k^*,k}\eta_k\big\|_{v_k,L_1^p(A_{h,\delta_{k^*},k})} \le C'(b)\epsilon_{k^*}. \tag{4.9}$$

Thus, from equations (4.7)-(4.9) and the second assumption of (4.4), we conclude that for all $k > N_{k^*}$

$$\big\|\tilde{P}_{k^*,k}\eta_k - P_{v_k}\eta_k\big\|_{v_k,p,1} \le C(b)\epsilon_{k^*}\left(1 + \|\eta^*\|_{v^*,p}\right). \tag{4.10}$$

Since $\|dq_{\tilde{v}_k}^{-1}\|_{C^0} \le C(b)$ on $\Sigma^*_{\delta_{k^*}}$, by equations (4.5), (4.6), and (4.10),

$$\big\|q_{\tilde{v}_k}^{-1*}P_{v_k}\eta_k - P_{v^*}\eta^*\big\|_{v^*,L_1^p(\Sigma^*_{\delta_{k^*}})} \le C(b)\epsilon_{k^*}\left(1 + \|\eta^*\|_{v^*,p}\right); \tag{4.11}$$

$$\big\|P_{v_k}\eta_k\big\|_{v_k,L_1^p(A_{h,\delta_{k^*},k})} \le C(b)\epsilon_{k^*}\left(1 + \|\eta^*\|_{v^*,p}\right) \quad \forall h \in H. \tag{4.12}$$

By equations (4.11) and (4.12), $P_{v_k}\eta_k$ converges to $P_{v^*}\eta^*$.

Lemma 4.3 *There exist $\tilde{C}, \delta \in C^\infty(\mathcal{M}_{\mathcal{T}}^{(0)}; \mathbb{R}^+)$ such that for all $v^* \in F^{(H)}\mathcal{T}_\delta$ and $h \in H$,*

$$\big\|P_{v^*}\bar{\partial}u_{v^*}\big\|_{g_{v^*},C^1(A^*_{h,\delta(b_{v^*})})} \le \tilde{C}(b_{v^*}).$$

Proof: For each $h \in H$, this lemma is obtained by pasting $(P_{v^*}\bar{\partial}u_{v^*})|(A_{h,\delta(b^*_{v^*})} \cap \Sigma_{v^*,\iota_h})$ and $(P_{v^*}\bar{\partial}u_{v^*})|(A_{h,\delta(b^*_{v^*})} \cap \Sigma_{v^*,h})$ onto $\Sigma_{b_{v^*},\iota_h}$ and $\Sigma_{b_{v^*},h}$ via a cutoff function. We then use the usual elliptic estimates and Sobolev inequalities on $\Sigma_{b_{v^*},\iota_h}$ and $\Sigma_{b_{v^*},h}$ along with

$$\big\|P_{v^*}\bar{\partial}u_{v^*}\big\|_{v^*,p,1} \le C(b_{v^*})|v^*|^{\frac{1}{p}}.$$

The bound obtained in this way is actually $C(b_{v^*})|v^*|^{\frac{1}{p}}$.

Corollary 4.4 *There exist* $C,\delta\in C^\infty(\mathcal{M}_{\mathcal{T}}^{(0)};\mathbb{R}^+)$ *such that for any sequence* $v_k\in F^{(\emptyset)}\mathcal{T}_\delta$ *converging to* $v^*\in F^{(H)}\mathcal{T}_\delta$ *as above,*

$$\big\|q_{\tilde{v}_k}^{-1*}\eta_{v_k}-\eta_{v^*}\big\|_{v^*,L^p(\Sigma^*_{2|v_k-v^*|^{1/2}})}\le C(b)\big|v_k-v^*\big|^{\frac{1}{p}};$$
$$\big\|\eta_{v_k}\big\|_{v_k,L^p(A_{h,2|v_k-v^*|^{1/2},k})}\le C(b)\big|v_k-v^*\big|^{\frac{1}{p}}\quad\forall h\in H.$$

Proof: Let $\delta_k=2|v_k-v^*|^{\frac{1}{2}}$ and $\epsilon_k=(2\|\beta'\|_{C^0}+\tilde{C}(b))|v_k-v^*|^{\frac{1}{p}}$, where $\tilde{C}$ is the function given by Lemma 4.3. Put

$$\eta^{(0)}=-\bar{\partial}u_{v^*},\qquad \eta^{(m+1)}=-\bar{\partial}u_{v^*}-N_{v^*}P_{v^*}\eta^{(m)}\quad m\ge 0;$$
$$\eta_k^{(0)}=-\bar{\partial}u_{v_k},\qquad \eta_k^{(m+1)}=-\bar{\partial}u_{v_k}-N_{v_k}P_{v_k}\eta_k^{(m)}\quad m\ge 0.$$

By Lemma 4.3 and the explicit description of $\bar{\partial}q_{v_k}$ in Lemma 2.3, ϵ_k, δ_k, $\eta^{(0)}$, and $\eta_k^{(0)}$ satisfy (4.3) and (4.4). Suppose $\epsilon_k^{(m)}$ is such that $\epsilon_k^{(m)}$, δ_k, $\eta^{(m)}$, and $\eta_k^{(m)}$ satisfy (4.3) and (4.4). Since the map $q_{\tilde{v}_k}$ is holomorphic on $q_{\tilde{v}_k}^{-1}(\Sigma^*_{\delta_k})$, by (4.11), (4.12), the estimates in the proof of Lemma 3.18 and the derivation of equation (3.11) in Section A.1.4,

$$\begin{aligned}&\big\|q_{v_k}^{-1*}N_{v_k}P_{v_k}\eta_k^{(m)}-N_{v^*}P_{v^*}\eta^{(m)}\big\|_{v^*,L^p(\Sigma^*_{\delta_k})}=\big\|N_{v^*}q_{v_k}^{-1*}P_{v_k}\eta_k^{(m)}-N_{v^*}P_{v^*}\eta^{(m)}\big\|_{v^*,L^p(\Sigma^*_{\delta_k})}\\
&\le C(b)\Big(\big\|q_{v_k}^{-1*}P_{v_k}\eta_k^{(m)}\big\|_{v^*,L^p(\Sigma^*_{\delta_k})}+\big\|P_{v_k}\eta_k^{(m)}\big\|_{v^*,L^p(\Sigma^*_{\delta_k})}\Big)\big\|q_{v_k}^{-1*}P_{v_k}\eta_k^{(m)}-P_{v^*}\eta^{(m)}\big\|_{v^*,L^p(\Sigma^*_{\delta_k})}\\
&\le C'(b)\big(\epsilon_{k^*}^{(m)}+|v_k|^{\frac{1}{p}}\big)\epsilon_{k^*}^{(m)};\\
&\qquad\big\|N_{v_k}P_{v_k}\eta_k^{(m)}\big\|_{v_k,L^p(A_{h,\delta_k,k})}\le C(b)|v_k|^{\frac{1}{p}}\big\|P_{v_k}\eta_k^{(m)}\big\|_{v_k,L^p_1(A_{h,\delta_k,k})}\le C'(b)|v_k|^{\frac{1}{p}}\epsilon_{k^*}^{(m)}.\end{aligned}$$

Thus, we can take $\epsilon_k^{(m+1)}=\epsilon_k^{(m)}+C'(b)\big(\epsilon_{k^*}^{(m)}+|v_k|^{\frac{1}{p}}\big)\epsilon_{k^*}^{(m)}$. This sequence is bounded as long as $|v_k|^{\frac{1}{p}}$ is sufficiently small (depending only on b). Since η_{v^*} is the limit in the (v^*,p)-norm of the sequence $\eta^{(m)}$ and η_{v_k} is the limit in the (v_k,p)-norm of the sequence $\eta_k^{(m)}$, the claim follows.

Corollary 4.5 *If* $\mathcal{T}$ *is a simple regular bubble type, there exist* $\delta,C\in C^\infty(\mathcal{M}_{\mathcal{T}};\mathbb{R}^+)$ *such that for any sequence* $\{v_k\in F^{\emptyset}\mathcal{T}_\delta\}$ *converging to* $v^*\in F^H\mathcal{T}_\delta$, $\tilde{\gamma}(v_k)$ *converges to* $\tilde{\gamma}(v^*)$ *with respect to the Gromov topology. Furthermore,*

$$\begin{aligned}&d_V\big(ev(\tilde{\gamma}(v^*)),ev(\tilde{\gamma}(v_k))\big)\le C(b_{v^*})\big|v_k-v^*\big|^{\frac{1}{p}} && \textit{if } S=S^2;\\
&d_V\big(ev_l(\tilde{\gamma}(v^*)),ev_l(\tilde{\gamma}(v_k))\big)\le C(b_{v^*})\big|v_k-v^*\big|^{\frac{1}{p}} && \forall l\in M;\\
&\big|\Psi_{\langle\mathcal{T}\rangle,\hat{0}}(\tilde{\gamma}(v_k))-\Psi_{\mathcal{T}(v^*),\hat{0}}(\tilde{\gamma}(v^*))\big|\le C(b_{v^*})\big|v_k-v^*\big|^{\frac{1}{p}} && \textit{if } S=S^2.\end{aligned}$$

Proof: It is sufficient to consider the case $\pi_h(v_k)=\pi_h(v^*)$ for all $h\notin H$ if $v^*\in F^{(H)}\mathcal{T}_\delta$. In such a case, $q_{\tilde{v}_k}$ maps the marked points of Σ_{v_k} to the marked points of Σ_{v^*} and $u_{v_k}=u_{v^*}\circ q_{v_k}$. By construction,

$$\tilde{u}_{v_k}=\exp_{b_{v^*},u_{v_k}}P_{v_k}\eta_{v_k},\qquad \tilde{u}_{v_k}=\exp_{b_{v^*},u_{v^*}}P_{v^*}\eta_{v^*}.$$

By Corollary 4.5 and the proof of Lemma 4.2,

$$\left\|q_{\tilde{v}_k}^{-1*}P_{v_k}\eta_{v_k} - P_{v^*}\eta_{v^*}\right\|_{v^*,L_1^p(\Sigma^*_{2|v_k-v^*|^{1/2}})} \le C(b_{v^*})\left|v_k - v^*\right|^{\frac{1}{p}}; \tag{4.13}$$

$$\left\|P_{v_k}\eta_{v_k}\right\|_{v_k,L_1^p(A_{h,2|v_k-v^*|^{1/2},k})} \le C(b_{v^*})\left|v_k - v^*\right|^{\frac{1}{p}} \quad \forall h\in H. \tag{4.14}$$

Let $\zeta_k\in\Gamma(\Sigma^*;\tilde{u}_{v^*})$ be given by

$$\exp_{b_{v^*},\tilde{u}_{v^*}}\zeta_k = \tilde{u}_{v_k}\circ q_{\tilde{v}_k}, \qquad \|\zeta_k\|_{C^0} < \text{inj }g_{V,b_{v^*}}.$$

By equation (4.13) and the proof of (2) of Lemma 3.5,

$$\|\zeta_k\|_{C^0(\Sigma^*_{2|v_k-v^*|^{1/2}})} \le C(b_{v^*})\|\zeta_k\|_{v^*,L_1^p(\Sigma^*_{2|v_k-v^*|^{1/2}})} \le C'(b_{v^*})\left|v_k - v^*\right|^{\frac{1}{p}}. \tag{4.15}$$

On the other hand, by (4.14) and by the same argument as in (3) of the proof of Lemma B.12, the variations of $P_{v^*}\eta_{v^*}$ on $A^*_{h,2|v_k-v^*|^{1/2}}$ and $P_{v_k}\eta_{v_k}$ on $A^*_{h,2|v_k-v^*|^{1/2},k}$ are bounded $C(b_v)|v_k-v^*|^{\frac{1}{p}}$. This can be seen from equation (B.33); observe that an argument similar to the proof Lemma 4.3 shows that we can take δ to be any small number bigger than $2|v_k-v^*|^{\frac{1}{2}}$. Equation (4.15) and the small variation on the annuli imply that

$$\sup_{z\in\Sigma_{v_k}} d_V\big(\tilde{u}_{v^*}(q_{\tilde{v}_k}(z)),\tilde{u}_{v_k}(z)\big) \le C(b_{v^*})\left|v_k - v^*\right|^{\frac{1}{p}}.$$

It follows that $\tilde{\gamma}_{\mathcal{T}}(v_k)$ converges to $\tilde{\gamma}_{\mathcal{T}}(v^*)$ in the Gromov topology. The estimate on the evaluation maps is immediate from the above bound. The last estimate follows from equations (4.13) and (4.14), along with a Sobolev estimate on a neighborhood of $\infty\in\Sigma_{v^*,\hat{0}}$ which implies that the C^1-norm of ζ_k there is bounded by $C(b_{v^*})|v_k-v^*|^{\frac{1}{p}}$.

4.2 Injectivity of the Gluing Map

The goal of this section is to prove that the gluing maps of (3.15) and (3.16) are injective, as long as $\delta\in C^\infty(\mathcal{M}_{\mathcal{T}}^{(0)};\mathbb{R}^+)$ is sufficiently small. We start by showing local injectivity on the subspaces $F^H\mathcal{T}_\delta$ of $F\mathcal{T}_\delta$, where H is a subset of $\hat{I}$.

If $\mathcal{T}$ is regular, we are only interested in the case $t=0$. If $\mathcal{T}$ is semiregular, we only consider the case $H=\emptyset$. We use the same notation as in Section 3.4. If $\|\varpi\|_v$ is sufficiently small, define $\tilde{\zeta}_{\varpi,t\nu}\in\Gamma(\tilde{u}_{v,t\nu})$ by

$$\exp_{b_v,\tilde{u}_{v,t\nu}}\tilde{\zeta}_{\varpi,t\nu} = u_{\varpi,t\nu}, \quad \|\tilde{\zeta}_{\varpi,t\nu}\|_{b_v,C^0} < \text{inj }g_{V,b}.$$

Lemma 4.6 *There exist $\delta,C\in C^\infty(\mathcal{M}_{\mathcal{T}}^{(0)};\mathbb{R}^+)$ such that for all $v\in F^{(H)}\mathcal{T}_\delta$, where $H=\emptyset$ if $\mathcal{T}$ is semiregular, and $\varpi\in\tilde{T}_vF^H\mathcal{T}_{\delta(b_v)}$,*
(1) $\left\|S_\varpi N_{\varpi,t\nu}R_\varpi\xi - N_{v,t\nu}\xi\right\|_{v,p} \le C(b_v)\|\varpi\|_v\|\xi\|_{v,p,1}^2$ for all $\xi\in\Gamma(u_v)$ with $\|\xi\|_{v,p,1}\le\delta(b_v)$ and $t\in[0,1]$;
(2) $\left\|S_\varpi\tilde{\pi}_{\varpi,\pm}R_\varpi\xi - \tilde{\pi}_{v,\pm}\xi\right\|_{v,p,1} \le C(b_v)\|\varpi\|_v\|\xi\|_{v,p,1}$ for all $\xi\in\Gamma(u_v)$;
(3) $\left\|S_\varpi P_\varpi R_\varpi\eta - P_v\eta\right\|_{v,p,1} \le C(b_v)\|\varpi\|_v\|\eta\|_{v,p}$ for all $\xi\in\Gamma^{0,1}(u_v)$.

Proof: Claim (1) follows from (2) of Lemma 3.6 and Riemannian geometry estimates similar to Appendix A.1. Claim (2) is a consequence of (5) of Lemma 3.6 and (2) of Definition 3.11. Finally, (3) is obtained from (1), (2), Definitions 3.11 and 3.13, and Lemmas 3.6 and 3.16 as follows. Writing $\triangle_\varpi P$ for $S_\varpi P_\varpi R_\varpi - P_\upsilon$, etc.,

$$\begin{aligned}
\triangle_\varpi P &= P_\upsilon \pi^{0,1}_{\upsilon,+} D_\upsilon \tilde{\pi}_{\upsilon,+} \triangle_\varpi P + \tilde{\pi}_{\upsilon,-} S_\varpi P_\varpi R_\varpi \\
&= P_\upsilon \pi^{0,1}_{\upsilon,+} D_\upsilon \triangle_\varpi P - \big(P_\upsilon \pi^{0,1}_{\upsilon,+} D_\upsilon - 1\big) \triangle_\varpi \tilde{\pi}_{\cdot,+} S_\varpi P_\varpi R_\varpi \\
&= P_\upsilon \pi^{0,1}_{\upsilon,+} \triangle_\varpi \tilde{\pi}^{0,1}_{\cdot,+} - \Big(P_\upsilon \pi^{0,1}_{\upsilon,+} \triangle_\varpi D + \big(P_\upsilon \pi^{0,1}_{\upsilon,+} D_\upsilon - 1\big) \triangle_\varpi \tilde{\pi}_{\cdot,+}\Big) S_\varpi P_\varpi R_\varpi \\
&= -P_\upsilon \triangle_\varpi \pi^{0,1}_{\cdot,+} S_\varpi \tilde{\pi}^{0,1}_{\varpi,+} R_\varpi - \Big(P_\upsilon \pi^{0,1}_{\upsilon,+} \triangle_\varpi D + \big(P_\upsilon \pi^{0,1}_{\upsilon,+} D_\upsilon - 1\big) \triangle_\varpi \tilde{\pi}_{\varpi,+} S_\varpi P_\varpi R_\varpi\Big).
\end{aligned}$$

Corollary 4.7 *There exist $\delta, C \in C^\infty(\mathcal{M}^{(0)}_{\mathcal{T}}; \mathbb{R}^+)$ such that for all $t \in [0; \delta(b_\upsilon)]$, $\upsilon \in F^{(H)}\mathcal{T}_\delta$, where $H = \emptyset$ if $\mathcal{T}$ is semiregular, and $\varpi \in \tilde{T}_\upsilon F^H \mathcal{T}_{\delta(b_\upsilon)}$,*

$$C(b_\upsilon)^{-1} \|\varpi\|_\upsilon \leq \|\tilde{\zeta}_{\varpi,t\nu}\|_{\upsilon,p,1} + \sum_{l \in M} |w_l(\varpi)|_{g_\upsilon} \leq C(b_\upsilon) \|\varpi\|_\upsilon .$$

Furthermore, $\big\|S_\varpi \xi_{\varpi,t\nu} - \xi_{\upsilon,t\nu}\big\|_{\upsilon,p,1} \leq C(b_\upsilon)\big(t + |\upsilon|^{\frac{1}{p}}\big)\|\varpi\|_\upsilon$.

Proof: The first claim of the lemma is immediate from the second and (1) of Lemma 3.6. On the other hand, by construction in Section 3.6,

$$\xi_{\varpi,t\nu} = tP_\varpi \nu - P_\varpi \bar{\partial} u_\varpi - P_\varpi N_{\varpi,t\nu} \xi_{\varpi,t\nu}.$$

Thus, if t and $|\upsilon|$ are sufficiently small (depending on b_υ), the second claim follows from Lemmas 3.6, 4.6, Corollary 3.19, and equation (3.11).

Corollary 4.8 *If $\mathcal{T}$ is a simple bubble type and K is an open subset of $\mathcal{M}_{\mathcal{T}}$ with compact closure, there exists $\delta > 0$ such that for any $t \in [0, \delta]$, the map*

$$\tilde{\gamma}_{\mathcal{T},t\nu} \colon F^\emptyset \mathcal{T}_\delta | K \longrightarrow C^\infty_{(\lambda;M)}(S;V), \qquad \tilde{\gamma}_{\mathcal{T},t\nu}(\upsilon) = \tilde{b}_{t\nu}(\upsilon),$$

is a differentiable embedding.

Proof: We first deduce from Corollary 4.7 that $\tilde{\gamma}_{\mathcal{T},t\nu}$ is injective if δ is sufficiently small. Suppose not, i.e. there exist sequences $\upsilon_k, \upsilon_{k'} \in F^\emptyset \mathcal{T}_\delta | K$ such that

$$\upsilon_k \longrightarrow b \in \bar{K}, \quad \upsilon'_k \longrightarrow b' \in \bar{K}, \quad \text{and} \quad \tilde{b}_{t\nu}(\upsilon_k) = \tilde{b}_{t\nu}(\upsilon'_k).$$

It follows that $b = b'$, after possibly modifying the sequence $\{\upsilon_{k'}\}$ the action of an element of $\big(\mathcal{A}(\mathcal{T}) \ltimes G_{\mathcal{T}}\big)$. If for some k, $\upsilon'_k = \upsilon_k(\varpi_k)$ with $\|\varpi_k\|_{\upsilon_k}$ sufficiently small, then by Corollary 4.7, $\upsilon'_k = \upsilon_k$. Otherwise, the difference between q_{υ_k} and $q_{\upsilon'_k}$ is uniformly bounded below outside of the preimage of the zeroth component and the necks $A_{\upsilon_k,h}$. Thus, the bubble maps $b(\upsilon_k)$ and $b(\upsilon'_k)$ are far apart unless b has an automorphism. In the latter case, υ'_k can be replaced by an equivalent element of $F^\emptyset \mathcal{T}_\delta$. In the former case, $\tilde{u}_{\upsilon_k}$ and $\tilde{u}_{\upsilon'_k}$ cannot be the same because

$$\big\|P_{\upsilon_k} \eta_{\upsilon_k,t\nu}\big\|_{C^0} \leq C\big(t + |\upsilon_k|^{\frac{1}{p}}\big) \leq C' \delta^{\frac{1}{p}} \quad \text{and} \quad \big\|P_{\upsilon'_k} \eta_{\upsilon'_k,t\nu}\big\|_{C^0} \leq C\big(t + |\upsilon'_k|^{\frac{1}{p}}\big)\delta^{\frac{1}{p}}.$$

Thus, $\tilde{\gamma}_{\mathcal{T},t\nu}$ is injective on $F^{\emptyset}\mathcal{T}_\delta|K$ provided δ is sufficiently small (depending on K). The smoothness of $\tilde{\gamma}_{\mathcal{T},t\nu}$ follows from the smooth dependence of solutions of equation (3.12) on the parameters. Finally, the differential of $\tilde{\gamma}_{\mathcal{T},t\nu}$ is nondegenerate by Corollary 4.7.

Corollary 4.9 *If $\mathcal{T}$ is regular, there exists $\delta \in C^\infty(\mathcal{M}_\mathcal{T};\mathbb{R}^+)$ such that for all m, the map*

$$\tilde{\gamma}_\mathcal{T}\colon \bigcup_{|H|=m} F^H\mathcal{T}_\delta \longrightarrow \bigcup_{|H|=m} \mathcal{M}_{\mathcal{T}(H)}, \qquad \tilde{\gamma}_\mathcal{T}(v)=\tilde{b}(v),$$

is injective.

Proof: The same argument as in the proof of Corollary 4.8 shows that map

$$\tilde{\gamma}_\mathcal{T}\colon F^H\mathcal{T}_\delta \longrightarrow \mathcal{H}_{\mathcal{T}(H)}$$

is an embedding if δ is sufficiently small. It remains to see that $\tilde{\gamma}_\mathcal{T}^{(0)}(v) \neq g\cdot\tilde{\gamma}_\mathcal{T}^{(0)}(v')$ for any $g\in G_{\mathcal{T}(H)}$ whenever $[v]\neq[v']$. For each $v\in F^{(H)}\mathcal{T}_\delta$ and $i\in H$, we construct $\big(c_i(v),r_i(v)\big)\in\mathbb{C}\times\mathbb{R}$ such that

$$\big(c(v),r(v)\big)\cdot\tilde{\gamma}_\mathcal{T}^{(0)}(v)\in\mathcal{M}^{(0)}_{\mathcal{T}(H)}.$$

We define $c_i(v)\in\mathbb{C}$ and $r_i(v)\in\mathbb{R}$ by

$$\begin{aligned}\tilde{\Psi}\big((c_i(v),0)\cdot\tilde{u}_{v,i}\big)+\sum_{\iota_h(v)=i} d_h\big(\mathcal{T}(H)\big)\big(x_h(v)+c_i(v)\big)+\sum_{j_l(v)=i}\big(y_l(v)+c_i(v)\big)=0;\\ \Psi^{(3)}\big((c_i(v),r_i(v))\cdot\tilde{u}_{v,i}\big)+\sum_{\iota_h(v)=i} d_h\big(\mathcal{T}(H)\big)\beta\big((1+r_i(v))|x_h(v)+c_i(v)|\big)\\ +\sum_{j_l(v)=i}\beta\big((1+r_i(v))|y_l(v)+c_i(v)|\big)=\frac{1}{2}.\end{aligned}$$

Since the metric $g_{v,i}$ for $i>0$ agrees with the standard metric on S^2 on a neighborhood of the south pole and $\Psi_{\mathcal{T},i}(b_v)=0$, by Corollary 4.7 for any $\varpi\in T_vF^H\mathcal{T}$ with $\|\varpi\|_v$ sufficiently small,

$$\big|c_i(\varpi)-c_i(v)\big|\le C(b_v)|v|^{\frac{1}{p}}\|\varpi\|_v, \qquad \big|r_i(\varpi)-r_i(v)\big|\le C(b_v)|v|^{\frac{1}{p}}\|\varpi\|_v. \tag{4.16}$$

Let $\bar{b}(v)=(c(v),r(v))\cdot\tilde{b}(v)$. Write

$$\bar{b}(v)=\big(S,M,H\cup\{\hat{0}\};\bar{x}(v),(j(v),\bar{y}),\bar{u}_v\big).$$

If $\|\varpi\|_v$ is sufficiently small, define $\bar{\zeta}_\varpi\in\Gamma(\bar{u}_v)$ by

$$\exp_{b_v,\bar{u}_v}\bar{\zeta}_\varpi=u_\varpi, \quad \|\bar{\zeta}_\varpi\|_{b_v,C^0}<\text{inj }g_{V,b_v}.$$

Similarly, for $h\in H$ and $l\in M$, define $\bar{w}_h(\varpi)\in T_{\bar{x}_h(v)}\Sigma_{v,\iota_h(v)}$ and $\bar{w}_l(\varpi)\in T_{\bar{y}_l}\Sigma_{v,j_l(v)}$ by

$$\begin{aligned}\exp_{g_v,\bar{x}_h(v)}\bar{w}_h(\varpi)=\bar{x}_h(\varpi), \quad |\bar{w}_h(\varpi)|\equiv|\bar{w}_h(\varpi)|_{g_v}<\text{inj}(g_v,x_h(v));\\ \exp_{g_v,\bar{y}_l(v)}\bar{w}_l(\varpi)=\bar{y}_l(\varpi), \quad |\bar{w}_l(\varpi)|\equiv|\bar{w}_l(\varpi)|_{g_v}<\text{inj}(g_v,y_l(v)).\end{aligned}$$

Then by equation (4.16) and Corollary 4.7,

$$C''(b_v)^{-1}\|\varpi\|_v \le \|\bar{\zeta}_\varpi\|_{v,p,1} + \sum_{h\in H}|\bar{w}_h(\varpi)| + \sum_{l\in M}|\bar{w}_l(\varpi)| \le C(b_v)\|\varpi\|_v. \tag{4.17}$$

It follows that the map

$$F^H\mathcal{T}_\delta \longrightarrow \mathcal{M}^{(0)}_{\mathcal{T}(H)}, \qquad \upsilon \longrightarrow \bar{b}(\upsilon),$$

is a local embedding. By the same argument as in the proof of Lemma 4.8, we can conclude that this map is injective as long as $\delta\in C^\infty(\mathcal{M}^{(0)}_{\mathcal{T}};\mathbb{R}^+)$ is sufficiently small. Since this map is $G_{\mathcal{T}(H)}$-equivariant by construction, it follows that the induced map on the quotient, i.e. the map of Corollary 4.9, is injective.

Corollary 4.10 *If $S=S^2$, there exists $\delta\in C^\infty(\mathcal{M}_{\mathcal{T}};\mathbb{R}^+)$ such that the map*

$$\tilde{\gamma}_{\mathcal{T}}\colon F\mathcal{T}_\delta|\mathcal{M}_{\mathcal{T}} \longrightarrow \bar{\mathcal{M}}_{\langle\mathcal{T}\rangle}$$

is injective. Furthermore, the restriction

$$\tilde{\gamma}_{\mathcal{T}}\colon F^\emptyset\mathcal{T}_\delta|\mathcal{M}_{\mathcal{T}} \longrightarrow \mathcal{M}_{\langle\mathcal{T}\rangle}$$

is a differentiable embedding.

In order to adjust the gluing procedure in the presence of constraints, below we state the analogue of Corollary 4.7 for $\varpi\in\mathcal{K}_{b_v}\mathcal{T}\subset T_vF^\emptyset\mathcal{T}$. It is obtained in the same way as Corollary 4.7, except the analogue of Lemma 4.6 would make use of Lemma 3.8, instead of Lemma 3.6, and of (2b), instead of (2a), of Definitions 3.11 and 3.13. We also use (3) of Lemma 3.5.

Corollary 4.11 *There exist $\delta, C\in C^\infty(\mathcal{M}^{(0)}_{\mathcal{T}};\mathbb{R}^+)$ such that for all $t\in[0;\delta(b_v)]$, $\upsilon\in F^{(\emptyset)}\mathcal{T}_\delta$, and $\varpi\in\mathcal{K}_{b_v}\mathcal{T}_{\delta(b_v)}$,*

$$C(b_v)^{-1}\|\varpi\| \le \|\bar{\zeta}_{\varpi,t\nu}\|_{v,p,1} + \sum_{l\in M}|w_l(\varpi)|_{g_v} \le C(b_v)\|\varpi\|.$$

Furthermore, $\big\|S'_\varpi\xi_{\varpi,t\nu} - \xi_{v,t\nu}\big\|_{v,C^0} \le C(b_v)\big(t+|v|^{\frac{1}{p}}\big)\|\varpi\|$.

4.3 The Basic Gluing Map and the Space of Balanced Maps

Our next goal is to show that the gluing map of Section 3.6 is surjective in the appropriate sense. More precisely, if $\mathcal{T}$ is a regular bubble type, we show that the image of $\tilde{\gamma}_{\mathcal{T}}$ contains a neighborhood of $\mathcal{M}_{\mathcal{T}}$ in $\bar{\mathcal{M}}_{\langle\mathcal{T}\rangle}$. If $\mathcal{T}$ is a semiregular, we show that all elements in $\mathcal{M}_{\Sigma,t\nu,\lambda}$ that are close to any given compact subset of $\mathcal{M}_{\mathcal{T}}$ are in the image of the gluing map $\tilde{\gamma}_{\mathcal{T},t\nu}$ if t is sufficiently small. The major difficulty in doing this is the following. If $\upsilon\in F\mathcal{T}$, a small change in the singular points of b_v may lead to a very large change in the map u_v. This is precisely the reason we used the norm $\|\varpi\|_v$ on $T_vF^H\mathcal{T}$ instead of just $\|\varpi\|$ in Section 3.4. In order to deal with this issue, we need Corollary 4.13, which is proved in this section. We continue to assume that $\mathcal{T}$ is a simple bubble type.

Recall that $\mathcal{H}_\mathcal{T}$ is the set of tuples $b=(S,M,I;x,(j,y),u)$ such that $u_{\iota_h}(x_h)=u_h(\infty)$ for all $h\in\hat{I}$ and $\bar{\partial}u_i=0$ for all $i\in I$. Furthermore, $\mathcal{M}_\mathcal{T}^{(0)}$ is the subset of $\mathcal{H}_\mathcal{T}$ consisting of the tuples b such that $\Psi_{\mathcal{T},h}(b)=0$ for all $h\in\hat{I}$. It is convenient to make the following definitions. If H is a subset of $\hat{I}$ and $\epsilon\geq 0$, let

$$\mathcal{M}_{\mathcal{T},\epsilon}^{(H)}=\big\{b=(S,M,I;x,(j,y),u):\bar{\partial}u_i=0\ \forall i\in I;\ d_V\big(u_{\iota_h}(x_h),u_h(\infty)\big)\leq\epsilon\ \forall h\in\hat{I};\\ \big|\Psi_{\mathcal{T},h}(b)\big|\leq\epsilon\ \forall h\in\hat{I}-H\big\}.$$

Lemma 4.12 *There exist* $\delta,C\in C^\infty(\mathcal{M}_\mathcal{T}^{(0)};\mathbb{R}^+)$ *with the following property. Suppose* $b^*\in\mathcal{M}_\mathcal{T}^{(0)}$, $\epsilon<\delta(b^*)$, $b\in\mathcal{M}_{\mathcal{T},\epsilon}^{(H)}$ *is such that* $d(b^*,b)\leq\delta(b^*)$, *and* $\upsilon=(b,v_{\hat{I}})\in F_b^{(H)}\mathcal{T}_{\delta(b^*)}$. *Then there exist*

$$\tilde{b}\in\mathcal{M}_{\mathcal{T},\epsilon^2}^{(H)}\quad\textit{and}\quad\tilde{\upsilon}=(\tilde{b},\tilde{v}_{\hat{I}})\in F_{\tilde{b}}^{(H)}\mathcal{T}\qquad\textit{such that}$$

(1) $d(b,\tilde{b})\leq C(b^*)\epsilon$ *and* $\big|\tilde{v}_h-v_h\big|_b\leq C(b^*)\epsilon|v_h|_b$ *for all* $h\in\hat{I}$*;*
(2) if $q_\upsilon(z)\in\Sigma_{\mathcal{T},i}$, $r_{b,h}(q_\upsilon z)\geq 2|v_h|^{\frac{1}{2}}$ *for all* $h\in\hat{I}-H$ *such that* $\iota_h=i$ *and* $\big|q_S^{-1}(q_\upsilon z)\big|\geq 2|v_i|^{\frac{1}{2}}$ *if* $i\in\hat{I}-H$, *then* $d_b\big(q_\upsilon z,q_{\tilde{\upsilon}}z\big)\leq\delta(b^*)\epsilon$.

Proof: (1) Let $b=(S,M,I;x,(j,y),u)$. If δ is sufficiently small, by Proposition 3.3, we can choose $\xi_i\in\Gamma(u_i)$ such that $\|\xi_i\|_{g_{b,i},C^1}\leq C(b^*)\epsilon$ and

$$b'\equiv\big(S,M,I;x,(j,y),u'\big)\in\mathcal{H}_\mathcal{T},$$

where $u_i'=\exp_{u_i}\xi_i$. The C^1-bound on ξ_i and the assumption $b\in\mathcal{M}_{\mathcal{T},\epsilon}^{(H)}$ imply that $|\Psi_{\mathcal{T},h}(b')|\leq C'(b^*)\epsilon$ for all $h\in\hat{I}-H$.
(2) We now define $\tilde{b}'\equiv\big(S,M,I;\tilde{x}',(j,\tilde{y}'),\tilde{u}'\big)\in\mathcal{H}_\mathcal{T}$ and $\tilde{\upsilon}'=(\tilde{b}',\tilde{v}_{\hat{I}}')$ as follows. Suppose $i^*\in I$ and for all $i\in\hat{I}$ with $i>i^*$, $h\in\hat{I}$ with $\iota_h=i$, and $l\in M$ with $j_l=i$, we have constructed
(i) $(c_i,r_i)\in\mathbb{C}\times\mathbb{R}$ such that $|(c_i,r_i)|\leq C(b^*)\epsilon$;
(ii) $\tilde{x}_h',\tilde{y}_l'\in\Sigma_{\mathcal{T},i}$ such that $|r_{b,h}(\tilde{x}_h')|\leq C(b^*)\epsilon$ and $|\phi_{y_l}\tilde{y}_l'|\leq C(b^*)\epsilon$;
(iii) $\tilde{v}_h'\in\mathbb{C}$ such that $|\tilde{v}_h'-v_h|\leq C(b^*)\epsilon|v_h|$;
(iv) if $x_i\in S^2$, $\bar{x}_i\in S^2$, such that $|r_{b,i}(\bar{x}_i)|\leq C(b^*)\epsilon|v_i|$;
(v) if $x_i\in S^2$, $\bar{v}_i\in\mathbb{C}$ such that $|\bar{v}_i-v_i|\leq C(b^*)\epsilon|v_i|$,
such that
(I1) if $i\notin H$, $\Psi_{\mathcal{T},i}(\tilde{b}')=0$ where $\tilde{u}_i'=(c_i,r_i)\cdot u_i'$;
(I2) if $\Sigma_{\mathcal{T},\iota_i}=S^2$, $z\in\Sigma_{\mathcal{T},\iota_i}$, and $\big|\phi_{\bar{x}_h}q_{i,(x_i,v_i)}(z)\big|\leq\frac{2}{3}|\bar{v}_h|^{\frac{1}{2}}$ for some $h\in\hat{I}-H$, then

$$\Big|\phi_{\tilde{x}_h'}q_{i,(\bar{x}_i,\bar{v}_i)}(z)\Big|\leq|\tilde{v}_h'|^{\frac{1}{2}}\quad\text{and}\quad q_{h,(\tilde{x}_h',\tilde{v}_h')}\big(q_{i,(\bar{x}_i,\bar{v}_i)}(z)\big)=q_{h,(\bar{x}_h,\bar{v}_h)}\big(q_{i,(x_i,v_i)}(z)\big),$$

where $q_{i,(x_i,v_i)}$, etc., are the maps defined in Section 2.3.
Note that while we have not defined $\tilde{b}'$ completely yet, (I1) is still a well-defined statement. The function $\Psi_{\mathcal{T},i}$ depends only the ith bubble component of $\tilde{b}'$, which has already been constructed by the induction assumptions.

If $i^*\in H$, we take $c_{i^*}=0$ and $r_{i^*}=0$. If $i^*\in\hat{I}-H$, let $(c_{i^*},r_{i^*})\in\mathbb{C}\times\mathbb{R}$ be given by

$$\tilde{\Psi}\left((c_{i^*},0)u'_{i^*}\right)+\sum_{\iota_h=i^*}d_h(\mathcal{T})(\bar{x}_h+c_{i^*})+\sum_{j_l=i^*}(y_l+c_{i^*})=0;$$

$$\Psi^{(3)}\left((c_{i^*},r_{i^*})u'_{i^*}\right)+\sum_{\iota_h=i^*}d_h(\mathcal{T})\beta\big((1+r_{i^*})|\bar{x}_h+c_{i^*}|\big)+\sum_{j_l=i^*}\beta\big((1+r_{i^*})|y_l+c_{i^*}|\big)=\frac{1}{2}.$$

If ϵ is sufficiently small, by the proof of Lemma 3.3 such $(c_{i^*},r_{i^*})\in\mathbb{C}\times\mathbb{R}$ exists and satisfies $|c_{i^*}|,|r_{i^*}|\le C(b^*)\epsilon$. For all $h\in\hat{I}$ with $\iota_h=i^*$ and $l\in M$ with $j_l=i^*$, put

$$\tilde{x}'_h=(1+r_{i^*})\left(\bar{x}_h+c_{i^*}\right),\quad \tilde{v}'_h=(1+r_{i^*})\bar{v}_h,\quad \tilde{y}'_l=(1+r_{i^*})\left(y_l+c_{i^*}\right);$$
$$\bar{x}_{i^*}=x_{i^*}-c_{i^*}v_i,\quad \bar{v}_{i^*}=(1+r_{i^*})^{-1}v_i\quad\text{if}\quad x_{i^*}\in S^2.$$

Continuing in this way, for all $i\in\hat{I}$, $h\in\hat{I}$ with $\iota_h=i$, and $l\in M$ with $j_l=i$, we obtain elements (i)-(v) satisfying (I1),(I2). Let $\tilde{u}'_{\hat{0}}=u'_{\hat{0}}$. If $l\in M$ and $j_l=\hat{0}$, take $\tilde{y}'_l=y_l$.
(3) If $S=S^2$, let $(\tilde{x}'_h,\tilde{v}'_h)=(\bar{x}_h,\bar{v}_h)$ if $\iota_h=\hat{0}$, $\tilde{b}=\tilde{b}'$, and $\tilde{v}=\tilde{v}'$. By the inductive construction, $\tilde{b}$ and $\tilde{v}$ satisfy the requirements of the lemma. In fact, $\tilde{b}\in\mathcal{M}^{(H)}_{\mathcal{T},0}$. If $S=\Sigma$, we could extend the above construction to the principal component Σ as we did for $S=S^2$ if $q_{\tilde{v}'}$ were defined using the metric $g_{b,\hat{0}}$ on Σ, which may differ slightly from $g_{\tilde{b}',\hat{0}}$. This problem is fixed below.
(4) If $l\in M$ and $j_l=\hat{0}$, we take $\tilde{y}_l=y_l$ as before. For all $h\in\hat{I}$ with $\iota_h=\hat{0}$, let $\tilde{x}_h\in\Sigma$, $\tilde{v}_h\in T_{\tilde{x}_h}\Sigma$, and $\Theta_h\colon B_{2|v_h|_b^{-\frac{1}{2}}}(0;\mathbb{C})\longrightarrow\mathbb{C}$ be such that
$(\Sigma_{\hat{0}}1)$ $d_b(x_h,\tilde{x}_h)\le C(b^*)\epsilon|v_h|$, $\big||\tilde{v}_h|_{\tilde{b}}-|v_h|_b\big|\le C(b^*)\epsilon|v_h|_b$;
$(\Sigma_{\hat{0}}2)$ for all $z\in B_b\big(x_h,2|v_h|_b^{\frac{1}{2}}\big)$,

$$\frac{\phi_{\tilde{b},h}z}{\tilde{v}_h}=(1+r_h)\Big\{c_h+\frac{\phi_{b,h}z}{v_h}+\Theta_h\Big(\frac{\phi_{b,h}z}{v_h}\Big)\Big\};$$

$(\Sigma_{\hat{0}}3)$ Θ_h is holomorphic, $\Theta_h(0)=0$, $\Theta'_h(0)=0$, and $\|\Theta''_h\|_{C^0}\le C(b^*)|v|^2\epsilon$.
Note that even though we have not defined $\tilde{b}$ completely yet, $(\Sigma_{\hat{0}}1)$ and $(\Sigma_{\hat{0}}2)$ are still well-defined statements, since the metric $g_{\tilde{b},\hat{0}}$ on Σ depends only on the singular points $\{\tilde{x}_h\colon\iota_h=\hat{0}\}$ on Σ. Existence of such $\tilde{x}_h$, $\tilde{v}_h$, and Θ_h follows from Corollary B.5, provided δ is sufficiently small.

If $\iota_i=\hat{0}$ and $j_l=i$, let $(i,\tilde{y}_l)=q_{\tilde{v},i}q_{v,i}^{-1}(i,y_l)$. The map $q_{\tilde{v},i}$ is well-defined even though $\tilde{v}$ has not been defined completely yet. By $(\Sigma_{\hat{0}}2)$,

$$\tilde{y}_l=q_{\tilde{v},i}q_{v,i}^{-1}(y_l)=\frac{\phi_{\tilde{b},\tilde{x}_i}\phi_{b,x_i}^{-1}(y_lv_i)}{\tilde{v}_i}=\big(1+r_i\big)\big\{c_i+y_l+\Theta_i(y_l)\big\}.\tag{4.18}$$

Since $\tilde{y}'_l=(1+r_i)(y_l+c_i)$, $|\tilde{y}_l-\tilde{y}_l|\le C(b^*)|v|^2\epsilon$ by $(\Sigma_{\hat{0}}3)$.

Suppose $h\in\hat{I}$, $\iota_h\in\hat{I}$, and for every $i\in\hat{I}$ with $i<h$ and $j\in M$ with $j_l=i$, we have defined

$$\tilde{x}_i\in\Sigma_{\mathcal{T},\iota_i},\quad \tilde{y}_l\in\Sigma_{\mathcal{T},i},\quad \tilde{v}_i\in\begin{cases}T_{\tilde{x}_i}\Sigma, & \text{if }\iota_i=\hat{0};\\ \mathbb{C}, & \text{if }\iota_i\ne\hat{0};\end{cases}\quad \tilde{c}_i\in\mathbb{C},\quad \Theta_i\colon B_{2|v_i|_b^{-\frac{1}{2}}}(0;\mathbb{C})\longrightarrow\mathbb{C}$$

such that
(Σ1) $|\phi_{\tilde{b}',i}\tilde{x}_i|_{\tilde{b}'} \le C(b^*)|v|^2\epsilon$ if $\iota_i \neq \hat{0}$ and $|\phi_{\tilde{b}',\tilde{y}'_l}\tilde{y}_l|_{\tilde{b}'} \le C(b^*)|v|^2\epsilon$;
(Σ2) $\big||\tilde{v}_i|_{\tilde{b}} - |v_i|_b\big| \le C(b^*)\epsilon|v_i|_b$;
(Σ3) $\big|\tilde{c}_i - c_i\big| \le C(b^*)|v|^2\epsilon$;
(Σ4) for all $z \in \Sigma$ such that $r_{b,i}q_{v,\iota_i}(z) \le 2|v_i|_b^{\frac{1}{2}}$,

$$\frac{\phi_{\tilde{b},i}q_{\tilde{v},\iota_i}z}{\tilde{v}_i} = (1+r_i)\Big\{\tilde{c}_i + \frac{\phi_{b,i}q_{v,\iota_i}z}{v_i} + \Theta_i\Big(\frac{\phi_{b,i}q_{v,\iota_i}z}{v_i}\Big)\Big\}$$

(Σ5) Θ_i is holomorphic, $\Theta_i(0)=0$, $\Theta'_i(0)=0$, and $\|\Theta''_i\|_{C^0} \le C(b^*)|v|^2\epsilon$.
If $h \in H$, we take $\tilde{x}_h = \tilde{x}'_h$, $\tilde{v}_h = \tilde{v}'_h = 0$, $\tilde{y}_l = \tilde{y}'_l$ if $j_l = h$, $\tilde{c}_h = c_h = 0$, and $\Phi_h(z) = 0$. If $h \notin H$, let

$$(\iota_h, \tilde{x}_h) = q_{\tilde{v},\iota_h}q_{v,\iota_h}^{-1}(h, \bar{x}_h).$$

By an argument similar to (4.18), from (Σ4) we obtain

$$\tilde{x}_h = \big(1 + r_{\iota_h}\big)\big\{\tilde{c}_{\iota_h} + \bar{x}_h + \Theta_{\iota_h}(\bar{x}_h)\big\}. \tag{4.19}$$

Since $\tilde{x}'_h = (1+r_{\iota_h})(\bar{x}_h + c_{\iota_h})$, (4.19), ($\Sigma$3), and ($\Sigma$5) imply the first part of (Σ1) with $i = h$. Furthermore, by assumption (Σ4),

$$\begin{aligned}\phi_{\tilde{b},h}q_{\tilde{v},\iota_h}(z) = q_{\tilde{v},\iota_h}(z) - \tilde{x}_h &= \frac{\phi_{\tilde{b},\iota_h}q_{\tilde{v},\iota_{\iota_h}}(z)}{\tilde{v}_{\iota_h}} - \tilde{x}_h \\ &= \big(1+r_{\iota_h}\big)\Big\{\Big(\frac{\phi_{b,\iota_h}q_{v,\iota_{\iota_h}}(z)}{v_{\iota_h}} - \bar{x}_h\Big) + \Big(\Theta_{\iota_h}\Big(\frac{\phi_{b,\iota_h}q_{v,\iota_{\iota_h}}(z)}{v_{\iota_h}}\Big) - \Theta_{\iota_h}(\bar{x}_h)\Big)\Big\}.\end{aligned} \tag{4.20}$$

Since Θ_{ι_h} is holomorphic, and

$$\frac{\phi_{b,\iota_h}q_{v,\iota_{\iota_h}}(z)}{v_{\iota_h}} - \bar{x}_h = \phi_{b,h}q_{v,\iota_h}(z) + c_h v_h,$$

we can rewrite (4.20) as

$$\phi_{\tilde{b},h}q_{\tilde{v},\iota_h}(z) = \big(1+r_{\iota_h}\big)\big(1+a_h\big)v_h\Big\{\tilde{c}_h + \frac{\phi_{b,h}q_{v,\iota_h}(z)}{v_h} + \Theta_h\Big(\frac{\phi_{b,h}q_{v,\iota_h}(z)}{v_h}\Big)\Big\}, \tag{4.21}$$

with the complex numbers $a_h, \tilde{c}_h \in \mathbb{C}$ and holomorphic function $\Theta_h \colon B_{2|v_h|^{-\frac{1}{2}}}(0,\mathbb{C}) \longrightarrow \mathbb{C}$ given by

$$a_h = \frac{d}{dz}\Theta_{\iota_h}(z)\Big|_{z=x_h}, \qquad (1+a_h)\tilde{c}_h = c_h + \frac{\Theta_{\iota_h}(x_h) - \Theta_{\iota_h}(x_h - c_h v_h)}{v_h}, \tag{4.22}$$

$$\Theta_h(z) = \frac{\Theta_{\iota_h}\big(v_h z + x_h\big) - v_h z\Theta'_{\iota_h}(x_h) - \Theta_{\iota_h}(x_h)}{(1+a_h)v_h}. \tag{4.23}$$

By (4.23), $\Theta_h(0)=0$ and $\Theta'_h(0)=0$. By our assumptions on Θ_{ι_h} and (4.22), (4.23),

$$|a_h| \le C(b^*)|v|^2\epsilon|x_{\iota_h}| \le C'(b^*)|v|^2\epsilon; \tag{4.24}$$

$$|\tilde{c}_h - c_h| \le C(b^*)\big(\epsilon|a_h| + |v_h|^{-1}|v|^2\epsilon|c_h v_h|\big) \le C'(b^*)|v|^2\epsilon, \tag{4.25}$$

$$\|\Theta''_h\|_{C^0} \le C(b^*)|v_h|^{-1}|v|^2\epsilon|v_h|^2 \le C'(b^*)|v|^2\epsilon. \tag{4.26}$$

We now take

$$\tilde{v}_h = \big(1+a_h\big)\tilde{v}'_h = \big(1+a_h\big)\big(1+r_{\iota_h}\big)\big(1+r_h\big)^{-1}v_h.$$

It follows from (4.24)-(4.26) that the induction hypotheses (Σ2)-(Σ5) with $i=h$ are satisfied. If $j_l=h$, let $(h,\tilde{y}_l)=q_{\tilde{v},h}q^{-1}_{v,h}(h,y_l)$. By the same argument as in the case $\iota_h=\hat{0}$ above, (Σ3)-(Σ5) of the $i=\iota_h$ case imply that the second part of (Σ1) with $i=h$ is satisfied. Continuing in this way, we obtain tuples

$$\tilde{b} = \big(\Sigma, M, I; \tilde{x}, (j,\tilde{y}), \tilde{u}'\big), \quad \tilde{c} = c_{\hat{I}}, \quad \tilde{v} = \big(\tilde{b}, \tilde{v}_{\hat{I}}\big),$$

satisfying (Σ1)-(Σ5). Since $\tilde{b}' \in \mathcal{M}^{(H)}_{\mathcal{T},0}$, by ($\Sigma$1) $\tilde{b} \in \mathcal{M}^{(H)}_{\mathcal{T},\epsilon}$ if δ is sufficiently small. Finally, (Σ1)-(Σ5) along with (I1) and (I2) show that $\tilde{b}$ and $\tilde{v}$ satisfy the two requirements of the lemma.

Corollary 4.13 *If $\mathcal{T}$ is a simple bubble type, there exist $\delta, C \in C^\infty(\mathcal{M}^{(0)}_{\mathcal{T}};\mathbb{R}^+)$ with the following property. Suppose $b^* \in \mathcal{M}^{(0)}_{\mathcal{T}}$, $\epsilon < \delta(b^*)$, $b \in \mathcal{M}^{(H)}_{\mathcal{T},\epsilon}$ is such that $d(b^*,b) \le \delta(b^*)$, and $v=(b,v_{\hat{I}}) \in F^{(H)}_b\mathcal{T}_{\delta(b^*)}$. Then there exist $\tilde{b} \in \mathcal{M}^{(H)}_{\mathcal{T},0}$ and $\tilde{v}=(\tilde{b},\tilde{v}_{\hat{I}}) \in F^{(H)}_{\tilde{b}}\mathcal{T}$ such that*
(1) $d(b,\tilde{b}) \le C(b^*)\epsilon$ and $|\tilde{v}_h - v_h| \le C(b^*)\epsilon|v_h|$ for all $h \in \hat{I}$;
(2) if $q_v(z) \in \Sigma_{\mathcal{T},i}$, $r_{b,h}(q_v z) \ge 3|v_h|^{\frac{1}{2}}$ for all $h \in \hat{I}-H$ such that $\iota_h = i$ and $\big|q_S^{-1}(q_v z)\big| \ge 3|v_i|^{\frac{1}{2}}$ if $i \in \hat{I}-H$, then $d_b\big(q_v z, q_{\tilde{v}}(z)\big) \le \epsilon$.

Proof: If $S=S^2$, the tuples $\tilde{b}$ and $\tilde{v}$ constructed in the first half of the proof of Lemma 4.12 satisfy the requirements of the corollary. In fact, $d_b\big(q_v(z), q_{\tilde{v}}(z)\big)=0$ if z is as in (2) above. If $S=\Sigma$, let

$$\tilde{\epsilon} = \epsilon^2 \prod_{h\in[I]-H} |v_h|_b^2 > 0.$$

If $C(b^*)\delta(b^*)$ is sufficiently small, by repeated applications of Lemma 4.12, we can replace the tuples b and v by $b' \in \mathcal{M}^{(H)}_{\mathcal{T},\tilde{\epsilon}}$ and $v'=(b',v'_{\hat{I}}) \in F^{(H)}\mathcal{T}$ such that
(1) $d(b,b') \le C'(b^*)\epsilon$ and $|v'_h - v_h| \le C'(b^*)\epsilon|v_h|_b$ for all $h \in \hat{I}$;
(2) if $q_v(z) \in \Sigma_{\mathcal{T},i}$, $r_{b,h}\big(q_v z\big) \ge \frac{5}{2}|v_h|^{\frac{1}{2}}$ for all $h \in \hat{I}-H$ such that $\iota_h = i$ and $\big|q_S^{-1}(q_v z)\big| \ge \frac{5}{2}|v_i|^{\frac{1}{2}}$ if $i \in \hat{I}-H$, then $d_b\big(q_v z, q_{\tilde{v}} z\big) \le 2\delta(b)\epsilon$.
Applying the construction of the first half of the proof of Lemma 4.12 to the tuples b' and v', we obtain tuples $\tilde{b} \in \mathcal{M}^{(0)}_{\mathcal{T}}$ and $\tilde{v}=(\tilde{b},\tilde{v}_{\hat{I}}) \in F^{(H)}_{\tilde{b}}\mathcal{T}$ such that

$$d(b',\tilde{b}) \le C(b^*)\tilde{\epsilon} \quad \text{and} \quad |\tilde{v}_h - v'_h| \le C(b^*)\tilde{\epsilon}|v'_h|_{b'} \quad \forall h \in \hat{I}.$$

Then if z is as in the requirement (2) of the corollary,

$$d_{b'}\big(q_{v'}(z), q_{\tilde{v}}(z)\big) \le C(b^*)\tilde{\epsilon}\Big(\prod_{h\in\hat{I}-H} |v'_h|\Big)^{-1} \le \epsilon^2$$

if δ is sufficiently small. Thus, the tuples $\tilde{b}$ and $\tilde{\upsilon}$ satisfy both requirements of the corollary.

4.4 Gromov Convergence and Norms of the Differential

Let $b_k = \big(S, M, I; x, (j, y_k), u_k\big)$ be a sequence of smooth maps converging to

$$b^* = \big(S, M, I^*; x^*, (j^*, y^*), u^*\big) \in \mathcal{M}^{(0)}_{\mathcal{T}^*}$$

with respect to the Gromov topology such that $\bar{\partial} u_{k,\hat{0}} = t_k \nu$ with $t_k \longrightarrow 0$ and $\bar{\partial} u_{k,h} = 0$ if $h \in \hat{I}$. We assume that $\mathcal{T}^*$ is a simple bubble type. In the next section, it is proved that b_k lies in the image of the gluing map $\tilde{\gamma}_{\mathcal{T}, t_k \nu}$ for some k. In this section, we show the differentials of $du_{k,i}$ satisfy a certain condition which holds for all bubble maps in the image of $\tilde{\gamma}_{\mathcal{T}, t_k \nu}$.

By definition of convergence, for all k sufficiently large, we can choose
(a) curves $\mathcal{C}_k = \big(S, M, I^*; x'_k, (j^*, y^*)\big)$ with $\lim_{k \longrightarrow \infty} x'_{k,h} = x^*_h$ for all $h \in \hat{I}^*$, and
(b) vectors $v_k \in F^{(0)}_{\mathcal{C}_k}$ with $16|v_k|_{g_b} \le r_{\mathcal{C}_k} g_b$,
such that $\lim_{k \longrightarrow \infty} |v_k| = 0$, $\mathcal{C}(v_k) = \big(S, M, I; x_k, (j_k, y(v_k))\big)$, and

$$\lim_{k \longrightarrow \infty} \sup_{z \in \Sigma_{\mathcal{C}(v_k)}} d_V\big(u_{b^*}(q_{\upsilon_k}(z)), u_{b_k}(z)\big) = 0, \qquad \lim_{k \longrightarrow \infty} q_{\upsilon_k}(j_{k,l}, y_{k,l}) = (j^*_l, y^*_l) \ \ \forall l \in M,$$

where $\upsilon_k = (\mathcal{C}_k, v_k)$ and g_b denotes the standard metric on $\Sigma_{\mathcal{C}_k}$ if $S = S^2$. Let

$$\phi_{k,h} = \begin{cases} \phi_{x'_{k,h}}, & \text{if } x'_{k,h} \in S^2; \\ \phi_{g_{b,\hat{0}}, x'_{k,h}}, & \text{if } x'_{k,h} \in \Sigma; \end{cases} \qquad r_{k,h} = \begin{cases} r_{x'_{k,h}}, & \text{if } x'_{k,h} \in S^2; \\ r_{g_{b,\hat{0}}, x'_{k,h}}, & \text{if } x'_{k,h} \in \Sigma. \end{cases}$$

Let g_{υ_k} be the metric on $\Sigma_{b_k} = \Sigma_{\upsilon_k}$ defined as in Section 3.3, using the metric $g_{b,\hat{0}}$ on Σ if $S = \Sigma$.

For any element in the image $\tilde{\gamma}_{\mathcal{T}, t\nu}$ that lies near b^*, the modified (L^p, g_{υ_k})-norm of $d\tilde{u}_\upsilon$ is bounded above by a constant dependent only on b^*. Furthermore, as $\upsilon \longrightarrow 0$ and the size of the necks is reduced, the modified (L^p, g_υ)-norm of $d\tilde{u}_\upsilon$ on such necks tends to zero. The modified (L^p, g_υ)-norm is bounded above by the usual (L^{2p}, g_υ)-norm times some constant dependent only on b^*. In this section, we show that the (L^{2p}, g_{υ_k})-norm of du_{b_k} is uniformly bounded and tends to zero on the "necks." Instead of using our usual cutoff function β, we will use the family of cutoff functions provided by the following lemma. The proof can be found in [MS, p166]. The statement below is somewhat sharper than in [MS], but the proof of [MS] suffices.

Lemma 4.14 *For every $\epsilon > 0$, there exists a smooth function $\tilde{\beta}_\epsilon \colon \mathbb{R} \longrightarrow [0,1]$ such that*

$$\int_{\mathbb{C}} |\tilde{\beta}'_\epsilon(r)|^2 r dr d\theta \le 8\epsilon, \quad \textit{and} \quad \tilde{\beta}_\epsilon(r) = \begin{cases} 1, & \textit{if } r \ge 1; \\ 0, & \textit{if } r \le e^{-1/\epsilon}. \end{cases}$$

Given $r > 0$, we denote by $\tilde{\beta}_{\epsilon,r}$ the cutoff function defined by $\tilde{\beta}_{\epsilon,r}(t) = \tilde{\beta}_\epsilon(r^{-\frac{1}{2}} t)$.

We now define nearly holomorphic maps $f_{k,i}\in C^\infty(\Sigma_{C_k,i};V)$. In order to simplify computations, we fix a finite family of J-invariant metrics on V such that for some fixed $\varepsilon>0$ and for every $q\in V$ there exists a metric $g_{V,q}$ in this family such that $\big(B_{g_{V,q}}(q,\varepsilon),J,g_{V,q}\big)$ is isomorphic to a ball in $\mathbb{C}^n$. Since V is compact and the family of metrics is finite, all estimates below that depend on a particular metric $g_{V,q}$ will involve bounds dependent only on V. We denote by $\exp_q$ the exponential map of (the Levi-Civita connection of) the metric $g_{V,q}$ and by $B_q(\epsilon)$ the $g_{V,q}$-geodesic ball about q of radius ϵ. If $\delta>0$ and $h\in I^*-I$, let

$$\begin{aligned} B^+_{h,k}(\delta) &= \big\{(\iota^*_h,z)\in\Sigma_{C_k,\iota^*_h}: r_{k,h}(\iota^*_h,z)\le\delta\big\},\\ B^-_{h,k}(\delta) &= \big\{(h,z)\in\Sigma_{C_k,h}: |q_S^{-1}(z)|\le\delta\big\}. \end{aligned} \tag{4.27}$$

Choose a sequence $\epsilon_k\in\mathbb{R}^+$ converging to zero. Let $r_k=\Big(2\sum_{i\in I^*}\|du^*_i\|_{b^*,C^2}\Big)^{-1}\epsilon_k$. By taking a subsequence if necessary, it can be assumed that

$$\begin{aligned} |t_k|\le\epsilon_k,\qquad & d_V\big(u_{b^*}(q_{v_k}(z)),u_{b_k}(z)\big)\le\epsilon_k\quad\forall z\in\Sigma_{b_k},\\ r_{b^*,h}(\iota^*_h,x'_{k,h})\le r_k,\qquad & e^{\frac{2p}{\epsilon_k}}|v_{k,h}|^{\frac12}_{b^*}\le r_k. \end{aligned} \tag{4.28}$$

Let $q_h=u^*_h(\infty)$ and

$$\tilde A^\pm_{h,k}=B^\pm_{h,k}\Big(|v_{k,h}|^{\frac12}_{b^*}\Big)-B^\pm_{h,k}\Big(e^{-\frac{1}{\epsilon_k}}|v_{k,h}|^{\frac12}_{b^*}\Big).$$

By (4.28), $u_{b_k}\Big(q^{-1}_{v_k}\big(B^\pm_{h,k}(e^{\frac{1}{\epsilon_k}}|v_{k,h}|^{\frac12}_{b})\big)\Big)\subset B_{q_h}(C(b^*)\epsilon_k)$. We define $\xi^\pm_{k,h}\in C^\infty\big(\tilde A^\pm_{h,k};T_{q_h}V\big)$ by

$$\begin{aligned} \exp_{q_h,q_h}\xi^+_{k,h}(z) &= u_{b_k}\big(q^{-1}_{v_k,\iota^*_h}(\iota^*_h,z)\big), & |\xi^+_{k,h}(z)|_{g_{V,q_h}}<\varepsilon;\\ \exp_{q_h,q_h}\xi^-_{k,h}(z) &= u_{b_k}\big(q^{-1}_{v_k,\iota^*_h}(\iota^*_h,\phi^{-1}_{k,h}(zv_{k,h}))\big), & |\xi^-_{k,h}(z)|_{g_{V,q_h}}<\varepsilon, \end{aligned} \tag{4.29}$$

provided k is sufficiently large (depending on b^*). Let $\bar\xi^\pm_{k,h}\in T_{q_h}V$ be given by

$$\bar\xi^\pm_{k,h}=\frac{1}{\mathrm{Area}(\tilde A^\pm_{k,h})}\int_{\tilde A^\pm_{k,h}}\xi^\pm_{k,h}, \tag{4.30}$$

where the area and the integral are computed using the metric g_{b^*,ι^*_h} on Σ_{b^*,ι^*_h} and $g_{b^*,h}$ on $\Sigma_{b^*,h}$. Define $f_{k,i}\in C^\infty(\Sigma_{b^*,i};V)$ by

$$f_{k,i}(z)=\begin{cases} \exp_{q_h,q_h}\Big\{\bar\xi^+_{k,h}+\tilde\beta_{\epsilon_k,|v_{k,h}|_{b^*}}\big(r_{k,h}(z)\big)\big(\xi^+_{k,h}(z)-\bar\xi^+_{k,h}\big)\Big\}, & \text{if } r_{k,h}(z)\le|v_{k,h}|^{\frac12}_{b^*};\\ \exp_{q_i,q_i}\Big\{\bar\xi^-_{k,i}+\tilde\beta_{\epsilon_k,|v_{k,i}|_{b^*}}\big(|q_S^{-1}(z)|\big)\big(\xi^-_{k,i}(z)-\bar\xi^-_{k,i}\big)\Big\}, & \text{if } i\in I^*-I\ \&|q_S^{-1}(z)|\le|v_{k,i}|^{\frac12}_{b^*}\\ u_{b_k}\big(q^{-1}_{v_k}(i,z)\big), & \text{otherwise.} \end{cases}$$

Let $\zeta'_{k,i}\in\Gamma(u^*_i)$ be given by

$$\exp_{b^*,u^*_i}\zeta'_{k,i}=f_{k,i},\qquad\|\zeta'_{k,i}\|_{b^*,C^0}<\mathrm{inj}\,g_{V,b^*}.$$

Lemma 4.15 *There exists $C>0$ such that for all k sufficiently large and $i\in I^*$,*

$$\|\zeta'_{k,i}\|_{b^*,C^0}\le C\epsilon_k,\qquad\|\bar\partial f_{k,i}\|_{g_{b^*,i},2p}\le C\epsilon_k^{\frac{1}{2p}}\big(\|df_{k,i}\|_{g_{b^*,i},2p}+1\big).$$

Proof: The first statement is clear from (4.28) and the construction of $f_{k,i}$ above. Suppose $z \in \Sigma_{b^*,i}$. If $z \notin B^+_{h,k}\big(|v_{k,h}|^{\frac{1}{2}}_{b^*}\big)$ for all $h \in I^* - I$ and $z \notin B^-_{i,k}\big(|v_{k,i}|^{\frac{1}{2}}_{b^*}\big)$ if $i \in I^* - I$, then

$$|\bar{\partial} f_{k,i}|_{g_{b^*,i},z} \leq C_\nu t_k \leq C_\nu \epsilon_k. \tag{4.31}$$

Suppose $z \in \tilde{A}^+_{h,k}$ with $h \in I^* - I$. Since the metric $g_{\mathbb{P}^n,q_h}$ is flat near q_h,

$$\bar{\partial} f_{k,i}\big|_z = d\exp_{q_h,q_h} \bar{\partial}\Big\{\bar{\xi}^+_{k,h} + \tilde{\beta}_{\epsilon_k,|v_{k,h}|_{b^*}}\left(r_{k,h}(\cdot)\right)\left(\xi^+_{k,h} - \bar{\xi}^+_{k,h}\right)\Big\}_z. \tag{4.32}$$

It follows from (4.32) that

$$|\bar{\partial} f_{k,i}|_{g_{b^*,i},z} \leq C\Big(|v_{k,h}|^{-\frac{1}{2}}_{b^*}|d\tilde{\beta}_\epsilon|_{|v_{k,h}|^{-\frac{1}{2}}_{b^*} r_{k,h}(z)}\big|\xi^+_{k,h} - \bar{\xi}^+_{k,h}\big|_z + \big|\bar{\partial}\xi^+_{k,h}\big|_z\Big) \tag{4.33}$$

By Poincare Lemma, see Lemma A.6 applied with $r = |v_{h,k}|^{-\frac{1}{2}}_{b^*}$ and $2p$ instead of p,

$$\begin{aligned}\Big\|\,|v_{k,h}|^{-\frac{1}{2}}_{b^*}|d\tilde{\beta}_{\epsilon_k}|_{|v_{k,h}|^{-\frac{1}{2}}_{b^*} r_{k,h}(\cdot)}\big|\xi^+_{k,h} - \bar{\xi}^+_{k,h}\big|\Big\|_{g_{b^*,i},L^{2p}(\tilde{A}^+_{h,k})} &\leq C|v_{k,h}|^{-\frac{p-1}{2p}}_{b^*}\|d\tilde{\beta}_{\epsilon_k}\|^{\frac{1}{p}}_{2}\big\|\xi^+_{k,h} - \bar{\xi}^+_{k,h}\big\|_{b^*,C^0}\\ &\leq C'\epsilon_k^{\frac{1}{2p}}\big\|d\xi^+_{k,h}\big\|_{g_{b^*,i},L^{2p}(\tilde{A}^+_{h,k})} \leq C'\epsilon_k^{\frac{1}{2p}}\|df_{k,h}\|_{g_{b^*,i},2p}.\end{aligned}$$

The last two equations give

$$\big\|\bar{\partial} f_{k,i}\big\|_{g_{b^*,i},L^{2p}(\tilde{A}^+_{h,k})} \leq C\big(\epsilon_k^{\frac{1}{2p}}\,\big\|df^+_{k,h}\big\|_{g_{b^*,i},2p} + \epsilon_k\big). \tag{4.34}$$

The same estimate applies to $\big\|\bar{\partial} f_{k,i}\big\|_{g_{b^*,i},L^{2p}(\tilde{A}^-_{i,k})}$ if $i \in I^* - I$. Here the exponent of $\frac{2p}{\epsilon_k}$ in (4.28) is crucial:

$$\begin{aligned}\|\bar{\partial}\xi^-_{k,i}\|^{2p}_{g_{b^*,i},L^{2p}(\tilde{A}^-_{i,k})} &\leq \int_{|v_{k,i}|^{-\frac{1}{2}}_{b^*} \leq r \leq e^{\frac{1}{\epsilon_k}}|v_i|^{-\frac{1}{2}}_{b^*}} t_k^{2p}|\nu \circ dq^{-1}_{v_k,\iota^*_i}|^{2p}_{g_{b^*,\iota^*_i}}|v_{k,i}|^{2p}_{b^*}(1+r^2)^{2p-2}r\,dr\,d\theta\\ &\leq Ct_k^{2p}|v_{k,i}|^{2p}_{b^*}\big(|v_{k,i}|^{-\frac{1}{2}}e^{\frac{1}{\epsilon_k}}\big)^{4p-2} \leq Ct_k^{2p}.\end{aligned} \tag{4.35}$$

Since $f_{k,i}$ is constant on $B^+_{h,k}\big(e^{-\frac{1}{\epsilon_k}}|v_{k,h}|^{\frac{1}{2}}_{b^*}\big)$ for $h \in I^* - I$ with $\iota^*_h = i$ and on $B^-_{i,k}\big(e^{-\frac{1}{\epsilon_k}}|v_{k,i}|^{\frac{1}{2}}_{b^*}\big)$ if $i \in I^* - I$, the second claim is proved.

Corollary 4.16 *There exists $C > 0$ such that for all k sufficiently large,*

$$\|df_{k,i}\|_{g_{b^*,i},2p} \leq C \quad \textit{and} \quad \|\zeta'_{k,i}\|_{g_{b^*,i},2p,1} \leq C\epsilon_k^{\frac{1}{2p}}.$$

Proof: By the quadratic expansion of $\bar{\partial}_{u^*_i}\zeta'_{k,i}$ as in Section 3.6,

$$D_{b^*,u^*_i}\zeta'_{k,i} + N_{\bar{\partial},u^*_i}\zeta'_{k,i} = \bar{\partial}_{u^*_i}\zeta'_{k,i}, \tag{4.36}$$

where

$$\|\bar{\partial}_{u^*_i}\zeta'_{k,i}\|_{g_{b^*,i},2p} \leq C\epsilon_k^{\frac{1}{2p}}\left(\|df_{k,i}\|_{g_{b^*,i},2p} + 1\right) \tag{4.37}$$

by Lemma 4.15 and

$$\|N_{\bar\partial,u_i^*}\zeta'_{k,i}\|_{g_{b^*,i},2p} \le C\|\zeta'_{k,i}\|_{C^0}\|\zeta'_{k,i}\|_{g_{b^*,i},2p,1} \le C\epsilon_k\|\zeta'_{k,i}\|_{g_{b^*,i},2p,1}, \tag{4.38}$$

by Proposition A.11 and Lemma 4.15. Thus, by standard elliptic estimates for u_{b^*} and (4.36)-(4.38),

$$\begin{aligned}\|\zeta'_{k,i}\|_{g_{b^*,i},2p,1} &\le C\big(\|D_{b^*,u_i}\zeta'_{k,i}\|_{g_{b^*,i},2p} + \|\zeta'_{k,i}\|_{g_{b^*,i},2p}\big)\\ &\le C'\epsilon_k^{\frac{1}{2p}}\big(\|\zeta'_{k,i}\|_{g_{b^*,i},2p,1} + \|df_{k,i}\|_{g_{b^*,i},2p} + 1\big).\end{aligned} \tag{4.39}$$

On the other hand, since $f_{k,i}=\exp_{b^*,u_i^*}\zeta'_{k,i}$,

$$\|df_{k,i}\|_{g_{b^*,i},2p} \le C\big(\|du_i^*\|_{g_{b^*,i},2p} + \|\zeta'_{k,i}\|_{g_{b^*,i},2p,1}\big). \tag{4.40}$$

If ϵ_k is sufficiently small, the claim follows from equations (4.39) and (4.40).

Corollary 4.17 *There exists $C>0$ such that for all k sufficiently large, $h\in\hat I^*$, and $\delta>0$,*

$$\|du_{b_k}\|_{g_{v_k},L^{2p}(q_{v_k}^{-1}(B^{\pm}_{h,k}(\delta)))} \le C\big(\epsilon_k^{\frac{1}{2p}} + \delta^{\frac{1}{p}}\big).$$

Proof: If $h\in\hat I$, the statement is immediate from Corollary 4.16; so we assume $h\in I^*-I$. The metric g_v on $q_{v_k}^{-1}(B^+_{h,k}(\delta))$ differs by a bounded factor from the metric $q^*_{v_k,\iota_h^*}g_{b^*,i}$. Thus,

$$\begin{aligned}\big\|du_{b_k}\big\|_{g_{v_k},L^{2p}\left(q_{v_k}^{-1}(B^+_{h,k}(\delta))\right)} &\le C\big\|d(f_k\circ q^{-1}_{v_k,\iota_h^*})\big\|_{g_{b^*,\iota_h^*},L^{2p}\left(B^+_{h,k}(\delta)-B^+_{h,k}(|v_{k,h}|^{\frac{1}{2}}_{b^*})\right)}\\ &= C\big\|df_{k,\iota_h^*}\big\|_{g_{b^*,\iota_h^*},L^{2p}\left(B^+_{h,k}(\delta)-B^+_{h,k}(|v_{k,h}|^{\frac{1}{2}}_{b^*})\right)}\\ &\le C\big\|df_{k,\iota_h^*}\big\|_{g_{b^*,\iota_h^*},L^{2p}(B^+_{h,k}(\delta))}.\end{aligned} \tag{4.41}$$

Since $f_{k,\iota_h^*}=\exp_{b^*,u_{\iota_h^*}}\zeta'_{k,\iota_h^*}$, by Corollary 4.16,

$$\begin{aligned}\big\|df_{k,\iota_h^*}\big\|_{g_{b^*,\iota_h^*},L^{2p}(B^+_{h,k}(\delta))} &\le C\Big(\big\|du^*_{\iota_h^*}\big\|_{g_{b^*,\iota_h^*},L^{2p}(B^+_{h,k}(\delta))} + \big\|\zeta'_{k,\iota_h^*}\big\|_{g_{b^*,\iota_h^*},2p,1}\Big)\\ &\le C'\big(\delta^{\frac{1}{p}} + \epsilon^{\frac{1}{2p}}\big).\end{aligned} \tag{4.42}$$

The claim for $B^+_{h,k}(\delta)$ follows from (4.41) and (4.42). The metric g_v on $q_{v_k}^{-1}(B^-_{h,k}(\delta))$ differs by a bounded factor from the metric which is the pullback of the metric $g_{b^*,h}$ by the map

$$z\longrightarrow q_N\Big(\frac{\phi_{k,h}q_{v_k,\iota_h^*}(z)}{v_{k,h}}\Big).$$

Thus, similarly to the above,

$$\big\|du_{b_k}\big\|_{g_{v_k},L^{2p}\left(q_{v_k}^{-1}(B^-_{h,k}(\delta))\right)} \le C\big\|df_{k,h}\big\|_{g_{b^*,h},L^{2p}(B^-_{h,k}(\delta))}; \tag{4.43}$$

$$\|df_{k,h}\|_{g_{b^*,h},L^{2p}(B^-_{h,k}(\delta))} \le C\big(\delta^{\frac{1}{p}} + \epsilon^{\frac{1}{2p}}\big). \tag{4.44}$$

The claim for $B^-_{h,k}(\delta)$ follows from (4.43) and (4.44).

4.5 Surjectivity of the Gluing Map

We continue with the notation of Section 4.4. In this section, for k sufficiently large, we use Corollary 4.13 to construct

$$\tilde{\upsilon}_k = \big(\tilde{b}_k, (\tilde{\upsilon}_k)_{\hat{I}^*}\big) \in F^{(0)}\mathcal{T}_\delta$$

and $\tilde{\zeta}_k \in \Gamma(u_{\tilde{\upsilon}_k})$ such that $\tilde{b}_k$ is very close to b in $\mathcal{M}_{\mathcal{T}}^{(0)}$, $\|\tilde{\zeta}_k\|_{\tilde{\upsilon}_k,p,1}$ is small, and $u_{b_k} = \exp_{\tilde{\upsilon}_k} \tilde{\zeta}_k$. We then look at the elements of $F^{(0)}\mathcal{T}_\delta$ near $\tilde{\upsilon}_k$ to find $\tilde{\upsilon}'_k$ and $\tilde{\zeta}_k \in \tilde{\Gamma}_+(\tilde{\upsilon}'_k)$ such that $u_{b_k} = \exp_{\tilde{\upsilon}'_k} \tilde{\zeta}'_k$. If $\mathcal{T}$ is semiregular, we consider only the case $\hat{I} = \emptyset$; if $\mathcal{T}$ is regular, we assume $t = 0$.

Let $H = \hat{I} \subset \hat{I}^*$. If $\delta > 0$ and $i \in I^*$, put

$$\Sigma_{i,\delta} = \big\{(i,z) \in \Sigma_{b^*,i} : r_{b^*,h}(i,z) \leq \delta \ \forall h \in \hat{I}^* - H \text{ s.t. } \iota_h = i,\ |q_S^{-1}(z)| \geq \delta \text{ if } i \in \hat{I}^* - H\big\}.$$

In addition to (4.28), we can assume that our sequence satisfies

$$\|\zeta_{k,i}\|_{g_{b^*,i},C^2(\Sigma_{i,r_k})} \leq \epsilon_k. \tag{4.45}$$

Let $b'_k = \big(S, M, I^*; x'_k, (j^*, y^*), u^*\big)$. By the second assumption in (4.28),

$$d(b^*, b'_k) \leq C\epsilon_k \Longrightarrow b'_k \in \mathcal{M}_{\mathcal{T}^*, C\epsilon_k}^{(H)},$$

since $b^* \in \mathcal{M}_{\mathcal{T}^*}^{(0)}$, where $C > 0$ depends only on b^*. By the last assumption of (4.28), $|\upsilon_k|_{b'_k} \leq C\epsilon_k$. Thus, if $\epsilon_k > 0$ is sufficiently small, by Corollary 4.13, there exist

$$\tilde{b}_k \in \mathcal{M}_{\mathcal{T}^*}^{(H)} \quad \text{and} \quad \tilde{\upsilon}_k = \big(\tilde{b}_k, (\tilde{\upsilon}_k)_{\hat{I}^*}\big) \in F^{(H)}\mathcal{T}^*$$

such that
(1) $d(b, \tilde{b}) \leq C'\epsilon_k$ and $|\tilde{\upsilon}_{k,h} - \upsilon_{k,h}| \leq C'\epsilon_k |\upsilon_{k,h}|_b$ for all $h \in \hat{I}^*$;
(2) if $q_{\upsilon_k}(z) \in \Sigma_{\mathcal{T}^*,i}$, $r_{b^*,h}\big(q_{\upsilon_k} z\big) \geq 3|\upsilon_{k,h}|^{\frac{1}{2}}$ for all $h \in \hat{I}^* - H$ such that $\iota^*_h = i$ and $\big|q_S^{-1}(q_{\upsilon_k} z)\big| \geq 3|\upsilon_{k,i}|^{\frac{1}{2}}$ if $i \in I^* - H$, then $d_b\big(q_{\upsilon_k} z, q_{\tilde{\upsilon}_k} z\big) \leq \epsilon_k$.
It then follows from the second and third assumptions of (4.28) that there exist $\tilde{\zeta}_k \in \Gamma(u_{\tilde{\upsilon}_k})$, $\tilde{w}_{k,h} \in T_{x_h(\tilde{\upsilon}_k)}\Sigma_{\tilde{\upsilon}_k,\iota_h}$ for $h \in H$, and $\tilde{w}_{k,l} \in T_{y_l(\tilde{\upsilon}_k)}\Sigma_{\tilde{\upsilon}_k,j_l}$ for $l \in M$ such that

$$\exp_{\tilde{\upsilon}_k} \tilde{\zeta}_k = u_{b_k}, \quad \exp_{g_{\tilde{\upsilon}_k}, x_{k,h}(\tilde{\upsilon}_k)} \tilde{w}_{k,h} = x_{k,h}, \quad \exp_{g_{\tilde{\upsilon}_k}, y_{k,l}(\tilde{\upsilon}_k)} \tilde{w}_{k,l} = y_{k,l};$$
$$\|\tilde{\zeta}_k\|_{b^*,C^0}, |\tilde{w}_{k,h}|_{g_{\tilde{\upsilon}_k}, x_{k,h}(\tilde{\upsilon}_k)} |\tilde{w}_{k,l}|_{g_{\tilde{\upsilon}_k}, y_{k,l}(\tilde{\upsilon}_k)} \leq C'\epsilon_k.$$

Lemma 4.18 *There exists $C > 0$ such that for all k,*

$$\|\tilde{\zeta}_k\|_{\tilde{\upsilon}_k,p,1} \leq C\epsilon_k^{\frac{1}{2p}}.$$

Proof: By (4.45), (1), and (2), $\|\tilde{\zeta}_k\|_{g_v,C^1} \leq C\epsilon_k$ outside of the necks

$$\tilde{A}_{k,h} = q_{\upsilon_k}^{-1}\Big(B^+_{k,h}(r_k) \cup B^-_{k,h}(r_k)\Big).$$

On the other hand, $\|du_{\tilde{v}_k}\|_{\tilde{v}_k,C^0} \le C$ by Lemma 3.5 and

$$\|du_{\tilde{v}_k}\|_{\tilde{v}_k,L^p(\tilde{A}_{k,h})} \le C(\epsilon_k^{\frac{1}{2p}} + r_k^{\frac{1}{p}}) \le C'\epsilon_k^{\frac{1}{2p}}$$

by Corollary 4.17. The three estimates imply the claim.

Suppose $\hat{I}=\emptyset$ and thus $H=\emptyset$. If k is sufficiently large and $\varpi \in T_{\tilde{v}_k}F^{\emptyset}\mathcal{T}^*$ is such that $2\|\varpi\|_{\tilde{v}_k} < \delta(b^*)$, where δ is as in Lemmas 3.6 and 4.6, let

$$\tilde{b}_k(\varpi) = \tilde{b}_{t\nu}(\tilde{v}_k(\varpi)) = \big(S, M, \{\hat{0}\};, (\hat{0}, \bar{y}(\varpi)), \tilde{u}_{\varpi,t\nu}\big)$$

be the tuple defined as in Sections 3.4 and 3.6. Let $\tilde{\zeta}_k(\varpi) \in \Gamma(\tilde{u}_{\varpi,t\nu})$ and $\tilde{w}_{k,l}(\varpi) \in T_{y_l(\varpi)}\Sigma_{\varpi,j_l}$ for $l \in M$ be given by

$$\begin{aligned} &\exp_{\tilde{v}_k(\varpi)} \tilde{\zeta}_k(\varpi) = u_{b_k}, && \|\tilde{\zeta}_k(\varpi)\|_{b^*,C^0} \le 2C'\epsilon_k; \\ &\exp_{g_{\tilde{v}_k(\varpi)},y_{k,l}(\varpi)} \tilde{w}_{k,l}(\varpi) = y_{k,l}, && |\tilde{w}_{k,l}(\varpi)|_{g_{\tilde{v}_k},y_{k,l}(\varpi)} \le 2C'\epsilon_k. \end{aligned}$$

We need to find ϖ such that $\tilde{\pi}_{\varpi,-}\tilde{\zeta}_k(\varpi)=0$ and $y_l(\varpi)=y_{k,l}$, or equivalently

$$S_\varpi \tilde{\pi}_{\varpi,-}\tilde{\zeta}_k(\varpi) = 0 \quad \text{and} \quad S_\varpi \tilde{w}_{k,l}(\varpi) = 0, \tag{4.46}$$

where $S_\varpi \tilde{w}_{k,l}(\varpi)$ denotes the parallel transport of $\tilde{w}_{k,l}(\varpi)$ back to $y_l(\tilde{v}_k)$ along the g_v-geodesic

$$s \longrightarrow \exp_{y_l(\tilde{v}_k)} s w_l(\varpi).$$

Lemma 4.19 *There exists $C>0$ such that for all k sufficiently large and $\varpi, \varpi' \in T_{\tilde{v}_k}F^{(\emptyset)}\mathcal{T}^*$ with $2\|\varpi\|_{\tilde{v}_k} < \delta(b^*)$,*

$$\begin{aligned} S_\varpi \tilde{\pi}_{\varpi,-}\tilde{\zeta}_k(\varpi) &= \tilde{\pi}_{v,-}\tilde{\zeta}_k + \tilde{N}^{(0)}(\tilde{\zeta}_k, \varpi) - \tilde{\pi}_{v,-}\zeta_\varpi + N^{(0)}(\varpi), \\ S_\varpi \tilde{w}_{k,l}(\varpi) &= \tilde{w}_{k,l} + \tilde{N}^{(l)}(\tilde{w}_{k,l}, \varpi) - w_l(\varpi) + N^{(l)}(\varpi) \quad \forall l \in M, \end{aligned}$$

where ζ_ϖ is as in Section 3.4 and $\tilde{N}^{(l)}$ and $N^{(l)}$ satisfy

$$\begin{aligned} &\big\|\tilde{N}^{(0)}(\tilde{\zeta}_k, \varpi) - \tilde{N}^{(0)}(\tilde{\zeta}_k, \varpi')\big\|_{\tilde{v}_k,2} \le C\|\tilde{\zeta}_k\|_{\tilde{v}_k,2}\|\varpi - \varpi'\|_{\tilde{v}_k}; \\ &\big|\tilde{N}^{(l)}(\varpi, \tilde{w}_{k,l}) - \tilde{N}^{(l)}(\varpi', \tilde{w}_{k,l})\big|_{g_{\tilde{v}_k},y_l(\tilde{v}_k)} \le C\|\tilde{w}_{k,l}\|_{g_{v_k},y_l(\tilde{v}_k)}\|\varpi - \varpi'\|_{\tilde{v}_k} \quad \forall l \in M; \\ &\|N^{(0)}(\varpi) - N^{(0)}(\varpi')\|_{\tilde{v}_k,2} \le C\left(\|\varpi\|_{\tilde{v}_k} + \|\varpi'\|_{\tilde{v}_k}\right)\|\varpi - \varpi'\|_{\tilde{v}_k}; \\ &\big|N^{(l)}(\varpi) - N^{(l)}(\varpi')\big|_{g_{\tilde{v}_k},y_l(\tilde{v}_k)} \le C\big(\|\varpi\|_{\tilde{v}_k} + \|\varpi'\|_{\tilde{v}_k}\big)\|\varpi - \varpi'\|_{\tilde{v}_k} \quad \forall l \in M. \end{aligned} \tag{4.47}$$

Proof: This lemma follows from a pointwise estimate on $S_\varpi\tilde{\zeta}_k(\varpi) - (\tilde{\zeta}_k - \tilde{\zeta}_\varpi)$ and the fact that all statements in Lemmas 3.6 and 4.6 can be written in a form similar to (4.47), e.g. for all $\xi \in \Gamma(\tilde{v}_k)$

$$\big\|S_\varpi \tilde{\pi}_{\varpi,-}R_\varpi\xi - S_{\varpi'}\tilde{\pi}_{\varpi',-}R_{\varpi'}\xi\big\|_{\tilde{v}_k,2} \le C\|\varpi - \varpi'\|_{\tilde{v}_k,2}\|\xi\|_{\tilde{v}_k,2}.$$

The latter fact can be seen from the two lemmas and the definitions of R_ϖ and S_ϖ in Section 3.4.

Lemma 4.20 *There exist* $C,\delta\in C^\infty(\mathcal{M}^{(0)}_{\mathcal{T}^*},\mathbb{R}^+)$ *such that for all* $\upsilon\in F^{(H)}\mathcal{T}^*_\delta$ *and* $\varpi\in T_\upsilon F^{(H)}\mathcal{T}^*_\delta$ *with* $\|\varpi\|_{\tilde{\upsilon}_k}\leq\delta(b)$,

$$\|\zeta_\varpi\|_{\upsilon,2}\leq C(b)\|\tilde{\pi}_{\upsilon,-}\zeta_\varpi\|_{\upsilon,2}.$$

Proof: It can be seen directly from the definitions that

$$\|\zeta_\varpi\|_{\upsilon,2}\leq\big(1+C(b_\upsilon)|\upsilon|\big)\|\pi_{\upsilon,-}\zeta_\varpi\|_{\upsilon,2}.$$

The claim then follows from the proof of (2b) of Lemma 3.12.

Corollary 4.21 *There exist a neighborhood* U *of* b^* *in* $\mathcal{M}^{(0)}_{\mathcal{T}^*}$ *and* $\delta,\epsilon>0$ *such that for all* $\upsilon\in F^{(\emptyset)}\mathcal{T}^*_\delta|U$, $\xi\in\tilde{\Gamma}_-(\upsilon)$ *with* $\|\xi\|_{\upsilon,2}<\delta$, *and* $w_l\in T_{y_l(\upsilon)}\Sigma_{\upsilon,j_l}$ *for* $l\in M$ *with* $|w_l|_{g_\upsilon,y_l(\upsilon)}<\delta$, *the system of equations*

$$\tilde{\pi}_{\upsilon,-}\zeta_\varpi-N^{(0)}(\varpi)=\xi,\qquad w_l(\varpi)-N^{(l)}(\varpi)=w_l\quad\forall l\in M,$$

has a (unique) solution $\varpi\in T_\upsilon F^{(\emptyset)}\mathcal{T}$ *with* $\|\varpi\|_\upsilon<\epsilon$.

Proof: By Lemmas 3.6 and 4.20,

$$C^{-1}\|\varpi\|_\upsilon\leq\|\tilde{\pi}_{\upsilon,-}\xi\|_{\upsilon,2}+\sum_{l\in M}|w_l(\varpi)|_{g_\upsilon,y_l(\upsilon)}\leq C\|\varpi\|_\upsilon.$$

whenever b_υ lies near b^*. Thus, the claim follows from (4.47) by the usual contraction-principle argument.

Corollary 4.22 *Let* $\mathcal{T}^*=(S,M,I^*;j^*,\underline{\lambda}^*)$ *be a simple bubble type. If* $\mathcal{T}^*$ *is regular, the map*

$$\tilde{\gamma}_{\mathcal{T}^*}:F\mathcal{T}^*_\delta\longrightarrow\bar{\mathcal{M}}_{\langle\mathcal{T}^*\rangle}$$

contains a neighborhood of $\mathcal{M}_{\mathcal{T}^*}$ *in* $\bar{\mathcal{M}}_{\langle\mathcal{T}^*\rangle}$. *If* $\mathcal{T}^*$ *is semiregular,* $H=\emptyset$, *and* k *is sufficiently large, there exists* $\tilde{\upsilon}_k\in F^{(\emptyset)}\mathcal{T}^*_\delta$ *such that* $b_k=\tilde{\gamma}_{\mathcal{T}^*,t_k\nu}(\tilde{\upsilon}_k)$.

Proof: The second statement is immediate from Lemmas 4.18 and 4.19 and Corollary 4.21. If $\mathcal{T}^*$ is regular, what we have shown is that the image of $\tilde{\gamma}_{\mathcal{T}^*}$ contains a neighborhood of $\mathcal{M}_{\mathcal{T}^*}$ in $\mathcal{M}_{\langle\mathcal{T}^*\rangle}\cup\mathcal{M}_{\mathcal{T}^*}$. Furthermore, there exists a sequence of neighborhoods $U_1\supset U_2\supset\ldots$ of b^* in $\bar{\mathcal{M}}_{\langle\mathcal{T}^*\rangle}$ such that $\bigcap U_k=\{[b^*]\}$. If $[b_k]\in\mathcal{M}_{\mathcal{T}}$ is a sequence of bubble maps converging to $[b^*]\in\mathcal{M}_{\mathcal{T}^*}$, it can be assumed that $[b_k]\in U_k$. By the above statement applied to $\mathcal{T}$, we can choose sequences

$$\{[b_{kr}]\}\subset\mathcal{M}_{\langle\mathcal{T}^*\rangle}=\mathcal{M}_{\langle\mathcal{T}\rangle}$$

such that for each fixed k the sequence $\{[b_{kr}]\}$ converges to $[b_k]$. Since U_k is an open neighborhood of $[b_k]$, it can be assumed that $[b_{kr}]\in U_k$ for all r. By the above, the image of $\tilde{\gamma}_{\mathcal{T}^*}|F\mathcal{T}^*_{\frac{1}{2}\delta}$ contains $U_k\cap\mathcal{M}_{\langle\mathcal{T}^*\rangle}$ if k is sufficiently large. Thus, for all r there exists $\upsilon_{kr}\in F\mathcal{T}^*_{\frac{1}{2}\delta}$ such that $\tilde{\gamma}_{\mathcal{T}^*}(\upsilon_{kr})=[b_{kr}]$. Let $\tilde{\upsilon}_k\in F\mathcal{T}^*_\delta$ be the limit of the sequence υ_{kr} with k fixed. Then, by continuity of the map $\tilde{\gamma}_{\mathcal{T}^*}$, see Corollary 4.5,

$$\tilde{\gamma}_{\mathcal{T}^*}(\tilde{\upsilon}_k)=\lim_{r\longrightarrow\infty}\tilde{\gamma}_{\mathcal{T}^*}(\tilde{\upsilon}_{kr})=\lim_{r\longrightarrow\infty}[b_{kr}]=[b_k].$$

Thus, the image of $\tilde{\gamma}_{\mathcal{T}^*}$ contains a neighborhood of $\mathcal{M}_{\mathcal{T}^*}$ in $\bar{\mathcal{M}}_{\langle\mathcal{T}^*\rangle}$.

Corollary 4.23 *If* $\mathcal{T}^* = (S, M, I^*; j^*, \underline{\lambda}^*)$ *is a simple regular bubble type, the map*

$$\tilde{\gamma}_{\mathcal{T}^*} : F\mathcal{T}^*_\delta \longrightarrow \bar{\mathcal{M}}_{\langle \mathcal{T}^* \rangle}$$

is a homeomorphism onto an open neighborhood of $\mathcal{M}_{\mathcal{T}^*}$ *in* $\bar{\mathcal{M}}_{\langle \mathcal{T}^* \rangle}$ *provided* $\delta \in C^\infty(\mathcal{M}_{\mathcal{T}^*}; \mathbb{R}^+)$ *is sufficiently small.*

Proof: By Corollaries 4.5, 4.10, 4.22, the map $\tilde{\gamma}_{\mathcal{T}^*} : F\mathcal{T}^*_\delta \longrightarrow \bar{\mathcal{M}}_{\langle \mathcal{T}^* \rangle}$ is a continuous bijection onto a neighborhood of $\mathcal{M}_{\mathcal{T}^*}$ in $\bar{\mathcal{M}}_{\langle \mathcal{T}^* \rangle}$. In addition, the proof of Corollary 4.22 shows that $\tilde{\gamma}_{\mathcal{T}^*}$ is an open map.

Chapter 5

Topological Tools

This chapter describes the topological tools that along with the gluing theorems of Chapter 3 lead to enumeration of curves of certain nature in $\mathbb{P}^2$ and $\mathbb{P}^3$ in Chapters 6, 9, and 10. Section 5.2 defines the notion of contribution to the euler class of a vector bundle from a (not necessarily closed) subset of the zero set of a section. In Section 5.3, we discuss how one can determine the number of zeros of an affine map between vector bundles. These concepts are closely intertwined, with the technical aspects of the link presented in Section 5.1. We give an example which is used in later computations.

5.1 Maps Between Vector Bundles

Throughout the rest of this thesis, all vector bundles are assumed to be complex and normed. Let $\mathcal{I}$ denote the unit interval $[0,1]$. If $\mathcal{Z}$ is a compact oriented zero-dimensional manifold, we denote the signed cardinality of $\mathcal{Z}$ by ${}^{\pm}|\mathcal{Z}|$.

Definition 5.1 *Suppose $F, \mathcal{O} \longrightarrow \mathcal{M}$ are smooth vector bundles.*
(1) If $F = \bigoplus_{i=1}^{i=k} F_i$, bundle map $\alpha: F \longrightarrow \mathcal{O}$ is a polynomial of degree $d_{[k]}$ if for each $i \in [k]$ there exists

$$p_i \in \Gamma(\mathcal{M}; F_i^{*\otimes d_i} \otimes \mathcal{O}) \quad \textit{for } i \in [k] \qquad \textit{s.t.} \qquad \alpha(v) = \sum_{i=1}^{i=k} p_i\big(v_i^{d_i}\big) \quad \forall v = (v_i)_{i\in[k]} \in \bigoplus_{i=1}^{i=k} F_i.$$

(2) If $\alpha: F \longrightarrow \mathcal{O}$ is a polynomial, the rank of α is the number

$$rk\ \alpha \equiv \max\{rk_b\alpha : b \in \mathcal{M}\}, \quad \textit{where} \quad rk_b\alpha = \dim_{\mathbb{C}}\big(Im\ \alpha_b\big).$$

Polynomial $\alpha: F \longrightarrow \mathcal{O}$ is of constant rank if $rk_b\alpha = rk\ \alpha$ for all $b \in \mathcal{M}$; α is nondegenerate if $rk_b\alpha = rk\ F$ for all $b \in \mathcal{M}$.
(3) If Ω is an open subset of $\mathcal{I}\times F$, and

$$\{\phi_t\} = \{\phi_t : \{v \in F : (t,v) \in \Omega\} \longrightarrow \mathcal{O}\}$$

is a family of smooth bundle maps, bundle map $\alpha: F \longrightarrow \mathcal{O}$ is a dominant term of $\{\phi_t\}$ if *there exists* $\varepsilon\in C^0(\mathcal{I}\times F;\mathbb{R})$ *such that*

$$\big|\phi_t(v)-\alpha(v)\big|\le\varepsilon(t,v)\big(t+|\alpha(v)|\big)\quad\forall(t,v)\in\Omega\quad and\quad\lim_{(t,v)\longrightarrow 0}\varepsilon(t,v)=0.$$

Dominant term $\alpha\colon F\longrightarrow\mathcal{O}$ *of* $\{\phi_t\}$ *is the* *resolvent* *of* $\{\phi_t\}$ *if* α *is a polynomial of constant rank.*

In (2) above, by $\dim_{\mathbb{C}}\big(\mathrm{Im}\ \alpha_b\big)$ we mean the dimension of the image of α_b as an analytic subvariety of the fiber $\mathcal{O}_b$. Note that if $\bar{\Omega}\subset\mathcal{I}\times F$ contains a neighborhood of $\{0\}\times\mathcal{M}$, the resolvent of $\{\phi_t\}$ is unique (if it exists).

Lemma 5.2 *Suppose* $\mathcal{M}$ *is a smooth manifold,*
(1) $F\equiv F^-\oplus F^+,\mathcal{O}\equiv\mathcal{O}^-\oplus\mathcal{O}^+\longrightarrow\mathcal{M}$ *are smooth vector bundles;*
(2) Ω *is an open subset of* $\mathcal{I}\times F$ *and* $\big\{\phi_t\colon\{v\in F\colon(t,v)\in\Omega\}\longrightarrow\mathcal{O}\big\}$ *is a family of smooth maps;*
(3) $\alpha:F\longrightarrow\mathcal{O}$ *is a dominant term of* $\{\phi_t\}$ *such that* $\alpha(F^+)\subset\mathcal{O}^+$, $\alpha^-\equiv\pi^-\circ(\alpha|F^-)$ *is a constant-rank polynomial, where* $\pi^-:\mathcal{O}^-\oplus\mathcal{O}^+\longrightarrow\mathcal{O}^-$ *is the projection map, and* $(\dim\mathcal{M}+2rk\ \alpha^-)<2rk\ \mathcal{O}^-$*;*
(4) $\bar{\nu}=(\bar{\nu}^-,\bar{\nu}^+)\in\Gamma(\mathcal{M};\mathcal{O}^-\oplus\mathcal{O}^+)$ *is generic with respect to* α^-.
Then for every compact subset K *of* $\mathcal{M}$*, there exist* $\delta_K>0$ *and a neighborhood* $U_F(K)$ *of* K *in* F *such that the map*

$$\psi_t\colon\{v\in F\colon(t,v)\in\Omega\}\longrightarrow\mathcal{O},\qquad\psi_t(v)=t\bar{\nu}_v+\phi_t(v),$$

has no zeros on $\{v\in U_F(K)\colon(t,v)\in\Omega\}$ *for all* $t\in(0,\delta_K)$.

Proof: (1) Suppose $\tilde{v}\in\Omega_{\delta_K}|K$ and $\psi_t(\tilde{v})=0$. Then by our assumptions on ϕ_t,

$$|\alpha(\tilde{v})|\le C_K\big(t+\bar{\varepsilon}_K(\delta_K)|\alpha(\tilde{v})|\big),$$

where $C_K>0$ depends only on K (and $\bar{\nu}$) and $\bar{\varepsilon}_K$ is a continuous function vanishing at zero. Thus, if $\delta_K>0$ is sufficiently small,

$$|\alpha(\tilde{v})|\le 2C_Kt\quad\forall t<\delta_K,\ \tilde{v}\in F_{\delta_K}|K\text{ s.t. }\psi_t(\tilde{v})=0.\tag{5.1}$$

(2) Let $F^-=\bigoplus_{i=1}^{i=k}F_i\longrightarrow\mathcal{M}$ be the bundles and $p_i\in\Gamma(\mathcal{M};F_i^{*\otimes d_i}\otimes\mathcal{O}^-)$ the sections as in (1) of Definition 5.1 corresponding to α^-. Define

$$\varphi_t\in\Gamma\big(\mathcal{M};\mathrm{End}(F^-)\big)\quad\text{by}\quad\varphi_t(v_i)=t^{-1/d_i}v_i\ \text{ if }v_i\in F_i.$$

Then by our assumption on ϕ_t and equation (5.1),

$$\big|\bar{\nu}^-+\alpha^-\big(\varphi_t(\tilde{v}^-)\big)\big|\le\tilde{C}_K\bar{\varepsilon}_K(\delta_K)\quad\forall t<\delta_K,\ \tilde{v}\in F_{\delta_K}|K\text{ s.t. }\psi_t(\tilde{v})=0,\tag{5.2}$$

where $\tilde{C}_K$ is determined by K. Since α^- has constant rank, the image of α^- is closed and is the total space of a bundle of affine analytic varieties of complex dimension

$$\mathrm{rk}\,\alpha^-<\mathrm{rk}\,\mathcal{O}^--\frac{1}{2}\dim\mathcal{M}.$$

Thus, by assumption (4) of the lemma, $\bar{\nu}^-$ does not intersect the image of α^-, and there exists $\epsilon_K > 0$ such that

$$\left|\bar{\nu}^- + \alpha^-(v^-)\right| \geq \epsilon_K \quad \forall v \in F^-|K. \tag{5.3}$$

If $\epsilon_K > \tilde{C}_K \bar{\epsilon}_K(\delta_K)$, by (5.2) and (5.3), $\pi^- \circ \psi_t$ has no zeros on $F_{\delta_K}|K$.

We will call family $\{\phi_t : \{v \in F : (t,v) \in \Omega\} \longrightarrow \mathcal{O}\}$ of smooth maps *hollow* if it admits a dominant term α that satisfies hypothesis (3) of Lemma 5.2. In many applications, family $\{\phi_t\}$ will be independent of t, i.e. $\phi_t = \phi$ for all t and some $\phi\colon F_\delta \longrightarrow \mathcal{O}$. In such cases, all terminology defined in this section will be applied directly to ϕ, rather than $\{\phi\}$.

Definition 5.3 *Suppose $\mathcal{M}$ is a smooth manifold and $F \longrightarrow \mathcal{M}$ is a smooth vector bundle. (1) Subset Y of F is* small *if Y contains no fiber of F and there exists a smooth manifold Z of dimension* $(\dim F - 1)$ *and a smooth map $f : Z \longrightarrow F$ such that the image of f is closed in F and contains Y.*
(2) If $F, \tilde{F} \longrightarrow \mathcal{M}$ are smooth vector bundles, $\rho \in \Gamma(\mathcal{M}; F^{*\otimes d} \otimes \tilde{F})$ induces a $\tilde{d}$-to-1 cover $F \longrightarrow \tilde{F}$ if the map

$$F_b \longrightarrow \tilde{F}_b, \qquad v \longrightarrow \rho(v) \equiv \rho(v^d),$$

is $\tilde{d}$-to-1 on a dense open subset of every fiber F_b of F.

Lemma 5.4 *Suppose $\mathcal{M}$ is a smooth manifold, $F = \bigoplus_{i=1}^{i=k} F_i$ and $\mathcal{O}$ are smooth vector bundles over $\mathcal{M}$, and*

$$\alpha = \sum_{i=1}^{i=k} p_i : F \longrightarrow \mathcal{O}, \quad \textit{where} \quad p_i \in \Gamma\big(\mathcal{M}; F_i^{*\otimes d_i} \otimes \mathcal{O}\big),$$

is a nondegenerate polynomial. Then there exists a small subset Y_α of $F = \bigoplus_{i=1}^{i=k} F_i$, which is invariant under scalar multiplication in each component separately, with the following property. If K is a compact subset of $\mathcal{O} - \alpha(Y_\alpha)$, there exists $C_K > 0$ such that

$$|v| \leq C_K |\alpha(v)| \quad \forall\, v \in F \ \ \textit{s.t.}\ \alpha(v) \in K.$$

Proof: (1) Let $Y_\alpha \subset F$ be the closed subset on which the differential of the fiberwise map $v \longrightarrow \alpha(v)$ does not have full rank, i.e. its rank is less than rk F. Since α is nondegenerate, Y_α contains no fiber of F. By our assumptions on α,

$$D(\alpha|_{F_b})\big|_v = \big(D(p_1|_{F_{1,b}})\big|_{v_1}, \ldots, D(p_k|_{F_{k,b}})\big|_{v_k}\big) : F_1 \oplus \ldots \oplus F_k \longrightarrow \mathcal{O}, \ \ \forall b \in \mathcal{M}, \ v = v_{[k]} \in \bigoplus_{i=1}^{i=k} F_i.$$

Since $p_i|_{F_{i,b}}$ is a homogeneous polynomial of degree d_i, its derivative is a homogeneous polynomial of degree $(d_i - 1)$. Thus, Y_α is preserved under scalar multiplication in each component separately. It also clearly satisfies the second condition of (1) of Definition 5.3.
(2) On $F - Y_\alpha$, α is a covering map onto its image with the number of leaves bounded by some number N_α. Thus, if K is any compact subset of $\mathcal{O} - \alpha(Y_\alpha)$, $\alpha^{-1}(K)$ is a compact subset of F. Therefore, there exists C_K such that

$$|v| \leq C_K |\alpha(v)| \quad \forall v \in F \ \text{ s.t. } \alpha(v) \in K.$$

Note that if $0\notin\alpha(Y_\alpha)$, then α is a linear injection on every fiber, and the above inequality holds on all of F.

Lemma 5.5 *Suppose $\mathcal{M}$ is a smooth manifold,*
(1) $F=\bigoplus_{i=1}^{i=k}F_i$ and $\mathcal{O}$ are smooth vector bundles over $\mathcal{M}$ with $rk\ F+\frac{1}{2}\dim\mathcal{M}=rk\ \mathcal{O}$;
(2) Y is a small subset of $F=\bigoplus_{i=1}^{i=k}F_i$, which is invariant under the scalar multiplication in each component separately;
(3) Ω is an open subset of $\mathcal{I}\times F$ such that $\Omega\cup(\{0\}\times X)$ contains a neighborhood of $\{0\}\times X$ in $\mathcal{I}\times\big(F-(Y-X)\big)$;
(4) $\{\phi_t\colon\{v\in F\colon(t,v)\in\Omega\}\longrightarrow\mathcal{O}\}$ is a family of smooth maps;
(5) nondegenerate polynomial $\alpha\colon F\longrightarrow\mathcal{O}$ is the resolvent of $\{\phi_t\}$;
(6) $\bar{\nu}\in\Gamma(\mathcal{M};\mathcal{O})$ is generic with respect to (Y,α), and the map

$$F\longrightarrow\mathcal{O},\qquad v\longrightarrow\bar{\nu}_v+\alpha(v),\tag{5.4}$$

has a finite number of (transverse) zeros.
If ψ_t is transversal to zero for all t, there exists a compact subset $K_{\alpha,\bar{\nu}}$ of $\mathcal{M}$ with the following property. If K is a precompact open subset of $\mathcal{M}$ containing $K_{\alpha,\bar{\nu}}$, there exist $\delta_K,\epsilon_K>0$ such that for all $t\in(0,\epsilon_K)$,

$$^{\pm}\big|\{v\in F_{\delta_K}|K\colon(t,v)\in\Omega,\ \psi_t(v)=0\}\big|={}^{\pm}\big|\{v\in F\colon\bar{\nu}_v+\alpha(v)=0\}\big|,$$

where $\psi_t(v)=t\bar{\nu}_v+\phi_t(v)$ as before. Furthermore, all the zeros of $\psi_t\big|(F_{\delta_K}|K)$ lie over $K_{\alpha,\bar{\nu}}$.

Proof: (1) Since the map in (5.4) has a finite number of zeros, all of them lie in the interior of $F_{C_{\alpha,\bar{\nu}}}|K_{\alpha,\bar{\nu}}$ for some compact subset $K_{\alpha,\bar{\nu}}$ of $\mathcal{M}$ and number $C_{\alpha,\bar{\nu}}>0$. Suppose $K\subset\mathcal{M}$ is a precompact open subset containing $K_{\alpha,\bar{\nu}}$, $\delta_K>0$ is such that $\big(F_{\delta_K}|K-Y\big)\subset\Omega$, and $\tilde{v}\in\Omega_{\delta_K}|K$ is such that $\psi_t(\tilde{v})=0$. By the same argument as in the proof of Lemma 5.2, if $\delta_K>0$ is sufficiently small,

$$\big|\alpha(\tilde{v})\big|\le C_Kt\quad\text{and}\quad\big|t\bar{\nu}_{\tilde{v}}+\alpha(\tilde{v})\big|\le\bar{\varepsilon}_K(\delta_K)t\quad\forall t<\delta_K,\ \tilde{v}\in F_{\delta_K}|K\text{ s.t. }\psi_t(\tilde{v})=0,\tag{5.5}$$

where C_K and $\bar{\varepsilon}_K=\bar{\varepsilon}_K(\delta_K)$ depend only on K, and $\bar{\varepsilon}_K(\delta_K)$ tends to zero with δ_K. Let $\phi_t\colon F\longrightarrow F$ be the map defined in (2) of the proof of Lemma 5.2, with F^- replaced by F. By (5.5),

$$\begin{aligned}\alpha\big(\phi_t(\tilde{v})\big)\in\mathcal{K}_{\bar{\nu}}\big(K;C_K,\bar{\varepsilon}_K(\delta_K)\big)\equiv\{\varpi\in\mathcal{O}_{C_K}\colon|\bar{\nu}_\varpi+\varpi|\le\bar{\varepsilon}_K(\delta_K)\}\\ \forall t<\delta_K,\ \tilde{v}\in F_{\delta_K}|K\text{ s.t. }\psi_t(\tilde{v})=0.\end{aligned}\tag{5.6}$$

(2) If $\bar{\nu}$ is generic, the map in (5.4) does not vanish on Y_α, where Y_α is as in Lemma 5.4. Since $\alpha(Y_\alpha)$ is a closed subset of $\mathcal{O}$, there exists $\epsilon_K>0$ such that

$$\big|\bar{\nu}_v+\alpha(v)\big|>\epsilon_K\quad\forall v\in Y_\alpha|K.$$

Thus, if $\bar{\varepsilon}_K(\delta_K)<\epsilon_K$, $\mathcal{K}_{\bar{\nu}}\big(K;C_K,\bar{\varepsilon}_K(\delta_K)\big)$ is a compact subset of $\mathcal{O}$ disjoint from $\alpha(Y_\alpha)$. Then by (5.6) and Lemma 5.4,

$$\big|\phi_t(\tilde{v})\big|\le C_K^*\quad\forall t<\delta_K,\ \tilde{v}\in F_{\delta_K}|K\text{ s.t. }\psi_t(\tilde{v})=0,\tag{5.7}$$

where C_K^* depends only on K.
(3) There is a one-to-one sign-preserving correspondence between the zeros of ψ_t on $\Omega_{\delta_K}|K$ and the zeros of

$$\tilde{\psi}_t\colon \Omega_{\delta_K}(K,t) \equiv \big\{v\in F\colon \big(t,\phi_t^{-1}(v)\big)\in\Omega_{\delta_K}|K\big\} \longrightarrow \mathcal{O}, \quad \tilde{\psi}_t(v) = t^{-1}\psi_t(\phi_t^{-1}(v)).$$

By (5.7), all the zeros of $\tilde{\psi}_t$ on $\Omega_{\delta_K}(K,t)$ are in fact contained in $F_{C_K^*}|K$. We can assume that $C_K^* > C_{\alpha,\bar{\nu}}$. By our assumptions on ϕ_t,

$$\big|\tilde{\psi}_t(v) - \big(\bar{\nu}_v + \alpha(v)\big)\big| \le C_K\bar{\epsilon}_K(\delta_K) \quad \forall v\in\Omega_{\delta_K}(K,t)\cap(F_{C_K^*}|K), \tag{5.8}$$

where $C_K > 0$ depends only on K. We define a cobordism between the zeros of $\tilde{\psi}_t$ and the zeros of $\bar{\nu}+\alpha$ on $\Omega_{\delta_K}(K,t)\cap(F_{C_K^*}|K)$ by

$$\Psi\colon \mathcal{I}\times\Omega_{\delta_K}(K,t)\cap(F_{C_K^*}|K) \longrightarrow \mathcal{O}, \quad \Psi_\tau(v) = \tau\tilde{\psi}_t(v) + (1-\tau)\big(\bar{\nu}_v+\alpha(v)\big) + \eta_\tau(v),$$

where $\eta : \mathcal{I}\times\Omega_{\delta_K}(K,t) \longrightarrow \mathcal{O}$ is any smooth function with very small C^0-norm such that $\eta_0=\eta_1=0$ and Ψ is transversal to zero. It remains to see that $\Psi^{-1}(0)$ is compact. Suppose $\Psi_{\tau_r}(v_r) = 0$ and (τ_r, v_r) converges $(\tilde{\tau},\tilde{v}) \in \mathcal{I}\times F_{2C_K^*}|\bar{K}$; we need to show that $\tilde{v}\in\Omega_{\delta_K}(K,t)\cap(F_{C_K^*}|K)$. By equation (5.8),

$$\big|\bar{\nu}_{v_r}+\alpha(v_r)\big| \le C_K\bar{\epsilon}_K(\delta_K) + \|\eta\|_{C^0} \quad \forall r \Longrightarrow \big|\bar{\nu}_{\tilde{v}}+\alpha(\tilde{v})\big| \le C_K\bar{\epsilon}_K(\delta_K) + \|\eta\|_{C^0}. \tag{5.9}$$

On the other hand, since $\bar{\nu}$ is generic, the map in (5.4) does not vanish on Y. Furthermore, all the zeros of this map are contained in the interior of $F_{C_{\alpha,\bar{\nu}}}|K_{\alpha,\bar{\nu}}$. Thus, by compactness,

$$\tilde{\epsilon}_K \equiv \inf\Big\{\big|\bar{\nu}_v+\alpha(v)\big| : v\in\big(Y\cap F_{2C_K^*}\big)\cup\big(F_{C_K^*}|K - F_{C_{\alpha,\bar{\nu}}}|K_{\alpha,\bar{\nu}}\big)\Big\} > 0, \tag{5.10}$$

where $\tilde{\epsilon}_K$ depends only on K. If $\tilde{\epsilon}_K > C_K\bar{\epsilon}_K(\delta_K)+\|\eta\|_{C^0}$, by (5.9) and (5.10),

$$\tilde{v} \in F_{C_{\alpha,\bar{\nu}}}|K_{\alpha,\bar{\nu}} \subset F_{C_K^*}|K - Y \subset \Omega_{\delta_K}(K,t).$$

The last inclusion follows from the very first assumption on δ_K above. We conclude that $\Psi^{-1}(0)$ is compact.

Corollary 5.6 *Suppose $\mathcal{M}$ is a smooth oriented manifold,*
(1) $F\equiv F^-\oplus F^+$, $\tilde{F}^-$, and $\mathcal{O}\equiv\mathcal{O}^-\oplus\mathcal{O}^+$ are smooth vector bundles over $\mathcal{M}$ with

$$rk\ F^- = rk\,\tilde{F}^- = rk\,\mathcal{O}^- - \frac{1}{2}\dim\mathcal{M} \quad \text{and} \quad rk\,F^+ = rk\,\mathcal{O}^+;$$

(2) $\rho\in\Gamma(\mathcal{M};F^{-*\otimes k}\otimes\tilde{F}^-)$ induces a $\tilde{d}$-to-1 cover $F\longrightarrow\tilde{F}$, and $\alpha^-\in\Gamma(\mathcal{M};\tilde{F}^{-*}\otimes\mathcal{O}^-)$;
(3) $\alpha\colon F\longrightarrow\mathcal{O}$ is a nondegenerate polynomial such that $\alpha^+\equiv\alpha|_{F^+}\colon F^+\longrightarrow\mathcal{O}^+$ is linear and $\pi^-\circ\alpha=\alpha^-\circ\rho$;
(4) Y is a small subset of F, which is invariant under the scalar multiplication in each component separately;
(5) Ω is an open subset of $\mathcal{I}\times F$ such that $\Omega\cup X$ contains a neighborhood of $\{0\}\times X$ in $\mathcal{I}\times\big(F-(Y-X)\big)$;
(6) $\{\phi_t\colon \{v\in F\colon (t,v)\in\Omega\}\longrightarrow\mathcal{O}\}$ is a family of smooth maps with resolvent α;

(7) $\bar{\nu}=(\bar{\nu}^-,\bar{\nu}^+)\in\Gamma(\mathcal{M};\mathcal{O}^-\oplus\mathcal{O}^+)$ *is generic with respect to* $(\alpha^+,\alpha^-,\rho,Y)$*, and the map*

$$\tilde{F}^- \longrightarrow \mathcal{O}^-, \quad \varpi \longrightarrow \bar{\nu}^-_{\varpi}+\alpha^-(\varpi), \tag{5.11}$$

has a finite number of (transverse) zeros.
If ψ_t *is transversal to zero for all* t*, there exists a compact subset* $K_{\alpha,\bar{\nu}}$ *of* $\mathcal{M}$ *with the following property. If* K *is precompact open subset of* $\mathcal{M}$ *containing* $K_{\alpha,\bar{\nu}}$*, there exist* $\delta_K,\epsilon_K>0$ *such that for all* $t\in(0,\epsilon_K)$*,*

$$^{\pm}\big|\{v\in F_{\delta_K}|K\colon (t,v)\in\Omega,\ \psi_t(v)=0\}\big| = \tilde{d}\cdot{}^{\pm}\big|\{\varpi\in\tilde{F}^-\colon \bar{\nu}^-_{\varpi}+\alpha^-(\varpi)=0\}\big|,$$

where $\psi_t(v)=t\bar{\nu}_v+\phi_t(v)$*. Furthermore, all the zeros of* $\psi_t\big|(F_{\delta_K}|K)$ *lie over* $K_{\alpha,\bar{\nu}}$*.*

Proof: Let $K_{\alpha,\bar{\nu}}$ and $\delta_K>0$ be as in Lemma 5.5. Then if K is a precompact open subset of $\mathcal{M}$, for all $t\in(0,\epsilon_K)$ the signed number of zeros of $\psi_t|(\Omega_{\delta_K}|K)$ is the same as the signed number of solutions of

$$F|K \longrightarrow \mathcal{O}, \qquad \begin{cases} \bar{\nu}^-_v+\alpha^-\big(\rho(v^-)\big)=0\in\mathcal{O}^-; \\ \bar{\nu}^+_v+\alpha^+(v^+)+\pi^+\big(\alpha(v^-)\big)=0\in\mathcal{O}^+. \end{cases} \tag{5.12}$$

For every solution of the first equation, there is a unique solution of the second equation. Since α^+ is complex-linear on the fibers, the signed number of solutions of (5.12) is the same as the signed number of solutions of the first equation. Since the first equation has no solutions on Y_{α^-} if $\bar{\nu}$ is generic and ρ is $\tilde{d}$-to-1 outside of Y_{α^-}, ρ induces a $\tilde{d}$-to-1 sign-preserving map from the set of zeros of (5.11) to the set of solutions of the first equation.

5.2 Contributions to the Euler Class

If $\bar{\mathcal{M}}$ is a smooth oriented compact manifold of dimension $2n$, and $V\longrightarrow\bar{\mathcal{M}}$ is a vector bundle of rank n, then the euler class of V is the number of zeros of any section $s\colon\bar{\mathcal{M}}\longrightarrow V$ which is transverse to the zero set. In this section, under slightly more topological assumptions on $\bar{\mathcal{M}}$ and V, we discuss a relationship between subsets of the zero set of a non-transverse section and the euler class of V.

Definition 5.7 *(1) Compact oriented topological manifold* $\bar{\mathcal{M}}=\mathcal{M}_n\sqcup\bigsqcup_{i=0}^{i=n-2}\mathcal{M}_i$ *of dimension* n *is* <u>*mostly smooth*</u>*, or* <u>*ms*</u>*, if*
(1a) each $\mathcal{M}_i$ *is a smooth manifold of dimension* i*, and* $\mathcal{M}\equiv\mathcal{M}_n$ *is a dense open subset of* $\bar{\mathcal{M}}$*;*
(1b) for each $i\in[n-2]$*,* $\bar{\mathcal{M}}_i-\mathcal{M}_i\subset\bigcup_{j=0}^{j-2}\mathcal{M}_j$*;*
(2) If $\bar{\mathcal{Z}}=\mathcal{Z}\sqcup\bigsqcup\mathcal{Z}_j$ *and* $\bar{\mathcal{M}}=\mathcal{M}\sqcup\bigsqcup\mathcal{M}_i$ *are ms-manifolds, continuous map* $\pi\colon\mathcal{Z}\longrightarrow\mathcal{M}$ *is an* <u>*ms-map*</u> *if for each* j *there exists* i *such that* $\pi\colon\mathcal{Z}_j\longrightarrow\mathcal{M}_i$ *is a smooth map.*
(3) If $\bar{\mathcal{M}}$ *is an ms-manifold, topological vector bundle* $V\longrightarrow\bar{\mathcal{M}}$ *is an* <u>*ms-bundle*</u> *if* $V|\mathcal{M}_i$ *is a smooth vector bundle for* $i=n$ *and all* $i\in[n-2]$*.*
(4) If $V\longrightarrow\bar{\mathcal{M}}$ *is an ms-bundle, continuous section* $s\colon\mathcal{M}\longrightarrow V$ *is an* <u>*ms-section*</u> *if* $s|_{\mathcal{M}_i}$ *is* C^2*-smooth for* $i=n$ *and all* $i\in[n-2]$*.*

The dense open submanifold $\mathcal{M}$ of $\bar{\mathcal{M}}$ will be called the *smooth base* of $\bar{\mathcal{M}}$. Note that if $E\longrightarrow\bar{\mathcal{M}}$ is an ms-bundle, then the (complex) projectivization $\mathbb{P}E$ of E is an ms-manifold.

Furthermore, the projection map $\pi_E : \mathbb{P}E \longrightarrow \bar{\mathcal{M}}$ is an ms-map, and the tautological line bundle $\gamma_E \longrightarrow \mathbb{P}E$ is an ms-bundle.

If $V \longrightarrow \bar{\mathcal{M}}$ is an ms-bundle, we denote the space of ms-sections of V by $\Gamma(\bar{\mathcal{M}}; V)$. Using (4) of Definition 5.7, we define an ms-polynomial map between two ms-bundles analogously to (1) of Definition 5.1. We topologize $\Gamma(\bar{\mathcal{M}}; V)$ as follows. If $s_k, s \in \Gamma(\bar{\mathcal{M}}; V)$, the sequence $\{s_k\}$ *converges to* s if s_k converges to s in the C^0-norm on all of $\bar{\mathcal{M}}$ and in the C^2-norm on compact subsets of $\mathcal{M}_i$ for $i=n$ and all $i\in[n-2]$. The C^0-norm is defined with respect to the norm on $V \longrightarrow \bar{\mathcal{M}}$. In order to define the C^2-norm on compact subsets of $\mathcal{M}_i$, we fix a connection in each smooth bundle in $V \longrightarrow \mathcal{M}_i$.

Definition 5.8 *Let $\bar{\mathcal{M}}$ be an ms-manifold as in Definition 5.7.*
(1) If $\mathcal{Z} \subset \mathcal{M}_i$ is a smooth oriented submanifold, a normal-bundle model for $\mathcal{Z}$ is a tuple (F, Y, ϑ), where
(1a) $F \longrightarrow \mathcal{Z}$ is a smooth vector bundle and Y is a small subset of F;
(1b) for some $\delta \in C^\infty(\mathcal{Z}; \mathbb{R}^+)$, $\vartheta: F_\delta - (Y-X) \longrightarrow \bar{\mathcal{M}}$ is a continuous map such that
(1b-i) $\vartheta: F_\delta - (Y-X) \longrightarrow \bar{\mathcal{M}}$ is a homeomorphism onto an open neighborhood of $\mathcal{Z}$ in $\mathcal{M} \cup \mathcal{Z}$;
(1b-ii) $\vartheta|_{\mathcal{Z}}$ is the identity map, and $\vartheta: F_\delta - (Y-X) \longrightarrow \mathcal{M}$ is an orientation preserving diffeomorphism on an open subset of $\mathcal{M}$.
(2) A closure of normal-bundle model (F, Y, ϑ) is a tuple $(\bar{\mathcal{Z}}, \tilde{F}, \pi)$, where
(2a) $\bar{\mathcal{Z}}$ is an ms-manifold with smooth base $\mathcal{Z}$;
(2b) $\pi: \bar{\mathcal{Z}} \longrightarrow \bar{\mathcal{M}}$ is an ms-map such that $\pi|_{\mathcal{Z}}$ is the identity;
(2c) $\tilde{F} \longrightarrow \bar{\mathcal{Z}}$ is an ms-bundle such that $\tilde{F}|_{\mathcal{Z}} = F$.

If $\mathcal{Z}$ is a smooth submanifold of $\mathcal{M}$, an identification of the normal bundle $\mathcal{N}\mathcal{Z}$ of $\mathcal{Z}$ in $\mathcal{M}$ with a neighborhood of $\mathcal{Z}$ in $\mathcal{M}$ induces a normal bundle model for $\mathcal{Z}$. Definition 5.8 extends this standard construction to the ms-category.

Definition 5.9 *Suppose $E, \mathcal{O} \longrightarrow \bar{\mathcal{M}}$ are ms-bundles and $\alpha: E \longrightarrow \mathcal{O}$ is an ms-polynomial.*
(1) Subset $\mathcal{Z}$ of $\mathcal{M}$ is α-regular if there exist a normal bundle model (F, Y, ϑ) for $\mathcal{Z}$, constant-rank polynomial $p: F \oplus E \longrightarrow \mathcal{O}$ over $\mathcal{Z}$, smooth bundle isomorphisms $\vartheta_E: \vartheta^* E \longrightarrow \pi_F^* E$ and $\vartheta_{\mathcal{O}}: \vartheta^* \mathcal{O} \longrightarrow \pi_F^* \mathcal{O}$ covering the identity on $F_\delta - (Y-X)$, and $\varepsilon \in C(F; \mathbb{R})$ such that
(1a) $\vartheta_E|_{F_\delta - Y - X}$ is smooth and $\vartheta_E|_{\mathcal{Z}}$ is the identity;
(1b) $\vartheta_{\mathcal{O}}|_{F_\delta - Y - X}$ is smooth and $\vartheta_{\mathcal{O}}|_{\mathcal{Z}}$ is the identity;
(1c) $\lim_{w \longrightarrow 0} \varepsilon(w) = 0$;
(1d) $\big|\vartheta_{\mathcal{O}} \alpha(\vartheta_E^{-1}(w, v)) - p(w, v)\big| \le \varepsilon(w) \big|p(w, v)\big|$ for all $w \in F_\delta - (Y-X)$, $v \in E$.
(2) α is a regular polynomial if $\bar{\mathcal{M}}$ is the union of finitely many α-regular subsets.

Lemma 5.10 *Suppose $E, \mathcal{O} \longrightarrow \bar{\mathcal{M}}$ are ms-bundles, such that rk $E + \frac{1}{2} \dim \bar{\mathcal{M}} =$ rk $\mathcal{O}$, and $\alpha: E \longrightarrow \mathcal{O}$ is a regular polynomial, such that α is nondegenerate on $\mathcal{M}$. Let $\nu \in \Gamma(\bar{\mathcal{M}}; \mathcal{O})$ be an ms-section such that the map*

$$\psi_{\alpha,\nu}: E \longrightarrow \mathcal{O}, \qquad \psi_{\alpha,\nu}(v) = \nu_v + \alpha(v),$$

does not vanish on $E|(\bar{\mathcal{M}} - \mathcal{M})$ and is transversal to the zero set in $\mathcal{O}|\mathcal{M}$. Then $\psi_{\alpha,\nu}^{-1}(0)$ is finite, and $N(\alpha) \equiv^{\pm} |\psi_{\alpha,\nu}^{-1}(0)|$ is independent of the choice of ν as above.

Proof: (1) We first show that for every $x \in \bar{\mathcal{M}} - \mathcal{M}$ there exists a neighborhood U of x in $\bar{\mathcal{M}}$ such that $\psi_{\alpha,\nu}$ does not vanish on $E|U$. By (2) of Definition 5.9, there exists an α-regular subset $\mathcal{Z}$ of $\bar{\mathcal{M}}$ containing x. Let (F,Y,ϑ), δ, p, ϑ_E, $\vartheta_{\mathcal{O}}$, and ε be as in (1) of Definition 5.9. It can be assumed that δ is such that

$$\varepsilon(w) < \frac{1}{2} \quad \text{and} \quad \big|\nu_{\vartheta(w)}\big| \le 2\big|\nu_w\big| \equiv 2\big|\nu_{b_w}\big| \qquad \forall w \in F_\delta - (Y-X).$$

Then, if $\psi_{\alpha,\nu}\big(\vartheta_E^{-1}(w,v)\big)=0$ for some $(w,v)\in F\oplus E$ with $w\in F_\delta-(Y-X)$, $|\alpha(w,v)|\le 4|\nu_w|$ by (1c) of Definition 5.9. Thus, if $\{(w_k,v_k)\}\subset F\oplus E$ is such that $\psi_{\alpha,\nu}\big(\vartheta_E^{-1}(w_k,v_k)\big)=0$ and $w_k\longrightarrow x\in F$, a subsequence of $\{\alpha(w_k,v_k)\}$ converges to an element $\varpi\in\mathcal{O}_x$. Since α is a polynomial map of constant rank, there exists $(0,v)\in F\oplus E$ such that $\alpha(0,v)=\varpi$. Since $\alpha(0,v)=p(0,v)$, it follows that $\psi_{\alpha,\nu}(v)=0$ contrary to the assumption.
(2) By (1), there exists a compact subset $K_{\alpha,\nu}$ of $\mathcal{M}$ such that $\psi_{\alpha,\nu}^{-1}(0)\subset E|K_{\alpha,\nu}$. Since $\psi_{\alpha,\nu}$ is transversal to zero, $\nu(\mathcal{M})\cap\alpha(Y_\alpha)=\emptyset$, where $Y_\alpha\subset E|\mathcal{M}$ is as in Lemma 5.4. It follows that $\psi_{\alpha,\nu}^{-1}(0)$ is a finite subset of $E|\mathcal{M}$.
(3) The final claim of the lemma is obtained by constructing a cobordism between $\psi_{\alpha,\nu}$ and $\psi_{\alpha,\nu'}$. More precisely, we take a smooth family $\{\nu_\tau\colon \tau\in\mathcal{I}\}$ of ms-sections of $\mathcal{O}$ such that $\nu_0=\nu$, $\nu_1=\nu'$, $\psi_{\alpha,\nu_\tau}^{-1}(0)\subset E|\mathcal{M}$, and the section

$$\Psi_\alpha\colon \mathcal{I}\times E \longrightarrow \mathcal{O}, \qquad \Psi_\alpha(\tau,v)=\psi_{\alpha,\nu_\tau}(v),$$

is transversal to the zero set in $\mathcal{O}$. Such a family can always be chosen, since $\bar{\mathcal{M}}-\mathcal{M}$ has codimension two in $\bar{\mathcal{M}}$. Then, by the same argument as in (1) and (2), $\Psi_\alpha^{-1}(0)$ is a smooth compact oriented submanifold of $E|\mathcal{M}$ with boundary $\psi_{\alpha,\nu_1}^{-1}(0)-\psi_{\alpha,\nu_0}^{-1}(0)$.

Definition 5.11 *Suppose $\bar{\mathcal{M}}$ is an ms-manifold of dimension $2n$, $V\longrightarrow\bar{\mathcal{M}}$ is an ms-bundle of rank n, $s\in\Gamma(\bar{\mathcal{M}};V)$, and $\mathcal{Z}\subset\mathcal{M}_i\cap s^{-1}(0)$.*
(1) $\mathcal{Z}$ is <u>s-hollow</u> if there exist a normal bundle model (F,Y,ϑ) for $\mathcal{Z}$ and a bundle isomorphism ϑ_V: $\vartheta^*V\longrightarrow\pi_F^*V$, covering the identity on $F_\delta-(Y-X)$, such that
(1a) $\vartheta_V|_{F_\delta-Y-X}$ is smooth and $\vartheta_V|_{\mathcal{Z}}$ is the identity;
(1b) $\phi_0\equiv\vartheta_V\circ\vartheta^*s\colon F_\delta-(Y-X)\longrightarrow V$ is hollow.
(2) $\mathcal{Z}$ is <u>s-regular</u> if there exist a normal bundle model (F,Y,ϑ) for $\mathcal{Z}$ with closure $(\bar{\mathcal{Z}},\tilde{F},\pi)$, regular polynomial $\alpha\colon\tilde{F}\longrightarrow\pi^*V$, and a bundle isomorphism ϑ_V: $\vartheta^*V\longrightarrow\pi_F^*V$ covering the identity on $F_\delta-(Y-X)$, such that
(2a) $\vartheta_V|_{F_\delta-Y-X}$ is smooth and $\vartheta_V|_{\mathcal{Z}}$ is the identity;
(2b) $\alpha|_{\mathcal{Z}}$ is nondegenerate and is the resolvent for $\phi_0\equiv\vartheta_V\circ\vartheta^*s\colon F_\delta-(Y-X)\longrightarrow V$, and Y is preserved under scalar multiplication in each of the components of F for the splitting corresponding to α as in (1) of Definition 5.1.

Lemma 5.12 *If $(\bar{\mathcal{M}},V,s)$ and $(\mathcal{Z},F,Y,\vartheta)$ are as in Definition 5.11, there exist a number $\mathcal{C}_{\mathcal{Z}}(s)\in\mathbb{Z}$, which equals zero if $\mathcal{Z}$ is s-hollow, and a dense open subset $\Gamma_{\mathcal{Z}}(s)\subset\Gamma(\bar{\mathcal{M}};V)$ with the following properties. For every $\nu\in\Gamma_{\mathcal{Z}}(s)$,*
(1) there exists $\epsilon_\nu>0$ such that for all $t\in(0,\epsilon_\nu)$, all the zeros of $t\nu+s$ are contained in $\mathcal{M}$ and $(t\nu+s)\big|_{\mathcal{M}}$ is transversal to the zero set in V;
(2) there exist a compact subset $K_\nu\subset\mathcal{Z}$, open neighborhood $U_\nu(K)$ of K in $\bar{\mathcal{M}}$ for each compact subset $K\subset\mathcal{Z}$, and $\epsilon_\nu(U)\in(0,\epsilon_\nu)$ for each open subset U of $\bar{\mathcal{M}}$ such that

$$^{\pm}\big|\{b\in U\colon t\nu(b)+s(b)=0\}\big| = \mathcal{C}_{\mathcal{Z}}(s) \qquad \text{if } t<\epsilon_\nu(U),\ K_\nu\subset K\subset U\subset U_\nu(K).$$

Proof: It is clear that we can choose a dense open subset $\Gamma'_{\mathcal{Z}}(s)\subset\Gamma(\bar{\mathcal{M}};V)$ such that every $\nu\in\Gamma'_{\mathcal{Z}}(s)$ satisfies requirement (1) of the lemma. If $\mathcal{Z}$ is s-hollow, we also need that

$\bar{\nu}\equiv\nu|\mathcal{Z}$ is generic with respect to the corresponding polynomial α^- in the sense of the proof of Lemma 5.2. We can then take $K_\nu=\emptyset$. If $\mathcal{Z}$ is s-regular, let $\bar{\nu}=\pi^*\nu\in\Gamma(\bar{\mathcal{Z}};\pi^*V)$. By Lemma 5.10, the second part of (6) of Lemma 5.5 is satisfied, as long as $t\nu{+}s$ is transversal to the zero set on each smooth strata. The other requirements on $\bar{\nu}$ in Lemma 5.5 are finitely many transversality properties. We then take

$$\mathcal{C}_{\mathcal{Z}}(s)={}^{\pm}\big|\{v\in F:\bar{\nu}_v+\alpha(v)=0\}\big|.$$

By Lemma 5.10, this number is well-defined.

The total number of zeros of a section $t\nu+s$ satisfying condition (1) of Lemma 5.12 is precisely the euler class $e(V)$ of the bundle $V\longrightarrow\bar{\mathcal{M}}$. Thus, due to (2) of Lemma 5.12, we call $\mathcal{C}_{\mathcal{Z}}(s)$ the *s-contribution* (or simply contribution) of $\mathcal{Z}$ to $e(V)$. If $\mathcal{Z}$ is any subset of $\bar{\mathcal{M}}$ such that $\mathcal{Z}\cap s^{-1}(0)$ satisfies the requirements of Definition 5.11, let $\mathcal{C}_{\mathcal{Z}}(s)=\mathcal{C}_{\mathcal{Z}(s)\cap s^{-1}(0)}(s)$. In addition, if Z is a closed subset of $\bar{\mathcal{M}}$ such that $s^{-1}(0)-Z$ is also closed, we can easily define $\mathcal{C}_Z(s)$ by Lemma 5.12.

Proposition 5.13 *Let $V\longrightarrow\bar{\mathcal{M}}$ be an ms-bundle of rank n over an ms-manifold of dimension $2n$. Suppose $\mathcal{U}$ is an open subset of $\mathcal{M}$ and $s\in\Gamma(\bar{\mathcal{M}};V)$ is such that $s|_{\mathcal{U}}$ is transversal to the zero set.*
(1) If $s^{-1}(0)\cap\mathcal{U}$ is a finite set, ${}^{\pm}|s^{-1}(0)\cap\mathcal{U}|=\langle e(V),[\bar{\mathcal{M}}]\rangle-\mathcal{C}_{\bar{\mathcal{M}}-\mathcal{U}}(s)$.
(2) If $\bar{\mathcal{M}}-\mathcal{U}=\bigsqcup_{i=1}^{i=k}\mathcal{Z}_i$, where each $\mathcal{Z}_i$ is s-hollow or s-regular, then $s^{-1}(0)\cap\mathcal{U}$ is finite, and

$${}^{\pm}|s^{-1}(0)\cap\mathcal{U}|=\langle e(V),[\bar{\mathcal{M}}]\rangle-\mathcal{C}_{\bar{\mathcal{M}}-\mathcal{U}}(s)=\langle e(V),[\bar{\mathcal{M}}]\rangle-\sum_{i=1}^{i=k}\mathcal{C}_{\mathcal{Z}_i}(s).$$

If $\mathcal{Z}_i$ is s-hollow, $\mathcal{C}_{\mathcal{Z}_i}(s)=0$. If $\mathcal{Z}_i$ is s-regular and $\alpha_i\colon\tilde{F}_i\longrightarrow V$ is the corresponding polynomial,

$$\mathcal{C}_{\mathcal{Z}_i}(s)={}^{\pm}\,\big|\{v\in\tilde{F}_i:\ \bar{\nu}_v+\alpha_i(v)=0\}\big|\equiv N(\alpha_i),$$

where $\bar{\nu}\in\Gamma(\bar{\mathcal{Z}}_i;V)$ is a generic section. Finally, if $\alpha_i\in\Gamma(\bar{\mathcal{Z}}_i;\tilde{F}_i^{*\otimes k}\otimes\pi^*V)$ has constant rank over $\bar{\mathcal{Z}}_i$ and factors through a $\tilde{k}$-to-1 cover $\rho_i\colon\tilde{F}_i\longrightarrow\tilde{F}_i^{\otimes k}$,

$$\mathcal{C}_{\mathcal{Z}_i}(s)=\tilde{k}\big\langle e\big(\pi^*V/\alpha_i(\tilde{F}_i)\big),[\bar{\mathcal{Z}}_i]\big\rangle.$$

All statements of this corollary have already been proved. A splitting of the zero set as in (2) of Proposition 5.13 always exists in the complex-analytic category. It should be possible to generalize the constructions of this section to an arbitrary compact oriented topological manifold. However, Lemma 5.10 will no longer be valid, and another approach will be needed to deal with the zeros of $\psi_{\alpha,\nu}$ that tend to infinity. For the cases that we encounter in later chapters, the version of s-regularity of Definition 5.11 suffices.

5.3 Zeros of Polynomial Maps

We now present a procedure for computing the number of zeros of a polynomial map between two complex vector bundles over a compact oriented manifold. All the polynomials we encounter in actual computations in this thesis are of degree-one. Thus, we focus on the degree-one case, but discuss the general case at the end for the sake of completeness.

Suppose $\bar{\mathcal{M}}$ is an ms-manifold, $E, \mathcal{O} \longrightarrow \bar{\mathcal{M}}$ are ms-bundles such that $\mathrm{rk}\ E + \frac{1}{2}\dim \bar{\mathcal{M}} = \mathrm{rk}\ \mathcal{O}$, and $\alpha \in \Gamma(\bar{\mathcal{M}}; E^* \otimes \mathcal{O})$ is an ms-section. Let $\bar{\nu} \in \Gamma(\bar{\mathcal{M}}; \mathcal{O})$ be such that $\bar{\nu}$ has no zeros, the map

$$\psi_{\alpha,\bar{\nu}} \equiv \bar{\nu} + \alpha \colon E \longrightarrow \mathcal{O}$$

is transversal to the zero set in $\mathcal{O}$ on $E|\mathcal{M}$, and all its zeros are contained in $E|\mathcal{M}$. The first step in our procedure of determining the number of zeros of $\psi_{\alpha,\bar{\nu}}$ reduces this issue to the case E is a line bundle. Let $\mathbb{P}E$ be the projectivization of E (over $\mathbb{C}$) and $\gamma_E \longrightarrow \mathbb{P}E$ the tautological line bundle. Then α induces an ms-section $\alpha_E \in \Gamma(\mathbb{P}E; \gamma_E^* \otimes \pi_E^* \mathcal{O})$, where $\pi_E \colon \mathbb{P}E \longrightarrow \bar{\mathcal{M}}$ is the bundle projection map. The number of zeros of $\psi_{\alpha,\bar{\nu}}$ is the same as the number of zeros of the induced map

$$\psi^E_{\alpha,\bar{\nu}} \equiv \pi_E^* \bar{\nu} + \alpha_E \colon \gamma_E \longrightarrow \pi_E^* \mathcal{O}.$$

Thus, we can always reduce the computation to the case E is a line bundle.

The second step describes the number of zeros of $\psi_{\alpha,\bar{\nu}}$ topologically in the case E is a line bundle. Since $\bar{\nu}$ has no zeros, it spans a trivial subbundle $\mathbb{C}\bar{\nu}$ of $\mathcal{O}$. Let $\mathcal{O}^\perp$ be the quotient of $\mathcal{O}$ by this trivial subbundle. Denote the $\mathbb{C}\bar{\nu}$- and $\mathcal{O}^\perp$-components of α by α^t and $\alpha^\perp$, respectively. Then the zeros of $\psi_{\alpha,\bar{\nu}}$ are described by

$$\begin{cases} \bar{\nu}_b + \alpha_b^t(v) = 0 \in \mathbb{C}\bar{\nu}; \\ \alpha_b^\perp(v) = 0 \in \mathcal{O}^\perp; \end{cases} \qquad b \in X,\ v \in E_b. \tag{5.13}$$

Since $\bar{\nu}$ does not vanish, all solutions of the first equations (5.13) are nonzero. The solution of the second equation with nonzero v is

$$\big(E - \bar{\mathcal{M}}\big)\big|\alpha^{\perp^{-1}}(0).$$

Furthermore, if $b \in \alpha^{\perp^{-1}}(0)$ and $\alpha(b) \neq 0$, $\alpha^t \colon E \longrightarrow (\mathbb{C}\bar{\nu})_b$ is an isomorphism. Thus, for every $b \in \alpha^{\perp^{-1}}(0) - \alpha^{-1}(0)$, there exists a unique $v \in E_b$ solving the first equation in (5.13), and the sign of (b, v) as a zero of $\psi_{\alpha,\bar{\nu}}$ agrees with the sign of b as a zero of $\alpha^\perp$. On the other hand, (5.13) has no solutions on $E|\alpha^{-1}(0)$. It follows that the number of zeros of $\psi_{\alpha,\bar{\nu}}$ is the number of zeros of $\alpha^\perp$ on $\bar{\mathcal{M}} - \alpha^{-1}(0)$, i.e.

$${}^{\pm}\big|\psi^{-1}(0)\big| = \big\langle e(E^* \otimes \mathcal{O}^\perp), [\bar{\mathcal{M}}]\big\rangle - \mathcal{C}_{\alpha^{-1}(0)}(\alpha^\perp); \tag{5.14}$$

see Proposition 5.13.

As discussed in the previous section, computing $\mathcal{C}_Z(s)$ in reasonably good cases reduces to counting the number of zeros of polynomial maps between vector bundles over ms-manifolds, but with the rank of the target bundle one less than the rank of the bundle $\mathcal{O}$ we started with. Thus, this process will eventually terminate. The lemma below summarizes the last two paragraphs.

Lemma 5.14 *Suppose $\bar{\mathcal{M}}$ is an ms-manifold and $E, \mathcal{O} \longrightarrow \bar{\mathcal{M}}$ are ms-bundles such that*

$$rk\ E + \frac{1}{2}\dim\bar{\mathcal{M}} = rk\ \mathcal{O}.$$

If $\alpha \in \Gamma(\bar{\mathcal{M}}; E^*\otimes\mathcal{O})$ and $\bar{\nu}\in\Gamma(\bar{\mathcal{M}};\mathcal{O})$ are such that α is regular, $\bar{\nu}$ has no zeros, the map

$$\psi_{\alpha,\bar{\nu}} \equiv \bar{\nu}+\alpha \colon E \longrightarrow \mathcal{O}$$

is transversal to the zero set on $E|\mathcal{M}$, and all its zeros are contained in $E|\mathcal{M}$, then $\psi_{\alpha,\bar{\nu}}^{-1}(0)$ is a finite set, ${}^{\pm}|\psi_{\alpha,\bar{\nu}}^{-1}(0)|$ depends only on α, and

$$N(\alpha) \equiv^{\pm} |\psi_{\alpha,\bar{\nu}}^{-1}(0)| = \langle c(\mathcal{O})c(E)^{-1}, [\bar{\mathcal{M}})]\rangle - \mathcal{C}_{\alpha_E^{-1}(0)}(\alpha_E^{\perp}).$$

Proof: Let $n = \text{rk}\,E$, $m = \text{rk}\,\mathcal{O}$, and $\lambda_E = c_1(\gamma_E^*)$. From Lemma 5.10, equation (5.14), and the construction above, we obtain the first two claims of the lemma along with

$$\begin{aligned} N(\alpha) &= \sum_{k=0}^{k=m-1} \langle c_k(\mathcal{O}^{\perp})\lambda_E^{m-1-k}, [\mathbb{P}E]\rangle - \mathcal{C}_{\alpha_E^{-1}(0)}(\alpha_E^{\perp}) \\ &= \sum_{k=0}^{k=m-1} \langle c_k(\mathcal{O})\lambda_E^{m-1-k}, [\mathbb{P}E]\rangle - \mathcal{C}_{\alpha_E^{-1}(0)}(\alpha_E^{\perp}). \end{aligned} \tag{5.15}$$

On the other hand,

$$\begin{aligned} &\lambda_E^n + \sum_{k=1}^{k=n} c_k(E)\lambda_E^{n-k} = 0 \in H^{2n}(\mathbb{P}E) \qquad \text{and} \\ &\langle \mu\lambda_E^{n-1}, [\mathbb{P}E]\rangle = \langle \mu, [\bar{\mathcal{M}}]\rangle \quad \forall \mu\in H^{2m-2n}(\bar{\mathcal{M}}); \end{aligned} \tag{5.16}$$

see [BT] for example. The last statement of the lemma follows from (5.15) and (5.15).

Remark: If $\alpha\colon E \longrightarrow \mathcal{O}$ is a polynomial, and not just a linear map, the first step in computing the number of zeros of the map $\psi_{\alpha,\bar{\nu}} = \bar{\nu}+\alpha$ would be to reduce to the case α is a linear map via a projectivization construction similar to the one in the second paragraph of this section. For example, suppose $\alpha = p_1 + p_2$, where $p_i \in \Gamma(\bar{\mathcal{M}}; E_i^{*\otimes d_i}\otimes\mathcal{O})$ and $E = E_1\oplus E_2$. Then the number of zeros of $\psi_{\alpha,\bar{\nu}}$ is the same as the number of zeros of

$$\psi_{\alpha,\bar{\nu}}^{E_1} \equiv \pi_{E_1}^*\bar{\nu} + p_{1,E_1} + \pi_{E_1}^* p_2 \colon \gamma_{E_1} \oplus \pi_{E_1}^* E_2 \longrightarrow \pi_{E_1}^*\mathcal{O}$$

over $\mathbb{P}E_1$, where $p_{1,E_1} \in \Gamma(\mathbb{P}E_1; \gamma_{E_1}^{*\otimes d_1})$ is the section induced by p_1. If $\bar{\nu}$ is generic, this number is d_1-times the number of zeros of the map

$$\tilde{\psi}_{\alpha,\bar{\nu}}^{E_1} \equiv \pi_{E_1}^*\bar{\nu} + p_{1,E_1} + \pi_{E_1}^* p_2 \colon \gamma_{E_1}^{\otimes d_1} \oplus \pi_{E_1}^* E_2 \longrightarrow \pi_{E_1}^*\mathcal{O}.$$

Note that p_{1,E_1} is *linear* on $\gamma_{E_1}^{\otimes d_1}$. Taking the projection of $\pi_{E_1}^* E_2$ over $\mathbb{P}E_1$ and repeating the above procedure, we obtain a linear map

$$\psi_{\alpha,\bar{\nu}}^{E_1,E_2} \colon \pi_{E_2}^*\gamma_{E_1}^{\otimes d_1} \oplus \gamma_{\pi_{E_1}^* E_2}^{\otimes d_2} \longrightarrow \pi_{\pi_{E_1}^* E_2}^* \pi_{E_1}^*\mathcal{O}.$$

5.4 Example

In this section, we give a topological formula for the number of zeros of an affine map of a very special form. We use Corollary 5.17, the main result of this section, in Chapters 9 and 10.

Suppose $\bar{\mathcal{Z}}$ and Σ are topological spaces, $L_{\mathcal{Z}}, V_{\mathcal{Z}}, E_{\mathcal{Z}} \longrightarrow \bar{\mathcal{Z}}$ and $L_\Sigma, V_\Sigma \longrightarrow \Sigma$ are vector bundles, and $\alpha_{\mathcal{Z}} \in \Gamma(\bar{\mathcal{Z}}; \mathrm{Hom}(E_{\mathcal{Z}}; L_{\mathcal{Z}}^* \otimes V_{\mathcal{Z}}))$, $s \in \Gamma(\Sigma; L_\Sigma^* \otimes V_\Sigma)$ and $\bar{\nu} \in \Gamma(\Sigma \times \bar{\mathcal{Z}}; V_\Sigma \otimes V_{\mathcal{Z}})$ are sections such that $\bar{\nu}$ does not vanish. Then we define

$$\alpha^s_{\mathcal{Z},\bar{\nu}} \in \Gamma\big(\Sigma \times \bar{\mathcal{Z}}; \mathrm{Hom}(E_{\mathcal{Z}}; L_\Sigma^* \otimes L_{\mathcal{Z}}^* \otimes (V_\Sigma \otimes V_{\mathcal{Z}})^\perp)\big), \quad \text{where} \quad (V_\Sigma \otimes V_{\mathcal{Z}})^\perp = (V_\Sigma \otimes V_{\mathcal{Z}})/\mathbb{C}\bar{\nu},$$
$$\text{by} \qquad \{\alpha^s_{\mathcal{Z},\bar{\nu}}(e)\}(w \otimes v) = \pi^\perp_{\bar{\nu}}\big(\{\alpha_{\mathcal{Z}}(e)\}(v) \otimes s(w)\big) \in (V_\Sigma \otimes V_{\mathcal{Z}})^\perp,$$

Here and in the rest of this thesis, we denote by

$$\pi^\perp_{\bar{\nu}} : \mathcal{O} \longrightarrow \mathcal{O}^\perp = \mathcal{O}/\mathbb{C}\bar{\nu}$$

the quotient project map, whenever $\bar{\nu}$ is a non-vanishing section of vector bundle $\mathcal{O}$.

Lemma 5.15 *Suppose $L_{\mathcal{Z}}, V_{\mathcal{Z}}, E_{\mathcal{Z}} \longrightarrow \bar{\mathcal{Z}}$ are ms-bundles of rank one, two, and $\big(2 - \frac{1}{2}\dim \bar{\mathcal{Z}}\big)$, respectively, and*

$$\alpha_{\mathcal{Z}} \in \Gamma\big(\bar{\mathcal{Z}}; \mathit{Hom}(E_{\mathcal{Z}}; L_{\mathcal{Z}}^* \otimes V_{\mathcal{Z}})\big)$$

is a regular polynomial. Let Σ be a smooth compact oriented two-manifold, $L_\Sigma, V_\Sigma \longrightarrow \Sigma$ smooth vector bundles of rank one and two, respectively, and $s \in \Gamma(\Sigma; L_\Sigma^* \otimes V_\Sigma)$ a nonvanishing section. Then for an open collection of nonvanishing sections, $\bar{\nu} \in \Gamma(\Sigma \times \bar{\mathcal{Z}}; V_\Sigma \otimes V_{\mathcal{Z}})$
(1) $\alpha^s_{\mathcal{Z},\bar{\nu}}$ is a regular polynomial;
(2) if $V_{\mathcal{Z}} \approx \bar{\mathcal{Z}} \times \mathbb{C}^2$, $N(\alpha^s_{\mathcal{Z},\bar{\nu}}) = \langle 3c_1(L_\Sigma^*) + 2c_1(V_\Sigma), [\Sigma]\rangle N(\alpha_{\mathcal{Z}})$.

Proof: (1) The first claim is clear. If $V_{\mathcal{Z}} \approx \bar{\mathcal{Z}} \times \mathbb{C}^2$, we can choose section $\bar{\nu} \in \Gamma(\Sigma \times \bar{\mathcal{Z}}; V_\Sigma \otimes V_{\mathcal{Z}})$ that does not intersect $V_\Sigma^+ \otimes V_{\mathcal{Z}}$, where $V_\Sigma^+ = \mathrm{Im}\, s$. Then $\pi^\perp_{\bar{\nu}}(V_\Sigma^+ \otimes V_{\mathcal{Z}})$ is a rank-two subbundle of $(V_\Sigma \otimes V_{\mathcal{Z}})^\perp$. Let

$$\mathcal{O}^+ = L_\Sigma^* \otimes L_{\mathcal{Z}}^* \otimes \pi^\perp_{\bar{\nu}}(V_\Sigma^+ \otimes V_{\mathcal{Z}}), \qquad \mathcal{O}^- \equiv \big(L_\Sigma^* \otimes L_{\mathcal{Z}}^* \otimes (V_\Sigma \otimes V_{\mathcal{Z}})^\perp\big)/\mathcal{O}^+.$$

We identify $\mathcal{O}^-$ with a complement of $\mathcal{O}^+$ in $L_\Sigma^* \otimes L_{\mathcal{Z}}^* \otimes (V_\Sigma \otimes V_{\mathcal{Z}})^\perp$.
(2) By definition, $N(\alpha_{\mathcal{Z}})$ is the number of zeros of the affine map

$$\psi_{\alpha_{\mathcal{Z}},\nu} : E_{\mathcal{Z}} \longrightarrow L_{\mathcal{Z}}^* \otimes V_{\mathcal{Z}}, \qquad \psi_{\alpha_{\mathcal{Z}},\nu}(b; v) = \nu_b + \alpha_{\mathcal{Z}}(v),$$

for a generic section $\nu \in \Gamma(\bar{\mathcal{Z}}; L_{\mathcal{Z}}^* \otimes V_{\mathcal{Z}}^*)$. Via the construction preceding the lemma, ν induces a section $\tilde{\nu}^+ \in \Gamma(\Sigma \times \bar{\mathcal{Z}}; \mathcal{O}^+)$. If ν and $\tilde{\nu}^- \in \Gamma(\Sigma \times \bar{\mathcal{Z}}; \mathcal{O}^-)$ are generic, $N(\alpha^s_{\mathcal{Z},\bar{\nu}})$ is the number of zeros of the affine map

$$\psi_{\alpha^s_{\mathcal{Z},\bar{\nu}}, \tilde{\nu}^+ + \tilde{\nu}^-} : E_{\mathcal{Z}} \longrightarrow \mathcal{O}^+ \oplus \mathcal{O}^-, \quad \psi_{\alpha^s_{\mathcal{Z},\bar{\nu}}, \tilde{\nu}^+ + \tilde{\nu}^-}(x, b; v) = \tilde{\nu}^+_{(x,b)} + \tilde{\nu}^-_{(x,b)} + \alpha^s_{\mathcal{Z},\bar{\nu}}(x, b; v).$$

The solution of the $\mathcal{O}^+$-part of this equation is precisely $\Sigma \times \psi^{-1}_{\alpha_{\mathcal{Z}},\nu}(0)$. Thus,

$$N(\alpha^s_{\mathcal{Z},\bar{\nu}}) =^{\pm} \big|\psi^{-1}_{\alpha^s_{\mathcal{Z},\bar{\nu}}, \tilde{\nu}^+ + \tilde{\nu}^-}(0)\big| = \langle c_1(\mathcal{O}^-), [\Sigma]\rangle N(\alpha_{\mathcal{Z}}),$$

as claimed.

Lemma 5.16 *Suppose $\bar{\mathcal{Z}}$ is an ms-manifold of dimension two, and Σ, $L_{\mathcal{Z}}$, $V_{\mathcal{Z}}$, $E_{\mathcal{Z}}$, L_Σ, V_Σ, $\alpha_{\mathcal{Z}}$ and s are in Lemma 5.15. Then for an open collection of sections $\bar{\nu}\in\Gamma(\Sigma\times\bar{\mathcal{Z}};V_\Sigma\otimes V_{\mathcal{Z}})$,*

$$\mathcal{C}_{\alpha^{s\,-1}_{\mathcal{Z},\bar{\nu}}(0)}\big(\alpha^{s\perp}_{\mathcal{Z},\bar{\nu}}\big) = \big\langle 3c_1(L^*_\Sigma)+2c_1(V_\Sigma),[\Sigma]\big\rangle\mathcal{C}_{\alpha^{-1}_{\mathcal{Z}}(0)}\big(\alpha^{\perp}_{\mathcal{Z}}\big);$$

$$N\big(\alpha^s_{\mathcal{Z},\bar{\nu}}\big) = \big\langle 3c_1(L^*_\Sigma)+2c_1(V_\Sigma),[\Sigma]\big\rangle N\big(\alpha_{\mathcal{Z}}\big) + \big\langle c_1(L^*_\Sigma)+c_1(V_\Sigma),[\Sigma]\big\rangle\big\langle c_1(V_{\mathcal{Z}}),[\bar{\mathcal{Z}}]\big\rangle.$$

Proof: (1) Since $\alpha_{\mathcal{Z}}$ is regular and $\bar{\mathcal{Z}}$ is two-dimensional, $\alpha^{-1}_{\mathcal{Z}}(0)$ is a finite set of points. Thus, we can trivialize the bundles $E_{\mathcal{Z}}$, $L_{\mathcal{Z}}$, and $V_{\mathcal{Z}}$ near $\alpha^{-1}_{\mathcal{Z}}(0)$, so that $\nu_z=(0,1)\in L^*_{\mathcal{Z}}\otimes V_{\mathcal{Z}}$, where ν is as in the proof of Lemma 5.15. Furthermore, for each $z\in\alpha^{-1}_{\mathcal{Z}}(0)$, there exists $d_z\geq 1$ such that

$$\big|\alpha_{\mathcal{Z}}(u)-(u^{d_z},0)\big|\leq\varepsilon(u)|u|^{d_z}\qquad\forall u\in\mathbb{C}_\delta,\qquad\text{with}\ \lim_{u\longrightarrow 0}\varepsilon(u)=0.$$

Then $\mathcal{C}_z\big(\alpha^{\perp}_{\mathcal{Z}}\big)=d_z$.
(2) By transversality and dimension-counting, we can choose $\bar{\nu}\in\Gamma(\Sigma\times\bar{\mathcal{Z}};V_\Sigma\otimes V_{\mathcal{Z}})$ such that

$$\mathrm{Im}\ \bar{\nu}\cap\big(s(L_\Sigma)\otimes\alpha_{\mathcal{Z}}(E_{\mathcal{Z}}\otimes L_{\mathcal{Z}})\big)=\emptyset\quad\text{and}\quad\mathrm{Im}\ \bar{\nu}\cap\big(s(L_\Sigma)\otimes\big(V_{\mathcal{Z}}|\Sigma\times\alpha^{-1}_{\mathcal{Z}}(0)\big)\big)=\emptyset.$$

Then, $\alpha^{s\,-1}_{\mathcal{Z},\bar{\nu}}(0)=\Sigma\times\alpha^{-1}_{\mathcal{Z}}(0)$. Furthermore, on a neighborhood of $\Sigma\times\alpha^{-1}_{\mathcal{Z}}(0)$, we can define a splitting

$$L^*_\Sigma\otimes(V_\Sigma\otimes\mathbb{C}^2)^{\perp}=\mathcal{O}^+\oplus\mathcal{O}^-\approx\mathbb{C}^2\oplus\mathcal{O}^-,$$

as in the proof of Lemma 5.15. Let

$$\tilde{\nu}\in\Gamma\big(\Sigma\times\mathcal{Z};L^*_\Sigma\otimes L^*_{\mathcal{Z}}\otimes(V_\Sigma\otimes V_{\mathcal{Z}})^{\perp}\big)$$

be a nonvanishing section such that on a neighborhood of $\Sigma\times\alpha^{-1}_{\mathcal{Z}}(0)$, $\tilde{\nu}=\tilde{\nu}^++\tilde{\nu}^-$, with $\tilde{\nu}^+$ as in the proof of Lemma 5.15. Then,

$$\big(\mathcal{O}^+\oplus\mathcal{O}^-\big)/\mathbb{C}\tilde{\nu}\approx\mathbb{C}\oplus\mathcal{O}^-,\quad\text{and}\quad\big|\alpha^{s\perp}_{\mathcal{Z},\bar{\nu}}-(u^{d_z},0)\big|\leq\varepsilon(u)|u|^{d_z}\quad\forall u\in\mathbb{C}_\delta.$$

Thus, by Proposition 5.13,

$$\mathcal{C}_{\Sigma\times\{z\}}\big(\alpha^{s\perp}_{\mathcal{Z},\bar{\nu}}\big)=d_z\big\langle c_1(\mathcal{O}^-),[\Sigma]\big\rangle=\mathcal{C}_z\big(\alpha_{\mathcal{Z}}\big)\big\langle 3c_1(L^*_\Sigma)+2c_1(V_\Sigma),[\Sigma]\big\rangle.$$

The second claim follows from the first, since by Lemma 5.14 and Proposition 5.13,

$$N\big(\alpha_{\mathcal{Z}}\big)=\big\langle c_1\big(L^*_{\mathcal{Z}}\otimes V_{\mathcal{Z}}\big)+c_1(E_{\mathcal{Z}}),[\bar{\mathcal{Z}}]\big\rangle-\mathcal{C}_{\alpha^{-1}_{\mathcal{Z}}(0)}\big(\alpha_{\mathcal{Z}}\big);$$

$$N\big(\alpha^s_{\mathcal{Z},\bar{\nu}}\big)=\big\langle c_2(\mathcal{O})+c_1(E^*)c_1(\mathcal{O}),\big[\Sigma\times\bar{\mathcal{Z}}\big]\big\rangle-\mathcal{C}_{\alpha^{s\,-1}_{\mathcal{Z},\bar{\nu}}(0)}\big(\alpha^{s\perp}_{\mathcal{Z},\bar{\nu}}\big),$$

where $\mathcal{O}=L^*_\Sigma\otimes L^*_{\mathcal{Z}}\otimes(V_\Sigma\otimes V_{\mathcal{Z}})^{\perp}$.

Corollary 5.17 *Suppose $\bar{\mathcal{M}}$ is an ms-manifold of dimension four, $L_{\mathcal{M}},V_{\mathcal{M}}\longrightarrow\bar{\mathcal{M}}$ are ms-bundles of rank one and two, respectively, and $\alpha\in\Gamma\big(\bar{\mathcal{M}};Hom(L_{\mathcal{M}},V_{\mathcal{Z}})\big)$ is a regular polynomial. If Σ is a compact smooth oriented manifold of dimension two, $L_\Sigma,V_\Sigma\longrightarrow\Sigma$ are smooth vector bundles of rank one and two, respectively, and $s\in\Gamma\big(\Sigma;Hom(L_\Sigma,V_\Sigma)\big)$ is a* *nonvanishing section, then*

$$N(\alpha\otimes s) = \Big\langle\big(c_1(L_\Sigma^*)+c_1(V_\Sigma)\big)\big(c_1(L_\mathcal{M}^*)c_1(V_\mathcal{M})+c_1^2(V_\mathcal{M})\big) - c_1(L_\Sigma^*)c_2(V_\mathcal{M}), \big[\Sigma\times\bar{\mathcal{M}}\big]\Big\rangle$$
$$- \big\langle c_1(L_\Sigma^*)+c_1(V_\Sigma), [\Sigma]\big\rangle \sum_{\alpha^{-1}(0)=\bigsqcup \mathcal{Z}_i} \big\langle c_1(V_\mathcal{M}), [\bar{\mathcal{Z}}_i]\big\rangle,$$

where the sum is taken over all α-regular subsets $\mathcal{Z}_i$ in a decomposition of $\alpha^{-1}(0)$ as in Proposition 5.13.

Proof: By Lemma 5.14,

$$N(\alpha\otimes s) = \big\langle c_3\big(L_\Sigma^*\otimes L_\mathcal{M}^* \otimes (V_\Sigma\otimes V_\mathcal{M})^\perp\big), \big[\Sigma\times\bar{\mathcal{M}}\big]\big\rangle - \mathcal{C}_{\Sigma\times\alpha^{-1}(0)}\big((\alpha\otimes s)^\perp\big).$$

The last term can be written as the sum of terms as in the second equation of Lemma 5.16. On the other hand, by Proposition 5.13,

$$\sum_{\alpha^{-1}(0)=\bigsqcup \mathcal{Z}_i} N(\alpha_{\mathcal{Z}_i}) = \big\langle c_2\big(L_\mathcal{M}^*\otimes V_\mathcal{M}\big), [\bar{\mathcal{M}}]\big\rangle.$$

Thus, the claim follows from Lemma 5.16.

Chapter 6

Rational Curves with Singularities

In this chapter, we use the topological tools of Chapter 5 and the analytic estimates of Theorem 6.2, deduced from the descriptive statement of Theorem 3.33, to count rational curves with certain singularities in $\mathbb{P}^2$ and $\mathbb{P}^3$. In particular, we prove Lemma 6.3 and Proposition 1.1, enumerating plane curves with a simple cusp and plane curves with a (3, 4)-cusp, respectively. Lemma 6.5 gives a practical topological formula for determining the number of two-component rational plane curves with a node common to both components at which one of the components has a cusp. Lemma 6.4 gives such a formula for the number of two-component rational curves in $\mathbb{P}^3$ connected at a tacnode. Of course, in all cases, the curves are required to pass through an appropriate collection of constraints, depending on the total degree of each curve, so that the expected answer is finite. All formulas involve top intersections of tautological classes of moduli spaces of stable rational maps into $\mathbb{P}^n$. Section 6.10 describes an algorithm for computing such intersection numbers in the cases we need. The formula of Lemma 6.3 is known in algebraic geometry. Its proof is included for the sake of completeness. It also provides a simple illustration of applications of Theorem 6.2 and Chapter 5. The result of Lemma 6.5 in the degree-four case agrees with the number obtained by P. Aluffi based on a formula in [AF2].

We also compute top intersections of tautological classes on certain spaces of stable maps parameterizing positive-dimensional spaces of rational curves with singularities. All of the results of this chapter are used in Chapters 9 and 10. These results represent only a tiny portion of the range of applications of Theorem 6.2 and Chapter 5.

6.1 Preliminaries

The rest of this thesis concerns enumerative geometry of projective spaces. We identify $H_2(\mathbb{P}^n;\mathbb{Z})$ with $\mathbb{Z}$ in the usual way. Thus, bubble types of maps into $\mathbb{P}^n$ will be written as $\mathcal{T}=(S,M,I;j,\underline{d})$, where $d_i\in\mathbb{Z}$ for all $i\in I$. It will always be assumed that all the numbers d_i are nonnegative; otherwise, the spaces $\mathcal{M}_{\mathcal{T}}$, $\mathcal{U}_{\mathcal{T}}$, etc. are empty.

If $\mathcal{T}$ is as above, for every $h\in I$, we define $\chi_{\mathcal{T}}h\in\mathbb{Z}$ by

$$\chi_{\mathcal{T}}h=\begin{cases}0, & \text{if } \forall i\in I \text{ s.t. } h\in\bar{D}_i\mathcal{T},\ d_i=0;\\ 1, & \text{if } d_h\neq 0, \text{ but } \forall i\in I \text{ s.t. } h\in D_i\mathcal{T},\ d_i=0;\\ 2, & \text{otherwise.}\end{cases}$$

Let $\chi(\mathcal{T})=\chi_{\mathcal{T}}^{-1}(1)$. This subset of I will play a central role in many explicit computations of this thesis.

Theorem 3.33 requires us to specify a smooth family $\{g_{\mathbb{P}^n,b}\}$ of Kahler metrics on $\mathbb{P}^n$. For computational reasons, we do not take $g_{\mathbb{P}^n,b}$ to be the standard Fubini-Study metric for all b. Instead, whenever $S=S^2$, we let $g_{\mathbb{P}^n,b}=g_{\mathbb{P}^n,\mathrm{ev}(b)}$ for all $b\in\mathcal{U}_{\mathcal{T}}$, where $\{g_{\mathbb{P}^n,q}:q\in\mathbb{P}^n\}$ is the family of metrics described by Lemma 6.1.

Lemma 6.1 *There exist $r_{\mathbb{P}^n}>0$ and a smooth family of Kahler metrics $\{g_{\mathbb{P}^n,q}:q\in\mathbb{P}^n\}$ on $\mathbb{P}^n$ with the following property. If $B_q(q',r)\subset\mathbb{P}^n$ denotes the $g_{\mathbb{P}^n,q}$-geodesic ball about q' of radius r, the triple $(B_q(q,r_{\mathbb{P}^n}),J,g_{\mathbb{P}^n,q})$ is isomorphic to a ball in $\mathbb{C}^n$ for all $q\in\mathbb{P}^n$.*

Proof: On the open set $U_0=\{[X_0:\ldots:X_n]\in\mathbb{P}^n:X_0\neq0\}$, the Fubini-Study symplectic form is given by

$$\omega_{\mathbb{P}^n}=\frac{i}{2\pi}\partial\bar{\partial}\ln(1+f_0),\quad\text{where}\quad f_0([X_0:\ldots:X_n])=\sum_{k\in[n]}|X_k/X_0|^2; \tag{6.1}$$

see [GH, p31]. Let $q=[1:0:\ldots:0]$. Set

$$\omega_{\mathbb{P}^n,q,\epsilon}=\frac{i}{2\pi}\partial\bar{\partial}\big\{f_0+(\beta_{\epsilon^2}\circ f_0)\big(\ln(1+f_0)-f_0\big)\big\}. \tag{6.2}$$

Note that $\omega_{\mathbb{P}^n,q,\epsilon}$ agrees with $\omega_{\mathbb{P}^n}$ outside of the set $\{f_0\le2\epsilon\}$ and with the standard symplectic form $\omega_{\mathbb{C}^n}$ on $\{f_0\le\epsilon\}$. Here we view $\omega_{\mathbb{C}^n}$ as a form on U_0 via the coordinates $z_{0,i}=X_i/X_0$, $i\in[n]$. In particular, $\omega_{\mathbb{P}^n,q,\epsilon}$ is globally defined, and the corresponding Riemannian metric on $\{f_0\le\epsilon\}$ is flat. Furthermore,

$$\begin{aligned}\omega_{\mathbb{P}^n,q,\epsilon}=&\big\{(1-\beta_{\epsilon^2}\circ f_0)\omega_{\mathbb{C}^n}+(\beta_{\epsilon^2}\circ f_0)\omega_{\mathbb{P}^n}\big\}\\&+\frac{i}{2\pi}\Big\{\big(\partial(\beta_{\epsilon^2}\circ f_0)\big)\big(\bar{\partial}\tilde{f}_0\big)-\big(\bar{\partial}(\beta_{\epsilon^2}\circ f_0)\big)\big(\partial\tilde{f}_0\big)+\big(\partial\bar{\partial}(\beta_{\epsilon^2}\circ f_0)\big)\tilde{f}_0\Big\},\end{aligned} \tag{6.3}$$

where $\tilde{f}_0=\ln(1+f_0)-f_0$. On the set $\{f_0\le2\epsilon\}$ with $\epsilon\le\frac{1}{2}$,

$$\big\|\tilde{f}_0\big\|_{C^0}\le C\epsilon^2\quad\text{and}\quad\big\|d\tilde{f}_0\big\|_{C^0}\le C\epsilon^{\frac{3}{2}}, \tag{6.4}$$

where $\|d\tilde{f}_0\|_{C^0}$ denotes the C^0-norm with respect to the standard metric on $\mathbb{C}^n$. Furthermore, by (2.2),

$$\big\|d(\beta_{\epsilon^2}\circ f_0)\big\|_{C^0}\le C\epsilon^{-1}\epsilon^{\frac{1}{2}},\quad\big\|\nabla^2(\beta_{\epsilon^2}\circ f_0)\big\|_{C^0}\le C\big(\epsilon^{-2}\epsilon^{\frac{1}{2}}\epsilon^{\frac{1}{2}}+\epsilon^{-1}\big), \tag{6.5}$$

where again all the norms are computed with respect to the standard metric on $\mathbb{C}^n$. Equations (6.4) and (6.5) imply that the term on the second line of (6.3) tends to 0 as ϵ goes to 0. Thus by (6.3), we can choose $\epsilon>0$ such that $\omega_{\mathbb{P}^n,q}\equiv\omega_{\mathbb{P}^n,q,\epsilon}$ is a symplectic form on all of $\mathbb{P}^n$. Note that $\omega_{\mathbb{P}^n,q}$ is invariant under the action of the stabilizer of q in SU_{n+1}, which is the subgroup

$$\mathrm{Stab}_p(SU_{n+1})=\left\{\begin{pmatrix}\overline{\det(h)}&0\\0&h\end{pmatrix}:h\in U_n\right\}\subset SU_{n+1}.$$

We can define a smooth family of symplectic Kahler forms on $\mathbb{P}^n$ by

$$\omega_{\mathbb{P}^n, g\cdot q} = g^* \omega_{\mathbb{P}^n, q}, \qquad g \in SU_{n+1}.$$

The above invariance property of $\omega_{\mathbb{P}^n,q}$ insures that $\omega_{\mathbb{P}^n,g\cdot q}$ depends only on $g\cdot q$. We can now take $g_{\mathbb{P}^n,g\cdot q}$ to be the metric corresponding to the symplectic form $\omega_{\mathbb{P}^n,g\cdot q}$ and the standard complex structure J on $\mathbb{P}^n$.

6.2 Main Structure Theorem, $S = S^2$

Given a bubble type $\mathcal{T}=(S^2, M, I; j, \underline{d})$, we define sections $\mathcal{D}^{(k)}_{\mathcal{T},i}$, of the bundle $L_i^*\mathcal{T}^{\otimes m} \otimes \mathrm{ev}^* T\mathbb{P}^n$ over $\mathcal{U}_{\mathcal{T}}(\mu)$, where i is an element of I and k is a positive integer. The zero sets of these sections describe rational curves with certain singularities. Often such sections patch together to define a continuous section over $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$. Theorem 6.2 and its proof describe the behavior of the sections $\mathcal{D}^{(k)}_{\mathcal{T}^*,i}$ near the boundary strata,

$$\partial\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu) \equiv \bar{\mathcal{U}}_{\mathcal{T}^*}(\mu) - \mathcal{U}_{\mathcal{T}^*}(\mu),$$

of $\mathcal{U}_{\mathcal{T}^*}(\mu)$ whenever $\mathcal{T}^*$ is a basic bubble type. The description of Theorem 6.2 is used to apply the topological tools of Chapter 5 to specific problems in enumerative geometry.

If $b=\big(S^2, M, I; x, (j,y), u\big) \in \mathcal{B}_{\mathcal{T}}$, $k \geq 1$, and $i \in I$, let

$$\mathcal{D}^{(k)}_{\mathcal{T},i} b = \frac{2}{(k-1)!} \frac{D^{k-1}}{ds^{k-1}} \frac{d}{ds} (u_i \circ q_S)\Big|_{(s,t)=0},$$

where the covariant derivatives are taken with respect to the metric $g_{\mathbb{P}^n,b}$ and $s+it \in \mathbb{C}$. If $\mathcal{T}^*$ is a basic bubble type, the maps $\mathcal{D}^{(k)}_{\mathcal{T},i}$ with $\mathcal{T} < \mathcal{T}^*$ and $i \in I - \hat{I}$ induce a continuous section of $\mathrm{ev}^* T\mathbb{P}^n$ over $\bar{\mathcal{U}}^{(0)}_{\mathcal{T}^*}$ and a continuous section of the bundle $L_i^*\mathcal{T}^{*\otimes k} \otimes \mathrm{ev}^* T\mathbb{P}^n$ over $\bar{\mathcal{U}}_{\mathcal{T}^*}$, described by

$$\mathcal{D}^{(k)}_{\mathcal{T}^*,i}[b, c_i] = c_i^k \mathcal{D}^{(k)}_{\mathcal{T},i} b, \quad \text{if } b \in \mathcal{U}^{(0)}_{\mathcal{T}}, \ c_i \in \mathbb{C}.$$

We will often write $\mathcal{D}_{\mathcal{T},i}$ instead of $\mathcal{D}^{(1)}_{\mathcal{T},i}$. If $\mathcal{T}$ is simple, we will abbreviate $\mathcal{D}^{(k)}_{\mathcal{T},i}$ as $\mathcal{D}^{(k)}$. If $\mathcal{T}=(\Sigma, [N], I; j, \underline{d})$ is a simple bubble type such that $d_{\hat{0}}=0$, let

$$\bar{\mathcal{T}} = \big(S^2, [N] - M_{\hat{0}}\mathcal{T}, \hat{I}; j|([N]-M_{\hat{0}}\mathcal{T}), \underline{d}|\hat{I}\big)$$

and denote by $\mathcal{D}^{(k)}_{\mathcal{T},i}$ denote the section $\mathcal{D}^{(k)}_{\bar{\mathcal{T}},i}$ for any $i \in \hat{I}$.

Theorem 6.2 *If $\mathcal{T}^* = (S^2, M, I^*; j^*, \underline{d}^*)$ is a basic bubble type and μ is an $\tilde{M}$-tuple of constraints in general position, the spaces $\bar{\mathcal{U}}^{(0)}_{\mathcal{T}^*}(\mu)$ and $\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$ are oriented topological orbifolds. If $\mathcal{T} < \mathcal{T}^*$, there exist $G_{\mathcal{T}^*}$-invariant functions $\delta, C \in C^\infty\big(\mathcal{U}^{(0)}_{\mathcal{T}}(\mu); \mathbb{R}^+\big)$ and a $G_{\mathcal{T}^*}$-equivariant homeomorphism,*

$$\tilde{\gamma}^{\mu}_{\mathcal{T}} \colon F\mathcal{T}_{\delta}\Big|\mathcal{U}^{(0)}_{\mathcal{T}}(\mu) \longrightarrow \bar{\mathcal{U}}^{(0)}_{\mathcal{T}^*}(\mu),$$

onto an open neighborhood of $\mathcal{U}^{(0)}_{\mathcal{T}}(\mu)$ in $\bar{\mathcal{U}}^{(0)}_{\mathcal{T}^*}(\mu)$ such that $\tilde{\gamma}^{\mu}_{\mathcal{T}}|\mathcal{U}^{(0)}_{\mathcal{T}}(\mu)$ is the identity and

$\tilde{\gamma}^{\mu}_{\mathcal{T}}|\mathcal{F}^{(\emptyset)}\mathcal{T}_\delta$ *is an orientation-preserving diffeomorphism onto an open subset of* $\mathcal{U}^{(0)}_{\mathcal{T}}(\mu)$. *Furthermore, for all* $k \in I^*$ *and any*

$$v = \big[(b, v_h)_{h\in\hat{I}}\big] = \big[\big(S^2, M, I; x, (j, y), u\big), (v_h)_{v\in\hat{I}}\big] \in F\mathcal{T}_\delta\Big|\mathcal{U}^{(0)}_{\mathcal{T}}(\mu),$$

$$\Big|\Pi^{-1}_{b_v, ev(\tilde{\gamma}^{\mu}_{\mathcal{T}}(v))}\big(\mathcal{D}_{\mathcal{T}^*,k}\tilde{\gamma}^{\mu}_{\mathcal{T}}(v)\big) - \sum_{h\in I_k\cap\chi(\mathcal{T})}\Big(\prod_{i\in\hat{I}, h\in\bar{D}_i\mathcal{T}} v_i\Big)\big(\mathcal{D}_{\mathcal{T},h}b\big)\Big|$$
$$\le C(b_v)|v|^{\frac{1}{p}} \sum_{h\in I_k\cap\chi(\mathcal{T})}\Big(\prod_{i\in\hat{I}, h\in\bar{D}_i\mathcal{T}} |v_i|\Big),$$

where $I_k \subset I$ *is the rooted tree containing* k.

Remark: This theorem states that there exists an identification, $\gamma^{\mu}_{\mathcal{T}}\colon \mathcal{F}\mathcal{T}_\delta \longrightarrow \bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$, of neighborhoods of $\mathcal{U}_{\mathcal{T}}$ in $\mathcal{F}\mathcal{T}$ and in $\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$. Furthermore, with appropriate identifications,

$$\Big|\mathcal{D}_{\mathcal{T}^*,k}\gamma^{\mu}_{\mathcal{T}}(v) - \alpha_{\mathcal{T}}\big(\rho_{\mathcal{T}}(v)\big)\Big| \le C(b_v)|v|^{\frac{1}{p}}\big|\rho_{\mathcal{T}}(v)\big|, \qquad \text{where} \tag{6.6}$$

$$\rho_{\mathcal{T}}(v) = \big(b, (\tilde{v}_h)_{h\in\chi(\mathcal{T})}\big) \in \tilde{\mathcal{F}}\mathcal{T} \equiv \bigoplus_{h\in\chi(\mathcal{T})} L_h\mathcal{T}\otimes L^*_{\iota_h}\mathcal{T};\ \ \tilde{v}_h = \prod_{i\in\hat{I}, h\in\bar{D}_i\mathcal{T}} v_i;\ \ \iota_h \in I - \hat{I},\ h \in \bar{D}_{\iota_h}\mathcal{T};$$

$$\alpha_{\mathcal{T}}\big(b, (\tilde{v}_h)_{h\in\chi_{\mathcal{T}}}\big) = \sum_{h\in I_k\cap\chi_{\mathcal{T}}h} \mathcal{D}_{\mathcal{T},h}\tilde{v}_h.$$

This estimate is used frequently in the rest of this thesis. Note that if $\mathcal{T}$ is a semiprimitive bubble type, the bundle $\mathcal{F}\mathcal{T}$ is defined over $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$. However, $\mathcal{F}\mathcal{T}$ is *not* the normal bundle of $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ in $\bar{\mathcal{U}}_{\langle\mathcal{T}\rangle}(\mu)$ unless $M_{\hat{0}}\mathcal{T}+H_{\hat{0}}\mathcal{T}$ is a two-element set; see [P2]. The theorem implies only that the restrictions of the normal bundle of $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ in $\bar{\mathcal{U}}_{\langle\mathcal{T}\rangle}(\mu)$ and of $F\mathcal{T}$ to $\mathcal{U}_{\mathcal{T}}(\mu)$ are isomorphic.

Proof: (1) All statements of this theorem, except for the analytic estimate, follow immediately from Theorem 3.33. We deduce the analytic estimate from (2) of Theorem 3.33. Let

$$\gamma^{\mu}_{\mathcal{T}}(v) = \big(S^2, M, I(v); x(v), (j(v), y(v)), \tilde{u}_v\big).$$

By Theorem 3.33, there exist a holomorphic bubble map

$$b' = \big[S^2, M, I; x', (j, y'), u'\big]$$

such that $d_{C^k}(b, b') \le C(b_v)|v|^{\frac{1}{p}}$ and with appropriate identifications, $\tilde{u}_v = \exp_{b', u_{b'}\circ q_v}\xi$ for some $\xi \in \Gamma(u_{b'}\circ q_v)$ with $\|\xi\|_{b,C^0} \le C(b_v)|v|^{\frac{1}{p}}$. Thus, for the purposes of proving the analytic estimate, we can assume that $u_v = \exp_{b, u_b\circ q_v}\xi_v$ for $\xi \in \Gamma(u\circ q_v)$ with $\|\xi_v\|_{b,C^0} \le C(b_v)|v|^{\frac{1}{p}}$, i.e. it is enough to prove the estimate for the map $\tilde{\gamma}_{\mathcal{T}}$ defined in Section 3.6 with $\mathcal{T}$ a simple bubble type. If $d_k \ne 0$, the claim is immediate from the usual Sobolev and elliptic estimates near (k, ∞). Thus, we assume that $d_{\hat{0}} = 0$. For future use, we obtain equations describing the behavior of $\big(\mathcal{D}^{(m)}\tilde{\gamma}_{\mathcal{T}}(v)\big)$ for all $m \ge 1$.

(2) We identify $B_{g_{\mathbb{P}^n},b}\big(\mathrm{ev}(b), \frac{1}{2}r_{\mathbb{P}^n}\big)$ with an open subset of $\mathbb{C}^n$ via the $g_{\mathbb{P}^n,\mathrm{ev}(b)}$-parallel trans-

port along the geodesics from $\mathrm{ev}(b)$. We assume that $\delta\in C^\infty(\mathcal{U}_{\mathcal{T}}^{(0)};\mathbb{R}^+)$ satisfies

$$C(b)\delta(b)^{\frac{1}{2p}}+\delta(b)^{\frac{1}{2}}\Big(\sum_{i\in M}\|du_i\|_{b,C^0}\Big)<\frac{1}{2}r_{\mathbb{P}^n}.$$

Let $q\colon B_1(0;\mathbb{C})\longrightarrow S^2$ be the local stretching map as in Section 2.3 with $v=1$ defined with respect to the standard metric on $\mathbb{C}$. Let $f_v=u_v\circ q$ and $\tilde{f}_v=\tilde{u}_v\circ q$. We denote the usual complex coordinate on $\mathbb{C}$ by z. For any $z\in B_1(0;\mathbb{C})$, let $i_v(z)$ be such that $q_v(q(z))\in\Sigma_{b,i_v(z)}$. If $X\in T_{\mathrm{ev}(b)}\mathbb{P}^n$ and $m\geq 1$, define $R_vX\psi^{(m)}\in\Gamma^{0,1}(\tilde{f}_v)$ by

$$R_vX\psi^{(m)}\big|_z=\begin{cases}X\bar{z}^{m-1}d\bar{z}, & \text{if } \chi_{\mathcal{T}}i_v(z)=0;\\ \beta\big(\delta(b_v)|q_v(q(z))|\big)X\bar{z}^{m-1}d\bar{z}, & \text{if } \chi_{\mathcal{T}}i_v(z)=1;\\ 0, & \text{if } \chi_{\mathcal{T}}i_v(z)=2.\end{cases}$$

Note that if $\chi_{\mathcal{T}}i_v(z)=0$, or $\chi_{\mathcal{T}}i_v(z)=1$ and $\beta\big(\delta(b_v)|q_v(q(z))|\big)$ is nonzero, $\tilde{f}_v(z)$ lies in $B_{g_{\mathbb{P}^n},b}\big(\mathrm{ev}(b),\frac{1}{2}r_{\mathbb{P}^n}\big)$. Thus, $R_vX\psi^{(m)}$ is well-defined. We now compute $\langle\langle\bar{\partial}\tilde{f}_v,R_vX\psi^{(m)}\rangle\rangle$ in two ways and compare the results. First, note that the map $\tilde{f}_v$ is holomorphic outside of the annulus

$$A_{\hat{0}}(v)\equiv B_1(0;\mathbb{C})-B_{\frac{1}{2}}(0;\mathbb{C}).$$

Thus, by the same computation as in the proof of Lemma 7.5, we see that

$$\langle\langle\bar{\partial}\tilde{f}_v,R_vX\psi^{(m)}\rangle\rangle=-\frac{\pi}{m}\Big\langle\mathcal{D}^{(m)}\tilde{\gamma}_{\mathcal{T}}(v),X\Big\rangle. \tag{6.7}$$

(3) Since $\tilde{f}_v=\exp_{\mathrm{ev}(b),f_v}(\xi_v\circ q)$ and f_v is constant on $A_{\hat{0}}(v)$,

$$2i\langle\langle\bar{\partial}\tilde{f}_v,R_vX\psi^{(m)}\rangle\rangle=\int_{A_{\hat{0}}(v)}\Big\langle\frac{\bar{\partial}}{\partial\bar{z}}(\xi_v\circ q),X\Big\rangle z^{m-1}d\bar{z}\wedge dz \tag{6.8}$$

Denote by $A_{\hat{0}}^+(v)$ and $A_{\hat{0}}^-(v)$ the outer and inner boundary of $A_{\hat{0}}(v)$, respectively. For every $h\in\hat{I}\cap\chi(\mathcal{T})$, let

$$A_h(v)=q_{v,\iota_h}^{-1}\Big(\big\{z\in\Sigma_{b_v,\iota_h}\colon 4\delta(b_v)^{-1}|v_h|\leq|\phi_{b,h}^{-1}z|\leq|v_h|^{\frac{1}{2}}\big\}\Big)\subset\Sigma_{b_v,\hat{0}}.$$

Denote by $A_h^{\pm}(v)$ the outer and inner boundary of $A_h(v)$. Let w be the complex coordinate on $\mathbb{C}\subset\Sigma_{b_v,\hat{0}}=S^2$. Note that q is holomorphic inside of $A_{\hat{0}}^-(v)$ and outside of $q^{-1}(A_h^-(v))$. Furthermore, since u_b and $\tilde{u}_v$ are both holomorphic, on the image of this set under q

$$\frac{\bar{\partial}}{\partial\bar{w}}\xi_v=-\frac{\bar{\partial}}{\partial\bar{w}}u_v.$$

The last quantity vanishes outside of the annuli $A_h(v)$. Thus by integration by parts,

$$\begin{aligned}
&\int_{A_{\hat{0}}(v)} \Big\langle \frac{\bar{\partial}}{\partial \bar{z}}(\xi_v \circ q), X \Big\rangle z^{m-1} d\bar{z} \wedge dz \\
&= \sum_{h \in \hat{I} \cap \chi(\mathcal{T})} \Big(\int_{q^{-1}(A_h(v))} \Big\langle \Big(\frac{\bar{\partial} u_v}{\partial \bar{w}}\Big) \overline{\Big(\frac{\partial q}{\partial z}\Big)}, X \Big\rangle z^{m-1} d\bar{z} \wedge dz + \int_{q^{-1}(A_h^-(v))} \langle \xi_v \circ q, X \rangle z^{m-1} dz \Big) \\
&= \sum_{h \in \hat{I} \cap \chi(\mathcal{T})} \Big(\int_{A_h(v)} \Big\langle \frac{\bar{\partial} u_v}{\partial \bar{w}}, X \Big\rangle g d\bar{w} \wedge dw + \int_{A_h^-(v)} \langle \xi_v, X \rangle g dw \Big), \qquad (6.9)
\end{aligned}$$

where $g(w) = w^{m-1}$. Since $\xi_v \circ q$ is constant on $A_{\hat{0}}^+(v)$, the second boundary term is zero. Note that the radius of $A_h^-(v)$ in $\mathbb{C} \subset S^2$ is bounded by $C(b_v)|\tilde{v}_h|$. Furthermore, $|g| \le C_m(b_v)$ on $A_h^-(v)$. It follows that

$$\Big| \int_{A_h^-(v)} \langle \xi_v, X \rangle g dw \Big| \le C_m(b_v) |v|^{\frac{1}{p}} |\tilde{v}_h|. \qquad (6.10)$$

On the other hand, by the same computation as in the proof of Lemma 7.5,

$$\int_{A_h(v)} \Big\langle \frac{\bar{\partial} u_v}{\partial \bar{w}}, X \Big\rangle g d\bar{w} \wedge dw = -2i \sum_{m'=1}^{m'=m} \frac{\pi a_{m',h}(v)}{m'} \tilde{v}_h^{m'} \big(\mathcal{D}_{\mathcal{T},h}^{(m')} b\big), \qquad (6.11)$$

where the numbers $a_{m',h}(v) \in \mathbb{C}$ are defined by

$$g\big(q_{v,\iota_h}^{-1}(\iota_h, y)\big) = \sum_{m'=1}^{m'=m} a_{m',h}(v) \cdot \big(\phi_{b_v,h} y\big)^{m'-1}$$

whenever $y \in S^2$ is close to x_h. Combining equations (6.7)-(6.11), we see that

$$\Big| \Big\langle \mathcal{D}^{(m)} \tilde{\gamma}_{\mathcal{T}}(v), X \Big\rangle - m \sum_{h \in \hat{I} \cap \chi(\mathcal{T})} a_{1,h}(v) \tilde{v}_h \big(\mathcal{D}_{\mathcal{T},h} b\big) \Big| \le C(b_v) |v|^{\frac{1}{p}} \Big(\sum_{h \in \hat{I} \cap \chi(\mathcal{T})} |\tilde{v}_h| \Big). \qquad (6.12)$$

Since $a_{1,h}(v) = \big(q_{v,\iota_h}(\iota_h, y)\big)^{m-1}$, equation (6.12) gives the analytic estimate of the theorem.

6.3 Notation for Final Formulas

The final answers for a number of enumerative problems are stated in this thesis in terms of top intersections of cohomology classes that are closely related to tautological classes of moduli spaces of stable rational maps into $\mathbb{P}^n$. In this section, we describe the spaces and the cohomology classes involved.

Suppose $\mathcal{T} = (S^2, M, I; j, \underline{d})$ is a bubble type, $\{\mathcal{T}_k = (S^2, M_k, I_k; j_k, \underline{d}_k)\}$ are the corresponding simple types, $k \in I - \hat{I}$, and M_0 is nonempty subset of $M_k \mathcal{T}$. Let

$$\mathcal{T}/M_0 = \big(S^2, \hat{I}, M - M_0; j|(M - M_0), d|\hat{I}\big).$$

Let $I+_k\hat{1}$ be the linearly ordered set consisting of the elements of I along with one more element $\hat{1}$. The partial-order relations of $I+_k\hat{1}$ are the partial-order relations of I along with the relations

$$k<\hat{1}, \qquad \hat{1}<h \text{ if } h\in\hat{I}_k.$$

Define $\mathcal{T}(M_0)\equiv(S^2,M,I+_k\hat{1};j',\underline{d}')$ by

$$j'_l=\begin{cases}k, & \text{if } l\in M_0;\\ \hat{1}, & \text{if } l\in M_k\mathcal{T}-M_0;\\ j_l, & \text{otherwise;}\end{cases} \qquad d'_i=\begin{cases}0, & \text{if } i=k;\\ d_k, & \text{if } i=\hat{1};\\ d_i, & \text{otherwise.}\end{cases}$$

The tuples $\mathcal{T}/M_0$ and $\mathcal{T}(M_0)$ are bubble types as long as $d_k\neq0$ or $M_0\neq M_{\hat{0}}\mathcal{T}$. Then,

$$\bar{\mathcal{U}}_{\mathcal{T}(M_0)}(\mu)=\bar{\mathcal{M}}_{0,\{\hat{1}\}+M_0}\times\bar{\mathcal{U}}_{\mathcal{T}/M_0}(\mu), \tag{6.13}$$

where $\bar{\mathcal{M}}_{0,\{\hat{1}\}+M_0}$ is the Deligne-Mumford moduli space of rational curves with $(\{\hat{0},\hat{1}\}+M_0)$-marked points; see Section 2.4. If $l\in M_k\mathcal{T}$ for some $k\in I-\hat{I}$, we denote $\mathcal{T}(\{l\})$ by $\mathcal{T}(l)$. If $\mathcal{T}$ is a basic bubble type, by Theorem 6.2 and decomposition (6.13), $\bar{\mathcal{U}}_{\mathcal{T}(M_0)}(\mu)$ is an oriented topological suborbifold of $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ of (real) codimension two. Thus,

$$c_1(\mathcal{L}_k^*\mathcal{T})\equiv c_1(L_k^*\mathcal{T})-\sum_{M_0\subset M_k,M_0\neq\emptyset}PD_{\bar{\mathcal{U}}_{\mathcal{T}}(\mu)}\big[\bar{\mathcal{U}}_{\mathcal{T}(M_0)}(\mu)\big]\in H^2\big(\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\big), \tag{6.14}$$

where $PD_{\bar{\mathcal{U}}_{\mathcal{T}}(\mu)}\big[\bar{\mathcal{U}}_{\mathcal{T}(M_0)}(\mu)\big]$ denotes the Poincare Dual of $\big[\bar{\mathcal{U}}_{\mathcal{T}(M_0)}(\mu)\big]$ in $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$, is a well-defined cohomology class. Since our constraints μ are disjoint, $\bar{\mathcal{U}}_{\mathcal{T}(M_0)}(\mu)=\emptyset$ if $|M_0|\geq2$. Furthermore, it is well-known in algebraic geometry that for any $l\in M_k$ the normal bundle of $\bar{\mathcal{U}}_{\mathcal{T}(l)}(\mu)$ in $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ is $L_{\hat{1}}\mathcal{T}(l)$; see [P2]. Thus, if μ is an M-tuple of disjoint constraints,

$$\big[\bar{\mathcal{U}}_{\mathcal{T}(l)}(\mu)\big]\cap c_1(\mathcal{L}_k^*\mathcal{T})=\big[\bar{\mathcal{U}}_{\mathcal{T}(l)}(\mu)\big]\cap c_1\big(L_{\hat{1}}^*\mathcal{T}(l)\big)=\big[\bar{\mathcal{U}}_{\mathcal{T}(l)}(\mu)\big]\cap c_1\big(\mathcal{L}_{\hat{1}}^*\mathcal{T}(l)\big), \tag{6.15}$$

since $L_k\mathcal{T}|\bar{\mathcal{U}}_{\mathcal{T}(l)}$ is a trivial line bundle. The above fact from algebraic geometry is only used to simplify notation and is not really needed for our computations. In addition, (6.15) can deduced from Section 6.10.

We are now ready to explain the notation used in our final topological formulas. Let d and N be positive integers and μ an N-tuple of constraints in general position in $\mathbb{P}^n$. If k is a positive integer, let $\bar{\mathcal{V}}_k(\mu)$ denote the disjoint union of the spaces $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ taken over equivalence classes of basic bubble types $\mathcal{T}=(S^2,[N],I;j,\underline{d})$ with $|I|=k$ and $\sum d_k=d$. Similarly, we denote by $\mathcal{V}_k(\mu)$ the subspace of $\bar{\mathcal{V}}_k(\mu)$ consisting of the spaces $\mathcal{U}_{\mathcal{T}}(\mu)$ with $\mathcal{T}$ as above. Let

$$a=\mathrm{ev}^*c_1(\gamma_{\mathbb{P}^n}^*)\in H^2\big(\bar{\mathcal{V}}_k(\mu)\big), \qquad c_1(\mathcal{L}^*)=c_1\big(\mathcal{L}_{\hat{0}}^*\mathcal{T}^*\big)\in H^2\big(\bar{\mathcal{V}}_1(\mu)\big),$$

where $\mathcal{T}^*=(S^2,[N],\{\hat{0}\};\hat{0},d)$. While the components of $\bar{\mathcal{V}}_2(\mu)$ are unordered, we can still define the chern classes

$$c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*),\ c_1^2(\mathcal{L}_1^*)+c_1^2(\mathcal{L}_2^*),\ c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*)\in H^*\big(\bar{\mathcal{V}}_2(\mu)\big).$$

In the notation of the previous paragraph, $c_1(\mathcal{L}_i^*)$ denotes the cohomology class $c_1(\mathcal{L}_{k_i}^*\mathcal{T})$, where we write $I=\{k_1,k_2\}$.

We next describe a generalization of the splitting (6.13) which is used in later computations. If $\mathcal{T}=(S^2,I,[N]-M_0;j,\underline{d})$ is a bubble type, let

$$\bar{\mathcal{T}}=(S^2,\bar{I},[N]-\bar{M}_0;j|([N]-\bar{M}_0),\underline{d}|\bar{I}), \quad \text{where} \quad \bar{I}=I-\{i\in I-\hat{I}: d_i=0\}, \ \bar{M}_0=M_0\cup\bigcup_{i\in I-\bar{I}}M_i\mathcal{T}.$$

Note that if $\mathcal{T}$ is semiprimitive, $\bar{\mathcal{T}}$ is basic. Furthermore,

$$\mathcal{U}_{\mathcal{T}}(\mu)=\prod_{i\in I-\bar{I}}\mathcal{M}_{0,H_i\mathcal{T}+M_i\mathcal{T}}\times\mathcal{U}_{\bar{\mathcal{T}}}(\mu), \tag{6.16}$$

$$\bar{\mathcal{U}}_{\mathcal{T}}(\mu)=\prod_{i\in I-\bar{I}}\bar{\mathcal{M}}_{0,H_i\mathcal{T}+M_i\mathcal{T}}\times\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu), \tag{6.17}$$

where $\mathcal{M}_{0,H_i\mathcal{T}+M_i\mathcal{T}}$ denotes the main stratum of $\bar{\mathcal{M}}_{0,H_i\mathcal{T}+M_i\mathcal{T}}$. If $i\in I-\bar{I}$, by definition, the bundle $L_i\mathcal{T}\longrightarrow\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ is the pullback by the projection map of the bundle

$$L_{\hat{0}}\mathcal{T}_i^{(0)}\longrightarrow\bar{\mathcal{M}}_{0,H_i\mathcal{T}+M_i\mathcal{T}}=\bar{\mathcal{U}}_{\mathcal{T}_i^{(0)}}, \quad \text{where} \quad \mathcal{T}_i^{(0)}=(S^2,H_i\mathcal{T}+M_i\mathcal{T},\{\hat{0}\};\hat{0},0).$$

We call the latter bundle the *tautological line bundle* over $\bar{\mathcal{M}}_{0,H_i\mathcal{T}+M_i\mathcal{T}}$. This is the universal tangent line at the marked point $\hat{0}\in\bar{\mathcal{M}}_{0,H_i\mathcal{T}+M_i\mathcal{T}}$.

6.4 "Codimension-Two" Numbers

In this section, we give formulas for counts of rational curves with singularities in the simplest of the cases described at the beginning of this chapter. From the computational point of view, the curves counted here correspond to a finite subset of a two-dimensional (over $\mathbb{C}$) space $\mathcal{V}_k(\mu)$. Intrinsically, the numbers of Lemmas 6.3, 6.5, and 6.4 can be viewed as codimension-one, -two, and -three, respectively, enumerative numbers for rational curves.

Lemma 6.3 *If $d\geq1$, the number of rational degree-d cuspidal curves passing through a tuple μ of $3d-2$ points in general position in $\mathbb{P}^2$ is given by*

$$|\mathcal{S}_1(\mu)|=\langle 3a^2+3ac_1(\mathcal{L}^*)+c_1^2(\mathcal{L}^*),[\bar{\mathcal{V}}_1(\mu)]\rangle-|\mathcal{V}_2(\mu)|.$$

d	1	2	3	4	5	6	7
$\lvert\mathcal{S}_1(\mu)\rvert$	0	0	12	2,304	435,168	156,153,600	97,424,784,000

Proof: (1) Let $\mathcal{T}^*=(S^2,[N],\{\hat{0}\};\hat{0},d)$ be the unique basic simple bubble corresponding to the degree d and $N\equiv3d-2$ marked points, i.e. $\mathcal{V}_1(\mu)=\mathcal{U}_{\mathcal{T}^*}(\mu)$. Let

$$\mathcal{S}_1(\mu)=\{b\in\mathcal{V}_1(\mu): \mathcal{D}b=0\}=\mathcal{D}^{-1}(0)\cap\mathcal{V}_1(\mu),$$

where $\mathcal{D}\equiv\mathcal{D}_{\mathcal{T}^*,\hat{0}}\in\Gamma(\bar{\mathcal{V}}_1(\mu);L^*\otimes\text{ev}^*T\mathbb{P}^2)$ and $L=L_{\hat{0}}\mathcal{T}^*$. The set $\mathcal{S}_1(\mu)$ consists of the equivalence classes of normalizations of the degree-d cuspidal curves passing through the chosen $3d-2$ points in $\mathbb{P}^2$. By Proposition 5.13, with $\partial\bar{\mathcal{V}}_1(\mu)=\bar{\mathcal{V}}_1(\mu)-\mathcal{V}_1(\mu)$,

$$\begin{aligned}|\mathcal{S}_1(\mu)| &= \langle c_2(L^*\otimes \mathrm{ev}^*T\mathbb{P}^2), [\bar{\mathcal{V}}_1(\mu)]\rangle - \mathcal{C}_{\partial\bar{\mathcal{V}}_1(\mu)}(\mathcal{D}) \\ &= \langle 3a^2+3ac_1(\mathcal{L}^*)+c_1^2(L^*), [\bar{\mathcal{V}}_1(\mu)]\rangle - \mathcal{C}_{\partial\bar{\mathcal{V}}_1(\mu)}(\mathcal{D}),\end{aligned} \tag{6.18}$$

since $ac_1(L^*)=ac_1(\mathcal{L}^*)\in H^2(\bar{\mathcal{V}}_1(\mu))$.
(2) Suppose $\mathcal{T}=(S^2,[N],I;j,\underline{d})<\mathcal{T}^*$. By Corollaries C.3 and C.5, $\mathcal{U}_{\mathcal{T}}(\mu)$ has the expected dimension, i.e. at most one over $\mathbb{C}$. Thus, if $\mathcal{D}$ vanishes somewhere on $\mathcal{U}_{\mathcal{T}}(\mu)$, $d_{\hat{0}}=0$ by Corollary C.3. By Theorem 6.2 and equation (6.6), there exists an identification, $\gamma^{\mu}_{\mathcal{T}}\colon \mathcal{FT}\longrightarrow\bar{\mathcal{V}}_1(\mu)$, of neighborhoods of $\mathcal{U}_{\mathcal{T}}(\mu)$ in $\mathcal{FT}$ and in $\bar{\mathcal{V}}_1(\mu)$ and an identification of appropriate vector bundles such that

$$\Big|\mathcal{D}(\gamma^{\mu}_{\mathcal{T}}(v)) - \alpha_{\mathcal{T}}(\rho_{\mathcal{T}}(v))\Big| \le C(b_v)|v|^{\frac{1}{p}}|\rho_{\mathcal{T}}(v)| \quad \forall v\in\mathcal{FT}_{\delta}, \tag{6.19}$$

where $\rho_{\mathcal{T}}\colon\mathcal{FT}\longrightarrow\tilde{\mathcal{F}}\mathcal{T}$ is a polynomial map, and $\alpha_{\mathcal{T}}\colon\tilde{\mathcal{F}}\mathcal{T}\longrightarrow L^*\otimes\mathrm{ev}^*T\mathbb{P}^2$ is a linear map over $\mathcal{U}_{\mathcal{T}}(\mu)$. By Corollary C.3 and dimension counting, $\alpha_{\mathcal{T}}$ has full rank on every fiber of $\tilde{\mathcal{F}}\mathcal{T}$; see also the proof of Lemma 8.9. Thus, $\mathcal{U}_{\mathcal{T}}(\mu)$ is $\mathcal{D}$-hollow if $\tilde{\mathcal{F}}\mathcal{T}\neq\mathcal{FT}$; see (1) of Definition 5.11. Thus, by Proposition 5.13, $\mathcal{C}_{\mathcal{U}_{\mathcal{T}}}(\mu)(\mathcal{D})=0$ unless $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|$. In such cases, $\rho_{\mathcal{T}}$ is the identity map.
(3) By dimension-counting, it remains to consider only the possibilities $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|\in\{1,2\}$. If $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=2$, $\alpha_{\mathcal{T}}$ is an isomorphism on every fiber. Thus, by the estimate (6.19) and Proposition 5.13, $\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)}(\mathcal{D})=|\mathcal{U}_{\mathcal{T}}(\mu)|$. Suppose $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=1$. Then

$$\mathcal{T}=\mathcal{T}^*(l) \text{ for some } l\in[N] \qquad \text{and} \qquad \mathcal{FT}=L^*\otimes L_{\hat{1}}\mathcal{T}\approx L_{\hat{1}}\mathcal{T}.$$

Since $\alpha_{\mathcal{T}}$ has constant rank over $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$, by the estimate (6.19) and Proposition 5.13,

$$\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)}(\mathcal{D}) = \langle c_1(L^*\otimes\mathrm{ev}^*T\mathbb{P}^2) - c_1(L_{\hat{1}}\mathcal{T}), [\bar{\mathcal{U}}_{\mathcal{T}}(\mu)]\rangle = \langle c_1(L_{\hat{1}}^*\mathcal{T}), [\bar{\mathcal{U}}_{\mathcal{T}}(\mu)]\rangle.$$

Combining these contributions to the euler class of $L^*\otimes\mathrm{ev}^*T\mathbb{P}^2$ and using equation (6.15), we obtain

$$\begin{aligned}\mathcal{C}_{\partial\bar{\mathcal{V}}_1}(\mathcal{D}_{\mathcal{T}^*}) &= \sum_{l\in[N]}\langle c_1(L_{\hat{1}}^*\mathcal{T}^*(l)), [\mathcal{U}_{\mathcal{T}^*(l)}(\mu)]\rangle + \sum_{[\mathcal{T}],|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=2}|\bar{\mathcal{U}}_{\mathcal{T}}(\mu)| \\ &= \sum_{l\in[N]}\langle c_1(\mathcal{L}^*), [\mathcal{U}_{\mathcal{T}^*(l)}(\mu)]\rangle + |\mathcal{V}_2(\mu)|.\end{aligned} \tag{6.20}$$

The claim follows by plugging equation (6.20) into (6.18) and using equation (6.14).

Lemma 6.4 *If $d\ge 2$, the number of two-component rational degree-d curves connected at a tacnode and passing through a tuple μ of p points and q lines in general position in $\mathbb{P}^3$, where $2p+q=4d-3$, is given by*

$$|\mathcal{S}_{2;2}(\mu)| = \langle 6a^2+4a(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*))+(c_1^2(\mathcal{L}_1^*)+c_1^2(\mathcal{L}_2^*))+c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*), [\bar{\mathcal{V}}_2(\mu)]\rangle - 3|\mathcal{V}_3(\mu)|.$$

d	2			3				
(p,q)	(2,1)	(1,3)	(0,5)	(4,1)	(3,3)	(2,5)	(1,7)	(0,9)
$\lvert\mathcal{S}_{2;2}(\mu)\rvert$	0	0	0	0	42	336	2,532	20,040

Proof: (1) Let $\mathcal{T}^* = (S^2, [N], I^*; j^*, \underline{d}^*)$ be a basic bubble type such that $I^* = \{k_1, k_2\}$ is a two-element set, $d^*_{k_1}, d^*_{k_2} > 0$, $d^*_{k_1} + d^*_{k_2} = d$, and $N = p+q$. The proof is similar to that of Lemma 6.3, but we pass to the projectivization $\mathbb{P}E$ (over $\mathbb{C}$) of the bundle

$$E = L_1 \oplus L_2 \longrightarrow \bar{\mathcal{U}}_{\mathcal{T}^*}(\mu), \quad \text{where} \quad L_i = L_{k_i}\mathcal{T}^*.$$

Let $\mathcal{D} \in \Gamma(\mathbb{P}E; \gamma_E^* \otimes \mathrm{ev}^* T\mathbb{P}^3)$ be the section given by

$$\mathcal{D}(v_1, v_2) = \mathcal{D}_{\mathcal{T}^*, k_1} v_1 + \mathcal{D}_{\mathcal{T}^*, k_2} v_2.$$

The space of tacnodal curves described in the statement of the lemma can be identified with

$$\mathcal{S}_{2;2}(\mu) \equiv \bigcup_{[\mathcal{T}^*]} \mathcal{S}_{\mathcal{T}^*;2}(\mu), \quad \text{where} \quad \mathcal{S}_{\mathcal{T}^*;2}(\mu) = \mathcal{D}^{-1}(0) \cap \big(\mathbb{P}E_2 | \mathcal{U}_{\mathcal{T}^*}(\mu)\big),$$

and the sum is taken over all equivalence classes of basic bubble types as above. Let $\mathbb{P}E' = \mathbb{P}E|_{\partial\bar{\mathcal{U}}(\mu)}$. Then, by Proposition 5.13 and identities 5.16,

$$\begin{aligned} |\bar{\mathcal{S}}_{\mathcal{T}^*}(\mu)| &= \big\langle c_3\big(\gamma_E^* \otimes \mathrm{ev}^* T\mathbb{P}^3\big), [\mathbb{P}E] \big\rangle - \mathcal{C}_{\mathbb{P}E'}(\mathcal{D}) \qquad (6.21)\\ &= \big\langle 6a^2 + 4a\big(c_1(L_1^*) + c_1(L_2^*)\big) + \big(c_1^2(L_1^*) + c_1^2(L_2^*)\big) + c_1(L_1^*)c_1(L_2^*), \big[\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)\big] \big\rangle - \mathcal{C}_{\mathbb{P}E'}(\mathcal{D}). \end{aligned}$$

The second equality above is obtained by applying the second identity of (5.16).
(2) Suppose $\mathcal{T} = (S^2, [N], I; j, \underline{d}) < \mathcal{T}^*$ is a bubble type such that $\mathcal{D}$ vanishes somewhere on $\mathbb{P}E|_{\mathcal{U}_{\mathcal{T}}(\mu)}$. By an argument similar to the proof of Lemma 6.3, up to the relabeling of the indices, we must have $d_{k_1} = 0$ and $d_{k_2} \neq 0$. Furthermore, $\mathcal{D}_{\mathcal{T}^*, k_2}$ does not vanish on $\mathcal{U}_{\mathcal{T}}(\mu)$. Thus, $\mathcal{D}$ vanishes only the subspace

$$\mathcal{Z}_{\mathcal{T}} \equiv \mathbb{P}L_1\big|_{\mathcal{U}_{\mathcal{T}}(\mu)} = \big\{\big(b, L_1|_b\big) : b \in \mathcal{U}_{\mathcal{T}}(\mu)\big\} \subset \mathbb{P}E\big|_{\mathcal{U}_{\mathcal{T}}(\mu)}.$$

The map $\gamma_{\mathcal{T}}^{\mu}$ of Theorem 6.2 induces an identification of a neighborhood of 0 in

$$\mathcal{FS} \equiv \pi_E^* \mathcal{FT} \oplus \pi_E^* L_1^* \otimes \pi_E^* L_2 \longrightarrow \mathcal{Z}_{\mathcal{T}}$$

with a neighborhood of $\mathcal{Z}_{\mathcal{T}}$ in $\mathbb{P}E$ such that

$$\begin{aligned} &\Big|\mathcal{D}\big(\gamma_{\mathcal{T}}^{\mu}(v,u)\big) - \tilde{\alpha}_{\mathcal{T}}\big(\tilde{\rho}_{\mathcal{T}}(v,u)\big)\Big| \le C(b_v)|v|^{\frac{1}{p}}\big|\rho_{\mathcal{T}}(v)\big| \quad \forall (v,u) \in \mathcal{FS}_\delta, \qquad (6.22)\\ &\text{where} \quad \tilde{\rho}_{\mathcal{T}}(v,u) = \big(\rho_{\mathcal{T}}(v), u\big) \in \tilde{\mathcal{F}}\mathcal{S} \equiv \pi^* \tilde{\mathcal{F}}\mathcal{T} \oplus \pi_E^* L_2^* \otimes \pi_E^* L_1 \longrightarrow \mathcal{Z}_{\mathcal{T}},\\ &\qquad \tilde{\alpha}_{\mathcal{T}}\big(\tilde{\rho}_{\mathcal{T}}(v,u)\big) = \alpha_{\mathcal{T}}\big(\rho_{\mathcal{T}}(v)\big) + \mathcal{D}_{\mathcal{T}, k_2} \circ u. \end{aligned}$$

As before, the linear map $\tilde{\alpha}_{\mathcal{T}} : \tilde{\mathcal{F}}\mathcal{S} \longrightarrow \gamma_E^* \otimes \mathrm{ev}^* T\mathbb{P}^3$ has full rank over all of $\mathcal{Z}_{\mathcal{T}}$; again, see the proof of Lemma 8.9. Thus, we only need to consider the cases $\big|H_{k_1}\mathcal{T}\big| = |\hat{I}| \in \{1, 2\}$.
(3) If $\big|H_{k_1}\mathcal{T}\big| = |\hat{I}| = 2$, we again have

$$\mathcal{C}_{\mathcal{Z}_{\mathcal{T}}}(\mathcal{D}) = |\mathcal{Z}_{\mathcal{T}}| = \big|\mathcal{U}_{\mathcal{T}}(\mu)\big|. \qquad (6.23)$$

Note that the sum of the numbers $|\mathcal{U}_{\mathcal{T}}(\mu)|$ taken over all equivalence classes of bubble types $\mathcal{T}^*$ and $\mathcal{T} < \mathcal{T}^*$ is $3\big|\mathcal{V}_3(\mu)\big|$, since one of the three components of the image of each bubble map of $\mathcal{U}_{\mathcal{T}}(\mu)$ is distinguished by the bubble type $\mathcal{T}$. If $|H_{k_1}\mathcal{T}| = |\hat{I}| = 1$, $\tilde{\alpha}_{\mathcal{T}}$ has full rank over all of $\bar{\mathcal{Z}}_{\mathcal{T}}$. Thus, by Proposition 5.13,

$$\mathcal{C}_{\mathcal{Z}_{\mathcal{T}}}(\mathcal{D}) = \langle c_1(\gamma_E^*\otimes \mathrm{ev}^*T\mathbb{P}^3) - c_1(\mathcal{FS}), [\bar{\mathcal{Z}}_{\mathcal{T}}]\rangle = \langle 4a + c_1(L_1^*\mathcal{T}) + c_1(L_2^*), [\bar{\mathcal{U}}_{\mathcal{T}}(\mu)]\rangle; \quad (6.24)$$

note that $c_1(\gamma_E^*) = c_1(L_1^*) = 0$ over $\bar{\mathcal{Z}}_{\mathcal{T}}(\mu)$. As in the proof of Lemma 6.3, we now sum up equations (6.23) and (6.24) over all equivalence classes of bubble types $\mathcal{T} < \mathcal{T}^*$ of the appropriate form, plug the result back into (6.21) and use equations (6.14) and (6.15). The claim follows by summing the result over all equivalence classes of basic simple bubble types $\mathcal{T}^*$.

Lemma 6.5 *If d is a positive integer, the number of two-component rational degree-d curves passing through a tuple μ of $3d-4$ points in general position in $\mathbb{P}^2$ such that the two components meet at a node at which one of them has a cusp is given by*

$$|\mathcal{S}_{2;1}(\mu)| = \langle 6a^2 + 3a\big(c_1(\mathcal{L}_1^*) + c_1(\mathcal{L}_2^*)\big) + \big(c_1^2(\mathcal{L}_1^*) + c_1^2(\mathcal{L}_2^*)\big), [\bar{\mathcal{V}}_2(\mu)]\rangle - 3|\mathcal{V}_3(\mu)|.$$

d	2	3	4	5	6	7
$\lvert\mathcal{S}_{2;1}(\mu)\rvert$	0	0	528	91,872	26,055,360	12,596,219,904

Proof: The proof is similar to the proofs of Lemmas 6.3 and 6.4. We let

$$\mathcal{S}_{2;1}(\mu) = \bigcup_{[\mathcal{T}^*]}\bigcup_{i=1,2} \mathcal{S}_{\mathcal{T}^*;(1,i)}(\mu), \quad \text{where} \quad \mathcal{S}_{\mathcal{T}^*;(1,i)}(\mu) = \{b \in \mathcal{U}_{\mathcal{T}^*}(\mu) : \mathcal{D}_{\mathcal{T}^*,k_i} b = 0\},$$

and the sum is taken over all equivalence classes of basic bubble types $\mathcal{T}^* = (S^2, [N], I^*; j^*, \underline{d}^*)$ with $I^* = \{k_1, k_2\}$, $d_{k_1}^*, d_{k_2}^* > 0$, $d_{k_1}^* + d_{k_2}^* = d$, and $N = 3d-4$. The computations of $|\mathcal{S}_{\mathcal{T}^*;(1,i)}(\mu)|$ is analogous to the computation of $|\mathcal{S}_1(\mu)|$ in the proof of Lemma 6.3. The bubble types $\mathcal{T} < \mathcal{T}^*$ that may contribute to $\mathcal{C}_{\partial\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)}(\mathcal{D}_{\mathcal{T}^*,k_1})$ are described in the proof of Lemma 6.4. In fact, the above formula for $|\mathcal{S}_{2;1}(\mu)|$ can be deduced from the formula of Lemma 6.3, since its proof applies with no change to enumerate irreducible curves through $3d-3$ points with a cusp on a fixed line in $\mathbb{P}^2$.

6.5 Structure Theorem for a One-Dimensional Space $\bar{\mathcal{S}}_1(\mu)$

In Section 9.4, we compute the number of zeros of certain affine maps over the closure $\bar{\mathcal{S}}_1(\mu) \subset \bar{\mathcal{V}}_1(\mu)$ of the space

$$\mathcal{S}_1(\mu) = \{b \in \mathcal{V}_1 : \mathcal{D}b = 0\},$$

where μ is a tuple of p and q lines in general position in $\mathbb{P}^3$ with $2p+q = 4d-3$ and d is the degree of the maps. By Corollary C.3, $\mathcal{S}_1(\mu)$ is one-dimensional over $\mathbb{C}$. In this section, we describe the structure of $\bar{\mathcal{S}}_1(\mu)$ and the behavior of $\mathcal{D}^{(2)}$ and of $\mathcal{D}^{(3)}$ near $\partial\bar{\mathcal{S}}_1(\mu) \equiv \bar{\mathcal{S}}_1(\mu) - \mathcal{S}_1(\mu)$.

Let $\mathcal{T}^* = (S^2, [N], \{\hat{0}\}; \hat{0}, d)$, where $N = p+q$. If $\mathcal{T} = (S^2, [N], I; j, \underline{d}) < \mathcal{T}^*$, from Theorem 6.2 one should expect that the normal bundle of $\mathcal{S}_{\mathcal{T}}(\mu) \equiv \mathcal{U}_{\mathcal{T}}(\mu) \cap \bar{\mathcal{S}}_1(\mu)$ in $\bar{\mathcal{S}}_1(\mu)$ is given by

$$\mathcal{FS} = \Big\{[\upsilon = (b, (v_h)_{h\in\hat{I}})] \in \mathcal{FT}|\mathcal{S}_{\mathcal{T}}(\mu) : \sum_{h\in\chi(\mathcal{T})}\Big(\prod_{i\in\hat{I}, h\in\bar{D}_i\mathcal{T}} v_i\Big)(\mathcal{D}_{\mathcal{T},h}b) = 0\Big\}.$$

The next lemma shows that this is indeed the case. Let $\mathcal{NS} \longrightarrow \mathcal{FS}$ denote the normal bundle of $\mathcal{FS}$ in $\mathcal{FT} \longrightarrow \mathcal{U}_{\mathcal{T}}(\mu)$. While for the purposes of Lemma 6.6, we can use any identification of neighborhoods of $\mathcal{FS}$ in $\mathcal{NS}$ and in $\mathcal{FT} \longrightarrow \mathcal{U}_{\mathcal{T}}(\mu)$, in order to simplify the statement of Lemma 6.8, we choose a fairly natural one. More precisely, denote by $\mathcal{FS}^{\perp}$ a subspace of $\mathcal{FT} \longrightarrow \mathcal{S}_{\mathcal{T}}(\mu)$ complementary to $\mathcal{FS}$ and by $\pi_{\mathcal{S}} : \mathcal{NS}^{(1)} \longrightarrow \mathcal{S}_{\mathcal{T}}(\mu)$ the normal bundle of $\mathcal{S}_{\mathcal{T}}(\mu)$ in $\mathcal{U}_{\mathcal{T}}(\mu)$. Choose a norm on $\mathcal{NS}^{(1)}$ and an identification $\phi_{\mathcal{S}} : \mathcal{NS}^{(1)}_{\delta} \longrightarrow \mathcal{S}_{\mathcal{T}}(\mu)$ of neighborhoods of $\mathcal{S}_{\mathcal{T}}(\mu)$ in $\mathcal{NS}^{(1)}$ and in $\mathcal{U}_{\mathcal{T}}(\mu)$. Let $\Phi_{\mathcal{S}} : \pi_{\mathcal{S}}^*\mathcal{FT} \longrightarrow \mathcal{FT}$ be a lift of $\phi_{\mathcal{S}}$ such that $\Phi_{\mathcal{S}}$ restricts to the identity over $\mathcal{S}_{\mathcal{T}}(\mu) \subset \mathcal{NS}^{(1)}_{\delta}$. Let $\pi : \mathcal{FT} \longrightarrow \mathcal{S}_{\mathcal{T}}(\mu)$ be the bundle projection. Then

$$\mathcal{NS} = \pi^*\mathcal{NS}^{(1)} \oplus \mathcal{FS}^{\perp}, \quad \text{and} \quad \tilde{\phi}_{\mathcal{S}} : \mathcal{NS}_{\delta} \longrightarrow \mathcal{FT}, \ \tilde{\phi}_{\mathcal{S}}\big((b,v),(X,v^{\perp})\big) = \Phi_{\mathcal{S}}\big((b,X), v+v^{\perp}\big),$$

is an identification of neighborhoods of $\mathcal{FS}$ in $\mathcal{NS}$ and $\mathcal{FT} \longrightarrow \mathcal{U}_{\mathcal{T}}(\mu)$.

Lemma 6.6 *For any bubble type $\mathcal{T} = (S^2, [N], I; j, \underline{d}) < \mathcal{T}^*$, there exist $\delta, C \in C^{\infty}(\mathcal{S}_{\mathcal{T}}(\mu); \mathbb{R}^+)$ and a section $\varphi_{\mathcal{S}} \in \Gamma(\mathcal{FS}_{\delta}; \mathcal{NS})$ such that*

$$\|\varphi_{\mathcal{S}}(v)\| \le C(b_v)|v|^{\frac{1}{p}}, \qquad \|\varphi_{\mathcal{FS}^{\perp}}(v)\| \le C(b_v)|v|^{1+\frac{1}{p}},$$

where $\varphi_{\mathcal{FS}^{\perp}}$ denotes the $\mathcal{FS}^{\perp}$-component of $\varphi_{\mathcal{S}}$, and the map

$$\gamma_{\mathcal{S}} : \mathcal{FS}_{\delta} \longrightarrow \bar{\mathcal{S}}_1(\mu), \qquad \gamma_{\mathcal{S}}(v) = \gamma^{\mu}_{\mathcal{T}}\big(\tilde{\phi}_{\mathcal{S}}\varphi_{\mathcal{S}}(v)\big),$$

is a homeomorphism onto an open neighborhood of $\mathcal{S}_{\mathcal{T}}(\mu)$ in $\bar{\mathcal{S}}_1(\mu)$, which is smooth and orientation-preserving on the preimage of $\mathcal{S}_1(\mu)$.

Proof: (1) The proof is similar to that of Lemma 3.32, so we only describe the differences. If $\mathcal{S}_{\mathcal{T}}(\mu) \neq \emptyset$, $\mathcal{T}$ must have one of the three forms described by Lemma 6.8. In Case (1), we apply Section 3.7 to $\mathcal{D}_{\mathcal{T}^*,\hat{0}}$ instead of the evaluation maps. By Theorem 6.2,

$$\Big|\Pi^{-1}_{b,\tilde{\gamma}^{\mu}_{\mathcal{T}}(v)}\big(\mathcal{D}_{\mathcal{T}^*,\hat{0}}\tilde{\gamma}^{\mu}_{\mathcal{T}}(v)\big) - \big(\mathcal{D}_{\mathcal{T}^*,\hat{0}}b_v\big)\Big| \le C'(b)|v|^{\frac{1}{p}} \qquad \forall v \in \mathcal{FS}_{\delta}.$$

This estimate suffices for applying an argument similar to the proof of Lemma 3.32.
(2) In Case (2) of Lemma 6.8, instead of the section $\mathcal{D}_{\mathcal{T}^*,\hat{0}}$ of $L^*_{\hat{0}}\mathcal{T}^* \otimes \mathrm{ev}^*T\mathbb{P}^3$, we consider the section $\tilde{\mathcal{D}}$ of $\big(L_{\hat{0}}\mathcal{T}^* \otimes \mathcal{FT}\big)^* \otimes \mathrm{ev}^*T\mathbb{P}^3$ on a neighborhood of $\mathcal{U}_{\mathcal{T}}(\mu)$ in $\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$ defined by

$$\tilde{\mathcal{D}}\big|_{\tilde{\gamma}^{\mu}_{\mathcal{T}}(b,v_{\hat{1}})}\big(v_{\hat{0}} \otimes v_{\hat{1}}\big) = \big(\mathcal{D}_{\mathcal{T}^*,\hat{0}}\big|_{\tilde{\gamma}^{\mu}_{\mathcal{T}}(b,v_{\hat{1}})} v_{\hat{0}}\big) \in \mathrm{ev}^*T\mathbb{P}^3.$$

This section is well-defined outside of $\mathcal{U}_{\mathcal{T}}(\mu)$ and by Theorem 6.2 extends over $\mathcal{U}_{\mathcal{T}}(\mu)$ by

$$\tilde{\mathcal{D}}\big|_b\big(v_{\hat{0}} \otimes v_{\hat{1}}\big) = v_{\hat{0}}\big(\mathcal{D}_{\mathcal{T},\hat{1}} v_{\hat{1}}\big).$$

The restriction of this section to $\mathcal{U}_{\mathcal{T}}(\mu)$ vanishes transversally at $\mathcal{S}_{\mathcal{T}}(\mu)$ by Corollary C.3, while its zero set on $\mathcal{U}_{\mathcal{T}^*}(\mu)$ is the same as the zero set of $\mathcal{D}_{\mathcal{T}^*,\hat{0}}$. By Theorem 6.2, with appropriate identifications,

$$\Big|\tilde{\mathcal{D}}\big|_{\gamma^{\mu}_{\mathcal{T}}(b,v)} - \tilde{\mathcal{D}}\big|_b\Big| \le C'(b)|v|^{\frac{1}{p}} \qquad \forall v \in \mathcal{FS}_{\delta}.$$

(3) In the final case of Lemma 6.8, we replace $\mathcal{D}_{\mathcal{T}^*,\hat{0}}$ by a bundle section over the blowup

of $\mathcal{FT}$ along $\mathcal{U}_{\mathcal{T}}(\mu)$. Let

$$\Omega_{\mathcal{T}} = \big\{(b,v,\ell)\colon (b,v)\in\mathcal{FT},\ v\in\ell\in\mathbb{P}\mathcal{FT}|_b\big\}, \quad \Omega_{\mathcal{T}}^* = \big\{(b,v,\ell)\in\Omega_{\mathcal{T}}\colon v\neq 0\big\}, \quad \mathcal{E}_{\mathcal{T}} = \Omega_{\mathcal{T}} - \Omega_{\mathcal{T}}^*.$$

Denote by $\gamma \longrightarrow \Omega_{\mathcal{T}}$ the tautological line bundle. The normal bundle $\tilde{\mathcal{N}}\mathcal{S}$ of $\gamma\longrightarrow\mathbb{P}\mathcal{FS}$ in $\gamma\longrightarrow\mathcal{E}_{\mathcal{T}}$ is given by

$$\begin{gathered}\tilde{\mathcal{N}}\mathcal{S} = \pi_\gamma^*\pi_{\mathcal{FT}}^*\mathcal{NS}^{(1)} \oplus \pi_\gamma^*\big(\gamma^*\otimes\pi_{\mathcal{FT}}^*\mathcal{FS}^\perp\big),\\ \tilde{\phi}_{\tilde{\mathcal{N}}\mathcal{S}}\big((b,\ell,v),X,\sigma\big) = \big(\phi_{\mathcal{S}}(b,X),\big[\Phi_{\mathcal{S}}(v+\sigma(v))\big],v+\sigma(v)\big),\end{gathered}$$

where $\pi_\gamma\colon\gamma\longrightarrow\Omega_{\mathcal{T}}$ is the bundle projection map. The bundle $L_{\hat{0}}\mathcal{T}^*$ pulls back to a bundle $\tilde{L}$ over a neighborhood Ω_δ of $\mathcal{E}_{\mathcal{T}}$ in $\Omega_{\mathcal{T}}$. We define a section $\tilde{\mathcal{D}}$ of $\big(\tilde{L}\otimes\gamma\big)^*\otimes\mathrm{ev}^*T\mathbb{P}^3$ over Ω_δ by

$$\tilde{\mathcal{D}}\big|_{(b,v_{\hat{1}},v_{\hat{2}},\ell)}\big(v_{\hat{0}}\otimes(v_{\hat{1}},v_{\hat{2}})\big) = \big(\mathcal{D}_{\mathcal{T}^*,\hat{0}}\big|_{\tilde{\gamma}_{\mathcal{T}}^{\mu}(b,v_{\hat{1}},v_{\hat{2}})}v_{\hat{0}}\big) \in \mathrm{ev}^*T\mathbb{P}^3.$$

This section is well-defined outside of $\mathcal{E}_{\mathcal{T}}(\mu)$ and by Theorem 6.2 extends over $\mathcal{E}_{\mathcal{T}}(\mu)$ by

$$\tilde{\mathcal{D}}\big|_b\big(v_{\hat{0}}\otimes(v_{\hat{1}},v_{\hat{2}})\big) = v_{\hat{0}}\Big(\big(\mathcal{D}_{\mathcal{T},\hat{1}}v_{\hat{1}}\big) + v_{\hat{2}}\big(\mathcal{D}_{\mathcal{T},\hat{2}}v_{\hat{2}}\big)\Big).$$

The restriction of this section to $\mathcal{E}_{\mathcal{T}}(\mu)$ vanishes transversally at $\mathbb{P}\mathcal{FS}\longrightarrow\mathcal{S}_{\mathcal{T}}(\mu)$ by Corollary C.3, while its zero set on Ω^* corresponds to the zero set of $\mathcal{D}_{\mathcal{T}^*,\hat{0}}$ on $\gamma_{\mathcal{T}}^{\mu}\big(\mathcal{FT}_\delta - \mathcal{U}_{\mathcal{T}}(\mu)\big)$. By Theorem 6.2, with appropriate identifications,

$$\Big|\tilde{\mathcal{D}}\big|_{(b,v_{\hat{1}},v_{\hat{2}},\ell)} - \tilde{\mathcal{D}}\big|_{(b,\ell)}\Big| \le C'(b)|v|^{\frac{1}{p}}.$$

Thus, we can apply the arguments of Lemma 3.32 to $\tilde{\mathcal{D}}$ to describe its zero set near $\mathcal{E}_{\mathcal{T}}$. We obtain a section $\tilde{\varphi}_{\mathcal{S}}\in\Gamma(\gamma_\delta|\mathbb{P}\mathcal{FS};\tilde{\mathcal{N}}\mathcal{S})$ such that $\|\tilde{\varphi}_{\mathcal{S}}(v)\|\le C(b_v)|v|^{\frac{1}{p}}$, and the map

$$\tilde{\gamma}_{\mathcal{S}}\colon\gamma_\delta|\mathbb{P}\mathcal{FS}\longrightarrow\Omega_{\mathcal{T}}, \quad \tilde{\gamma}_{\mathcal{S}}(v) = \tilde{\phi}_{\tilde{\mathcal{N}}\mathcal{S}}\big(\varphi_{\mathcal{S}}(v)\big),$$

is a homeomorphism onto an open neighborhood of $\mathbb{P}\mathcal{FS}$ in $\mathcal{D}^{-1}(0)$. This section $\tilde{\varphi}_{\mathcal{S}}$ induces the required section $\varphi_{\mathcal{S}}$ with the claimed properties.

Corollary 6.7 *For every bubble type $\mathcal{T}=(S^2,[N],I;j,\underline{d})<\mathcal{T}^*$, there exist $\delta\in C^\infty(\mathcal{S}_{\mathcal{T}}(\mu);\mathbb{R}^+)$ and a map*

$$\gamma_{\mathcal{S}}\colon\big(\mathcal{NS}^{(1)}\oplus\mathcal{FT}\big)_\delta\big|\mathcal{S}_{\mathcal{T}}(\mu)\longrightarrow\bar{\mathcal{V}}_1(\mu)$$

such that $\gamma_{\mathcal{S}}$ is a homeomorphism onto an open neighborhood of $\mathcal{S}_{\mathcal{T}}(\mu)$ in $\bar{\mathcal{V}}_1(\mu)$, which is smooth and orientation-preserving on the preimage of $\mathcal{V}_1(\mu)$. Furthermore, with appropriate identifications,

$$\mathcal{D}\gamma_{\mathcal{S}}(X,v) = \begin{cases} X, & \textit{in Case (1) with } X\in L^*\otimes ev^*T\mathbb{P}^3;\\ Xv_{\hat{1}}, & \textit{in Case (2) with } X\in L_{\hat{1}}^*\mathcal{T}\otimes ev^*T\mathbb{P}^3,\end{cases}$$

where the cases are the ones described by Lemma 6.8.

Proof: The proof is just a modification of the proof of Lemma 6.6. We work with the sections $\bar{\mathcal{D}}\equiv\mathcal{D}$ and $\bar{\mathcal{D}}\equiv\tilde{\mathcal{D}}$ in Cases (1) and (2), respectively. Choose an identification $\gamma\colon\mathcal{NS}_\delta^{(1)}\longrightarrow\mathcal{U}_{\mathcal{T}}(\mu)$ of neighborhoods of $\mathcal{S}_{\mathcal{T}}(\mu)$ in $\mathcal{NS}^{(1)}$ and in $\mathcal{U}_{\mathcal{T}}(\mu)$ as well as of the appropriate line bundle over these neighborhoods such that

$$\bar{\mathcal{D}}\big|_{(b,X)} = X \quad \forall X \in \mathcal{NS}_\delta^{(1)}.$$

By the same argument as in the proof of Lemma 6.6, for any $(Y,v) \in N\mathcal{S}^{(1)} \oplus \mathcal{FT}$, there exists a unique $Z \in N\mathcal{S}^{(1)}$, such that

$$\bar{\mathcal{D}}\big|_{\gamma_{\mathcal{T}}^{\mu}(\Phi_{\mathcal{S}}(X+Y;v))} = \bar{\mathcal{D}}\big|_X = X.$$

Furthermore, $|Z| \le C(b)\big(|Y| + |v|^{\frac{1}{p}}\big)$.

Lemma 6.8 *If $d \geq 1$, μ is a tuple of p points and q lines in general position in $\mathbb{P}^3$ with $2p+q=4d-3$, and $N=p+q$, the set $\bar{\mathcal{S}}_1(\mu) - \mathcal{S}_1(\mu)$, is finite. Furthermore, if*

$$\mathcal{T} = (S^2, [N], I; j, \underline{d}) < \mathcal{T}^* \quad \textit{and} \quad \mathcal{S}_{\mathcal{T}}(\mu) \neq \emptyset,$$

(1) $\hat{I} = \{\hat{1}\}$, $d_{\hat{0}} > 0$, and the images of $\mathcal{D}^{(2)}_{\mathcal{T}^*,\hat{0}}$ and $\mathcal{D}^{(3)}_{\mathcal{T}^*,\hat{0}}$ are linearly independent in every fiber of $ev^* T\mathbb{P}^3$ over $\mathcal{S}_{\mathcal{T}}(\mu)$;
(2) OR $\hat{I} = H_{\hat{0}}\mathcal{T} = \{\hat{1}\}$, $d_{\hat{0}} = 0$, $d_{\hat{1}} = d$, and, with appropriate identification of vector bundles, for all $v = [b, v_{\hat{1}}] \in \mathcal{FS}_\delta$,

$$\Big|\mathcal{D}^{(2)}\big|_{\gamma_{\mathcal{S}}(v)} - \big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}} v^2\big)\Big| \le C|v|^{2+\frac{1}{p}};$$
$$\Big|\Big(\mathcal{D}^{(3)}\big|_{\gamma_{\mathcal{S}}(v)} - 3x_{\hat{1}}\mathcal{D}^{(2)}\big|_{\gamma_{\mathcal{S}}(v)}\Big) - \big(\mathcal{D}^{(3)}_{\mathcal{T},\hat{1}} v^3\big)\Big| \le C|v|^{3+\frac{1}{p}};$$

if $b = (S^2, [N], I; x, (j,y), u) \in \mathcal{B}_{\mathcal{T}}$.
(3) OR $\hat{I} = \{\hat{1}, \hat{2}\}$, $d_{\hat{0}} = 0$, and, with appropriate identification of vector bundles, for all $v = [b, v_{\hat{1}}, v_{\hat{2}}] \in \mathcal{FS}$

$$\Big|\mathcal{D}^{(2)}\big|_{\gamma_{\mathcal{S}}(v)} - 2\Big(x_{\hat{1}}\big(\mathcal{D}_{\mathcal{T},\hat{1}}\big|_b v_{\hat{1}}\big) + x_{\hat{2}}\big(\mathcal{D}_{\mathcal{T},\hat{2}}\big|_b v_{\hat{2}}\big)\Big)\Big| \le C|v|^{1+\frac{1}{p}};$$
$$\Big|\Big(2\mathcal{D}^{(3)}\big|_{\gamma_{\mathcal{S}}(v)} - 3(x_{\hat{1}}+x_{\hat{2}})\mathcal{D}^{(2)}\big|_{\gamma_{\mathcal{S}}(v)}\Big) - 3(x_{\hat{1}}-x_{\hat{2}})\Big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}} v_{\hat{1}}^2 - \big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{2}} v_{\hat{2}}^2\big)\Big)\Big| \le C|v|^{2+\frac{1}{p}}.$$

Proof: (1) The statement about the possible structures of $\mathcal{T}$ is easily seen from Theorem 6.2 and dimension count. The finiteness claim then also follows by dimension count. In Case (1), if $d_{\hat{0}} \geq 3$, by Corollary C.3, the images of $\mathcal{D}^{(2)}_{\mathcal{T}^*,\hat{0}}$ and $\mathcal{D}^{(3)}_{\mathcal{T}^*,\hat{0}}$ are transversal and thus linearly independent over the finite set $\mathcal{S}_{\mathcal{T}}(\mu)$. On the other hand, if $d_{\hat{0}} < 3$, $\mathcal{S}_{\mathcal{T}}(\mu) = \emptyset$; see Section 8.5.
(2) The four inequalities in the lemma will be deduced from the proof of the analytic estimate of Theorem 6.2. We use the same notation. Combining equations (6.7), (6.8), (6.9), and (6.11), we obtain

$$\big(\mathcal{D}^{(m)}\tilde{\gamma}_{\mathcal{T}}(v)\big) = m \sum_{h\in\chi(\mathcal{T})} \sum_{k=1}^{k=m} \frac{a_{k,h}(v)}{k} \tilde{v}^k \big(\mathcal{D}^{(k)}_{\mathcal{T},h} b\big) - \frac{m}{2\pi i} \sum_{h\in\chi(\mathcal{T})} \int_{A_h^-(v)} \xi_v w^{m-1} dw, \tag{6.25}$$

where the integral is computed using the same trivializations as before. This equality holds for any bubble type. If $\gamma_{\mathcal{T}}(v) \in \mathcal{S}_1(\mu)$ and $\mathcal{T}$ is as in (2) of the lemma, (6.25) with $m = 1, 2, 3$

gives

$$0 = \left(\mathcal{D}^{(1)}_{\mathcal{T},\hat{1}} v_{\hat{1}}\right) - \frac{1}{2\pi i}\int_{|x_{\hat{1}}-w|=\epsilon} \xi_v dw; \qquad (6.26)$$

$$\left(\mathcal{D}^{(2)}\tilde{\gamma}_{\mathcal{T}}(v)\right) = 2x_{\hat{1}}\left(\mathcal{D}^{(1)}_{\mathcal{T},\hat{1}} v_{\hat{1}}\right) + \left(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}} v_{\hat{1}}^2\right) - \frac{1}{\pi i}\int_{|x_{\hat{1}}-w|=\epsilon} \xi_v w dw; \qquad (6.27)$$

$$\left(\mathcal{D}^{(3)}\tilde{\gamma}_{\mathcal{T}}(v)\right) = 3x_{\hat{1}}^2\left(\mathcal{D}^{(1)}_{\mathcal{T},\hat{1}} v_{\hat{1}}\right) + 3x_{\hat{1}}\left(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}} v_{\hat{1}}^2\right) + 2\left(\mathcal{D}^{(3)}_{\mathcal{T},\hat{1}} v_{\hat{1}}^3\right) - \frac{3}{2\pi i}\int_{|x_{\hat{1}}-w|=\epsilon} \xi_v w^2 dw. \qquad (6.28)$$

where $\epsilon = 4\delta(b_v)^{-1}|v_{\hat{1}}|$. Subtracting $2x_{\hat{1}}$ times the first equation from the second, we obtain

$$\left|\left(\mathcal{D}^{(2)}\tilde{\gamma}_{\mathcal{T}}(v)\right) - \left(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}} v_{\hat{1}}^2\right)\right| \le C(b)|v_{\hat{1}}|^{2+\frac{1}{p}}. \qquad (6.29)$$

Similarly, subtracting $3x_{\hat{1}}$ times (6.27) from and adding $3x_{\hat{1}}^2$ times (6.26) to (6.28), we obtain

$$\left|\left(\left(\mathcal{D}^{(3)}\tilde{\gamma}_{\mathcal{T}}(v)\right) - 3x_{\hat{1}}\left(\mathcal{D}^{(2)}\tilde{\gamma}_{\mathcal{T}}(v)\right)\right) - 2\left(\mathcal{D}^{(3)}_{\mathcal{T},\hat{1}} v_{\hat{1}}^3\right)\right| \le C(b)|v_{\hat{1}}|^{3+\frac{1}{p}}. \qquad (6.30)$$

If $v \in \mathcal{FS}$ is sufficiently small, the claim in Case (2) follows from equations (6.29) and (6.30) along with Lemma 6.6 and our choice of $\tilde{\phi}_{\mathcal{S}}$. Note that if $v \in \mathcal{FS}$, we have to apply (6.29) and (6.30) with v replaced by $\Phi^{\mu}_{\mathcal{T}}\varphi^{\mu}_{\mathcal{T}}\tilde{\phi}_{\mathcal{S}}\varphi_{\mathcal{S}}(v)$, where $\Phi^{\mu}_{\mathcal{T}}$ and $\varphi^{\mu}_{\mathcal{T}}$ are as in Section 3.9. However, applying the bounds on $\varphi^{\mu}_{\mathcal{T}}$ and $\tilde{\phi}_{\mathcal{S}}$, we obtain the claimed estimates.
(3) In Case (3), we proceed similarly. The analog of equation (6.27) gives

$$\left|\left(\mathcal{D}^{(2)}\tilde{\gamma}_{\mathcal{T}}(v)\right) - 2\left(x_{\hat{1}}\left(\mathcal{D}_{\mathcal{T},\hat{1}} v_{\hat{1}}\right) + x_{\hat{2}}\left(\mathcal{D}_{\mathcal{T},\hat{2}} v_{\hat{2}}\right)\right)\right| \le C(b)|v|^{1+\frac{1}{p}}.$$

Subtracting $3(x_{\hat{1}}+x_{\hat{2}})$ times the analog of (6.27) from and adding $6x_{\hat{1}}x_{\hat{2}}$ times the analog of (6.26) to twice the analog of (6.28), we obtain

$$\left|\left(2\left(\mathcal{D}^{(3)}\tilde{\gamma}_{\mathcal{T}}(v)\right) - 3(x_{\hat{1}}+x_{\hat{2}})\left(\mathcal{D}^{(2)}\tilde{\gamma}_{\mathcal{T}}(v)\right)\right) - 3\left(x_{\hat{1}}-x_{\hat{2}}\right)\left(\left(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}} v_{\hat{1}}^2\right) - \left(\mathcal{D}^{(2)}_{\mathcal{T},\hat{2}} v_{\hat{2}}^2\right)\right)\right| \le C(b)|v|^{2+\frac{1}{p}}.$$

The estimates of Case 3 follow from the last two equations and Lemma 6.6. The finer bound on $\varphi_{\mathcal{FS}^{\perp}}$ of Lemma 6.6 is essential here.

6.6 Intersections in $H^*\left(\bar{\mathcal{S}}_1(\mu)\right)$

We now relate $\langle a, [\bar{\mathcal{S}}_1(\mu)]\rangle$ and $\langle c_1(\mathcal{L}^*), [\bar{\mathcal{S}}_1(\mu)]\rangle$ to intersection numbers on the spaces $\bar{\mathcal{V}}_1(\mu)$, $\bar{\mathcal{V}}_2(\mu)$, and $\bar{\mathcal{V}}_3(\mu)$, where μ is as in the previous section. The approach is similar to the proof of Lemma 6.3, but first we need to interpret $\langle a, [\bar{\mathcal{S}}_1(\mu)]\rangle$ and $\langle c_1(\mathcal{L}^*), [\bar{\mathcal{S}}_1(\mu)]\rangle$ as the zero sets of some bundle sections. In our case, the spaces $\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$ and $\bar{\mathcal{U}}_{\mathcal{T}^*(l)}(\mu)$ for all $l \in [N]$ are topological manifolds (not just orbifolds). Thus, $c_1(\mathcal{L}^*)$ represents the first chern class of some line bundle $\mathcal{L}^* \longrightarrow \bar{\mathcal{U}}_{\mathcal{T}}(\mu)$. It is well-known in algebraic geometry that a slightly weaker statement is in fact true for any choice of constraints, and

$$\mathcal{L}^* = L^* \otimes \mathcal{O}\Big(-\sum_{l\in[N]} \bar{\mathcal{U}}_{\mathcal{T}^*(l)}\Big).$$

Let $V_1 = \mathrm{ev}^*\mathcal{O}(1) \longrightarrow \bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$, $V_2 = \mathcal{L}^* \longrightarrow \bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$, and $\eta_i = c_1(V_i)$. Choose sections $s_i \in \Gamma(\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu); V_i)$ such that s_i is smooth and transversal to the zero set on all smooth strata $\mathcal{U}_{\mathcal{T}}(\mu) \subset \bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$ and on $\mathcal{S}_{\mathcal{T}}(\mu) \subset \bar{\mathcal{S}}_1(\mu)$. The second condition implies that s_i does not vanish on the finite set $\partial\bar{\mathcal{S}}_1$.

Lemma 6.9 *If $d \geq 1$, μ is a tuple of p points and q lines in general position in $\mathbb{P}^3$ with $2p+q=4d-3$,*

$$\langle a, [\bar{\mathcal{S}}_1(\mu)]\rangle = \big\langle 6a^3c_1(\mathcal{L}^*)+4a^2c_1^2(\mathcal{L}^*)+ac_1^3(\mathcal{L}^*), [\bar{\mathcal{V}}_1(\mu)]\big\rangle - \big\langle 4a^2+a\big(c_1(\mathcal{L}_1^*)+c_2(\mathcal{L}_1^*)\big), [\bar{\mathcal{V}}_2(\mu)]\big\rangle;$$
$$\langle c_1(\mathcal{L}^*), [\bar{\mathcal{S}}_1(\mu)]\rangle = \big\langle 4a^3c_1(\mathcal{L}^*) + 6a^2c_1^2(\mathcal{L}^*) + 4ac_1^3(\mathcal{L}^*) + c_1^4(\mathcal{L}^*), [\bar{\mathcal{V}}_1(\mu)]\big\rangle - \big|\mathcal{V}_3(\mu)\big|.$$

Proof: (1) Similarly to the proof of Lemma 6.3,

$$\langle \eta_i, [\bar{\mathcal{S}}_1(\mu)]\rangle = \big\langle \eta_i c_3\big(L^*\otimes \mathrm{ev}^*T\mathbb{P}^3\big), [\bar{\mathcal{V}}_1(\mu)]\big\rangle - \mathcal{C}_{\partial\bar{\mathcal{V}}_1(\mu)}\big(\mathcal{D}\oplus s_i\big). \tag{6.31}$$

Suppose $\mathcal{T} = (S^2, [N], I; j, \underline{d}) < \mathcal{T}^*$ is a bubble type such that $\mathcal{U}_{\mathcal{T}}(\mu) \neq \emptyset$. If $d_{\hat{0}} \neq 0$, by our assumptions on s_i, $\mathcal{D}\oplus s_i$ does not vanish on $\mathcal{U}_{\mathcal{T}}(\mu)$. Thus, for the purposes of computing $\mathcal{C}_{\partial\bar{\mathcal{V}}_1(\mu)}\big(\mathcal{D}\oplus s_i\big)$, we can assume $d_{\hat{0}}=0$.
(2) In order to compute the numbers $\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)}\big(\mathcal{D}\oplus s_i\big)$, we slightly modify the approach of Section 5.2, since we have a great amount of flexibility in choosing the section s_i. We consider a family $\psi_t = (t\nu + \mathcal{D}, s_i)$ of sections of $L^*\otimes \mathrm{ev}^*T\mathbb{P}^3 \oplus V_i$, with ν generic with respect to $\mathcal{D}$. Let $\pi\colon \mathcal{F}\mathcal{T} \longrightarrow \mathcal{U}_{\mathcal{T}}(\mu)$ be the bundle projection map and fix an identification of $\gamma_{\mathcal{T}}^{\mu *}V_i \longrightarrow \mathcal{F}\mathcal{T}_\delta$ with π^*V_i. It can be assumed that the section s_i has been chosen so that $\gamma_{\mathcal{T}}^{\mu *}s_i \in \Gamma(\mathcal{F}\mathcal{T}_\delta; \pi^*V_i)$ is constant on the fibers of $\mathcal{F}\mathcal{T}_\delta$ over an open subset $K_{\mathcal{T}}$ of $\mathcal{U}_{\mathcal{T}}(\mu)$ that contains all of the finitely many zeros of the affine map

$$\mathcal{F}\mathcal{T} \longrightarrow L^*\otimes \mathrm{ev}^*T\mathbb{P}^3 \oplus V_i, \quad (b, v) \longrightarrow \big(\nu_b + \alpha_{\mathcal{T}}\big(\rho_{\mathcal{T}}(v)\big), s_i(b)\big),$$

over $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$, where $\alpha_{\mathcal{T}}$ and $\rho_{\mathcal{T}}$ are as in equation (6.6). Note that by our assumptions on s_i and Lemma 6.8, $\alpha_{\mathcal{T}}$ has full rank on every fiber of $\bar{\mathcal{F}}\mathcal{T}|s_i^{-1}(0)$. Thus, by Theorem 6.2 and Proposition 5.13, $\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)}\big(\mathcal{D}\oplus s_i\big)=0$ if $H_{\hat{0}}\mathcal{T} \neq \hat{I}$. Furthermore, if $H_{\hat{0}}\mathcal{T} = \hat{I}$,

$$\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)}\big(\mathcal{D}\oplus s_i\big) = N(\alpha_{\mathcal{T}}), \quad \text{with} \quad \alpha_{\mathcal{T}} \in \Gamma\big(\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\cap s_i^{-1}(0); \mathrm{Hom}(\mathcal{F}\mathcal{T}; L^*\otimes \mathrm{ev}^*T\mathbb{P}^3)\big).$$

Thus, by Lemma 5.14,

$$\begin{aligned}
&\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)}\big(\mathcal{D}\oplus s_i\big)\\
&\quad = \big\langle c(L^*\otimes \mathrm{ev}^*T\mathbb{P}^3)c(\mathcal{F}\mathcal{T})^{-1}, [s_i^{-1}(0)\cap\bar{\mathcal{U}}_{\mathcal{T}}(\mu)]\big\rangle - \mathcal{C}_{\mathbb{P}\mathcal{F}\mathcal{T}|\left(s_i^{-1}(0)\cap\partial\bar{\mathcal{U}}_{\mathcal{T}}\right)}(\alpha_{\mathcal{T}}'^{\perp}),\\
&\quad = \big\langle \eta_i c(L^*\otimes \mathrm{ev}^*T\mathbb{P}^3)c(\mathcal{F}\mathcal{T})^{-1}, [\bar{\mathcal{U}}_{\mathcal{T}}(\mu)]\big\rangle - \mathcal{C}_{\mathbb{P}\mathcal{F}\mathcal{T}|\left(s_i^{-1}(0)\cap\partial\bar{\mathcal{U}}_{\mathcal{T}}\right)}(\alpha_{\mathcal{T}}'^{\perp})
\end{aligned} \tag{6.32}$$

where $\partial\bar{\mathcal{U}}_{\mathcal{T}} = \bar{\mathcal{U}}_{\mathcal{T}}(\mu) - \mathcal{U}_{\mathcal{T}}(\mu)$ and $\tilde{\alpha}_{\mathcal{T}} \in \Gamma(\mathbb{P}\mathcal{F}\mathcal{T}; \gamma_{\mathcal{F}\mathcal{T}}^*\otimes L^*\otimes \mathrm{ev}^*T\mathbb{P}^3)$ is the section induced by $\alpha_{\mathcal{T}}$.
(3) Suppose $i=1$, i.e. $\eta_i = a$. If $\mathcal{T} = \mathcal{T}^*(l)$ and $\mathcal{T}' = (S^2, [N], I'; j, \underline{d}') < \mathcal{T}$ is a bubble type such that $s_1^{-1}(0)\cap\mathcal{U}_{\mathcal{T}'}(\mu)\cap\alpha_{\mathcal{T}}^{-1}(0) \neq \emptyset$, $\mathcal{T}'$ must have the form

$$I' - I = H_{\hat{1}}\mathcal{T}' = \{\hat{2}, \hat{3}\}, \quad d'_{\hat{1}} = 0, \; d'_{\hat{2}} \neq 0, \; d'_{\hat{3}} \neq 0.$$

By Theorem 6.2 applied to $\bar{\mathcal{T}}' < \bar{\mathcal{T}}$ and Proposition 5.13,

$$\mathcal{C}_{s_1^{-1}(0)\cap\mathcal{U}_{\mathcal{T}'}(\mu)}(\alpha_{\mathcal{T}}'^{\perp}) = \big|\mathcal{U}_{\mathcal{T}'}(\mu)\cap s_1^{-1}(0)\big| = \langle a, [\bar{\mathcal{U}}_{\mathcal{T}'}(\mu)]\rangle.$$

Thus, summing up equation (6.32) over $\mathcal{T}=\mathcal{T}^*(l)$ with $l\in[N]$, we obtain

$$\sum_{l\in[N]} \mathcal{C}_{\mathcal{U}_{\mathcal{T}^*(l)}(\mu)}(\mathcal{D}\oplus s_1) \tag{6.33}$$

$$= \sum_{l\in[N]} \big\langle 6a^3+4a^2c_1\big(L_{\hat{1}}^*\mathcal{T}^*(l)\big)+ac_1^2\big(L_{\hat{1}}^*\mathcal{T}^*(l)\big),[\bar{\mathcal{U}}_{\mathcal{T}^*(l)}(\mu)]\big\rangle - \tau_2^{(1)}(\mu),$$

where $\tau_2^{(1)}(\mu)$ is the number of two-component connected degree-d curves passing through the constraints with the node at the intersection of one of the constraints with a generic plane in $\mathbb{P}^3$. If $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=2$, $|M_{\hat{0}}\mathcal{T}|=0$, and $\mathcal{T}'$ is as above, up to equivalence of bubble types,

$$I'-I=\{\hat{3}\},\quad \iota'_{\hat{3}}=\hat{1},\quad d'_{\hat{1}}=0,\ d'_{\hat{2}}\neq 0,\ d'_{\hat{3}}\neq 0,$$

i.e. $\bar{\mathcal{T}}'=\bar{\mathcal{T}}(l)$ for some $l\in[N]$. By Theorem 6.2 applied to $\bar{\mathcal{T}}'<\bar{\mathcal{T}}$ and Proposition 5.13,

$$\mathcal{C}_{\mathbb{P}\mathcal{F}\mathcal{T}|s_1^{-1}(0)\cap\mathcal{U}_{\mathcal{T}'}(\mu)}(\alpha_{\mathcal{T}}'^{\perp}) = \big|\mathcal{U}_{\mathcal{T}'}(\mu)\cap s_1^{-1}(0)\big| = \langle a,[\bar{\mathcal{U}}_{\mathcal{T}'}(\mu)]\rangle.$$

Thus, summing up equation (6.32) over $\mathcal{T}$ with $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=2$ and $|M_{\hat{0}}\mathcal{T}|=0$, we obtain

$$\sum_{|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=2,|M_{\hat{0}}\mathcal{T}|=0} \big\langle a\big(4a+c_1(L_{\hat{1}}^*\mathcal{T})+c_1(L_{\hat{2}}^*\mathcal{T})\big),[\bar{\mathcal{U}}_{\mathcal{T}}(\mu)]\big\rangle-2\tau_2^{(1)}(\mu) \tag{6.34}$$

$$= \big\langle 4a^2+a\big(c_1(\mathcal{L}_{\hat{1}}^*)+c_1(\mathcal{L}_{\hat{2}}^*)\big),[\bar{\mathcal{V}}_2(\mu)]\big\rangle.$$

If $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=2$ and $|M_{\hat{0}}\mathcal{T}|=1$, $\alpha_{\mathcal{T}}$ has full rank on all of $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$. Thus, by Proposition 5.13,

$$\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)}(\mathcal{D}\oplus s_i) = \big\langle c_1(L^*\otimes \mathrm{ev}^*T\mathbb{P}^3)-c_1(\mathcal{F}\mathcal{T}),[\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\cap s_1^{-1}(0)]\big\rangle = |\mathcal{U}_{\bar{\mathcal{T}}}(\mu)|.$$

Here we used $\mathcal{F}\mathcal{T}=L^*\otimes\big(L_{\hat{1}}\mathcal{T}\oplus L_{\hat{2}}\mathcal{T}\big)\approx L^*\oplus L^*$ along with the splitting (6.17) and Lemma 6.23. Thus, summing up equation (6.32) over $\mathcal{T}$ with $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=2$ and $|M_{\hat{0}}\mathcal{T}|=1$ gives

$$\sum_{|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=2,|M_{\hat{0}}\mathcal{T}|=1} \mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)}(\mathcal{D}\oplus s_i) = \tau_2^{(1)}(\mu). \tag{6.35}$$

Finally, if $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=3$, $\eta_1|\bar{\mathcal{U}}'_{\mathcal{T}}(\mu)=0$. The first claim follows by plugging the sum of equations (6.33)-(6.35) into (6.31). See also equations (6.14) and (6.15).
(4) Suppose $\eta_i=c_1(\mathcal{L}^*)$. We continue as in (3) above. If $\mathcal{T}=\mathcal{T}^*(l)$, $\alpha_{\mathcal{T}}$ does not vanish anywhere on $s_2^{-1}(0)\cap\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ for a good choice of s_2. Thus, by Proposition 5.13,

$$\sum_{l\in[N]} \mathcal{C}_{\mathcal{U}_{\mathcal{T}^*(l)}(\mu)}(\mathcal{D}\oplus s_2) = \sum_{l\in[N]} \big\langle c(L^*\otimes \mathrm{ev}^*T\mathbb{P}^3)c(L_{\hat{1}}\mathcal{T})^{-1},[\bar{\mathcal{U}}_{\mathcal{T}^*(l)}(\mu)\cap s_2^{-1}(0)]\big\rangle \tag{6.36}$$

$$= \sum_{l\in[N]} \big\langle c_1(\mathcal{L}^*)\big(6a^2+4ac_1\big(L_{\hat{1}}^*\mathcal{T}^*(l)\big)+c_1^2(L_{\hat{1}}^*\mathcal{T}^*(l))\big),[\bar{\mathcal{U}}_{\mathcal{T}^*(l)}(\mu)]\big\rangle.$$

If $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=3$, $\alpha_{\mathcal{T}}$ again does not vanish anywhere on $s_2^{-1}(0)\cap\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$, and thus

$$\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)}(\mathcal{D}\oplus s_2) = \big|\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\cap s_2^{-1}(0)\big| = \big\langle c_1(\mathcal{L}^*),[\bar{\mathcal{U}}_{\mathcal{T}}(\mu)]\big\rangle = \big|\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)\big|. \tag{6.37}$$

Here we used $\mathcal{FT} = L^*\otimes(L_{\hat{1}}\mathcal{T}\oplus L_{\hat{2}}\mathcal{T}\oplus L_{\hat{3}}\mathcal{T}) \approx L^*\oplus L^*\oplus L^*$ along with the splitting (6.17) and Lemma 6.23. Note that if $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=2$, $\eta_2|\bar{\mathcal{U}}_{\mathcal{T}}(\mu)=0$. This is immediate in the case $|M_{\hat{0}}\mathcal{T}|=0$ and follows from Lemma 6.23 and (6.14) in the case $|M_{\hat{0}}\mathcal{T}|=1$. The second claim of the lemma is obtained by summing (6.37) over all equivalence classes of bubble types $\mathcal{T}<\mathcal{T}^*$ with $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=3$, and plugging the result along with (6.36) into (6.31). Note that

$$a^3|\bar{\mathcal{U}}_{\mathcal{T}^*(l)}(\mu)=0 \quad \forall l\in[N] \Longrightarrow \langle 4a^3c_1(L^*),[\bar{\mathcal{V}}_1(\mu)]\rangle = \langle 4a^3c_1(\mathcal{L}^*),[\bar{\mathcal{V}}_1(\mu)]\rangle.$$

6.7 More Structural Descriptions

We now describe the structure of spaces of stable maps that correspond to singular rational degree-d curves passing through an N-tuple μ of $3d-4$ points in general position in $\mathbb{P}^2$. These descriptions are used in Section 6.9 and in Chapter 10.

If $\mathcal{T}=(S^2,M,I;j,\underline{d})$ is a bubble type and $i\in\chi(\mathcal{T})$, let

$$E_i\mathcal{T} = \begin{cases} L_i\mathcal{T}, & \text{if } i\in I-\hat{I}; \\ \tilde{\mathcal{F}}_i\mathcal{T}, & \text{if } i\in\hat{I}; \end{cases} \qquad E\mathcal{T} = \bigoplus_{i\in\chi(\mathcal{T})} E_i\mathcal{T}.$$

If $M\subset[N]$, and $\sum d_i=d$, put

$$\mathcal{S}_{\mathcal{T};1}(\mu) = \big\{b\in\mathcal{U}_{\mathcal{T}}(\mu)\colon \mathcal{D}_{\mathcal{T},i}b=0 \text{ for some } i\in\chi(\mathcal{T})\big\}.$$

If $|\chi(\mathcal{T})|\geq 2$, let

$$\mathcal{S}_{\mathcal{T};2}(\mu) = \Big\{\big[(\tilde{v}_i)_{i\in\chi(\mathcal{T})}\big]\in\mathbb{P}E\mathcal{T}\colon \sum_{i\in\chi(\mathcal{T})}\mathcal{D}_{\mathcal{T},i}\tilde{v}_i=0\Big\} - \mathcal{S}_{\mathcal{T};1}(\mu).$$

If $\mathcal{T}$ is basic and $|I|=1$, the set $\mathcal{S}_1(\mu)\equiv\mathcal{S}_{\mathcal{T};1}(\mu)$ of maps can be identified with a dense open subset of the set of irreducible rational degree-d curves that pass through the $3d-4$ points and have a cusp. We denote the closure of $\mathcal{S}_1(\mu)$ in $\bar{\mathcal{V}}_1(\mu)$ by $\bar{\mathcal{S}}_1(\mu)$. Let $\mathcal{S}_{2;2}(\mu)$ be the disjoint union of the spaces $\mathcal{S}_{\mathcal{T};2}(\mu)$ taken over all equivalence classes of basic bubble types $\mathcal{T}$ with $|I|=2$. The set $\mathcal{S}_{2;2}(\mu)$ can be identified with a dense open subset of the set of two-component rational degree-d curves that pass through the $3d-4$ points and have a tacnode as a node common to both components. We denote by $\bar{\mathcal{S}}_{2;2}(\mu)$ the closure of $\mathcal{S}_{2;2}(\mu)$ in $\mathbb{P}E_2$, where $E_2\longrightarrow\bar{\mathcal{V}}_2(\mu)$ is the bundle such that $E_2|\bar{\mathcal{U}}_{\mathcal{T}}(\mu)=E\mathcal{T}$. Similarly, we denote by $\mathcal{S}_{2;1}(\mu)$ the disjoint union of the spaces $\mathcal{S}_{\mathcal{T};1}(\mu)$ taken over all equivalence classes of basic bubble types $\mathcal{T}$ with $|I|=2$. This finite set can be identified with a subset of $\bar{\mathcal{S}}_{2;2}(\mu)$ as well as with the set of two-component rational degree-d curves passing through the $3d-4$ points such that the two components meet at a node at which one of them has a cusp.

Suppose M_0 is a subset of $[N]$ and $\mathcal{T}=(S^2,[N]-M_0,I;j,\underline{d})$ is a simple bubble type with $\sum d_i=d$. By Corollary C.3, the space $\mathcal{S}_{\mathcal{T};1}(\mu)$ is a smooth complex submanifold of $\mathcal{U}_{\mathcal{T}}(\mu)$. Let

$$\mathcal{NS}_{\mathcal{T};1} = L^*_{h_1}\mathcal{T}\otimes\mathrm{ev}^*T\mathbb{P}^2 \longrightarrow \mathcal{S}_{\mathcal{T};1}(\mu)$$

denote its normal bundle, where $h_1 \in \chi(\mathcal{T})$ is such that $\mathcal{D}_{\mathcal{T},h_1} b = 0$. Put

$$\mathcal{F}\mathcal{S}_{\mathcal{T};1} = \bigoplus_{h \in \hat{I}-\chi(\mathcal{T})} \mathcal{F}_h\mathcal{T} \;\;\oplus \begin{cases} \{0\}, & \text{if } h_1 \notin \hat{I};\\ \mathcal{F}_{h_1}\mathcal{T}, & \text{if } h_1 \in \hat{I} \;\&\; \chi(\mathcal{T}) = \{h_1\};\\ \mathcal{F}_{h_1}\mathcal{T} \oplus L^*_{h_1}\mathcal{T} \otimes L_{h_2}\mathcal{T}, & \text{if } \chi(\mathcal{T}) = \{h_1, h_2\}; \end{cases}$$

$$\tilde{\mathcal{F}}\mathcal{S}_{\mathcal{T};1} = \begin{cases} \mathcal{F}_{h_1}\mathcal{T}^{\otimes 2}, & \text{if } h_1 \in \hat{I} \;\&\; \chi(\mathcal{T}) = \{h_1\};\\ \mathcal{F}_{h_1}\mathcal{T}^{\otimes 2} \oplus \mathcal{F}_{h_2}\mathcal{T}, & \text{if } \chi(\mathcal{T}) = \{h_1, h_2\}. \end{cases}$$

We define $\rho_{\mathcal{T};1} \colon \mathcal{F}\mathcal{S}_{\mathcal{T};1} \longrightarrow \tilde{\mathcal{F}}\mathcal{S}_{\mathcal{T};1}$ and $\alpha_{\mathcal{T};1} \in \Gamma\big(\bar{\mathcal{S}}_{\mathcal{T};1}(\mu); \mathrm{Hom}(\tilde{\mathcal{F}}\mathcal{S}_{\mathcal{T};1}; L^*_{\hat{0}}\mathcal{T}^{\otimes 2} \otimes \mathrm{ev}^*T\mathbb{P}^2)\big)$ by

$$\rho_{\mathcal{T};1}(v) = \begin{cases} v_{h_1} \otimes v_{h_1}, & \text{if } \chi(\mathcal{T}) = \{h_1\};\\ \big(v_{h_1} \otimes v_{h_1}, v_{h_1} \otimes u\big), & \text{if } \chi(\mathcal{T}) = \{h_1, h_2\} \;\&\; u \in L^*_{h_1}\mathcal{T} \otimes L_{h_2}\mathcal{T}; \end{cases}$$

$$\alpha_{\mathcal{T};1}(\varpi) = \begin{cases} \mathcal{D}^{(2)}_{\mathcal{T},h_1}\varpi, & \text{if } \chi(\mathcal{T}) = \{h_1\};\\ \mathcal{D}^{(2)}_{\mathcal{T},h_1}\varpi_1 + x_{h_2}\mathcal{D}^{(1)}_{\mathcal{T},h_1}\varpi_2, & \text{if } \chi(\mathcal{T}) = \{h_1, h_2\},\; \varpi = (\varpi_1, \varpi_2). \end{cases}$$

If $|\chi(\mathcal{T})| \geq 2$, $\mathcal{S}_{\mathcal{T};2}(\mu)$ is a smooth submanifold of $\mathbb{P}E\mathcal{T} \longrightarrow \mathcal{U}_{\mathcal{T}}(\mu)$. We identify it with a subset of $\mathcal{U}_{\mathcal{T}}(\mu)$ via the bundle projection map $\pi_{E\mathcal{T}} : \mathbb{P}E\mathcal{T} \longrightarrow \mathcal{U}_{\mathcal{T}}(\mu)$. If $|\chi(\mathcal{T})| = 3$, $\mathcal{S}_{\mathcal{T};2}(\mu) = \mathcal{U}_{\mathcal{T}}(\mu)$, and we put $\mathcal{N}\mathcal{S}_{\mathcal{T};2} = \{0\}$. If $\chi(\mathcal{T}) = \{h_1, h_2\}$, $\mathcal{S}_{\mathcal{T};2}(\mu)$ is a smooth submanifold of $\mathcal{U}_{\mathcal{T}}(\mu)$ with normal bundle

$$\mathcal{N}\mathcal{S}_{\mathcal{T};2} = L^*_{h_2}\mathcal{T} \otimes \big(\mathrm{Im}\, \mathcal{D}_{\mathcal{T},h_1}\big)^{\perp}.$$

If $\iota_{h_1} = \iota_{h_2}$ for all $h_1, h_2 \in \chi(\mathcal{T})$, put

$$\mathcal{F}\mathcal{S}_{\mathcal{T};2} = \gamma_{\underset{h \in \chi(\mathcal{T})}{\oplus} \mathcal{F}_h\mathcal{T}} \oplus \bigoplus_{h \notin \chi(\mathcal{T})} \mathcal{F}_h\mathcal{T}, \quad \tilde{\mathcal{F}}\mathcal{S}_{\mathcal{T};2} = \gamma_{\underset{h \in \chi(\mathcal{T})}{\oplus} \mathcal{F}_h\mathcal{T}} \otimes \begin{cases} \mathbb{C}, & \text{if } \iota_h = \hat{0} \;\forall h \in \chi(\mathcal{T});\\ \mathcal{F}_{\hat{1}}\mathcal{T}^{\otimes 2}, & \text{if } \iota_h = \hat{1} \neq \hat{0} \;\forall h \in \chi(\mathcal{T}). \end{cases}$$

Let $\rho_{\mathcal{T};2} \colon \mathcal{F}\mathcal{S}_{\mathcal{T};2} \longrightarrow \tilde{\mathcal{F}}\mathcal{S}_{\mathcal{T};2}$ be the projection map followed by multiplication. Define

$$\alpha_{\mathcal{T};2} \in \Gamma\big(\bar{\mathcal{S}}_{\mathcal{T};2}(\mu); \mathrm{Hom}(\tilde{\mathcal{F}}\mathcal{S}_{\mathcal{T};2}; L^*_{\hat{0}}\mathcal{T}^{\otimes 2} \otimes \mathrm{ev}^*T\mathbb{P}^2)\big) \qquad \text{by}$$

$$\alpha_{\mathcal{T};2}\big((\tilde{v}_h)_{h \in \chi(\mathcal{T})}\big) = \sum_{h \in \chi(\mathcal{T})} x_h \mathcal{D}^{(1)}_{\mathcal{T},h}\tilde{v}_h \qquad \text{if } b_v = \big(S^2, M, I; x, (j, y), u\big).$$

If $\chi_{\mathcal{T}}\hat{1} = 0$ for some $\hat{1} \in \hat{I} - \chi(\mathcal{T})$, $H_{\hat{0}}\mathcal{T} = \{\hat{1}, h_1\}$, and $h_2 \in \chi(\mathcal{T}) - \{h_1\}$, let

$$\mathcal{F}\mathcal{S}_{\mathcal{T};2} = \mathcal{F}_{\hat{1}}\mathcal{T} \oplus \mathcal{F}_{h_2}\mathcal{T}, \qquad \tilde{\mathcal{F}}\mathcal{S}_{\mathcal{T};2} = \mathcal{F}_{\hat{1}}\mathcal{T} \otimes \mathcal{F}_{h_2}\mathcal{T}, \quad \text{and} \quad \rho_{\mathcal{T};2} \colon \mathcal{F}\mathcal{S}_{\mathcal{T};2} \longrightarrow \tilde{\mathcal{F}}\mathcal{S}_{\mathcal{T};2}$$

be the multiplication map. Define $\alpha_{\mathcal{T};2} \in \Gamma(\mathcal{S}_{\mathcal{T};2}(\mu); \mathrm{Hom}(\tilde{\mathcal{F}}\mathcal{S}_{\mathcal{T};2}; L^*_{\hat{0}}\mathcal{T}^{\otimes 2} \otimes \mathrm{ev}^*T\mathbb{P}^2))$ by

$$\alpha_{\mathcal{T};2}\big(v_{\hat{1}} \otimes v_{h_2}\big) = x_{\hat{1}} v_{\hat{1}} \sum_{h \in \chi(\mathcal{T}) - \{h_1\}} \mathcal{D}_{\mathcal{T},h} v'_h + x_{h_1}\mathcal{D}_{\mathcal{T},h_1} v_{h_1},$$

$$\text{if} \quad b_{v_{\hat{1}} \otimes v_{h_2}} = \big(S^2, M, I; x, (j, y), u\big), \quad \big[v_{\hat{1}} \otimes v', v_{h_1}\big] \in \mathcal{S}_{\mathcal{T};2}(\mu), \quad \text{where} \quad v' = (v_h)_{h \in \chi(\mathcal{T}) - \{h_1\}}.$$

In all other cases, $\mathcal{S}_{\mathcal{T};k}(\mu) = \emptyset$ as can be seen from dimension-counting and Corollary C.3. Denote by $\mathcal{F}^{\emptyset}\mathcal{S}_{\mathcal{T};k}$ the subset of $\mathcal{F}\mathcal{S}_{\mathcal{T};k}$ consisting of vectors with all components nonzero.

Proposition 6.10 *If $\mathcal{T}^*=(S^2,[N]-M_0,\{\hat{0}\};\hat{0},d)$ and $\mathcal{T}=(S^2,[N]-M_0,I;j,\underline{d})<\mathcal{T}^*$ are simple bubble types,*

$$\bar{\mathcal{S}}_{\mathcal{T}^*;1}(\mu)\cap\mathcal{U}_{\mathcal{T}}(\mu)=\begin{cases}\mathcal{S}_{\mathcal{T};1}(\mu), & \text{if } |\chi(\mathcal{T})|=1;\\ \mathcal{S}_{\mathcal{T};1}(\mu)\cup\mathcal{S}_{\mathcal{T};2}(\mu), & \text{if } |H_{\hat{0}}\mathcal{T})|=|\hat{I}|=2\ \&\ M_{\hat{0}}\mathcal{T}=\emptyset;\\ \mathcal{S}_{\mathcal{T};2}(\mu), & \text{otherwise.}\end{cases}$$

In addition, there exist $\delta,C\in C^\infty(\mathcal{S}_{\mathcal{T};k}(\mu);\mathbb{R}^+)$ and a continuous map

$$\gamma_{\mathcal{T};k}\colon \mathcal{F}\mathcal{S}_{\mathcal{T};k;\delta}\longrightarrow\bar{\mathcal{S}}_{\mathcal{T}^*;1}(\mu)$$

onto an open neighborhood of $\mathcal{S}_{\mathcal{T};k}(\mu)$ in $\bar{\mathcal{S}}_{\mathcal{T}^*;1}(\mu)$ such that $\gamma_{\mathcal{T};k}|\mathcal{S}_{\mathcal{T};k}(\mu)$ is the identity and $\gamma_{\mathcal{T};k}|\mathcal{F}^\emptyset\mathcal{S}_{\mathcal{T};k;\delta}$ is an orientation-preserving diffeomorphism onto an open subset of $\mathcal{S}_{\mathcal{T}^*;1}(\mu)$. Furthermore, if $d_{\hat{0}}\neq 0$, $\mathcal{D}^{(2)}_{\mathcal{T}^*,\hat{0}}$ does not vanish on $\mathcal{S}_{\mathcal{T};1}(\mu)$. If $d_{\hat{0}}=0$, with appropriate identifications,

$$\Big|\mathcal{D}^{(2)}_{\mathcal{T}^*,\hat{0}}\big(\gamma_{\mathcal{T};k}(\upsilon)\big)-\alpha_{\mathcal{T};k}\big(\rho_{\mathcal{T};k}(\upsilon)\big)\Big|\le C(b_\upsilon)|\upsilon|^{\frac{1}{p}}\big|\rho_{\mathcal{T};k}(\upsilon)\big|\qquad\forall\upsilon\in\mathcal{F}\mathcal{S}_{\mathcal{T};k;\delta}.$$

Lemma 6.11 *If $\mathcal{T}^*$ and $\mathcal{T}$ are as in Proposition 6.10, there exist $\delta\in C^\infty(\mathcal{S}_{\mathcal{T};k}(\mu);\mathbb{R}^+)$ and a continuous map*

$$\tilde{\gamma}_{\mathcal{T};k}\colon\big(\mathcal{N}\mathcal{S}_{\mathcal{T};k}\oplus\mathcal{F}\mathcal{T}\big)_\delta\longrightarrow\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$$

onto an open neighborhood of $\mathcal{S}_{\mathcal{T};k}(\mu)$ in $\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$ such that $\gamma_{\mathcal{T};k}|\mathcal{S}_{\mathcal{T};k}(\mu)$ is the identity and $\gamma_{\mathcal{T};k}$ is smooth and orientation-preserving on the preimage of $\mathcal{V}_{\mathcal{T}^*}(\mu)$. Furthermore, there exists a subbundle $\mathcal{O}_{\mathcal{T}}$ of $L^*\otimes ev^*T\mathbb{P}^2$ of rank less than the rank of $\mathcal{N}\mathcal{S}_{\mathcal{T};k}\oplus\mathcal{F}\mathcal{T}$ such that, with appropriate identifications,

$$\mathcal{D}_{\mathcal{T}^*,\hat{0}}|\tilde{\gamma}_{\mathcal{T};k}(X,\upsilon)\subset\mathcal{O}_{\mathcal{T}}\qquad\forall(X,\upsilon)\in\big(\mathcal{N}\mathcal{S}_{\mathcal{T};k}\oplus\mathcal{F}\mathcal{T}\big)_\delta.$$

The proof of Proposition 6.10, with the exception of the estimate on $\mathcal{D}^{(2)}_{\mathcal{T}^*,\hat{0}}$, is either the same or very similar to the proof of Lemma 6.6, depending on the bubble type $\mathcal{T}$. If $d_{\hat{0}}\neq 0$, we apply the analytic estimate of Theorem 6.2 and the Implicit Function Theorem to the section $\gamma_{\mathcal{T}}^{\mu*}\mathcal{D}_{\mathcal{T}^*,\hat{0}}$. If $d_{\hat{0}}=0$ and $k=1$, the two theorems are applied to a section of $(L_{\hat{0}}\mathcal{T}\otimes\mathcal{F}_{h_1}\mathcal{T})^*\otimes\mathrm{ev}^*T\mathbb{P}^2$ induced by $\gamma_{\mathcal{T}}^{\mu*}\mathcal{D}_{\mathcal{T}^*,\hat{0}}$. If $|\chi(\mathcal{T})|\ge 2$ and $H_{\hat{0}}\mathcal{T}=\chi(\mathcal{T})$, we work with a section of $(L_{\hat{0}}\mathcal{T}\otimes\gamma_{E\mathcal{T}})^*\otimes\mathrm{ev}^*T\mathbb{P}^2$ over the blowup of $E\mathcal{T}$ along $\mathcal{U}_{\mathcal{T}}(\mu)$. The case $\iota_h=\hat{1}$ for all $h\in\chi(\mathcal{T})$ is similar. If $\chi(\mathcal{T})=\{h_1,h_2\}$ is a two-element set, and $H_{\hat{0}}\mathcal{T}=\{\hat{1},h_1\}$, we use the same section, but given a small element $(\upsilon_{\hat{1}},\upsilon_{h_2})\in\mathcal{F}\mathcal{S}_{\mathcal{T};2}$, we start with the approximate solution $(\upsilon_{\hat{1}},\kappa\upsilon_{\hat{1}}\upsilon_{h_2},\upsilon_{h_2})$, with

$$\kappa\in(L_{\hat{1}}\mathcal{T}\otimes L_{h_2}\mathcal{T})^*\otimes L_{h_1}\mathcal{T}\quad\text{s.t.}\quad\big[\upsilon_{\hat{1}}\upsilon_{h_2},\kappa\upsilon_{\hat{1}}\upsilon_{h_2}\big]\in\mathcal{S}_{\mathcal{T};2}(\mu).$$

The approach to the remaining case is analogous. The estimate on $\mathcal{D}^{(2)}_{\mathcal{T}^*,\hat{0}}$ is obtained by the same argument as in the proof of Lemma 6.8. The first statement of Lemma 6.11 is an immediate consequence of Theorem 6.2. The second statement makes no claim unless $|\hat{I}|=1$. In such a case, the proof is exactly the same as the proof of Corollary 6.7.

We next describe the behavior of the section

$$\mathcal{D}_{2;2}\equiv c_1\mathcal{D}_1+c_2\mathcal{D}_2\in\Gamma\big(\bar{\mathcal{S}}_{2;2}(\mu);\gamma_{E_2}^*\otimes\mathrm{ev}^*T\mathbb{P}^2\big),$$

for $c_1, c_2 \in \mathbb{C}$ distinct, near $\partial\bar{\mathcal{S}}_{2;2}(\mu) \equiv \bar{\mathcal{S}}_{2;2}(\mu) - \mathcal{S}_{2;2}(\mu)$. As before, we identify $\mathcal{S}_{2;1}(\mu)$ with a subset of $\bar{\mathcal{S}}_{2;2}(\mu)$. Similarly, if $\mathcal{T} = (S^2, [N], I; j, \underline{d})$ is a bubble type such that $I - \hat{I} = \{k_1, k_2\}$ is a two-element set and $\sum d_i = d$, let

$$\mathcal{S}_{\mathcal{T};2}(\mu) = \{[b, L_{k_1}\mathcal{T}] : b \in \mathcal{U}_{\mathcal{T}}(\mu),\ \mathcal{D}_{\mathcal{T},k_1} b = 0\} \cup \{[b, L_{k_2}\mathcal{T}] : b \in \mathcal{U}_{\mathcal{T}}(\mu),\ \mathcal{D}_{\mathcal{T},k_2} b = 0\} \subset \mathbb{P}E_2.$$

Proposition 6.12 *Suppose d is a positive integer and μ is a tuple of $3d-4$ points in general position in $\mathbb{P}^2$. Then*

$$\partial\bar{\mathcal{S}}_{2;2}(\mu) = \mathcal{S}_{2;1}(\mu) \cup \bigcup_{[\mathcal{T}]} \mathcal{S}_{\mathcal{T};2}(\mu),$$

where the union is taken over all equivalence classes of non-basic types $\mathcal{T} = (S^2, [N], I; j, \underline{d})$ such that $I - \hat{I} = \{k_1, k_2\}$ is a two-element set and $\sum d_i = d$. Furthermore, there exist $\delta, C > 0$ and homeomorphism

$$\gamma_{2;2} : \{u \in \gamma_{E_2}^* \otimes (E_2/\gamma_{E_2}) | \partial\bar{\mathcal{S}}_{2;2}(\mu) : |u| < \delta\} \longrightarrow \bar{\mathcal{S}}_{2;2}(\mu)$$

onto an open neighborhood of $\partial\bar{\mathcal{S}}_{2;2}(\mu)$ in $\bar{\mathcal{S}}_{2;2}(\mu)$ such that $\gamma_{2;2}|\partial\bar{\mathcal{S}}_{2;2}(\mu)$ is the identity and $\gamma_{2;2}$ restricts to an orientation-preserving diffeomorphism from the complement of $\partial\bar{\mathcal{S}}_{2;2}(\mu)$ onto an open subset of $\mathcal{S}_{2;2}(\mu)$. Finally, with appropriate identifications,

$$\left|\mathcal{D}_{2;2}\gamma_{2;2}(u) - \alpha_{2;2}(u)\right| \le C|u|^{1+\frac{1}{p}} \quad \forall u \in \{u \in \gamma_{E_2}^* \otimes (E_2/\gamma_{E_2}) | \partial\bar{\mathcal{S}}_{2;2}(\mu) : |u| < \delta\} \longrightarrow \bar{\mathcal{S}}_{2;2}(\mu),$$

where $\alpha_{2;2} \in \Gamma(\partial\bar{\mathcal{S}}_{2;2}(\mu); Hom(\gamma_{E_2}^* \otimes (E_2/\gamma_{E_2}), \gamma_{E_2}^* \otimes ev^* T\mathbb{P}^2))$ is an injection on every fiber.

This proposition follows from Theorem 6.2 and the Implicit Function Theorem by an argument similar to the proof of Proposition 6.10. Note that with our choice of constraints, $\partial\bar{\mathcal{S}}_{2;2}(\mu)$ is a finite set. Thus, we are able to take δ and C to be positive real numbers rather than continuous functions $\partial\bar{\mathcal{S}}_{2;2}(\mu) \longrightarrow \mathbb{R}^+$.

6.8 Intersections in $H^*(\bar{\mathcal{S}}_{2;2}(\mu))$ and $H^*(\bar{\mathcal{S}}_1(\mu))$

We now relate top intersections of certain tautological classes in $\bar{\mathcal{S}}_{2;2}(\mu)$ and in $\bar{\mathcal{S}}_1(\mu)$ to intersections of tautological classes in $\bar{\mathcal{V}}_2(\mu)$ and $\bar{\mathcal{V}}_1(\mu)$, respectively.

Lemma 6.13 *If d is a positive integer and μ is a tuple of $3d-4$ points in general position in $\mathbb{P}^2$,*

$$\langle a, [\bar{\mathcal{S}}_{2;2}(\mu)]\rangle = \langle 3a^2 + a(c_1(\mathcal{L}_1^*) + c_1(\mathcal{L}_2^*)), [\bar{\mathcal{V}}_2(\mu)]\rangle,$$
$$\langle \lambda_{E_2}, [\bar{\mathcal{S}}_{2;2}(\mu)]\rangle = \langle 3a^2 + 3a(c_1(\mathcal{L}_1^*) + c_1(\mathcal{L}_2^*)) + (c_1^2(L_1^*) + c_1^2(L_2^*)) + c_1(L_1^*)c_1(L_2^*), [\bar{\mathcal{V}}_2(\mu)]\rangle,$$

with chern classes $c_1^2(L_1^*) + c_1^2(L_2^*)$ and $c_1(L_1^*)c_1(L_2^*)$ defined similarly to $c_1^2(\mathcal{L}_1^*) + c_1^2(\mathcal{L}_2^*)$ and $c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*)$.

Proof: We only sketch the argument, since the proof is analogous to that of Lemma 6.9. Let $\mathcal{D} \in \Gamma(\mathbb{P}E_2; \gamma_{E_2}^* \otimes \mathrm{ev}^* T\mathbb{P}^n)$ be the section induced by the section

$$\mathcal{D}_1 + \mathcal{D}_2 \in \Gamma(\bar{\mathcal{V}}_2(\mu); E_2^* \otimes \mathrm{ev}^* T\mathbb{P}^2),$$

defined similarly to $c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)$; see Section 6.3. Then

$$\mathcal{S}_{2;2}(\mu) = \mathcal{D}^{-1}(0) \cap \big(\mathbb{P}E_2|(\mathcal{V}_2(\mu) - \mathcal{S}_{2;1}(\mu))\big).$$

Let s be a section of $\mathrm{ev}^*\mathcal{O}(1_{\mathbb{P}^2}) \longrightarrow \mathbb{P}E_2$ such that s is smooth and transversal to the zero set of the bundle on all smooth strata of $\mathbb{P}E_2$ and of $\bar{\mathcal{S}}_{2;2}(\mu)$. In particular, $s^{-1}(0)\cap \mathcal{S}_{2;1}(\mu)=\emptyset$. Then, by Proposition 5.13,

$$\begin{aligned}\langle a, [\bar{\mathcal{S}}_{2;2}(\mu)]\rangle &=^{\pm} |s^{-1}(0)\cap \mathcal{S}_{2;2}(\mu)| =^{\pm} \big|\mathcal{D}^{-1}(0)\cap \big(\mathbb{P}E_2|(\mathcal{V}_2(\mu)-\mathcal{S}_{2;1}(\mu))\big)\cap s^{-1}(0)\big| \\ &= \big\langle e(\gamma_{E_2}^*\otimes \mathrm{ev}^*T\mathbb{P}^2), [s^{-1}(0)]\big\rangle - \mathcal{C}_{(\mathbb{P}E_2|\bar\partial\mathcal{V}_2(\mu))\cap s^{-1}(0)}(\mathcal{D}). \end{aligned} \tag{6.38}$$

If $\mathcal{U}_{\mathcal{T}}(\mu)\subset \mathcal{V}_2(\mu)$ and $\mathcal{T}$ is not a basic bubble type, it is easy to see that $\mathrm{ev}(\bar{\mathcal{U}}_{\mathcal{T}}(\mu))$ is a finite set of points. It follows that $\mathrm{ev}^*\mathcal{O}(1_{\mathbb{P}^2})|\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ is a trivial line bundle and we can assume that s does not vanish on $\mathbb{P}E_2|\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$. Thus, the last term in (6.38) is zero, and

$$\langle a, [\bar{\mathcal{S}}_{2;2}(\mu)]\rangle = \langle ac_2(\gamma_{E_2}^*\otimes \mathrm{ev}^*T\mathbb{P}^2), [\mathbb{P}E_2]\rangle.$$

The first claim then follows from (5.16) and from definitions of E_2 and of the relevant chern classes. Note that

$$a\big(c_1(L_1^*)+c_1(L_2^*)\big) = a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big).$$

For the second identity, observe that

$$\gamma_{E_2}|\big((\mathbb{P}E_2|\bar{\mathcal{U}}_{\mathcal{T}}(\mu))\cap \mathcal{D}^{-1}(0)\big) \approx \mathbb{C}.$$

Thus, the proof is similar.

Lemma 6.14 *If d is a positive integer and μ is a tuple of $3d-4$ points in general position in $\mathbb{P}^2$,*

$$\begin{aligned}\langle a^2, [\bar{\mathcal{S}}_1(\mu)]\rangle &= \langle a^2c_1^2(\mathcal{L}^*), [\bar{\mathcal{V}}_1(\mu)]\rangle - \langle a^2, [\bar{\mathcal{V}}_2(\mu)]\rangle, \\ \langle ac_1(\mathcal{L}^*), [\bar{\mathcal{S}}_1(\mu)]\rangle &= \langle 3a^2c_1^2(\mathcal{L}^*) + ac_1^3(\mathcal{L}^*), [\bar{\mathcal{V}}_1(\mu)]\rangle, \\ \langle c_1^2(\mathcal{L}^*), [\bar{\mathcal{S}}_1(\mu)]\rangle &= \langle 3a^2c_1^2(\mathcal{L}^*) + 3ac_1^3(\mathcal{L}^*) + c_1^4(\mathcal{L}^*), [\bar{\mathcal{V}}_1(\mu)]\rangle.\end{aligned}$$

Proof: (1) This lemma is proved similarly to Lemma 6.13. Let

$$E=\mathrm{ev}^*\mathcal{O}(1_{\mathbb{P}^2})\oplus \mathrm{ev}^*\mathcal{O}(1_{\mathbb{P}^2}), \qquad E=\mathrm{ev}^*\mathcal{O}(1_{\mathbb{P}^2})\oplus \mathcal{L}^*, \quad \text{or} \quad E=\mathcal{L}^*\oplus\mathcal{L}^*,$$

depending on which of the three identities we are proving. Let s be a section of $E\longrightarrow \bar{\mathcal{V}}_1(\mu)$ such that s is smooth and transversal to the zero set in E on all smooth strata of $\bar{\mathcal{V}}_1(\mu)$ and of $\bar{\mathcal{S}}_1(\mu)$. Then

$$\langle c_2(E), [\bar{\mathcal{S}}_1(\mu)]\rangle = \langle c_2(E)c_2(L^*\otimes \mathrm{ev}^*T\mathbb{P}^2), [\bar{\mathcal{V}}_1(\mu)]\rangle - \mathcal{C}_{\partial\bar{\mathcal{V}}_1(\mu)\cap s^{-1}(0)}(\mathcal{D}). \tag{6.39}$$

The bundle $L\longrightarrow \bar{\mathcal{V}}_1(\mu)$ and section $\mathcal{D}\in\Gamma(\bar{\mathcal{V}}_1(\mu); L^*\otimes \mathrm{ev}^*T\mathbb{P}^2)$ in (6.18) are defined as follows. Let $N=3d-4$ and $\mathcal{T}^*=(S^2,[N],\{\hat{0}\};\hat{0},d)$. Then $L=L_{\hat{0}}\mathcal{T}^*$ and $\mathcal{D}=\mathcal{D}_{\mathcal{T}^*,\hat{0}}$. Suppose

$$\mathcal{T}=(S^2,[N],I;j,\underline{d}) < \mathcal{T}^*.$$

If $d_{\hat{0}} \neq 0$, $\mathcal{D}$ is transversal to the zero set on $\mathcal{U}_{\mathcal{T}}(\mu)$, and $s^{-1}(0) \cap \mathcal{D}^{-1}(0) \cap \mathcal{U}_{\mathcal{T}}(\mu) = \emptyset$ by our assumptions on s and Corollary C.3. Thus, from now on, we assume that $d_{\hat{0}} = 0$. If $\mathcal{T}$ is a semiprimitive bubble type, either $\mathrm{ev}^*\mathcal{O}(1_{\mathbb{P}^2})|\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ or $\mathcal{L}|\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ is trivial. It follows that if $E = \mathrm{ev}^*\mathcal{O}(1_{\mathbb{P}^2}) \oplus \mathcal{L}^*$, we can choose s so that it does not vanish on $\partial\bar{\mathcal{V}}_1(\mu)$. The second equality is then immediate from (6.39) and

$$ac_1(L^*) = ac_1(\mathcal{L}^*) \in H^4\big(\bar{\mathcal{V}}_1(\mu)\big). \tag{6.40}$$

(2) Suppose $E = \mathrm{ev}^*\mathcal{O}(1_{\mathbb{P}^2}) \oplus \mathrm{ev}^*\mathcal{O}(1_{\mathbb{P}^2})$. If the complex dimension of $\mathcal{U}_{\mathcal{T}}(\mu)$ is at least two and $\mathrm{ev}^*\mathcal{O}(1_{\mathbb{P}^2})|\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ is not trivial, $|\hat{I}| = |H_{\hat{0}}\mathcal{T}| = 2$. For a good choice of s, the map $\gamma_{\mathcal{T}}^{\mu}$ of Theorem 6.2 identifies neighborhoods of $s^{-1}(0) \cap \mathcal{U}_{\mathcal{T}}(\mu)$ in $\mathcal{FT}$ and in $s^{-1}(0)$. Since $s^{-1}(0) \cap \mathcal{S}_{\mathcal{T};k}(\mu) = \emptyset$, the section $\alpha_{\mathcal{T}}$ of Theorem 3.33 defines an isomorphism between $\mathcal{FT}$ and $L^* \otimes \mathrm{ev}^*T\mathbb{P}^2$ over every point of $s^{-1}(0) \cap \mathcal{U}_{\mathcal{T}}(\mu)$. Thus, by Proposition 5.13 and the analytic estimate of Theorem 6.2,

$$\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)\cap s^{-1}(0)}(\mathcal{D}) =^{\pm} \big|\mathcal{U}_{\mathcal{T}}(\mu) \cap s^{-1}(0)\big| = \big\langle a^2, [\bar{\mathcal{U}}_{\mathcal{T}}(\mu)]\big\rangle.$$

Summing these contributions over all equivalence classes of such bubble types $\mathcal{T}$, we obtain

$$\mathcal{C}_{\partial\bar{\mathcal{V}}_1(\mu)\cap s^{-1}(0)}(\mathcal{D}) = \big\langle a^2, [\bar{\mathcal{V}}_2(\mu)]\big\rangle. \tag{6.41}$$

The first identity follows from equations (6.39), (6.40), and (6.41).
(3) Suppose $E = \mathcal{L}^* \oplus \mathcal{L}^*$. If the complex dimension of $\mathcal{U}_{\mathcal{T}}(\mu)$ is at least two and $\mathcal{L}|\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ is not trivial, $H_{\hat{0}}\mathcal{T} = \{\hat{1}\}$ is a one-element set and $|\hat{I}| \in \{1, 2\}$. If $|\hat{I}| = 2$, by an argument similar to (2) above, $\mathcal{U}_{\mathcal{T}}(\mu) \cap s^{-1}(0)$ is $\mathcal{D}$-hollow in the sense of Definition 5.11, where $\mathcal{D}$ is viewed as a section over $s^{-1}(0) \subset \bar{\mathcal{V}}_1(\mu)$. Thus, by Proposition 5.13, $\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)\cap s^{-1}(0)}(\mathcal{D}) = 0$. If $|\hat{I}| = 1$, $\mathcal{T} = \mathcal{T}^*(\{l\})$ for some $l \in [N]$. For the purposes of computing $\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)\cap s^{-1}(0)}(\mathcal{D})$, it can be assumed that the map of Theorem 6.2 still identifies neighborhoods of $s^{-1}(0) \cap \mathcal{U}_{\mathcal{T}}(\mu)$ in $\mathcal{FT}$ and in $s^{-1}(0)$; see the proof of Lemma 6.9. Since $\alpha_{\mathcal{T}}$ does not vanish on $\mathcal{U}_{\mathcal{T}}(\mu) \cap s^{-1}(0)$, by Proposition 5.13 and the analytic estimate of Theorem 6.2,

$$\begin{aligned}\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)\cap s^{-1}(0)}(\mathcal{D}) &= \big\langle c_1\big(L^* \otimes \mathrm{ev}^*T\mathbb{P}^2\big) - c_1(\mathcal{FT}), \big[\bar{\mathcal{U}}_{\mathcal{T}}(\mu) \cap s^{-1}(0)\big]\big\rangle \\ &= \big\langle c_1^2(\mathcal{L}^*)c_1(L_{\hat{1}}^*\mathcal{T}), [\bar{\mathcal{U}}_{\mathcal{T}}(\mu)]\big\rangle,\end{aligned}$$

since the restrictions of the bundles L^* and $\mathrm{ev}^*T\mathbb{P}^2$ to $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ are trivial. Summing up these contributions and using equation (6.15), we obtain

$$\mathcal{C}_{\partial\bar{\mathcal{V}}_1(\mu)\cap s^{-1}(0)}(\mathcal{D}) = \sum_{l\in[N]} \big\langle c_1^3(\mathcal{L}^*), [\bar{\mathcal{U}}_{\mathcal{T}^*(l)}(\mu)]\big\rangle. \tag{6.42}$$

The final identity of the lemma follows from equations (6.39), (6.40), and (6.42); see also equation (6.14).

6.9 Rational Curves with a $(3, 4)$-Cusp

In this section, we prove Proposition 1.1. It follows immediately from Lemmas 6.15, 6.5, and 6.14.

Lemma 6.15 *If d is a positive integer, the number of rational degree-d curves that pass through a tuple μ of $3d-4$ points in general position in $\mathbb{P}^2$ and have a $(3,4)$-cusp is given by*

$$\begin{aligned}\big|\mathcal{S}_{1;2}(\mu)\big| = \big\langle 3a^2+6ac_1(\mathcal{L}^*)+4c_1^2(\mathcal{L}^*),[\bar{\mathcal{S}}_1(\mu)]\big\rangle - 2\big|\mathcal{S}_{2;1}(\mu)\big| - 3\big|\mathcal{V}_3(\mu)\big| \\ -\big\langle 6a^2+3a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big)+c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*),[\bar{\mathcal{V}}_2(\mu)]\big\rangle.\end{aligned}$$

Proof: (1) We continue with the notation used in the proof of Lemma 6.14. By definition, Proposition 5.13, and equation (6.40),

$$\begin{aligned}\big|\mathcal{S}_{1;2}(\mu)\big| &=^{\pm} \big|\mathcal{D}^{(2)\,-1}(0)\cap\mathcal{S}_1(\mu)\big| = \big\langle e\big(L^{*\otimes 2}\otimes \mathrm{ev}^*T\mathbb{P}^2\big),[\bar{\mathcal{S}}_1(\mu)]\big\rangle - \mathcal{C}_{\partial\bar{\mathcal{S}}_1(\mu)}(\mathcal{D}^{(2)}) \\ &= \big\langle 3a^2+6ac_1(\mathcal{L}^*)+4c_1^2(L^*),[\bar{\mathcal{S}}_1(\mu)]\big\rangle - \mathcal{C}_{\partial\bar{\mathcal{S}}_1(\mu)}(\mathcal{D}^{(2)}).\end{aligned} \tag{6.43}$$

Suppose $\mathcal{T}=(S^2,[N],I;j,\underline{d})<\mathcal{T}^*$. If $d_{\hat{0}}\neq 0$, $\mathcal{D}^{(2)}$ does not vanish on $\mathcal{U}_{\mathcal{T}}(\mu)$ by Proposition 6.10. Thus, from now on we consider only bubble types $\mathcal{T}$ such that $d_{\hat{0}}=0$.
(2) Suppose $\chi(\mathcal{T})=\{\hat{1}\}$ is a one-element set. Then $\bar{\mathcal{S}}_1(\mu)\cap\mathcal{U}_{\mathcal{T}}(\mu)=\mathcal{S}_{\mathcal{T};1}(\mu)$ and with appropriate identifications

$$\Big|\mathcal{D}^{(2)}_{\mathcal{T}^*,\hat{0}}\big(\gamma_{\mathcal{T};1}(\upsilon)\big)-\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}\big(\tilde{\upsilon}_{\hat{1}}\otimes\tilde{\upsilon}_{\hat{1}}\big)\Big| \le C(b_\upsilon)|\upsilon|^{\frac{1}{p}}|\tilde{\upsilon}_{\hat{1}}|^2 \qquad \forall \upsilon\in\mathcal{F}\mathcal{S}_{\mathcal{T};1;\delta}=\mathcal{F}\mathcal{T}_\delta,$$

where $\gamma_{\mathcal{T};1}\colon \mathcal{F}\mathcal{T}_\delta\longrightarrow\bar{\mathcal{S}}_1(\mu)$ is the map of Proposition 6.10. Since $d_{\hat{1}}\neq 0$, $\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}$ does not vanish on $\mathcal{S}_{\mathcal{T};1}(\mu)$. Thus, if $\hat{I}\neq H_{\hat{0}}\mathcal{T}$, $\mathcal{T}$ is $\mathcal{D}^{(2)}$-hollow and $\mathcal{C}_{\bar{\mathcal{S}}_1(\mu)\cap\mathcal{U}_{\mathcal{T}}(\mu)}(\mathcal{D}^{(2)})=0$. If $\hat{I}=H_{\hat{0}}\mathcal{T}$, i.e. $\mathcal{T}=\mathcal{T}^*(l)$ for some $l\in[N]$, by Proposition 5.13 and the splitting (6.17),

$$\mathcal{C}_{\bar{\mathcal{S}}_1(\mu)\cap\mathcal{U}_{\mathcal{T}}(\mu)}(\mathcal{D}^{(2)}) = 2N\big(\mathcal{D}^{(2)}_{\bar{\mathcal{T}},\hat{1}}\big), \tag{6.44}$$

where $\mathcal{D}^{(2)}_{\bar{\mathcal{T}},\hat{1}}\in\Gamma(\bar{\mathcal{S}}_{\bar{\mathcal{T}};1}(\mu);\mathrm{Hom}(L_{\hat{1}}\bar{\mathcal{T}}^{\otimes 2};\mathrm{ev}^*T\mathbb{P}^2))$. By Lemma 5.14,

$$\begin{aligned}N\big(\mathcal{D}^{(2)}_{\bar{\mathcal{T}},\hat{1}}\big) &= \big\langle c_1(\mathrm{ev}^*T\mathbb{P}^2)-c_1\big(L_{\hat{1}}\bar{\mathcal{T}}^{\otimes 2}\big),[\bar{\mathcal{S}}_{\bar{\mathcal{T}};1}(\mu)]\big\rangle - \mathcal{C}_{\mathcal{D}^{(2)\,-1}_{\bar{\mathcal{T}},\hat{1}}(0)}\big(\mathcal{D}^{(2)\perp}_{\bar{\mathcal{T}},\hat{1}}\big) \\ &= 2\big\langle c_1\big(L_{\hat{1}}^*\mathcal{T}\big),[\bar{\mathcal{S}}_{\mathcal{T};1}(\mu)]\big\rangle - \mathcal{C}_{\mathcal{D}^{(2)\,-1}_{\bar{\mathcal{T}},\hat{1}}(0)}\big(\mathcal{D}^{(2)\perp}_{\bar{\mathcal{T}},\hat{1}}\big),\end{aligned} \tag{6.45}$$

since $\mathrm{ev}|\mathcal{U}_{\mathcal{T}^*(l)}(\mu)=\mu_l$. If $\mathcal{T}'=(S^2,[N]-\{l\},I';j',\underline{d}')\le\bar{\mathcal{T}}$ and $d'_{\hat{1}}\neq 0$, by Corollary C.3, $\mathcal{D}^{(2)}_{\bar{\mathcal{T}},\hat{1}}$ does not vanish on $\mathcal{U}_{\mathcal{T}'}(\mu)$ if the constraints μ are in general position. If $d'_{\hat{1}}=0$ and $\bar{\mathcal{S}}_{\bar{\mathcal{T}};1}(\mu)\cap\mathcal{U}_{\mathcal{T}'}(\mu)\neq\emptyset$, Proposition 6.10 implies that $H_{\hat{1}}\mathcal{T}'=\hat{I}'$ is a two-element set. Furthermore, in such a case,

$$\bar{\mathcal{S}}_{\bar{\mathcal{T}};1}(\mu)\cap\mathcal{U}_{\mathcal{T}'}(\mu)=\mathcal{S}_{\mathcal{T}';2}(\mu),$$

is a finite set and there exists an identification $\gamma_{\mathcal{T}';2}$ of neighborhoods of $\mathcal{S}_{\mathcal{T}';2}(\mu)$ in $\gamma_{E\mathcal{T}'}$ and in $\bar{\mathcal{S}}_{\bar{\mathcal{T}};1}(\mu)$ such that

$$\Big|\mathcal{D}^{(2)}_{\bar{\mathcal{T}},\hat{1}}\gamma_{\mathcal{T}';2}(\upsilon)-\alpha_{\mathcal{T}';k}(\upsilon)\Big| \le C|\upsilon|^{1+\frac{1}{p}} \qquad \forall\upsilon\in\gamma_{E\mathcal{T}',\delta}, \tag{6.46}$$

where $\alpha_{\mathcal{T}';k}\in\Gamma(\mathcal{S}_{\mathcal{T}';2}(\mu);\mathrm{Hom}(\gamma_{E\mathcal{T}'};L_{\hat{1}}^*\mathcal{T}'^{\otimes 2}\otimes\mathrm{ev}^*T\mathbb{P}^2))$ is an injection on every fiber. On the other hand, $\mathcal{D}^{(2)\perp}_{\mathcal{T},\hat{1}}=\pi_{\bar{\nu}}^{\perp}\circ\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}$, where

$$\pi_{\bar{\nu}}^{\perp}: L_{\hat{1}}^*\mathcal{T}'^{\otimes 2}\otimes\mathrm{ev}^*T\mathbb{P}^2\longrightarrow\big(L_{\hat{1}}^*\mathcal{T}'^{\otimes 2}\otimes\mathrm{ev}^*T\mathbb{P}^2\big)/\mathbb{C}\bar{\nu}$$

is the projection onto the quotient by a trivial subbundle $\mathbb{C}\bar{\nu}$; see Section 5.3. Since $\alpha_{\mathcal{T}';2}$ is an injection, $\pi_{\bar{\nu}}^{\perp}\circ\alpha_{\mathcal{T}';2}$ is an isomorphism between $\gamma_{E\mathcal{T}'}$ and $L^{*\otimes 2}_{\mathcal{T}',\hat{1}}\otimes\mathrm{ev}^*T\mathbb{P}^2/\mathbb{C}\bar{\nu}$ over every point of $\mathcal{S}_{\mathcal{T}';2}(\mu)$ if $\bar{\nu}$ is generic. Thus, by Proposition 5.13 and the estimate (6.46),

$$\mathcal{C}_{\mathcal{S}_{\mathcal{T}';2}(\mu)}(\mathcal{D}^{(2)\perp}_{\mathcal{T},\hat{1}})=\big|\mathcal{S}_{\mathcal{T}';2}(\mu)\big|. \tag{6.47}$$

Combining equations (6.44)-(6.47), using (6.15), and summing over the bubble types $\mathcal{T}$ with $\chi(\mathcal{T})=\{\hat{1}\}$, we obtain

$$\sum_{|\chi(\mathcal{T})|=1}\mathcal{C}_{\bar{\mathcal{S}}_1(\mu)\cap\mathcal{U}_{\mathcal{T}}(\mu)}(\mathcal{D}^{(2)})=\sum_{l\in[N]}\big\langle 4c_1(\mathcal{L}^*),[\bar{\mathcal{S}}_{\mathcal{T}^*(l);1}(\mu)]\big\rangle-2\big|\mathcal{S}_{2,1;2}(\mu)\big|, \tag{6.48}$$

where $\mathcal{S}_{2,1;2}(\mu)$ denotes the set of two-component rational degree-d curves that pass through the $3d-4$ points and have a tacnode at one of the points, which is a node common to both irreducible components.

(3) Suppose $\chi(\mathcal{T})=\{\hat{1},\hat{2}\}$ is a two-element set. If $\chi(\mathcal{T})=\hat{I}$ and $M_{\hat{0}}\mathcal{T}=\emptyset$,

$$\bar{\mathcal{S}}_1(\mu)\cap\mathcal{U}_{\mathcal{T}}(\mu)=\mathcal{S}_{\mathcal{T};1}(\mu)\cup\mathcal{S}_{\mathcal{T};2}(\mu);$$

see Proposition 6.10. By Corollary C.3, the images of $\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}$ and $\mathcal{D}^{(1)}_{\mathcal{T},\hat{2}}$ are transversal in $\mathrm{ev}^*T\mathbb{P}^2\longrightarrow\mathcal{U}_{\mathcal{T}}(\mu)$. Since $\mathcal{S}_{\mathcal{T};1}(\mu)$ is a finite set, it follows that the section $\alpha_{\mathcal{T};1}$ of Proposition 6.10 defines an isomorphism between $\mathcal{FS}_{\mathcal{T};1}$ and $L^*\otimes\mathrm{ev}^*T\mathbb{P}^2$ over every point of $\mathcal{S}_{\mathcal{T};1}(\mu)$. Thus, by Proposition 5.13 and the analytic estimate of Proposition 6.10,

$$\mathcal{C}_{\mathcal{S}_{\mathcal{T};1}(\mu)}(\mathcal{D}^{(2)})=2\big|\mathcal{S}_{\mathcal{T};1}(\mu)\big|\Longrightarrow\mathcal{C}_{\mathcal{S}_{2;1}(\mu)}(\mathcal{D}^{(2)})=2\big|\mathcal{S}_{2;1}(\mu)\big|.$$

Similarly, since $\alpha_{\mathcal{T};2}$ does not vanish on $\mathcal{S}_{\mathcal{T};2}(\mu)$ and extends naturally over $\bar{\mathcal{S}}_{\mathcal{T};2}(\mu)$,

$$\mathcal{C}_{\mathcal{S}_{\mathcal{T};2}(\mu)}(\mathcal{D}^{(2)})=N(\alpha_{\mathcal{T};2})=N(\mathcal{D}_{2;2}).$$

If $\chi(\mathcal{T})\neq\hat{I}$ or $M_{\hat{0}}\mathcal{T}\neq\emptyset$, by Proposition 6.10, $\bar{\mathcal{S}}_1(\mu)\cap\mathcal{U}_{\mathcal{T}}(\mu)=\mathcal{S}_{\mathcal{T};2}(\mu)$. The section $\alpha_{\mathcal{T};2}$ has full rank on every fiber in these cases. Thus, by Proposition 5.13 and the analytic estimate of Proposition 6.10, if $\hat{I}\neq H_{\hat{0}}\mathcal{T}$, $\mathcal{T}$ is $\mathcal{D}^{(2)}$-hollow and $\mathcal{C}_{\bar{\mathcal{S}}_1(\mu)\cap\mathcal{U}_{\mathcal{T}}(\mu)}(\mathcal{D}^{(2)})=0$. If $\chi(\mathcal{T})=\hat{I}$ and $M_{\hat{0}}\mathcal{T}\neq\emptyset$, $\alpha_{\mathcal{T};2}$ extends over $\bar{\mathcal{S}}_{\mathcal{T};2}(\mu)$ via the splitting (6.17) by

$$\alpha_{\mathcal{T};2}\big[x_{\hat{1}},x_{\hat{2}},y_l,b;v_{\hat{1}},v_{\hat{2}}\big]=x_{\hat{1}}\mathcal{D}_{\mathcal{T},\hat{1}}v_{\hat{1}}+x_{\hat{2}}\mathcal{D}_{\mathcal{T},\hat{2}}v_{\hat{2}}.$$

This extension vanishes only on the set $x_{\hat{1}}=x_{\hat{2}}$. Thus, by Proposition 5.13 and Lemma 5.14,

$$\begin{aligned}\mathcal{C}_{\bar{\mathcal{S}}_1(\mu)\cap\mathcal{U}_{\mathcal{T}}(\mu)}(\mathcal{D}^{(2)})&=\big\langle c_1(L^{*\otimes 2}\otimes\mathrm{ev}^*T\mathbb{P}^n)-c_1(\gamma_{L^*\otimes(L_{\hat{1}}\mathcal{T}\oplus L_{\hat{2}}\mathcal{T})}),[\bar{\mathcal{S}}_{\mathcal{T};2}(\mu)]\big\rangle-\mathcal{C}_{\alpha^{-1}_{\bar{\mathcal{T}};2}(0)}(\alpha^{\perp}_{\bar{\mathcal{T}};2})\\&=2\big|\mathcal{S}_{\bar{\mathcal{T}};2}(\mu)\big|.\end{aligned}$$

Here we used $\langle c_1(L^*), [\bar{\mathcal{M}}_{0,4}]\rangle = 1$; see Lemma 6.23. Summing over all bubble types $\mathcal{T}$ as above and using Lemma 6.16, we obtain

$$\sum_{|\chi(\mathcal{T})|=2} \mathcal{C}_{\bar{\mathcal{S}}_1(\mu)\cap\mathcal{U}_\mathcal{T}(\mu)}(\mathcal{D}^{(2)}) = \big\langle 6a^2+3a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big)+c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*), \big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle \\ +2\big|\mathcal{S}_{2;1}(\mu)\big| + 2\big|\mathcal{S}_{2,1;2}(\mu)\big|. \tag{6.49}$$

(4) Finally, suppose $\chi(\mathcal{T})=\{\hat{1},\hat{2},\hat{3}\}$ is a three-element set. By Proposition 6.10,

$$\bar{\mathcal{S}}_1(\mu)\cap\mathcal{U}_\mathcal{T}(\mu) = \mathcal{S}_{\mathcal{T};2}(\mu) = \mathcal{U}_\mathcal{T}(\mu).$$

The section $\alpha_{\mathcal{T};2}$ again has full rank. Thus, $\mathcal{S}_{\mathcal{T};2}(\mu)$ is $\mathcal{D}^{(2)}$-hollow if $\chi(\mathcal{T})\neq\hat{I}$. If $\chi(\mathcal{T})=\hat{I}$, $\alpha_{\mathcal{T};2}$ extends over $\bar{\mathcal{S}}_{\mathcal{T};2}(\mu)$ via the splitting (6.17) by

$$\alpha_{\mathcal{T};2}\big[x_{\hat{1}}, x_{\hat{2}}, x_{\hat{3}}, b; v_{\hat{1}}, v_{\hat{2}}, v_{\hat{3}}\big] = x_{\hat{1}}\mathcal{D}_{\mathcal{T},\hat{1}}v_{\hat{1}} + x_{\hat{2}}\mathcal{D}_{\mathcal{T},\hat{2}}v_{\hat{2}} + x_{\hat{3}}\mathcal{D}_{\mathcal{T},\hat{3}}v_{\hat{3}}.$$

This extension does not vanish on $\mathcal{F}\mathcal{S}_{\mathcal{T};2}(\mu)$, since $x_{\hat{1}}, x_{\hat{2}}, x_{\hat{3}}$ are never all the same. Thus, by Proposition 5.13,

$$\mathcal{C}_{\bar{\mathcal{S}}_1(\mu)\cap\mathcal{U}_\mathcal{T}(\mu)}(\mathcal{D}^{(2)}) = \big\langle c_1\big(L^{*\otimes 2}\otimes \mathrm{ev}^*T\mathbb{P}^n\big) - c_1\big(\gamma_{L^*\otimes(L_{\hat{1}}\mathcal{T}\oplus L_{\hat{2}}\mathcal{T}\oplus L_{\hat{3}}\mathcal{T})}\big), \big[\bar{\mathcal{S}}_{\mathcal{T};2}(\mu)\big]\big\rangle = 3\big|\mathcal{U}_{\bar{\mathcal{T}}}(\mu)\big|.$$

Summing over all such bubble types $\mathcal{T}$, we obtain

$$\sum_{|\chi(\mathcal{T})|=3} \mathcal{C}_{\bar{\mathcal{S}}_1(\mu)\cap\mathcal{U}_\mathcal{T}(\mu)}(\mathcal{D}^{(2)}) = 3\big|\mathcal{V}_3(\mu)\big|. \tag{6.50}$$

The claim follows by plugging the sum of equations (6.48), (6.49), and (6.50) into (6.43) and using (6.14).

Lemma 6.16 *If $d\geq 2$ and μ is a tuple of $3d-4$ points in general position in $\mathbb{P}^2$,*

$$N\big(\mathcal{D}_{2;2}\big) = \big\langle 6a^2 + 3a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big) + c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*), \big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle.$$

Proof: (1) Since $\mathcal{D}_{2;2}$ does not vanish on $\mathcal{S}_{2;2}(\mu)$, by Lemmas 5.14 and 6.13,

$$\begin{aligned} N\big(\mathcal{D}_{2;2}\big) &= \big\langle c_1(\mathrm{ev}^*T\mathbb{P}^2)-c_1(\gamma_{E_2}), \big[\bar{\mathcal{S}}_{2;2}(\mu)\big]\big\rangle - \mathcal{C}_{\partial\bar{\mathcal{S}}_{2;2}(\mu)}(\mathcal{D}_{2;2}^\perp) \\ &= \big\langle 3a+\lambda_{E_2}, \big[\bar{\mathcal{S}}_{2;2}(\mu)\big]\big\rangle - \mathcal{C}_{\partial\bar{\mathcal{S}}_{2;2}(\mu)}(\mathcal{D}_{2;2}^\perp) \\ &= \big\langle 12a^2+6a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big)+\big(c_1^2(L_1^*)+c_1^2(L_2^*)\big)+c_1(L_1^*)c_1(L_2^*), \big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle - \mathcal{C}_{\partial\bar{\mathcal{S}}_{2;2}(\mu)}(\mathcal{D}_{2;2}^\perp). \end{aligned} \tag{6.51}$$

In (6.51), $\mathcal{D}_{2;2}^\perp = \pi_{\bar{\nu}}^\perp\circ\mathcal{D}_{2;2}$, where $\pi_{\bar{\nu}}^\perp\colon \mathrm{ev}^*T\mathbb{P}^2\longrightarrow \mathrm{ev}^*T\mathbb{P}^2/\mathbb{C}\bar{\nu}$ is the projection onto the quotient by a trivial subbundle $\mathbb{C}\bar{\nu}$; see Section 5.3. If $\bar{\nu}$ is generic, by Proposition 6.12, $\pi_{\bar{\nu}}^\perp\circ\alpha_{2;2}$ is an isomorphism between $\gamma_{E_2}^*\otimes\big(E_2/\gamma_{E_2}\big)$ and $\mathrm{ev}^*T\mathbb{P}^2/\mathbb{C}\bar{\nu}$ over every point of $\partial\bar{\mathcal{S}}_{2;2}(\mu)$. Thus, by Proposition 5.13 and the analytic estimate of Proposition 6.12,

$$\mathcal{C}_{\partial\bar{\mathcal{S}}_{2;2}(\mu)}(\mathcal{D}_{2;2}^\perp) = \big|\partial\bar{\mathcal{S}}_{2;2}(\mu)\big| = \big|\mathcal{S}_{2;1}(\mu)\big| + \sum_{[\mathcal{T}]}\big|\mathcal{S}_{\mathcal{T};2}(\mu)\big|, \tag{6.52}$$

where the sum is taken over all equivalence classes of non-basic types $\mathcal{T}=\big(S^2,[N],I;j,\underline{d}\big)$ such that $I-\hat{I}=\{k_1,k_2\}$ is a two-element set and $\sum d_i=d$.

(2) Let $\mathcal{T}_i=(S^2, M_{k_i}, I_{k_i}; j, \underline{d})$, where $i=1,2$, be the simple bubble types corresponding to a bubble type $\mathcal{T}$ as above. If $\mathcal{S}_{\mathcal{T};2}(\mu)\neq\emptyset$, up to a re-ordering of indices, $\mathcal{T}$ must have one of two forms. The first possibility is that $\mathcal{T}_2$ is basic, while $d_{k_1}=0$ and $H_{k_1}\mathcal{T}=I_{k_1}$ is a two-element set. Then $\mathcal{S}_{\mathcal{T};2}(\mu)=\mathcal{U}_{\mathcal{T}}(\mu)$. The sum of the cardinalities of the sets $\mathcal{S}_{\mathcal{T};2}(\mu)$ taken over all equivalence classes of such bubble types is then $3|\mathcal{V}_3(\mu)|$, since one of the three irreducible components of the image of each map is distinguished. The other possibility is that $\mathcal{T}_2$ is basic, while $d_{k_1}=0$, $H_{k_1}\mathcal{T}=I_{k_1}$ is a one-element set, and $j_l=k_1$ for some $l\in[N]$. Since $\mathrm{ev}^*\mathcal{O}(1_{\mathbb{P}^2})$ is trivial on $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$,

$$\begin{aligned}|\mathcal{S}_{\mathcal{T};2}(\mu)| = \langle c_2(\gamma^*_{E\bar{\mathcal{T}}}\otimes\mathrm{ev}^*T\mathbb{P}^2), [E\bar{\mathcal{T}}]\rangle &= \langle c_1(L^*_{\hat{1}}\bar{\mathcal{T}})+c_1(L^*_{k_2}\bar{\mathcal{T}}), [\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)]\rangle\\ &= \langle c_1(L^*_{\hat{1}}\mathcal{T})+c_1(L^*_{k_2}\mathcal{T}), [\bar{\mathcal{U}}_{\mathcal{T}}(\mu)]\rangle,\end{aligned}$$

if $I=\{k_1,k_2,\hat{1}\}$. Summing over all equivalence classes of such bubble types and using equations (6.14) and (6.15), we obtain

$$\sum_{[\mathcal{T}]}|\mathcal{S}_{\mathcal{T};2}(\mu)| = \sum_{[\mathcal{T}^*]}\sum_{l\in[N]}\langle c_1(\mathcal{L}^*_1)+c_1(\mathcal{L}^*_2), [\bar{\mathcal{U}}_{\mathcal{T}^*(l)}(\mu)]\rangle + 3|\mathcal{V}_3(\mu)|,$$

where the second sum is taken over equivalence classes of basic bubble types $\mathcal{T}^*$ such that $|I^*|=2$ and $\sum d^*_i=d$. Combing equations (6.51) and (6.52) thus gives

$$\begin{aligned}N(\mathcal{D}_{2;2}) = \langle 12a^2+6a(c_1(\mathcal{L}^*_1)+c_1(\mathcal{L}^*_2))+(c_1^2(\mathcal{L}^*_1)+c_1^2(\mathcal{L}^*_2))+c_1(\mathcal{L}^*_1)c_1(\mathcal{L}^*_2), [\bar{\mathcal{V}}_2(\mu)]\rangle&\\ -|\mathcal{S}_{2;1}(\mu)|-3|\mathcal{V}_3(\mu)|.&\end{aligned}$$

The claim now follows by using Lemma 6.5.

6.10 Computation of Chern Classes

In this section, we show that all intersection numbers of the spaces $\bar{\mathcal{V}}_k(\mu)$ involving powers of a and powers of $c_1(\mathcal{L}^*_i)$ are computable. The computability of intersection numbers of tautological classes of $\bar{\mathcal{V}}_k(\mu)$, which include a and $c_1(\mathcal{L}^*_i)$, has been shown in [P2]. For the sake of completeness, a slightly different approach is presented below.

If $d_{\hat{0}}$ and $d_{\hat{1}}$ are nonnegative integers and μ is an N-tuple of any generic constraints in $\mathbb{P}^n$, let $\bar{\mathcal{M}}_{(d_{\hat{0}},d_{\hat{1}})}(\mu)$ denote the union of the spaces $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$, where $\mathcal{T}$ is a simple bubble type of the form

$$\mathcal{T}=(S^2,[N],\{\hat{0},\hat{1}\};j,\{d_{\hat{0}},d_{\hat{1}}\}).$$

Then $\bar{\mathcal{M}}_{\mathcal{T},(d_{\hat{0}},d_{\hat{1}})}(\mu)$ is a complex codimension-one homology class in $\bar{\mathcal{U}}_{(d_{\hat{0}}+d_{\hat{1}};[N])}$. If $d>0$, let

$$\sum^{\geq}_{d_{\hat{0}}+d_{\hat{1}}=d} f(d_{\hat{0}},d_{\hat{1}}) = \sum_{\substack{d_{\hat{0}}+d_{\hat{1}}=d\\ d_{\hat{0}},d_{\hat{1}}\geq 0}} f(d_{\hat{0}},d_{\hat{1}}), \qquad \sum^{>}_{d_{\hat{0}}+d_{\hat{1}}=d} f(d_{\hat{0}},d_{\hat{1}}) = \sum_{\substack{d_{\hat{0}}+d_{\hat{1}}=d\\ d_{\hat{0}},d_{\hat{1}}>0}} f(d_{\hat{0}},d_{\hat{1}}),$$

whenever f is any function defined on an appropriate subset of $\mathbb{Z}\times\mathbb{Z}$.

Lemma 6.17 *Let $\mathcal{T}^*=(S^2,[N],\{\hat{0}\};\hat{0},d)$ be a bubble type with $d>0$. Then in $H^*(\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu))$,*

$$c_1(L^*)=\frac{1}{d^2}\Big(\mathcal{H}-2da+\sum_{d_{\hat{0}}+d_{\hat{1}}=d}^{\geq} d_{\hat{1}}^2\bar{\mathcal{M}}_{(d_{\hat{0}},d_{\hat{1}})}(\mu)\Big),$$

where $\mathcal{H}$ denotes the subset of elements in $\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$ that pass through a generic codimension-two linear subspace of $\mathbb{P}^n$.

Proof: (1) We restate the proof of [I] in terms of the line bundle $L^{*\otimes d^2}\longrightarrow\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$, instead of passing to a cover of $\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$. Define a section $\psi\in\Gamma(\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu);L^{*\otimes d^2})$ as follows. Let H_0 and H_1 be two fixed hyperplanes in $\mathbb{P}^n$, generic with respect to the constraints $\mu_1,\ldots\mu_N$. Suppose

$$[b]=\big[(S^2,[N],\hat{0};,(\hat{0},y),u_{\hat{0}})\big]\in\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$$

is such that $u_{\hat{0}}$ is transversal to H_0 and H_1. Then,

$$u_{\hat{0}}^{-1}(H_i)=\big\{[x_1^{(i)},y_1^{[i]}],\ldots,[x_d^{(i)},y_d^{[i]}]\big\},\quad i=0,1,$$

for some $[x_k^{(i)},y_k^{[i]}]\in\mathbb{P}^1$. Define $\psi([b])$ by

$$\psi([b,c])=c^{d^2}\prod_{k,l\in[d]}\Big(\frac{x_k^{(0)}}{y_k^{(0)}}-\frac{x_k^{(1)}}{y_k^{(1)}}\Big). \tag{6.53}$$

While this section could be infinite, it is well-defined, i.e. independent of the choice of a representative $b\in\mathcal{B}_{\mathcal{T}^*}$ for $[b]$. With an appropriate coordinate change on $\mathbb{C}^{n+1}$, it can be assumed that $H_i=\{X_i=0\}$. The map $u_{\hat{0}}$ corresponds to $(n+1)$ homogeneous polynomials of degree d: $p_0,\ldots,p_n$. Since the right-hand side of (6.53) is symmetric in the roots of p_0 and separately in the roots of p_1, ψ is a rational function in the coefficients of p_0 and p_1. Thus, ψ extends over all of $\mathcal{U}_{\mathcal{T}^*}(\mu)$. Furthermore, this section extends by zero over $\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)-\mathcal{U}_{\mathcal{T}^*}(\mu)$.
(2) We now identify the zero set of the section ψ with multiplicities. From equation (6.53), it is clear that ψ vanishes with multiplicity one if p_0 and p_1 have a common root, i.e. if $u_{\hat{0}}$ passes through $H_0\cap H_1$. The section ψ also has a pole of order d along the sets of maps

$$X_0=\{b\colon y_k^{(0)}(b)=0\text{ for }!k,\ p_1(1,0)\neq0\},\qquad X_1=\{b\colon y_k^{(1)}(b)=0\text{ for }!k,\ p_0(1,0)\neq0\}.$$

Note that $\bar{X}_i=\mathrm{ev}^{-1}(H_i)$. Finally, while ψ vanishes outside of $\mathcal{U}_{\mathcal{T}^*}(\mu)$, $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ has (complex) codimension one in $\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$ if and only if $\mathcal{T}<\mathcal{T}^*$ is a two-bubble strata, i.e. as described just before the statement of the lemma. Let $d_{\hat{0}}$ and $d_{\hat{1}}$ be the corresponding degrees. It follows from equation (6.53) that ψ has a zero of order $d_{\hat{1}}^2$ along an open subset of $\mathcal{U}_{\mathcal{T}}(\mu)$. Thus, we obtain

$$c_1\big(L^{*\otimes d^2}\big)=\mathcal{H}-2da+\sum_{d_{\hat{0}}+d_{\hat{1}}=d}^{\geq} d_{\hat{1}}^2\bar{\mathcal{M}}_{(d_{\hat{0}},d_{\hat{1}})}(\mu).$$

Corollary 6.18 *With notation as in Lemma 6.17,*

$$c_1(\mathcal{L}^*)=\frac{1}{d^2}\Big(\mathcal{H}-2da+\sum_{d_{\hat{0}}+d_{\hat{1}}=d}^{>} d_{\hat{1}}^2\bar{\mathcal{M}}_{(d_{\hat{0}},d_{\hat{1}})}(\mu)\Big).$$

Proof: This is immediate from Lemma 6.17 and equation (6.14).

If $\mathcal{T} = (S^2, [N], I; j, \underline{d})$ is any bubble type, let $\mathcal{T}_{\hat{0}} = (S^2, M_{\hat{0}}\mathcal{T} + H_{\hat{0}}\mathcal{T}, \{\hat{0}\}; \hat{0}, d_{\hat{0}})$. Denote by $\mathcal{T}_k$ for $k \in H_{\hat{0}}\mathcal{T}$ the simple bubble types corresponding to $\mathcal{T}$. Then,

$$\bar{\mathcal{U}}_{\mathcal{T}}(\mu) = \bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}}(\mu) \times_{\prod_{k \in H_{\hat{0}}\mathcal{T}} (\mathrm{ev}_k \times \mathrm{ev})} \prod_{k \in H_{\hat{0}}\mathcal{T}} \bar{\mathcal{U}}_{\mathcal{T}_k}(\mu) \tag{6.54}$$

$$\equiv \Big\{ \big(b_{\hat{0}}, (b_k)_{k \in H_{\hat{0}}\mathcal{T}}\big) \in \bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}}(\mu) \times \prod_{k \in H_{\hat{0}}\mathcal{T}} \bar{\mathcal{U}}_{\mathcal{T}_k}(\mu) : \mathrm{ev}_k(b_{\hat{0}}) = \mathrm{ev}(b_h)\ \forall k \in H_{\hat{0}}\mathcal{T} \Big\}.$$

Lemma 6.19 *With notation as above, if $\mathcal{T} < \mathcal{T}^*$ and $d_{\hat{0}} \neq 0$,*

$$c_1(\mathcal{L}^*_{\mathcal{T}^*})\Big|\bar{\mathcal{U}}_{\mathcal{T}}(\mu) = \Big\{ c_1(\mathcal{L}^*_{\mathcal{T}_{\hat{0}}})\big|\bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}}(\mu) + \sum_{\emptyset \neq M_0 \subset M_{\hat{0}}\mathcal{T}_{\hat{0}}, M_0 \cap H_{\hat{0}}\mathcal{T} \neq \emptyset} \bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}(M_0)}(\mu) \Big\}$$
$$\times_{\prod_{k \in H_{\hat{0}}\mathcal{T}} (ev_k \times ev)} \prod_{k \in H_{\hat{0}}\mathcal{T}} \bar{\mathcal{U}}_{\mathcal{T}_k}(\mu).$$

Proof: Since $L_{\mathcal{T}^*}|\bar{\mathcal{U}}_{\mathcal{T}}(\mu) = L_{\mathcal{T}}$ and $\bar{\mathcal{U}}_{\mathcal{T}}(\mu) \cap \bar{\mathcal{U}}_{\mathcal{T}^*(M_0)}(\mu) = \emptyset$ unless $M_0 \subset M_{\hat{0}}\mathcal{T}$, by equation (6.14),

$$c_1(\mathcal{L}^*_{\mathcal{T}^*})\Big|\bar{\mathcal{U}}_{\mathcal{T}}(\mu) = c_1(L^*_{\mathcal{T}})\Big|\bar{\mathcal{U}}_{\mathcal{T}}(\mu) - \sum_{\emptyset \neq M_0 \subset M_{\hat{0}}\mathcal{T}} \bar{\mathcal{U}}_{\mathcal{T}}(\mu) \cdot \bar{\mathcal{U}}_{\mathcal{T}^*(M_0)(\mu)}$$
$$= c_1(L^*_{\mathcal{T}_{\hat{0}}})\Big|\bar{\mathcal{U}}_{\mathcal{T}}(\mu) - \sum_{\emptyset \neq M_0 \subset M_{\hat{0}}\mathcal{T}} \bar{\mathcal{U}}_{\mathcal{T}(M_0)}(\mu).$$

The claim follows by using equation (6.14) again.

Corollary 6.20 *All intersection numbers on $\bar{\mathcal{V}}_k(\mu)$ involving only the powers of a and $c_1(\mathcal{L}^*_k)$ are computable.*

Proof: Corollary 6.18 and Lemma 6.19 reduce the computation of such numbers to understanding the restrictions $c_1(\mathcal{L}^*_{\mathcal{T}^*})|\bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}(M_0)}(\mu)$, where M_0 is a subset of $M_{\hat{0}}\mathcal{T}_{\hat{0}}$ intersecting $H_{\hat{0}}\mathcal{T}$. By (6.13),

$$\bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}(M_0)} \approx \bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}+M_0} \times \bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}/M_0}.$$

We express $c_1(\mathcal{L}^*_{\mathcal{T}^*})|\bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}(M_0)}$ in terms of cohomology classes of $\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}+M_0}$. By definition, $L_{\mathcal{T}^*}|\bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}(M_0)}$ comes from a line bundle over $\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}+M_0}$. In fact,

$$c_1(L^*_{\mathcal{T}^*})\big|\big(\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}+M_0} \times \bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}/M_0}\big) = \psi_{\hat{0}} \times 1,$$

where $\psi_{\hat{0}}$ is the ψ-class of $\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}+M_0}$ corresponding to the marked point $\hat{0}$. Since the restriction of $L_{\mathcal{T}^*}$ to $\bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}(M_0)}$ is $L_{\mathcal{T}_{\hat{0}}(M_0)}$,

$$c_1(\mathcal{L}^*_{\mathcal{T}^*})|\bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}(M_0)} = c_1(L^*_{\mathcal{T}^*})|\bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}(M_0)} - \sum_{\emptyset \neq M_0' \subset M_{\hat{0}}\mathcal{T}} \bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}(M_0')} \cdot \bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}(M_0)}$$
$$= \psi_{\hat{0}} \times 1 - \sum_{\emptyset \neq M_0' \subset (M_0 - H_{\hat{0}}\mathcal{T})} \bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}(M_0'; M_0 - M_0')} = \tilde{\psi}_{M_0 - H_{\hat{0}}\mathcal{T}} \times 1|\bar{\mathcal{M}}_{\{\hat{0},\hat{1}\}+M_0} \times \bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}/M_0},$$

where $\mathcal{T}_{\hat{0}}(M_0'; M_0 - M_0') \equiv \{\mathcal{T}_{\hat{0}}(M_0)\}(M_0')$ and for any proper subset $\tilde{J}$ of J we define $\tilde{\psi}_{\tilde{J}} \in H^2(\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}+J})$ by

$$\tilde{\psi}_{\tilde{J}} = \psi_{\hat{0}} - \sum_{\emptyset \neq J' \subset \tilde{J}} \bar{\mathcal{M}}_{0,(\{\hat{0}\}+J',\{\hat{1}\}+(J-J'))}.$$

Here $\bar{\mathcal{M}}_{0,(\{\hat{0}\}+J',\{\hat{1}\}+(J-J'))}$ denotes the closure in $\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}+J}$ of the two-component strata such that the marked points on one of the components are $\{\hat{0}\}+J'$. The numbers

$$\chi(|J|,|\tilde{J}|) \equiv \langle \tilde{\psi}_{\tilde{J}}^{|J|-1}, [\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}+J}] \rangle$$

are given in Corollary 6.22, which is a consequence of the following well-known lemma; see [P2] for example.

Lemma 6.21 *(1) For any $j^* \in J$, $\tilde{\psi}_{J-\{j^*\}} = 0$ in $H^*(\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}+J})$.*
(2) If $\mathcal{N}\bar{\mathcal{M}}_{0,(\{\hat{0}\}+J',\{\hat{1}\}+(J-J'))}$ is the normal bundle of

$$\bar{\mathcal{M}}_{0,(\{\hat{0}\}+J',\{\hat{1}\}+(J-J'))} \approx \bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}+J'} \times \bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}+(J-J')}$$

in $\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}+J}$,

$$c_1(\mathcal{N}\bar{\mathcal{M}}_{0,(\{\hat{0}\}+J',\{\hat{1}\}+(J-J'))}) = -\psi_{\hat{1}} \times 1 - 1 \times \psi_{\hat{0}}.$$

Corollary 6.22 *If $m > 0$, $\chi(m,0) = 1$. If $m > k > 0$, $\chi(m,k) = 0$.*

For our purposes, we can assume that the constraints $\mu_1, \ldots, \mu_N$ are disjoint. If the dimension of the space $\bar{\mathcal{V}}_k(\mu)$ is two over $\mathbb{C}$ and $\tilde{\mathcal{U}}_{\mathcal{T}_{\hat{0}}(M_0)}(\mu)$ is nonempty and appears in the computation of the intersection numbers of Corollary 6.20 via Lemma 6.19, then $H_{\hat{0}}\mathcal{T}$ consists of a single element and $M_0 = H_{\hat{0}}\mathcal{T}$. The corresponding moduli space $\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}+M_0}$ is a single point and thus

$$\langle \tilde{\psi}_{M_0-H_{\hat{0}}\mathcal{T}}^{|M_0|-1}, [\bar{\mathcal{M}}_{\{\hat{0},\hat{1}\}+M_0}] \rangle = 1.$$

If $\bar{\mathcal{V}}_k(\mu)$ is four-dimensional, and we encounter two cases when $\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}+M_0}$ is positive-dimensional. One possibility is that $H_{\hat{0}}\mathcal{T}$ is still a single-element set, but M_0 contains one of the N marked points. In this case, by Corollary 6.22 or simply by the first statement of Lemma 6.21,

$$\langle \tilde{\psi}_{M_0-H_{\hat{0}}\mathcal{T}}^{|M_0|-1}, [\bar{\mathcal{M}}_{\{\hat{0},\hat{1}\}+M_0}] \rangle = \chi(2,1) = 0.$$

In fact, we can replace the first statement of Lemma 6.21 with the direct computation of the degree $\psi_{\hat{0}}$ on $\bar{\mathcal{M}}_{0,4}$ given by Lemma 6.23 below. The other case when $\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}+M_0}$ is positive-dimensional is $M_0 = H_{\hat{0}}\mathcal{T}$ is a two-element set. Then

$$\langle \tilde{\psi}_{M_0-H_{\hat{0}}\mathcal{T}}^{|M_0|-1}, [\bar{\mathcal{M}}_{\{\hat{0},\hat{1}\}+M_0}] \rangle = \chi(2,0) = 0.$$

Lemma 6.23 *If $\mathcal{T} = (S^2, [3], \{\hat{0}\}; \hat{0}, 0)$, $\langle c_1(L^*), [\tilde{\mathcal{U}}_{\mathcal{T}}] \rangle = 1$.*

Proof: By definitions in Chapter 2 and Section 6.3, $L \longrightarrow \tilde{\mathcal{U}}_{\mathcal{T}}$ is the line bundle associated to the quotient $\tilde{\mathcal{U}}_{\mathcal{T}} = \tilde{\mathcal{M}}_{\mathcal{T}}^{(0)}/S^1$, where

$$\tilde{\mathcal{M}}_{\mathcal{T}} = \{(y_1, y_2, y_3) \in \mathbb{C}^3 : y_1 + y_2 + y_3 = 0,\ \beta(|y_1|) + \beta(|y_2|) + \beta(|y_3|) = \frac{1}{2}\}$$

and S^1 acts diagonally on $\tilde{\mathcal{M}}_T^{(0)}$. Identify $\tilde{\mathcal{M}}_T^{(0)}$ with $S^3 \subset \mathbb{C}^2$ S^1-equivariantly by the map

$$(y_1, y_2, y_3) \longrightarrow \frac{(y_1, y_2)}{|y_1| + |y_2|}.$$

Our assumptions on β imply that this map is a diffeomorphism; see Section 2.1.

Chapter 7

Positive-Genus Curves

In this chapter, we present some aspects of an approach to counting positive-genus curves with a fixed complex structure in projective spaces based on Theorem 3.29. In the next three chapters, we describe the steps of the procedure that are specific to the genus-two case in $\mathbb{P}^2$ and $\mathbb{P}^3$ and the genus-three case in $\mathbb{P}^2$. More generally, constructions in this chapter and arguments as in the chapters that follow should lead to enumeration of curves with a fixed complex structure whenever stable maps with singular domains that are not constant on the principal components do not contribute to the corresponding symplectic invariant as defined in [RT]; see below.

Throughout this chapter, $\Sigma=(\Sigma,j)$ denotes a smooth Riemann surface of genus at least two. Let d be a positive integer and μ an N-tuple of complex subspaces of $\mathbb{P}^n$ in general position such that each subspace has codimension at least two and

$$\operatorname{codim}_{\mathbb{C}}\mu \equiv \sum_{l\in[N]} \operatorname{codim}_{\mathbb{C}}\mu_l = d(n+1)-n(g-1)+N. \tag{7.1}$$

We are interested in determining the number $n_{\Sigma,d}(\mu)$ of degree-d complex curves in $\mathbb{P}^n$ that pass through the constraints μ and admit a normalization isomorphic to Σ. Condition (7.1) insures that the expected answer is finite.

7.1 The General Approach

The symplectic invariant $\mathrm{RT}_{g,d}(\cdot;\mu)$ is the signed cardinality of the set $\mathcal{M}_{\Sigma,d,t\nu}(\mu)$ for any $t>0$ and a generic section

$$\nu\in\Gamma(\Sigma\times\mathbb{P}^n;\Lambda^{0,1}\pi_\Sigma^*T^*\Sigma\otimes\pi_{\mathbb{P}^n}^*T\mathbb{P}^n);$$

see Section 1.3. If $t_i\longrightarrow 0$ and $(\underline{y}_i;u_i)\in\mathcal{M}_{\Sigma,t_i\nu,d}(\mu)$, then a subsequence of $\{(\underline{y}_i;u_i)\}_{i=1}^{\infty}$ must converge in the Gromov topology to one of the following:

(1) an element of $\mathcal{H}_{\Sigma,d}(\mu)$;

(2) $(\Sigma_b,\underline{y},u)$, where Σ_b is a bubble tree of S^2's attached to Σ with marked points $y_1,\ldots,y_N$, and $u\colon\Sigma_b\longrightarrow\mathbb{P}^n$ is a holomorphic map such that $u(y_l)\in\mu_l$ for $l=1,\ldots,N$, and

(2a) $u|\Sigma$ is simple and the tree contains at least one S^2;

(2b) $u|\Sigma$ is multiply-covered;

(2c) $u|\Sigma$ is constant and the tree contains at least one S^2.

If $g(\Sigma) = 2$ and $n = 2, 3$, Cases (2a) and (2b) cannot occur, because the corresponding spaces of stable maps are empty by Proposition C.6. The same argument applies to the case $g(\Sigma)=3$ and $n=2$, and, at least for a generic complex structure on Σ, to a number of other cases. On the other hand, by Proposition 3.30 and Corollary C.5, there is a one-to-one correspondence between the elements of $\mathcal{H}_{\Sigma,d}(\mu)$ and the nearby elements of $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$, provided t is sufficiently small and $d+\chi(\Sigma) > 0$. In particular, such a bijection exists if $g(\Sigma) = 2$ and $d \geq 3$ or $g(\Sigma) = 3$ and $d \geq 5$. In fact, by standard arguments, it is enough to assume $d \geq 4$, at least if the complex structure on Σ is generic. If $d \leq 3$, $\mathcal{H}_\Sigma(\mu)=\emptyset$ by [ACGH, p116]. Thus, if $g(\Sigma)=2$ and $n=2,3$ or $g(\Sigma)=3$ and $n=2$,

$$\big|\mathcal{H}_{\Sigma,d}(\mu)\big| = RT_{g,d}(\cdot;\mu) - CR_g(\mu),$$

where $CR_g(\mu)$ is the number of elements of $\mathcal{M}_{\Sigma,d,t\nu}(\mu)$, for any $t>0$ sufficiently small, that lie near the set of stable maps of type (2c) with respect to the Gromov topology.

The set of stable maps of type (2c) is the union of spaces $\mathcal{M}_\mathcal{T}(\mu)$, where $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ is a simple bubble type such that $d_{\hat{0}}=0$ and $\sum d_i=d$; see Chapter 2. For such bubble types $\mathcal{T}$, there is well-defined evaluation map $\mathrm{ev}\colon \mathcal{M}_\mathcal{T}(\mu)\longrightarrow\mathbb{P}^n$ that sends each element $b\in\mathcal{M}_\mathcal{T}(\mu)$ to the image of the principal component Σ; see Section 2.6. By Corollaries C.3 and C.5, each space $\mathcal{M}_\mathcal{T}(\mu)$ is a smooth manifold. Furthermore, the cokernel bundle $\Gamma_-\longrightarrow\mathcal{M}_\mathcal{T}$, see Section 3.2, is naturally isomorphic to the bundle $\mathcal{H}^{0,1}_\Sigma\otimes\mathrm{ev}^*T\mathbb{P}^n$, where $\mathcal{H}^{0,1}_\Sigma$ is the g-dimensional vector space of harmonic $(0,1)$-forms on Σ.

Theorem 3.29 describes the number $N_\mathcal{T}(\mu)$ of elements of $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$ that lie near the space $\mathcal{M}_\mathcal{T}(\mu)$ as the zero set of a bundle map between $F^\emptyset\mathcal{T}_\delta$ and $\mathcal{H}^{0,1}_\Sigma\otimes\mathrm{ev}^*T\mathbb{P}^n$. However, before we can use this theorem, we need to specify an obstruction bundle setup. First, we choose any smooth family $\{g_{\Sigma,b}\colon b\in\mathcal{M}_\mathcal{T}\}$ of Kahler metrics on Σ such that $g_{\Sigma,b}$ is flat near all singular points of $\Sigma_{b,\hat{0}}$ and depends only on such singular points; see Section 3.1. We define a family $\{g_{\mathbb{P}^n,b}\colon b\in\mathcal{M}_\mathcal{T}\}$ of Kahler metrics on $\mathbb{P}^n$ by $g_{\mathbb{P}^n,b}=g_{\mathbb{P}^n,\mathrm{ev}(b)}$, where $\{g_{\mathbb{P}^n,q}\colon q\in\mathbb{P}^n\}$ is the smooth family of Kahler metrics on $\mathbb{P}^n$ provided by Lemma 6.1. The crucial property of this family of metrics is that $g_{\mathbb{P}^n,b}$ is hermitian on $B_{r_{\mathbb{P}^n}}\big(\mathrm{ev}(b)\big)$ for every $b\in\mathcal{M}_\mathcal{T}$.

In the next section, we describe our choice for an obstruction bundle for gluing along $\mathcal{M}_\mathcal{T}$; see Definition 3.13. Some consequences of this choice are described at the end of the section and in Section 7.3. The construction only requires that $d_{\hat{0}}=0$. While one can easily define a tangent-bundle model, see Definition 3.11, choosing one that will provide sufficiently good control over the last term of the map $\psi_{\mathcal{T},t\nu}$ in Theorem 7.2 requires care. It will be chosen in later chapters only in the cases we need.

7.2 The Obstruction Bundle

In this section, we choose an obstruction bundle over $F^{(\emptyset)}\mathcal{T}_\delta$ in the sense of Definition 3.13 with $\delta\in C^\infty(\mathcal{M}_\mathcal{T};\mathbb{R}^+)$ sufficiently small.

Let $\delta_\mathcal{T}\in C^\infty(\mathcal{M}_\mathcal{T};\mathbb{R}^+)$ such that

$$4\delta_\mathcal{T}(b)\|du_i\|_{b,C^0} < r_{\mathbb{P}^n} \qquad \forall b=\big(\Sigma,M,I;x,(j,y),u\big)\in\mathcal{M}_\mathcal{T},\ i\in I.$$

We assume that the above function δ is such that $8\delta^{\frac{1}{2}} < \delta_{\mathcal{T}}$. For all $\upsilon \in F^{(\emptyset)}\mathcal{T}_\delta$ and $X\psi \in T_{\mathrm{ev}(b_\upsilon)}\mathbb{P}^n \otimes \mathcal{H}_\Sigma^{0,1}$, define $R_\upsilon X\psi \in \Gamma^{0,1}(u_\upsilon)$ as follows. If $z \in \Sigma_\upsilon = \Sigma$ is such that $q_\upsilon(z) \in \Sigma_{b_\upsilon,h}$ for some $h \in \hat{I}$ with $\chi_{\mathcal{T}} h = 1$ and $|q_S^{-1}(q_\upsilon(z))| \leq 2\delta_{\mathcal{T}}(b_\upsilon)$, by our assumption on $\delta_{\mathcal{T}}$, we can define $\bar{u}_\upsilon(z) \in T_{\mathrm{ev}(b_\upsilon)}\mathbb{P}^n$ by

$$\exp_{\upsilon,\mathrm{ev}(b_\upsilon)} \bar{u}_\upsilon(z) = u_\upsilon(z), \quad |\bar{u}_\upsilon(z)| < r_{\mathbb{P}^n}.$$

Given $z \in \Sigma$, let $h_z \in I$ be such that $q_\upsilon(z) \in \Sigma_{b_\upsilon,h_z}$. If $w \in T_z\Sigma$, put

$$R_\upsilon X\psi|_z w = \begin{cases} 0, & \text{if } \chi_{\mathcal{T}} h_z = 2; \\ \beta\big(\delta_{\mathcal{T}}(b_\upsilon)|q_\upsilon z|\big)(\psi|_z w)\Pi_{\upsilon,\bar{u}_\upsilon(z)}X, & \text{if } \chi_{\mathcal{T}} h_z = 1; \\ (\psi|_z w)X, & \text{if } \chi_{\mathcal{T}} h_z = 0. \end{cases}$$

Let $\Gamma_-^{0,1}(\upsilon)$ be the image of $T_{\mathrm{ev}(b_\upsilon)}\mathbb{P}^n \otimes \mathcal{H}_\Sigma^{0,1}$ under the map R_υ. Denote by $\pi_{\upsilon,-}^{0,1}$ the (L^2, υ)-orthogonal projection of $L^p(\upsilon)$ onto $\Gamma_-^{0,1}(\upsilon)$.

The spaces $\Gamma_-^{0,1}(\upsilon)$ form our obstruction bundle over $F^{(\emptyset)}\mathcal{T}$. We need to show that these spaces satisfy the requirements of Definition 3.13. First, the rate of change of $\pi_{\upsilon,-}^{0,1}$ with respect to changes in υ should be controlled by a function of b_υ only. The proof of this is similar to the proof of statement (6) of Lemma 3.6. The next lemma implies that the remaining conditions are also satisfied. For any $h \in \hat{I}$, put

$$|\upsilon|_h = \prod_{i \in \hat{I}, h \in \bar{D}_i \mathcal{T}} |\upsilon_i|.$$

Lemma 7.1 *For any $\upsilon \in F^{(\emptyset)}\mathcal{T}_\delta$ and $X\psi \in T_{ev(b_\upsilon)}\mathbb{P}^n \otimes \mathcal{H}_\Sigma^{0,1}$, $D_\upsilon^* R_\upsilon X\psi$ vanishes outside of the annuli*

$$\tilde{A}_{\upsilon,h} \equiv q_\upsilon^{-1}\big(\{(h,z) \in \Sigma_{b_\upsilon,h} : \delta_{\mathcal{T}}(b_\upsilon) \leq |q_S^{-1}(z)| \leq 2\delta_{\mathcal{T}}(b_\upsilon)\}\big)$$

with $h \in \hat{I}$ such that $\chi_{\mathcal{T}} h = 1$. Furthermore, there exists $C \in C^\infty(\mathcal{M}_{\mathcal{T}}; \mathbb{R}^+)$ such that
(1) $\|D_\upsilon^* R_\upsilon X\psi\|_{\upsilon,C^0} \leq C(b_\upsilon)\Big(\sum_{\chi_{\mathcal{T}} h=1} |\upsilon|_h\Big)|X|_\upsilon \|\psi\|_2$;
(2) $\big(1 - C(b_\upsilon)^{-1}|\upsilon|^{\frac{2}{p}}\big)\|X\psi\|_{\upsilon,\tilde{p}} \leq \|R_\upsilon X\psi\|_{\upsilon,\tilde{p}} \leq \big(1 + C(b_\upsilon)^{-1}|\upsilon|^{\frac{2}{p}}\big)\|X\psi\|_{\upsilon,\tilde{p}}$, where $\tilde{p} = 2, p$.

Proof: The first statement and estimate (2) are immediate from the definition of $R_\upsilon X\psi$ and of the norms; see Section 3.3. Let (s,t) be the conformal coordinates on $\tilde{A}_{\upsilon,h}$ given by $q_\upsilon(s,t) = s + it \in \mathbb{C}$. Write $g_\upsilon = \theta^{-2}(s,t)(ds^2 + dt^2)$. Then

$$\theta = \frac{1}{2}\big(1 + s^2 + t^2\big). \tag{7.2}$$

Put

$$\xi(s,t) = \{R_\upsilon X\psi\}_{(s,t)} \partial_s = \beta\big(\delta_{\mathcal{T}}(b_\upsilon)\sqrt{s^2+t^2}\big)\big(\psi|_{(s,t)}\partial_s\big)\Pi_{\upsilon,\bar{u}_\upsilon(s,t)}X. \tag{7.3}$$

Then by [MS, p29],

$$D_\upsilon^* R_\upsilon X\psi|_z = \theta^2\Big(-\frac{D}{ds}\xi + J\frac{D}{dt}\xi\Big), \tag{7.4}$$

where $\frac{D}{ds}$ and $\frac{D}{dt}$ denote covariant differentiation with respect to the metric $g_{\mathbb{P}^n,\upsilon}$ on $\mathbb{P}^n$.

Since this metric is flat on the support of ξ and $\psi\in\mathcal{H}_\Sigma^{0,1}$, equations (7.2)-(7.4) give

$$D_v^*R_vX\psi|_z=\frac{(1+s^2+t^2)^2}{4}\Big\{\beta'|_{\delta_{\mathcal{T}}(b_v)\sqrt{s^2+t^2}}\delta_{\mathcal{T}}(b_v)\frac{-s+it}{\sqrt{s^2+t^2}}\Big\}\big(\psi|_{(s,t)}\partial_s\big)\Pi_{v,\tilde{u}_v(s,t)}X. \tag{7.5}$$

Since the right hand-side of (7.5) vanishes unless $\delta_{\mathcal{T}}(b_v)^{-1}\le\sqrt{s^2+t^2}\le 2\delta_{\mathcal{T}}(b_v)^{-1}$, it follows that

$$\big|D_v^*R_vX\psi\big|_{v,z}\le C(b_v)\big|\psi|_{(s,t)}\partial_s\big||X|_v\le C'(b_v)|v|_h\|\psi\|_2|X| \tag{7.6}$$

Claim (1) follows from (7.6).

Let $\tilde{R}_v\colon\mathcal{H}_\Sigma^{0,1}\otimes T_{\mathrm{ev}(b_v)}\mathbb{P}^n\longrightarrow\Gamma_-(v)$ be the adjoint of R_v^{-1}, i.e.

$$\big\langle\big\langle\tilde{R}_vX\psi,R_vX'\psi'\big\rangle\big\rangle_{v,2}=\big\langle\big\langle X\psi,X'\psi'\big\rangle\big\rangle_{b_v,2}=\langle X,X'\rangle_{b_v}\langle\psi,\psi'\rangle_2 \tag{7.7}$$

for all $X,X'\in T_{\mathrm{ev}(b_v)}\mathbb{P}^n$ and $\psi,\psi'\in\mathcal{H}_\Sigma^{0,1}$. By Lemma 7.1, $\|\tilde{R}_v-R_v\|_2\le C(b_v)|v|$.

Theorem 7.2 *Suppose d is a positive integer, $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ is a simple bubble type with $d_{\hat{0}}=0$ and $\sum_{i\in I}d_i=d$, $\mathcal{S}\subset\mathcal{M}_{\mathcal{T}}$ is a complex submanifold, and*

$$\nu\in\Gamma^{0,1}\big(\Sigma\times\mathbb{P}^n;\Lambda_{J,j}^{0,1}\pi_\Sigma^*T^*\Sigma\otimes\pi_{\mathbb{P}^n}^*T\mathbb{P}^n\big)$$

is a generic section. Let μ be an N-tuple of complex submanifolds of $\mathbb{P}^n$ in general position of total codimension

$$\mathrm{codim}_{\mathbb{C}}\mu=d(n+1)-n(g-1)+N.$$

and $(\Phi_{\mathcal{S}},\Phi_{\mathcal{S}}^\mu)$ a regularization of $\mathcal{S}(\mu)$. Then for every precompact open subset K of $\mathcal{S}(\mu)$, there exist a neighborhood U_K of K in $\bar{C}^\infty_{(d;N)}(\Sigma;\mu)$ and $\delta,\epsilon,C>0$ with the following property. For every $t\in(0,\epsilon)$, there exist a section

$$\varphi_{\mathcal{S},t\nu}^\mu\in\Gamma\big(F^\emptyset\mathcal{S}_\delta|K;\pi_{F\mathcal{S}}^*\mathcal{N}^\mu\mathcal{S}\big),\quad \textit{with}\quad \big|\varphi_{\mathcal{S},t\nu}^\mu(v)\big|_{b_v}\le C\big(t+|v|^{\frac{1}{p}}\big),$$

and a sign-preserving bijection between $\mathcal{M}_{\Sigma,t\nu,d}(\mu)\cap U_K$ and the zero set of the section $\psi_{\mathcal{S},t\nu}^\mu$ defined by

$$\psi_{\mathcal{S},t\nu}^\mu\in\Gamma\big(F^\emptyset\mathcal{S}_\delta|K;\pi_{F\mathcal{S}}^*(\mathcal{H}_\Sigma^{0,1}\otimes ev^*T\mathbb{P}^n)\big),\quad \Pi_{b_v,\phi_{\mathcal{S}}^\mu\varphi_{\mathcal{S},t\nu}^\mu(v)}\psi_{\mathcal{S},t\nu}^\mu(v)=\psi_{\mathcal{S},t\nu}\big(\Phi_{\mathcal{S}}^\mu(\varphi_{\mathcal{S},t\nu}^\mu(v))\big);$$

$$\psi_{\mathcal{S},t\nu}\in\Gamma\big(F^\emptyset\mathcal{S}_\delta\big|(\mathcal{S}\cap U_K);\pi_{F\mathcal{S}}^*(\mathcal{H}_\Sigma^{0,1}\otimes ev^*T\mathbb{P}^n)\big),\quad \Pi_{b_v,\phi_{\mathcal{S}}(v)}\psi_{\mathcal{S},t\nu}(v)=\psi_{\mathcal{T},t\nu}\big(\Phi_{\mathcal{S}}(v)\big);$$

$$\psi_{\mathcal{T},t\nu}\in\Gamma\big(F^\emptyset\mathcal{T}_\delta\big|(\mathcal{M}_{\mathcal{T}}\cap U_K);\pi_{F\mathcal{T}}^*(\mathcal{H}_\Sigma^{0,1}\otimes ev^*T\mathbb{P}^n)\big),\quad \tilde{R}_v\psi_{\mathcal{T},t\nu}(v)=\pi_{v,-}^{0,1}\big(t\nu_{v,t}-\bar{\partial}u_v-D_v\xi_{v,t\nu}\big),$$

where $\Pi_{b,b'}$ denotes the $g_{\mathbb{P}^n,b}$-parallel transport along the $g_{\mathbb{P}^n,b}$-geodesics from $ev(b)$ to $ev(b')$ whenever $d_{\mathbb{P}^n}\big(ev(b),ev(b')\big)<r_{\mathbb{P}^n}$, $\xi_{v,t\nu}\in\tilde{\Gamma}_+(v)$,

$$\big\|\nu_{v,t}-\nu\big\|_{v,2}\le C\big(t+|v|^{\frac{1}{p}}\big),\quad \textit{and}\quad \big\|\xi_{v,t\nu}\big\|_{v,p,1}\le C\big(t+|v|^{\frac{1}{p}}\big).$$

This theorem follows immediately from Theorem 3.29 applied to the obstruction bundle setup described above, once a tangent-bundle model is chosen, so that $\tilde{\Gamma}_+(v)$ is well-defined. The only refinement is that we drop the term $\tilde{\eta}_{v,t\nu}$ from the definition of $\psi_{\mathcal{T},t\nu}$. This is because it vanishes on the support of the $(0,1)$-forms in $\Gamma_-^{0,1}(v)$, provided δ is sufficiently small. Thus, $\pi^{0,1}_{\upsilon,-}\tilde{\eta}_{\upsilon,t\nu}=0$.

7.3 A Power Series Expansion for $\pi^{0,1}_{\upsilon,-}\bar{\partial}u_\upsilon$

Nearly all of this section is devoted to obtaining the power series expansion for $\pi^{0,1}_{\upsilon,-}\bar{\partial}u_\upsilon$ of Proposition 7.6. However, we first state an estimate for $\pi^{0,1}_{\upsilon,-}\nu_{\upsilon,t}$, which is immediate from Theorem 7.2. Both the estimate and the power series expansion are a consequence of our choice of the obstruction bundle.

Let $\{\psi_j\}$ denote an orthonormal basis for $\mathcal{H}^{0,1}_\Sigma$. Given $q\in\mathbb{P}^n$ and an orthonormal basis $\{X_i\}$ for $T_q\mathbb{P}^n$, put

$$\bar{\nu}_q=\sum_{i=1,j=1}^{i=n,j=g}\Big(\int_{z\in\Sigma}\langle\nu(z,q),X_i\psi_j\rangle_z\Big)X_i\psi_j\equiv\pi_{\mathcal{H}^{0,1}_\Sigma}\nu(\cdot,q)\in\mathcal{H}^{0,1}_\Sigma\otimes T_q\mathbb{P}^n.$$

Note that $\bar{\nu}$ is well-defined.

Lemma 7.3 *There exist $\delta,C\in C^\infty(\mathcal{M}^{(0)}_{\mathcal{T}};\mathbb{R}^+)$ such that for all $\upsilon\in F^{(\emptyset)}\mathcal{T}_\delta$ and $t\in(0,\delta(b_\upsilon))$,*

$$\big\|\pi^{0,1}_{\upsilon,-}\nu_{\upsilon,t}-\tilde{R}_\upsilon\bar{\nu}_{ev(b_\upsilon)}\big\|_{\upsilon,2}\le C(b_\upsilon)\big(t+|\upsilon|^{\frac{1}{p}}\big).$$

Suppose $\upsilon=\big((\Sigma,[N],I;x,(j,y),u),(v_h)_{h\in\hat{I}}\big)\in F^{(\emptyset)}\mathcal{T}$ is such that q_υ is defined. For any $h\in\hat{I}$, define $\tilde{h}(\mathcal{T})\in\hat{I}$ by $\iota_{\tilde{h}(\mathcal{T})}=\hat{0}$ and $h\in\bar{D}_{\tilde{h}(\mathcal{T})}\mathcal{T}$. By the basic gluing constriction, see Section 2.3,

$$\tilde{v}_h=d\phi_{b_\upsilon,\tilde{h}(\mathcal{T})}\big|_{\tilde{x}_h(\upsilon)}\big(dq^{-1}_{\upsilon,\iota_h}\big|_{\tilde{x}_h(\upsilon)}d\phi^{-1}_{b_\upsilon,h}\big|_0v_h\big)=\prod_{i\in\hat{I},h\in\bar{D}_i\mathcal{T}}v_i\in T_{x_{\tilde{h}(\mathcal{T})}}\Sigma.$$

Lemma 7.4 *For all $\upsilon\in F^{(\emptyset)}\mathcal{T}$ such that q_υ is defined, $\bar{\partial}u_\upsilon$ vanishes outside of the annuli $A^-_{\upsilon,h}$ with $\chi_{\mathcal{T}}h=1$ and $A^\pm_{\upsilon,h}$ with $\chi_{\mathcal{T}}h=2$. Furthermore, there exists $\delta\in C^\infty(\mathcal{M}_{\mathcal{T}};\mathbb{R}^+)$ such that for all $\upsilon\in F^{(\emptyset)}\mathcal{T}_\delta$ and $h\in\hat{I}$ with $\chi_{\mathcal{T}}h=1$, on $\tilde{A}^-_{\upsilon,h}\equiv\big\{z\in F^{(0)}_{h,b_\upsilon}:\frac{1}{2}|v_h|^{\frac{1}{2}}\le|z|_{b_\upsilon}\le|v_h|^{\frac{1}{2}}\big\}$,*

$$\begin{aligned}&\Pi^{-1}_{b_\upsilon,\bar{u}_\upsilon(z)}\bar{\partial}\big(u_\upsilon\circ q^{-1}_{\upsilon,\iota_h}\big)\circ d\phi^{-1}_{b_\upsilon,h}\Big|_z\\&\qquad=-|v_h|^{-\frac{1}{2}}\Big(\sum_{m\ge1}\big(1-\beta_{|v_h|}(2|z|)\big)^{(m-1)}\mathcal{D}^{(m)}_{\mathcal{T},h}\Big(\Big[b_\upsilon,\Big(\frac{v_h}{z}\Big)\Big]\Big)\Big)\bar{\partial}\beta\big|_{2|v_h|^{-\frac{1}{2}}z},\end{aligned}$$

where $\bar{u}_\upsilon(z)\in T_{ev(b_\upsilon)}\mathbb{P}^n$ is given by

$$\exp_{b_\upsilon,ev(b_\upsilon)}\bar{u}_\upsilon(z)=u_\upsilon\big(q^{-1}_{\upsilon,\iota_h}\phi^{-1}_{b_\upsilon,h}(z)\big)=u_h\big(q_{h,(x_h,v_h)}\phi^{-1}_{b_\upsilon,h}(z)\big),\quad|\bar{u}_\upsilon(z)|_{b_\upsilon}<r_{\mathbb{P}^n}.$$

This sum converges uniformly on $\tilde{A}^-_{\upsilon,h}$.

Proof: The first claim is immediate from Lemma 2.3. If $y\in\Sigma_{b_\upsilon,h}$ and $|q_S^{-1}(y)|\le2\delta_{\mathcal{T}}(b_\upsilon)$, define $\bar{u}_h(y)\in T_{\mathrm{ev}(b_\upsilon)}\mathbb{P}^n$ by

$$\exp_{b_\upsilon,\mathrm{ev}(b_\upsilon)}\bar{u}_h(y)=u_h(y),\quad|\bar{u}_h(y)|_{b_\upsilon}<r_{\mathbb{P}^n}.$$

By construction, $u_v\circ q_{v,\iota_h}^{-1}=u_h\circ q_v\circ q_{v,\iota_h}^{-1}$ on $q_{v,\iota_h}(A_{v,h}^-)$. Since $\Pi_{b_v,\bar{u}_v}^{-1}\circ du_h$ is $\mathbb{C}$-linear on $q_v(A_{v,h}^-)$, for any $z\in\tilde{A}_{v,h}^-$

$$\begin{aligned}\Pi_{b_v,\bar{u}_v(\cdot)}^{-1}\bar{\partial}(u_v\circ q_{v,\iota_h}^{-1})\circ d\phi_{b_v,h}^{-1}\Big|_z &= \Pi_{b_v,\bar{u}_v(\cdot)}^{-1}du_h\circ\bar{\partial}(q_v\circ q_{v,\iota_h}^{-1})\circ d\phi_{b_v,h}^{-1}\Big|_z \\ &= -2|v_h|^{-\frac{1}{2}}\Big(\frac{v_h}{z}\Big)\Pi_{b_v,\bar{u}_v(\cdot)}^{-1}(du_h\circ dq_S)\Big|_{p_{h,(x_h,v_h)}\phi_{v,h}^{-1}(z)}\circ\partial\beta\Big|_{2|v_h|^{-\frac{1}{2}}z};\end{aligned}\tag{7.8}$$

see Lemma 2.3. Since $g_{\mathbb{P}^n,b_v}$ is flat on $u_v(A_{v,h}^-)$ by our choice of metrics,

$$\Pi_{b_v,\bar{u}_v}^{-1}(du_h\circ dq_S)=d(\bar{u}_h\circ q_S)\tag{7.9}$$

on $q_S^{-1}q_v(A_{v,h}^-)$. Since $\bar{u}_h\circ q_S$ is antiholomorphic and the metric $g_{\mathbb{P}^n,b_v}$ is flat near $\mathrm{ev}(b_v)$,

$$\begin{aligned}(\bar{u}_h\circ q_S)\big|_x\Big(\frac{\partial}{\partial s}\Big) &= d(\bar{u}_h\circ q_S)\big|_x\Big(\frac{\partial}{\partial\bar{y}}\Big)=\sum_{m\ge1}\frac{\overline{x}^{m-1}}{(m-1)!}\frac{d^m}{d\bar{y}^m}(\bar{u}_h\circ q_S)\big|_{(s,t)=0} \\ &= \sum_{m\ge1}\frac{\overline{x}^{m-1}}{(m-1)!}\frac{D^{m-1}}{ds^{m-1}}\frac{d}{ds}(u_h\circ q_S)\big|_{(s,t)=0},\end{aligned}\tag{7.10}$$

for any $x\in q_v(A_{v,h}^-)$, where $y=s+it\in\mathbb{C}$ is the complex coordinate. The second claim follows from equations (7.8)-(7.10). For the last claim, note that the sum converges uniformly on $\tilde{A}_{v,h}^-$ as long as $q_v(A_{v,h}^-)$ is contained in the ball of convergence for the power series expansion for $\bar{u}_h$ at 0.

If $\psi\in\mathcal{H}_\Sigma^{0,1}$, $b\in\mathcal{M}_{\mathcal{T}}^{(0)}$, $m\ge1$, and the metric $g_{b,\hat{0}}$ is flat near x, we define $D_{b,x}^{(m)}\psi\in T_x^{0,1}\Sigma^{\otimes m}$ as follows. If (s,t) are conformal coordinates centered at x such that s^2+t^2 is the square of the $g_{b,\hat{0}}$-distance to x, let

$$\{D_{b,x}^{(m)}\psi\}\Big(\frac{\partial}{\partial s}\Big)\equiv\{D_{b,x}^{(m)}\psi\}\Big(\underbrace{\frac{\partial}{\partial s},\ldots,\frac{\partial}{\partial s}}_{m}\Big)=\frac{\pi}{m!}\Big\{\frac{D^{m-1}}{ds^{m-1}}\psi_j\Big|_{(s,t)=0}\Big\}\Big(\frac{\partial}{\partial s}\Big),$$

where the covariant derivatives are taken with respect to the metric $g_{b,\hat{0}}$. Since $\psi_j\in\mathcal{H}_\Sigma^{0,1}$, $\psi_j=f(ds-idt)$ for some anti-holomorphic function f. Since $g_{b,\hat{0}}$ is flat near x, it follows that $D_{b,x}^{(m)}\psi\in T_x^{0,1}\Sigma^{\otimes m}$. If $\{\psi_j\}$ is an orthonormal basis for $\mathcal{H}_\Sigma^{0,1}$, let $s_{b,x}^{(m)}\in T_x^*\Sigma^{\otimes m}\otimes\mathcal{H}_\Sigma^{0,1}$ be given by

$$s_{b,x}^{(m)}(v)\equiv s_{b,x}^{(m)}(\underbrace{v,\ldots,v}_{m})=\sum_{j\in[g]}\overline{\Big\{D_{b,x}^{(m)}\psi_j\Big\}(v)}\psi_j.$$

The section $s_{b,x}^{(m)}$ is always independent of the choice of a basis for $\mathcal{H}_\Sigma^{0,1}$, but is dependent on the choice of the metric $g_{b,\hat{0}}$ if $m>1$. However, $s_x\equiv s_{b,x}^{(1)}$ depends only on (Σ,j). By [GH, p246], s_x does not vanish and thus spans a subbundle of $\Sigma\times\mathcal{H}_\Sigma^{0,1}\longrightarrow\Sigma$. We denote this subbundle by $\mathcal{H}_\Sigma^+$ and its orthogonal complement by $\mathcal{H}_\Sigma^-$. A slightly different description of these bundles is given in Section 8.1. Let

$$\pi^+,\pi^-\in\Gamma(\Sigma;(\Sigma\times\mathcal{H}_\Sigma^{0,1})^*\otimes\mathcal{H}_\Sigma^\pm)$$

be the corresponding orthogonal projection maps and $s_{b,x}^{(m,\pm)} = \pi_x^{\pm} \circ s_{b,x}^{(m,\pm)}$.

Lemma 7.5 *There exists $\delta \in C^\infty(\mathcal{M}_\mathcal{T};\mathbb{R}^+)$ such that for all $v \in F^{(\emptyset)}\mathcal{T}_\delta$, $X \in T_{ev(b_v)}\mathbb{P}^n$, and $\psi \in \mathcal{H}_\Sigma^{0,1}$,*

$$\langle\langle \pi_{v,-}^{0,1}\bar\partial u_v, R_v X\psi\rangle\rangle_{v,2} = -\sum_{m\geq 1}\sum_{\chi_\mathcal{T}h=1}\langle \mathcal{D}_{\mathcal{T},h}^{(m)}b_v, X\rangle\overline{\Big(\big\{D_{b_v,\tilde x_h(v)}^{(m)}\psi\big\}\big((d\phi_{b_v,x_{\tilde h(\mathcal{T})}}|_{\tilde x_h(v)})^{-1}\tilde v_h\big)\Big)}.$$

Furthermore, the sum is absolutely convergent.

Proof: Since $\langle\bar\partial u_v, R_v X\psi\rangle = 0$ outside of the annuli $A_{v,h}^-$ with $\chi_\mathcal{T}h=1$,

$$\langle\langle \pi_{v,-}^{0,1}\bar\partial u_v, R_v X\psi\rangle\rangle = \langle\langle \bar\partial u_v, R_v X\psi\rangle\rangle = \sum_{\chi_\mathcal{T}h=1}\int_{A_{v,h}^-}\langle\bar\partial u_v, R_v X\psi\rangle. \tag{7.11}$$

Since $q_{v,\iota_h}^{-1}\circ\phi_{b_v,h}^{-1}$ is holomorphic on $\tilde A_{v,h}^-$, $\Pi_{b_v,\tilde u_v}^{-1}$ is unitary on $u_v(A_{v,h}^-)$, and the inner-product of one-forms is conformally invariant,

$$\begin{aligned}\int_{A_{v,h}^-}\langle\bar\partial u_v, R_v X\psi\rangle &= \int_{\tilde A_{v,h}^-}\langle\bar\partial(u_v\circ q_{v,\iota_h}^{-1})\circ d\phi_{b_v,h}^{-1}, R_v X\psi\circ dq_{v,\iota_h}^{-1}\circ d\phi_{b_v,h}^{-1}\rangle\\ &= \int_{\tilde A_{v,h}^-}\langle\Pi_{b_v,\tilde u_v}^{-1}\bar\partial(u_v\circ q_{v,\iota_h}^{-1})\circ d\phi_{b_v,h}^{-1}, X\psi\circ dq_{v,\iota_h}^{-1}\circ\phi_{b_v,h}^{-1}\rangle,\end{aligned} \tag{7.12}$$

since $\Pi_{b_v,\tilde u_v}^{-1}R_v X\psi = X\psi$ on $A_{v,h}^-$. If $\iota_h=\hat 0$, we identify $F_{h,b_v}^{(0)} = T_{x_h}\Sigma$ with $\mathbb{C}$ in a $g_{b_v,\hat 0}$-unitary way. In all cases, we can then write

$$\psi\circ dq_{v,\iota_h}^{-1}\circ d\phi_{b_v,h}^{-1} = f d\bar z.$$

Since ψ is harmonic and $q_{v,\iota_h}^{-1}\circ\phi_{b_v,h}^{-1}$ is holomorphic on $\tilde A_{v,h}$, f is anti-holomorphic. Using the change of variables $2|v_h|^{-\frac12}z = re^{i\theta}$, we obtain

$$\begin{aligned}&\int_{\tilde A_{v,h}}\Big\langle |v_h|^{-\frac12}\big(1-\beta_{|v_h|}(2|z|)\big)^{m-1}\mathcal{D}_{\mathcal{T},h}^{(m)}\Big(\Big[b_v,\frac{v_h}{z}\Big]\Big)\bar\partial\beta\big|_{2|v_h|^{-\frac12}z}, X\psi\circ dq_{v,\iota_h}^{-1}\circ d\phi_{b_v,h}^{-1}\Big\rangle\\ &= \Big\langle\mathcal{D}_{\mathcal{T},h}^{(m)}b_v, X\Big\rangle v_h^m\int_{\tilde A_{v,h}}\Big\{\big(1-\beta(2|v_h|^{-\frac12}|z|)\big)^{m-1}\beta'\big|_{2|v_h|^{-\frac12}|z|}\Big\}|v_h|^{-\frac12}z^{-m}\frac{z}{|z|}\bar f\\ &= \Big\langle\mathcal{D}_{\mathcal{T},h}^{(m)}b_v, X\Big\rangle v_h^m|v_h|^{-\frac{m-1}{2}}2^{m-2}\frac{1}{m}\int_1^2\int_0^{2\pi}\Big\{\big(1-\beta(r)\big)^m\Big\}'(re^{i\theta})^{-(m-1)}\bar f\Big(\frac12|v_h|^{\frac12}re^{i\theta}\Big)d\theta dr\end{aligned} \tag{7.13}$$

Since $\bar f$ is holomorphic, for any $r>0$,

$$\begin{aligned}\int_0^{2\pi}(re^{i\theta})^{-(m-1)}\bar f\Big(\frac12|v_h|^{\frac12}re^{i\theta}\Big)d\theta &= -i\int_{|z|=r}z^{-m}\bar f\Big(\frac12|v_h|^{\frac12}z\Big)dz\\ &= \frac{2\pi}{(m-1)!}\frac{d^{(m-1)}}{dz^{(m-1)}}\bar f\Big(\frac12|v_h|^{\frac12}z\Big)\Big|_{z=0} = \frac{2\pi}{(m-1)!}2^{-(m-1)}|v_h|^{\frac{m-1}{2}}\bar f^{(m-1)}(0).\end{aligned} \tag{7.14}$$

Since the metric $g_{b,\hat{0}}$ is flat near $\tilde{x}_h$,

$$\frac{\pi}{m!}v_h^m \bar{f}^{\langle m-1\rangle}(0) = \overline{\{D^{(m)}_{b_v,\tilde{x}_h(v)}\psi\}\big(dq^{-1}_{v,\iota_h}\big|_{x_h} d\phi^{-1}_{b_v,h}\big|_0 v_h\big)} \\ = \overline{\{D^{(m)}_{b_v,\tilde{x}_h(v)}\psi\}\big((d\phi_{b_v,x_{\tilde{h}(\mathcal{T})}}|_{\tilde{x}_h(v)})^{-1}\tilde{v}_h\big)}. \tag{7.15}$$

The claim follows from equations (7.11)-(7.15) and Lemma 7.4.

Proposition 7.6 *If $\mathcal{T} = (\Sigma, [N], I; j, \underline{d})$ is a simple bubble type with $d_{\hat{0}} = 0$, there exists $\delta \in C^\infty(\mathcal{M}_\mathcal{T}; \mathbb{R}^+)$ such that*

$$\pi^{0,1}_{v,-}\bar{\partial}u_v = -\tilde{R}_v \sum_{m\geq 1}\sum_{\chi_\mathcal{T} h=1} \big(\mathcal{D}^{(m)}_{\mathcal{T},h}b\big)\Big(s^{(m)}_{b,\tilde{x}_h(v)}\big(d\phi_{b,x_{\tilde{h}(\mathcal{T})}}|^{-1}_{\tilde{x}_h(v)}\tilde{v}_h\big)\Big) \quad \forall v = \big[b, (v_h)_{h\in\hat{I}}\big] \in F^\emptyset\mathcal{T}_\delta.$$

Furthermore, the sum is absolutely convergent.

Proof: This proposition follows from Lemma 7.5 and equation (7.7).

Chapter 8

Genus-Two Curves

In this chapter, we consider the case $g(\Sigma) = 2$ and $n = 2, 3$. We begin by describing a tangent-space model for gluing along the spaces $\mathcal{M}_{\mathcal{T}}(\mu)$. In the sections that follow, we use Theorem 7.2, Section 7.3, and properties of the tangent bundle model to relate the contribution $N_{\mathcal{T}}(\mu)$ of $\mathcal{M}_{\mathcal{T}}(\mu)$ to $\mathrm{RT}_{2,d}(\cdot;\mu)$ to the zeros of explicit affine maps between vector bundles over natural compactifications of $\mathcal{M}_{\mathcal{T}}(\mu)$ and its subspaces.

8.1 Tangent-Bundle Model

In this section, we describe our choice for a tangent-bundle model, which is the subject of Definition 3.11.

For any $\upsilon \in F^{(0)}\mathcal{T}$ sufficiently small and $\xi \in \Gamma(b_\upsilon)$, define $R_\upsilon \xi \in L^p_1(\upsilon)$ by $\{R_\upsilon \xi\}(z) = \xi\big(q_\upsilon(z)\big)$. Let $\Gamma_-(\upsilon)$ be the image of $\ker(D_{b_\upsilon})$ under the map R_υ. Denote by $\Gamma_+(\upsilon)$ its (L^2, υ)-orthogonal complement in $L^p_1(\upsilon)$. Let $\pi_{\upsilon,\pm}$ be the (L^2, υ)-orthogonal projection onto $\Gamma_\pm(\upsilon)$.

If $x \in \Sigma$, let $\mathcal{H}^-_\Sigma(x) = \{\psi \in \mathcal{H}^{0,1}_\Sigma : \psi|_x = 0\}$. This is a codimension-one subspace of $\mathcal{H}^{0,1}_\Sigma$ for all $x \in \Sigma$; see [GH]. Denote by $\mathcal{H}^+_\Sigma(x)$ its L^2-orthogonal complement. The space $\mathcal{H}^+_\Sigma(x)$ is independent of the choice of a Kahler metric on (Σ, j_Σ). For any $h \in \hat{I}$, we put $\tilde{x}_h(\upsilon) = q^{-1}_{\upsilon,\iota_h}(\iota_h, x_h)$. Fix $h^* \in \hat{I}$ such that $\chi_{\mathcal{T}} h^* = 1$. Let

$$\bar{\Gamma}_-(\upsilon) = D^*_\upsilon R_\upsilon \big(\mathcal{H}^+_\Sigma(\tilde{x}_{h^*}(\upsilon)) \otimes T_{\mathrm{ev}(b_\upsilon)}\mathbb{P}^n\big).$$

Denote by $\bar{\Gamma}_+(\upsilon)$ the (L^2, υ)-orthogonal complement of $\bar{\Gamma}_-(\upsilon)$ in $L^p_1(\upsilon)$ and by $\bar{\pi}_{\upsilon,\pm}$ the (L^2, g_υ)-orthogonal projections onto $\bar{\Gamma}_\pm(\upsilon)$. Let $\tilde{\Gamma}_-(\upsilon)$ be the (L^2, υ)-orthogonal complement of $\tilde{\Gamma}_+(\upsilon) \equiv \bar{\pi}_{\upsilon,+}\big(\Gamma_+(\upsilon)\big)$ in $L^p_1(u_\upsilon)$.

The spaces $\tilde{\Gamma}_-(\upsilon)$ will be our tangent-space model. We need to check that the requirements of Definition 3.11 are satisfied. Let

$$\{h \in \hat{I} : \chi_{\mathcal{T}}(h) = 1\} = \{h_1 = h^*, h_2, \ldots, h_m\}.$$

If $z \in \Sigma_{b,h_r}$ is such that $|q_S^{-1}(z)| \le 2\delta(b)$, define $\bar{u}_{h_r}(z) \in T_{\mathrm{ev}(b)}\mathbb{P}^n$ by

$$\exp_{b,\mathrm{ev}(b)} \bar{u}_{h_r}(z) = u_{h_r}(z), \quad |\bar{u}_{h_r}(z)|_b < r_{\mathbb{P}^n}.$$

If $X \in T_{\mathrm{ev}(b)}\mathbb{P}^n$, define $R_{b,h_r}X \in \Gamma(u_{h_r})$ by

$$R_{b,h_r}X(z) = \begin{cases} 0, & \text{if } |z| \ge 2\delta_{\mathcal{T}}(b)^{-1}; \\ \beta'|_{\delta_{\mathcal{T}}(b)|z|} \frac{(1+|z|^2)^2 z}{|z|} \Pi_{b,\tilde{u}_{h_r}(z)} X, & \text{otherwise.} \end{cases}$$

Since $R_{b,h_r}X$ vanishes at all the nodes of Σ_b by assumption on $\delta_{\mathcal{T}}$, we can extend $R_{b,h_r}X$ by zero to an element of $\Gamma(b)$. If $c = c_{[m]} \in \mathbb{C}^{[m]}$ is different from zero, let

$$\bar{\Gamma}_-(b;c) = \Big\{ \sum_{r \in [m]} c_r R_{b,h_r} X : X \in T_{\mathrm{ev}(b)}\mathbb{P}^n \Big\}.$$

Denote by $\bar{\Gamma}_+(b;c)$ the (L^2, b)-orthogonal complement of $\bar{\Gamma}_-(b;c)$ in $\Gamma(b)$. Let $\bar{\pi}_{(b;c),\pm}$ be the corresponding (L^2, b)-orthogonal projection maps. Let $\tilde{\Gamma}_+(b;c) = \bar{\pi}_{(b,c),+}\big(\Gamma_+(b)\big)$ and let $\tilde{\Gamma}_-(b;c)$ be its (b, L^2)-orthogonal complement.

Lemma 8.1 *There exist $\delta, C \in C^\infty(\mathcal{M}_{\mathcal{T}}^{(0)}; \mathbb{R}^+)$ such that for all $v \in F^{(\emptyset)}\mathcal{T}_\delta$ and $\xi \in \bar{\Gamma}_-(v)$,*

$$\|\xi\|_{v,p,1} \le C(b_v)\|\xi\|_{v,2}.$$

In addition, $\dim_{\mathbb{C}} \bar{\Gamma}_-(v) = \dim_{\mathbb{C}} \bar{\Gamma}_-(b_v;c) = n$ for any nonzero $c \in \mathbb{C}^m$. Furthermore, if $v_k \longrightarrow b \in \mathcal{M}_{\mathcal{T}}^{(0)}$ and $\xi_k \in \bar{\Gamma}_-(v)$ is such that $\|\xi_k\|_{v_k,2} = 1$, then there exists a nonzero $c \in \mathbb{C}^m$ and $\xi \in \bar{\Gamma}_-(b;c)$ with $\|\xi\|_{b,2} = 1$ such that a subsequence of $\{\xi_k\}$ C^0-converges to ξ.

Remark: The last statement means that a subsequence of $\{\xi_k\}$ C^0-converges to ξ on compact subsets of Σ_b^* and the norms $\|\xi_k\|_{v_k,p,1}$ are uniformly bounded; see Definition 3.9.

Proof: (1) Let ψ be a generator of $\mathcal{H}^{0,1}_{\Sigma,+}\big(\tilde{x}_{h_1}(v)\big)$. If $X \in T_{\mathrm{ev}(b_v)}\mathbb{P}^n$ and $r \in [m]$, define $R_{v,h_r}X \in \Gamma(u_v)$ as follows. If $q_v(z) \in \Sigma_{b_v,h_r}$, let

$$R_{v,h_r}X(z) = \Big(\sum_{r \in [m]} \big|\psi_{\tilde{x}_r(v)} d(q_{v,h_r}^{-1} \circ q_N)\partial_s\big| \Big)^{-1} \frac{(1+|q_v z|^2)^2 q_v z}{|q_v z|}$$
$$\times \beta'|_{\delta_{\mathcal{T}}(b_v)|q_v z|} \big(\psi_z d(q_{v,h_r}^{-1} \circ q_N)\partial_s\big) \Pi_{b_v,\tilde{u}_v(z)} X.$$

Note that the sum is not zero, since $\psi|_{\tilde{x}_{h_1}(v)} \ne 0$. If $q_v(z) \notin \Sigma_{b_v,h_r}$, we let $R_{v,h_r}X(z) = 0$. Since the modified Sobolev norms are equivalent to the standard ones away from the thin necks of (Σ_v, g_v),

$$\begin{aligned} \big\|R_{v,h_r}X\big\|_{v,p,1} &\le C(b_v)\Big(\sum_{r \in [m]} \big|\psi_{\tilde{x}_r(v)} d(q_{v,h_r}^{-1} \circ q_N)\partial_s\big| \Big)^{-1} \big|\psi_z d(q_{v,h_r}^{-1} \circ q_N)\partial_s\big| |X|_v \\ &\le C'(b_v)\|R_{v,h_r}X\|_{v,2}. \end{aligned} \tag{8.1}$$

By Lemma 7.1, if $\xi \in \bar{\Gamma}_-(v)$,

$$\xi = R_v X \equiv \sum_{r \in [m]} R_{v,h_r} X,$$

for some $X \in T_{\mathrm{ev}(b_v)}\mathbb{P}^n$. Thus, the first two statements of the lemma follow from (8.1).
(2) If $v_k \longrightarrow b$ and $\xi_k = R_v X_k \in \bar{\Gamma}_-(v_k)$ is such that $\|\xi_k\|_{v_k,2} = 1$, then it is immediate from

(1) that a subsequence of ξ_k C^0-converges to $\sum_{r\in[m]} c_r R_{b,h_r} X$, where

$$X=\lim_{k\longrightarrow\infty} X_k, \quad c_r=\lim_{k\longrightarrow\infty}\Big(\sum_{r\in[m]}\big|\psi_{\tilde{x}_r(v)}d(q_{v,h_r}^{-1}\circ q_N)\partial_s\big|\Big)^{-1}\big(\psi_{\tilde{x}_r(v)}d(q_{v,h_r}^{-1}\circ q_N)\partial_s\big). \quad (8.2)$$

The two limits in (8.2) exist after passing to a subsequence of the original sequence. This proves the last statement of the lemma.

Lemma 8.2 *There exist $\delta, C\in C^\infty(\mathcal{M}_{\mathcal{T}}^{(0)};\mathbb{R}^+)$ such that for all $v\in F^{(\emptyset)}\mathcal{T}_\delta$ and $\xi\in\tilde{\Gamma}_-(v)$,*

$$\|\xi\|_{v,p,1}\le C(b_v)\|\xi\|_{v,2}.$$

Proof: Let $\Gamma_{-+}(v)$ be the (v,L^2)-orthogonal complement of $\pi_{v,-}\big(\bar{\Gamma}_-(v)\big)$ in $\Gamma_-(v)$. Then

$$\tilde{\Gamma}_-(v)=\Gamma_{-+}(v)\oplus\bar{\Gamma}_-(v).$$

Since this decomposition is (L^2,v)-orthogonal, we can assume that either $\xi\in\Gamma_{-+}(v)$ or $\xi\in\bar{\Gamma}_-(v)$. In the first case, the statement is obvious, since $\Gamma_{-+}(v)\subset\Gamma_-(v)$. The second case is proved in Lemma 8.1.

Corollary 8.3 *Suppose $v_k\in F^{(0)}\mathcal{T}_\delta$ and $v_k\longrightarrow b\in\mathcal{M}_{\mathcal{T}}^{(0)}$. If $\{\xi_{v_k,l}\}$ is an (L^2,v)-orthonormal basis for $\tilde{\Gamma}_-(v_k)$, then there exists a nonzero $c\in\mathbb{C}^m$ and an (L^2,b)-orthonormal basis $\{\xi_{b,l}\}$ for $\tilde{\Gamma}_-(b;c)$ such that after passing to a subsequence $\xi_{v_k,l}$ C^0-converges to $\xi_{b,l}$ for all l.*

Proof: If $\xi_{k,l}\in\bar{\Gamma}_-(v_k)$, by Lemma 8.1 a subsequence of $\{\xi_{k,l}\}$ C^0-converges to an element of $\xi_l\in\bar{\Gamma}_-(b;c)$ for some nonzero $c\in\mathbb{C}^n$ dependent on the sequence $\{v_k\}$. Furthermore, orthonormal pairs of such elements C^0-converge to an orthonormal pair in $\bar{\Gamma}_-(b)$. If $\xi_{k,l}\in\bar{\Gamma}_{-+}(v_k)\subset\Gamma_-(v_k)$, then by definition of $\Gamma_-(v_k)$, a subsequence of $\{\xi_{k,l}\}$ C^0-converge to an element $\xi_l\in\Gamma_-(b)$, which must be orthogonal to $\bar{\Gamma}_-(b;c)$; see Lemma 3.10. Thus, a subsequence of $\big\{\{\xi_{k,l}\}\big\}$ C^0-converges to an orthonormal set of vectors in $\tilde{\Gamma}_-(b)$, which implies that $\dim_{\mathbb{C}}\tilde{\Gamma}_-(b;c)\ge\dim_{\mathbb{C}}\tilde{\Gamma}_-(v_k)$. However,

$$\begin{aligned}\dim_{\mathbb{C}}\tilde{\Gamma}_-(b;c)&=\dim_{\mathbb{C}}\Gamma_{-+}(b;c)+\dim_{\mathbb{C}}\bar{\Gamma}_-(b;c)\\&=\dim_{\mathbb{C}}\Gamma_-(b)+\big(\dim_{\mathbb{C}}\bar{\Gamma}_-(b;c)-\dim_{\mathbb{C}}\pi_{b,-}\bar{\Gamma}_-(b;c)\big);\\ \dim_{\mathbb{C}}\tilde{\Gamma}_-(v_k)&=\dim_{\mathbb{C}}\Gamma_{-+}(v_k)+\dim_{\mathbb{C}}\bar{\Gamma}_-(v_k)\\&=\dim_{\mathbb{C}}\Gamma_-(v_k)+\big(\dim_{\mathbb{C}}\bar{\Gamma}_-(v_k)-\dim_{\mathbb{C}}\pi_{v_k,-}\bar{\Gamma}_-(v_k)\big),\end{aligned}$$

where $\Gamma_{-+}(b;c)$ denotes the (L^2,b)-complement of $\pi_{b,-}\bar{\Gamma}_-(b;c)$ in $\Gamma_-(b)$. Since $\Gamma_-(v_k)$ and $\Gamma_-(b)$ have the same dimension, in order to conclude the proof, it is sufficient to show that

$$\pi_{b,-}:\bar{\Gamma}_-(b;c)\longrightarrow\Gamma_-(b;c)$$

is an isomorphism; see Lemma 8.4.

Lemma 8.4 *There exists $C\in C^\infty(\mathcal{M}_{\mathcal{T}}^{(0)};\mathbb{R}^+)$ such that for all $b\in\mathcal{M}_{\mathcal{T}}^{(0)}$, nonzero $c\in\mathbb{C}^m$, and $\xi\in\bar{\Gamma}_-(b;c)$*

$$\|\xi\|_{b,2}\le C(b_\omega)\|\pi_{b,-}\xi\|_{b,2}.$$

Proof: Let $X \in T_{\mathrm{ev}(b)}\mathbb{P}^n$. Outside of $\infty \in \Sigma_{b,h_r}$, by the same computation as in the proof of Lemma 7.1, $R_{b,h_r}X = D^*_{b,u_{h_r}} R_{b,h_r}X$, where we define $R_{b,h_r}X \in \Gamma^{0,1}(u_{h_r})$ as follows. Let $d\bar{z}$ denote the usual $(0,1)$-form on $\mathbb{C}$. Then

$$\delta_{\mathcal{T}}(b) R_b X\big|_x = 4\beta\big(\delta_{\mathcal{T}}(b)|q_N^{-1}(x)|\big)\big(q_N^{-1*}d\bar{z}\big)\Pi_{b,\bar{u}_{h_r}(x)}X.$$

Thus, if $\xi = \xi_{\tilde{f}} \in \ker D_b$ and $2\delta < \delta_{\mathcal{T}}(b)$, by integration by parts,

$$\begin{aligned}\langle\langle \xi, R_{b,h_r}X\rangle\rangle_b &= \langle\langle \xi_{h_r}, D^*_{b,u_{h_r}} R_{b,h_r}X\rangle\rangle_b \\ &= 2i\delta_{\mathcal{T}}(b)^{-1} \int\limits_{|q_N^{-1}(x)|=\delta^{-1}} \langle \xi_{h_r}(x), \Pi_{b,\bar{u}_{h_r}(x)}X\rangle_b q_N^{-1*}dz,\end{aligned} \tag{8.3}$$

since $D_{b,u_{h_r}}\xi_{h_r} = 0$. Using the change of variables with $x = q_N(w^{-1})$, we obtain

$$\begin{aligned}\int\limits_{|q_N^{-1}(x)|=\delta^{-1}} \langle \xi_{h_r}(x), \Pi_{b,\bar{u}_{h_r}(x)}X\rangle_b q_N^{-1*}dz &= -\int\limits_{|w|=\delta} \langle \xi_{h_r}|_{q_N(w^{-1})}, \Pi_{b,\bar{u}_{h_r}(q_N(w^{-1}))}X\rangle_b \frac{dw}{w^2} \\ &= -2\pi i \frac{d}{dw}\langle \xi_{h^*}|_{q_N(w^{-1})}, \Pi_{b,\bar{u}_{h_r}(q_N(w^{-1}))}X\rangle_b\Big|_{w=0} \\ &= -2\pi i \frac{d}{d\bar{z}}\langle \xi_{h_r}|_{q_S(z)}, \Pi_{b,\bar{u}_{h_r}(q_S(z))}X\rangle_b\Big|_{z=0} = -2\pi i\Big\langle \frac{D}{ds}(\xi_{h_r}\circ q_S)\Big|_{z=0}, X\Big\rangle,\end{aligned} \tag{8.4}$$

since $D_{b,u_{h_r}}\xi_{h_r} = 0$. It follows from (8.3) and (8.4) that for any $\xi = \xi_{[M]} \in \ker(D_b)$,

$$\Big\langle\Big\langle \xi, \sum_{r\in[m]} c_r R_{b,h_r}X\Big\rangle\Big\rangle_b = 4\pi\delta_{\mathcal{T}}(b)^{-1}\sum_{r\in[m]} c_r\Big\langle \frac{D}{ds}(\xi_{h_r}\circ q_S)\Big|_{z=0}, X\Big\rangle. \tag{8.5}$$

Along with Corollary C.3, equations (8.5) gives

$$\begin{aligned}\Big\|\pi_{b,-}\sum_{r\in[m]} c_r R_{b,h_r}X\Big\|_{b,2} &\geq C(b)|c_{r^*}| \sup_{\xi_{[M]}\in\ker(D_b),\|\xi_{[M]}\|=1}\Big\langle \frac{D}{ds}(\xi_{h_{r^*}}\circ q_S)\Big|_{z=0}, X\Big\rangle_b \\ &\geq C'(b)|c_{r^*}||X| \geq C''(b)\Big\|\sum_{r\in[m]} c_r R_{b,h_r}X\Big\|_{b,2},\end{aligned} \tag{8.6}$$

where $r^* \in [m]$ is such that $|c_{r^*}| = \sup_r |c_r|$. Since the right-hand side of (8.6) must be a continuous function of b, the claim follows.

The statement of Corollary 8.3 is precisely Condition (1) of Definition 3.11. The other two conditions require that the rate of change of the (L^2, υ)-orthogonal projection onto $\tilde{\Gamma}_-(\upsilon)$ be controlled by a function of b_υ only. This is a consequence of the convergence described in the Corollary 8.3, i.e. we can use the same argument as described in the remark following Lemma 3.6, but with $\Gamma_-(b)$ replaced by the appropriate space $\Gamma_-(b; c)$ (depending on υ).

8.2 First-Order Estimate for $\psi^{\mu}_{\mathcal{T},t\nu}$

For any $\upsilon = [(\Sigma,[N],I;x,(j,y),u),(v_h)_{h\in\hat{I}}] \in F\mathcal{T}$ and $h\in\hat{I}$ such that $\chi_{\mathcal{T}}h=1$, let

$$\alpha^{(k)}_{\mathcal{T},h}(\upsilon) = \big(\mathcal{D}^{(k)}_{\mathcal{T},h}b_\upsilon\big)s^{(k)}_{b_\upsilon,x_{\tilde{h}(\mathcal{T})}}(\tilde{v}_h), \qquad \alpha^{(k)}_{\mathcal{T},h}(\upsilon) = \sum_{\chi_{\mathcal{T}}h=1}\alpha^{(k)}_{\mathcal{T},h}(\upsilon).$$

We denote $\alpha^{(1)}_{\mathcal{T},h}$ and $\alpha^{(1)}_{\mathcal{T}}$ by $\alpha_{\mathcal{T},h}$ and $\alpha_{\mathcal{T}}$, respectively.

Lemma 8.5 *There exist $\delta, C\in C^\infty(\mathcal{M}_{\mathcal{T}};\mathbb{R}^+)$ such that for all $\upsilon\in F^{\emptyset}\mathcal{T}_\delta$,*

$$\big\|\pi^{0,1}_{\upsilon,-}\bar{\partial}u_\upsilon + \tilde{R}_\upsilon\alpha_{\mathcal{T}}(\upsilon)\big\|_2 \le C(b_\upsilon)|\upsilon|\sum_{\chi_{\mathcal{T}}h=1}|\upsilon|_h.$$

Proof: This is immediate from Proposition 7.6, since

$$\big\|s_{\tilde{x}_h(\upsilon)}\big(d\phi_{b_\upsilon,x_{\tilde{h}(\mathcal{T})}}|^{-1}_{\tilde{x}_h(\upsilon)}\tilde{v}_h\big) - s_{x_{\tilde{h}(\mathcal{T})}}(\tilde{v}_h)\big\|_2 \le C(b_\upsilon)\big|\phi_{b_\upsilon,x_{\tilde{h}(\mathcal{T})}}\tilde{x}_h(\upsilon)\big|_{b_\upsilon}|\tilde{v}_h| \le C'(b_\upsilon)|\upsilon|\big|\tilde{v}_h\big|_b;$$
$$\sum_{m\ge2}\big|\mathcal{D}^{(m)}_{\mathcal{T},h}b_\upsilon\big||\tilde{v}_h|^m \le C(b_\upsilon)|\tilde{v}_h|^2,$$

for all $h\in\hat{I}$ with $\chi_{\mathcal{T}}h=1$ and $\upsilon\in F\mathcal{T}_\delta$ with $\delta\in C^\infty(\mathcal{M}_{\mathcal{T}};\mathbb{R}^+)$ sufficiently small.

Lemma 8.6 *There exist $\delta, C\in C^\infty(\mathcal{M}_{\mathcal{T}}(\mu);\mathbb{R}^+)$ such that for all $\upsilon\in F^{\emptyset}\mathcal{T}_\delta$,*

$$\Big\|\psi^{\mu}_{\mathcal{T},t\nu}(\upsilon) - \big(t\bar{\nu}_{ev(b_\upsilon)} + \alpha_{\mathcal{T}}(\upsilon)\big)\Big\|_2 \le C(b_\upsilon)\big(t+|\upsilon|^{\frac{1}{p}}\big)\Big(t+\sum_{\chi_{\mathcal{T}}h=1}|\upsilon|_h\Big),$$

where $\psi^{\mu}_{\mathcal{T},t\nu}$ denotes $\psi^{\mu}_{\mathcal{M}_{\mathcal{T}},t\nu}$.

Proof: By Lemma 7.1 and Theorem 7.2,

$$\big\|\pi^{0,1}_{\upsilon,-}D_\upsilon\xi_{\upsilon,t\nu}\big\|_2 \le C(b_\upsilon)\Big(\sum_{\chi_{\mathcal{T}}h=1}|\upsilon|_h\Big)\|D_\upsilon\xi_{\upsilon,t\nu}\|_{\upsilon,p,1} \le C'(b_\upsilon)\big(t+|\upsilon|^{\frac{1}{p}}\big)\sum_{\chi_{\mathcal{T}}h=1}|\upsilon|_h.$$

Combining this estimate with Lemmas 7.3 and 8.5, we obtain

$$\Big\|\psi_{\mathcal{T},t\nu}(\upsilon) - \big(t\bar{\nu}_{\mathrm{ev}(b_\upsilon)} + \alpha_{\mathcal{T}}(\upsilon)\big)\Big\|_2 \le C(b_\upsilon)\big(t+|\upsilon|^{\frac{1}{p}}\big)\Big(t+\sum_{\chi_{\mathcal{T}}h=1}|\upsilon|_h\Big) \tag{8.7}$$

for all $\upsilon\in F^{\emptyset}\mathcal{T}_\delta$, provided $\delta\in C^\infty(\mathcal{M}_{\mathcal{T}};\mathbb{R}^+)$ is sufficiently small. On the other hand, if $b_\upsilon\in\mathcal{M}_{\mathcal{T}}(\mu)$,

$$\big\|\varphi^{\mu}_{\mathcal{T},t\nu}(\upsilon)\big\|_{b_\upsilon} \le C(b_\upsilon)\big(t+|\upsilon|^{\frac{1}{p}}\big) \Longrightarrow$$
$$\Big\|\big(t\bar{\nu}_{\mathrm{ev}(\phi^{\mu}_{\mathcal{T}}\varphi^{\mu}_{\mathcal{T},t\nu}(\upsilon))}+\alpha_{\mathcal{T}}(\Phi^{\mu}_{\mathcal{T}}\varphi^{\mu}_{\mathcal{T},t\nu}(\upsilon))\big) - \Pi_{b_\upsilon,\phi^{\mu}_{\mathcal{T}}\varphi^{\mu}_{\mathcal{T},t\nu}(\upsilon)}\big(t\bar{\nu}_{\mathrm{ev}(b_\upsilon)}+\alpha_{\mathcal{T}}(\upsilon)\big)\Big\|_2$$
$$\le C(b_\upsilon)\big(t+|\upsilon|^{\frac{1}{p}}\big)\Big(t+\sum_{\chi_{\mathcal{T}}h=1}|\upsilon|_h\Big), \tag{8.8}$$

where $\varphi^{\mu}_{\mathcal{T},t\nu}=\varphi^{\mu}_{\mathcal{M}_{\mathcal{T}},t\nu}$ is the section of Theorem 7.2 for any fixed regularization $(\Phi_{\mathcal{T}}\equiv Id,\Phi^{\mu}_{\mathcal{T}})$ of $\mathcal{M}_{\mathcal{T}}(\mu)$. The claim follows from (8.7) and (8.8).

Our next step is to apply Lemma 5.2 or Corollary 5.6 to the map $\psi^{\mu}_{\mathcal{T},t\nu}$ whenever possible. In terms of notation of Section 5.1, we take

$$F^{+}=\mathcal{O}^{+}=\{0\},\quad F^{-}=F\mathcal{T},\quad \mathcal{O}^{-}=\mathcal{H}^{0,1}_{\Sigma}\otimes \mathrm{ev}^{*}T\mathbb{P}^{n},\quad \tilde{F}^{-}=\bigoplus_{\chi_{\mathcal{T}}h=1}\ \bigotimes_{i\in\hat{I},h\in\bar{D}_i\mathcal{T}} F_i\mathcal{T};$$

$$\rho_h\big([b,v_{\hat{I}}]\big)=\Big[b,\bigotimes_{i\in\hat{I},h\in\bar{D}_i\mathcal{T}} v_i\Big]=\big[b,\tilde{v}_h\big],\quad \alpha^{-}(\rho(v))\equiv\alpha_{\mathcal{T}}(v),$$

where ρ_h denotes the hth component of $\rho\colon F^{-}\longrightarrow\tilde{F}^{-}$. Note that $\alpha^{-}\in\Gamma(\mathcal{M}_{\mathcal{T}};\tilde{F}^{-*}\otimes\mathcal{O}^{-})$ is well-defined. A priori, α^{-} may not have full rank on every fiber over $\mathcal{M}_{\mathcal{T}}(\mu)$. We will call a subset $K\subset\mathcal{M}_{\mathcal{T}}(\mu)$ *$\mathcal{T}$-regular* if α^{-} has full rank over K. From Theorem 7.2, Lemma 5.2, and Corollary 5.6, we then obtain

Corollary 8.7 *Suppose d is a positive integer, $\mathcal{T}=(\Sigma,[N],I;j_{[N]},\underline{d})$ is a simple bubble type, with $d_{\hat{0}}=0$ and $\sum_{i\in I}d_i=d$, and μ is an N-tuple of constraints in general position such that*

$$codim_{\mathbb{C}}\mu=d(n+1)-n(g-1)+N.$$

Let $\nu\in\Gamma(\Sigma\times\mathbb{P}^n;\Lambda^{0,1}\pi^*_{\Sigma}T^*\Sigma\otimes\pi^*_{\mathbb{P}^n}T\mathbb{P}^n)$ be a generic section. If $\iota_h\neq\hat{0}$ for some $h\in\hat{I}$, for every regular compact subset K of $\mathcal{M}_{\mathcal{T}}(\mu)$, there exist a neighborhood U_K of K in $\bar{C}^{\infty}_{(d;[N])}(\Sigma;\mu)$ and $\epsilon_K>0$ such that for any $t\in(0,\epsilon_K)$, $U_K\cap\mathcal{M}_{\Sigma,d,t\nu}(\mu)=\emptyset$. If $\iota_h=\hat{0}$ for all $h\in\hat{I}$, there exists a compact regular subset $K_{\mathcal{T}}$ of $\mathcal{M}_{\mathcal{T}}(\mu)$ with the following property. If K is a compact regular subset of $\mathcal{M}_{\mathcal{T}}(\mu)$ containing $K_{\mathcal{T}}$, there exist a neighborhood U_K of K in $\bar{C}^{\infty}_{(d;[N])}(\Sigma;\mu)$ and $\epsilon_K>0$ such that for all $t\in(0,\epsilon_K)$, the signed cardinality of $U_K\cap\mathcal{M}_{\Sigma,d,t\nu}(\mu)$ equals to the signed number of zeros of the map

$$F\mathcal{T}\big|\mathcal{M}_{\mathcal{T}}(\mu)\longrightarrow\mathcal{H}^{0,1}_{\Sigma}\otimes ev^{*}T\mathbb{P}^n,\quad v\longrightarrow\bar{\nu}_{ev(b_v)}+\alpha_{\mathcal{T}}(v).\tag{8.9}$$

Proof: In either case, by Theorem 7.2, there exist a neighborhood U_K of K in $\bar{C}^{\infty}_{(d;[N])}(\Sigma;\mu)$ and $\delta_K,\epsilon_K>0$ such that for any $t\in(0,\epsilon_K)$, there exists a sign-preserving bijection between $U_K\cap\mathcal{M}_{\Sigma,d,t\nu}(\mu)$ and the zeros of $\psi^{\mu}_{\mathcal{T},t\nu}$ on $F^{\emptyset}\mathcal{T}_{\delta_K}\big|(U_K\cap\mathcal{M}_{\mathcal{T}}(\mu))$, provided $U_K\cap\mathcal{M}_{\mathcal{T}}(\mu)$ is precompact in $\mathcal{M}_{\mathcal{T}}(\mu)$. Furthermore, δ_K can be required to be arbitrarily small. If K is regular, U_K can be chosen so that the closure of $U_K\cap\mathcal{M}_{\mathcal{T}}(\mu)$ in $\mathcal{M}_{\mathcal{T}}(\mu)$ is also regular. Then by Lemma 8.6,

$$\Big\|\psi^{\mu}_{\mathcal{T},t\nu}(v)-\big(t\bar{\nu}_{\mathrm{ev}(b_v)}+\alpha_{\mathcal{T}}(v)\big)\Big\|_2\le C_K\big(t+|v|^{\frac{1}{p}}\big)\big(t+|\alpha_{\mathcal{T}}(v)|\big)\quad\forall v\in F^{\emptyset}\mathcal{T}_{\delta_K}\big|K,$$

where $C_K>0$ depends only on K. Thus, the first claim follows from Lemma 5.2. The second follows from Corollary 5.6, provided that for a generic ν the set of zeros of the map in (8.9) is $\mathcal{T}$-regular and finite; see below.

The affine maps of Corollaries 8.7, 8.14, 8.18, and 8.22 extend over the natural compactifications of the spaces $\mathcal{M}_{\mathcal{T}}(\mu)$ and $\mathcal{S}_{\mathcal{T};k}(\mu)$ described in Section 8.9. Along with counting the zeros of these affine maps in Chapter 9, we also show that the linear part of each of the affine maps is regular in the sense of Definition 5.9. Thus, by Lemma 5.10 these affine maps have a finite numbers of transverse zeros, which must lie over the subspace of the base where the linear part of the affine map has full rank.

8.3 Consequences of the First-Order Estimate for $\psi^{\mu}_{\mathcal{T},t\nu}$

In this subsection, we show that $\mathcal{M}_{\mathcal{T}}(\mu)$ is $\mathcal{T}$-regular for most bubble types $\mathcal{T}$ under consideration, and nearly all of them fall under the first case of Corollary 8.7. We call $\mathcal{T}$ *effective*, if for some generic choice of ν and of the constraints $\mu_1,\ldots,\mu_N$, $\overline{\bigcup_{t<1}\mathcal{M}_{\Sigma,t\nu,d}(\mu)}$ intersects $\bar{\mathcal{M}}_{\mathcal{T}}(\mu)$. If K is a compact subset of $\bar{\mathcal{M}}_{\mathcal{T}}(\mu)$, we call K *effective* if $\overline{\bigcup_{t<1}\mathcal{M}_{\Sigma,t\nu,d}(\mu)}$ intersects K.

Lemma 8.8 *Let $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ be a simple bubble type. If $j_l=\hat{0}$ for some $l\in[N]$ and K is a $\mathcal{T}$-regular subset of $\mathcal{M}_{\mathcal{T}}(\mu)$, then K is not effective.*

Proof: By Corollary 8.7, it is sufficient to show that the map

$$\bar{\nu}+\alpha_{\mathcal{T}}\colon F\mathcal{T}\longrightarrow \mathcal{H}^{0,1}_{\Sigma}\otimes \mathrm{ev}^*T\mathbb{P}^n$$

has no zeros for a generic ν. For a generic ν, the zero set of this section is zero-dimensional. However, if $j_l=\hat{0}$ for some $l\in[N]$, we can move $y_l\in\Sigma$ freely, without changing the value of $\bar{\nu}+\alpha_{\mathcal{T}}$. Thus, if the zero-set of the section is nonempty, it must be at least one-dimensional, which is not the case for a generic ν.

Lemma 8.9 *Let $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ be a bubble type with $d_{\hat{0}}=0$. If*

$$ng-|\hat{I}|-\Big(|H_{\hat{0}}\mathcal{T}|+|M_{\hat{0}}\mathcal{T}|+\sum_{i\in\hat{I},d_i=0}\big(|H_i\mathcal{T}|+|M_i\mathcal{T}|-2\big)\Big)\le n-\big|\chi(\mathcal{T})\big|,$$

$\mathcal{M}_{\mathcal{T}}(\mu)$ is $\mathcal{T}$-regular. Furthermore, if the number on the left-hand side above is negative, then $\mathcal{M}_{\mathcal{T}}(\mu)$ is empty.

Proof: (1) The dimension of $\mathcal{M}(\mu)$ is given by

$$\dim\mathcal{M}_{\mathcal{T}}(\mu)=\big(d(n+1)+n+N-|\hat{I}|\big)-\big(d(n+1)-n(g-1)+N\big)=ng-|\hat{I}|.$$

However, given $b=\big(\Sigma,[N],I;x,(j,y),u\big)\in\mathcal{M}_{\mathcal{T}}(\mu)$, we are free to vary x_h if $\iota_h=\hat{0}$ (i.e. $x_h\in\Sigma$) and y_l if $j_l=\hat{0}$. Similarly, if $i\in\hat{I}$, $d_i=0$, and $|H_i\mathcal{T}|+|M_i\mathcal{T}|>2$, we can vary $|H_i\mathcal{T}|+|M_i\mathcal{T}|-2$ marked and singular points on $\Sigma_{b,i}$. Thus, the space $\mathcal{M}_{\mathcal{T}}(\mu)$ must have dimension at least

$$d_{\min}(\mathcal{T})\equiv|H_{\hat{0}}\mathcal{T}|+|M_{\hat{0}}\mathcal{T}|+\sum_{i\in\hat{I},d_i=0}\big(|H_i\mathcal{T}|+|M_i\mathcal{T}|-2\big),$$

if $\mathcal{M}_{\mathcal{T}}(\mu)$ is nonempty. Therefore, we can assume $\big|\chi(\mathcal{T})\big|\le n$.
(2) Let $h_1,\ldots,h_{|\chi(\mathcal{T})|}$ be the elements of $\chi(\mathcal{T})$. The section $s\in\Gamma(\Sigma;T^*\Sigma\otimes\mathcal{H}^{0,1}_{\Sigma})$ does not vanish; see [GH, p246]. Thus, the section α^- defined above has rank at least k if the section

$$\bar{\mathcal{D}}_{\mathcal{T};k}\in\Gamma\Big(\mathcal{M}_{\mathcal{T}}(\mu);\Big(\bigoplus_{m\le k}L^*_{h_m}\mathcal{T}\Big)\otimes\mathrm{ev}^*T\mathbb{P}^n\Big),\quad \bar{\mathcal{D}}_{\mathcal{T};k}\big([b,c_{\{h_m:m\le k\}}]\big)=\sum_{m\le k}\mathcal{D}_{\mathcal{T},h_m}\big([b,c_{h_m}]\big),$$

has rank k. We prove inductively that under the assumptions of the lemma this is the case for all $k\le|\chi(\mathcal{T})|$. If $k=0$, there is nothing to prove. So we can assume that $k>0$ and that the statement has been shown to be true for $k-1$. The $k-1$ statement shows that the image of $\bar{\mathcal{D}}_{\mathcal{T};k-1}$ is a rank $k-1$ subbundle of $\mathrm{ev}^*T\mathbb{P}^n$. Let $\pi_{k-1}^{\perp}$ denote the orthogonal projection onto the orthogonal complement of this rank $(k-1)$-subbundle in $\mathrm{ev}^*T\mathbb{P}^n$ with respect to the standard metric in $\mathbb{P}^n$. We need to show that the section

$$\pi_{k-1}^{\perp}\circ\mathcal{D}_{\mathcal{T};k}\in\Gamma\big(\mathcal{M}_{\mathcal{T}}(\mu);L_{h_k}^*\mathcal{T}\otimes\pi_{k-1}^{\perp}(\mathrm{ev}_{\mathcal{T}}^*T\mathbb{P}^n)\big)$$

does not vanish. By Corollary C.3, $\pi_{k-1}^{\perp}\circ\mathcal{D}_{\mathcal{T};k}$ is transverse to zero for a generic choice of the constraints $\mu_1,\ldots,\mu_N$. Its zero set must have dimension at least $d_{\min}(\mathcal{T})$, if nonempty, since the movements of points described in (1) do not effect $\pi_{k-1}^{\perp}\circ\mathcal{D}_{\mathcal{T};k}$. Thus, $\pi_{k-1}^{\perp}\circ\mathcal{D}_{\mathcal{T},k}$ does not vanish if

$$\dim(\mathcal{M}_{\mathcal{T}}(\mu))-d_{\min}(\mathcal{T})<n-(k-1).$$

By the assumption of the lemma, this is the case as long as $k\le|\chi(\mathcal{T})|$.

Corollary 8.10 *Let $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ be an effective bubble type with $d_{\hat{0}}=0$. If $g=2$ and $n=2$, then either*
(1) $|\hat{I}|=1$ and $j_l\neq\hat{0}$ for all $l\in[N]$, or
(2) $|\hat{I}|=2$, $H_{\hat{0}}\mathcal{T}=\hat{I}$, and $j_l\neq\hat{0}$ for all $l\in[N]$.
Furthermore, in Case (2) $\alpha_{\mathcal{T}}$ has full rank over all of $\mathcal{M}_{\mathcal{T}}(\mu)$.

Proof: (1) By Lemma 8.9, $\mathcal{M}_{\mathcal{T}}(\mu)$ is empty, unless $ng-|\hat{I}|\ge1$, i.e. $|\hat{I}|\le3$. Suppose $|\hat{I}|=3$. If $|H_{\hat{0}}\mathcal{T}|\ge2$,

$$ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|\le4-3-2<0,$$

and thus $\mathcal{M}_{\mathcal{T}}(\mu)$ is empty by Lemma 8.9. If $|H_{\hat{0}}\mathcal{T}|=1$, $\chi_{\mathcal{T}}h\neq1$ for some $h\in\hat{I}$, and thus

$$n-|\chi(\mathcal{T})|\ge2-(|\hat{I}|-1)=0=ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|,$$

and by Lemma 8.9 every compact subset of $\mathcal{M}_{\mathcal{T}}(\mu)$ is $\mathcal{T}$-semiregular. The space $\mathcal{M}_{\mathcal{T}}(\mu)$ is compact, since by the above $\mathcal{M}_{\mathcal{T}'}(\mu)=\emptyset$ if $\mathcal{T}'<\mathcal{T}$. Corollary 8.7 then implies that $\mathcal{M}_{\mathcal{T}}(\mu)$ is not effective, i.e. $\mathcal{T}$ is not effective.
(2) Suppose $|\hat{I}|=2$. If $|H_{\hat{0}}\mathcal{T}|=2$ and $j_l=\hat{0}$ for some $l\in[N]$,

$$ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|-|M_{\hat{0}}\mathcal{T}|\le4-2-2-1<0,$$

and thus $\mathcal{M}_{\mathcal{T}}(\mu)$ is empty by Lemma 8.9. If $|H_{\hat{0}}\mathcal{T}|=1$, then $\chi_{\mathcal{T}}h\neq1$ for some $h\in\hat{I}$, and thus

$$n-|\chi(\mathcal{T})|=2-1=ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|,$$

and it follows from Lemma 8.9 and Corollary 8.7, that every compact subset of $\mathcal{M}_{\mathcal{T}}(\mu)$ is not effective. Furthermore, $\bar{\mathcal{M}}_{\mathcal{T}}(\mu)-\mathcal{M}_{\mathcal{T}}(\mu)$ consists of three-bubble strata, all of which are not effective by (1) above. Thus, $\mathcal{T}$ is not effective, unless $\iota_h=\hat{0}$ for all $h\in\hat{I}$ and $j_l\neq\hat{0}$ for all $l\in[N]$. The second statement about the $|\hat{I}|=2$ case is immediate from Lemma 8.9.
(3) Finally, suppose $|\hat{I}|=1$ and $j_l=\hat{0}$ for some $l\in[N]$. Then,

$$n-|\chi(\mathcal{T})|=2-1\ge ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|-|M_{\hat{0}}\mathcal{T}|,$$

and thus by Lemmas 8.8 and 8.9, every compact subset of $\mathcal{M}_{\mathcal{T}}(\mu)$ is not effective. Furthermore, $\bar{\mathcal{M}}_{\mathcal{T}}(\mu) - \mathcal{M}_{\mathcal{T}}(\mu)$ consists of two- and three-bubble strata that by (1) and (2) are not effective. It follows that $\mathcal{T}$ is not effective.

Corollary 8.11 *Let $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ be an effective bubble type with $d_{\hat{0}}=0$. If $g=2$ and $n=3$, then either*
(1) $|\hat{I}|=1$, or
(2a) $|\hat{I}|=2$, $H_{\hat{0}}\mathcal{T}=\hat{I}$, and $j_l\neq\hat{0}$ for all $l\in[N]$, or
(2b) $|\hat{I}|=2$, $H_{\hat{0}}\mathcal{T}\neq\hat{I}$, and $j_l\neq\hat{0}$ for all $l\in[N]$, or
(3a) $|\hat{I}|=3$, $H_{\hat{0}}\mathcal{T}=\hat{I}$, and $j_l\neq\hat{0}$ for all $l\in[N]$, or
(3b) $|\hat{I}|=3$, $\iota_h=\hat{1}$ for some $\hat{1}\in\hat{I}$ and all $h\in\hat{I}-\{\hat{1}\}$, $d_{\hat{1}}=0$, and $j_l\neq\hat{0},\hat{1}$ for all $l\in[N]$.
Furthermore, in Case (3a) $\alpha_{\mathcal{T}}$ has full rank on all of $\mathcal{M}_{\mathcal{T}}(\mu)$.

Proof: (1) Similarly to the proof of Corollary 8.10, $\mathcal{M}_{\mathcal{T}}(\mu)$ is empty unless $|\hat{I}|\leq 5$. If $|\hat{I}|=5$, $\mathcal{M}_{\mathcal{T}}(\mu)$ is compact and $|H_{\hat{0}}\mathcal{T}|=1$. Let $\hat{1}\in\hat{I}$ be such that $\iota_{\hat{1}}=\hat{0}$. If $d_{\hat{1}}>0$,

$$n-\big|\chi(\mathcal{T})\big| = 3-1 > 0 = ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|,$$

and $\mathcal{M}_{\mathcal{T}}(\mu)$ is not effective by Lemma 8.9 and Corollary 8.7. Suppose $d_{\hat{1}}=0$. Then, $|H_{\hat{1}}\mathcal{T}|=2$; otherwise $\mathcal{M}_{\mathcal{T}}(\mu)$ is empty by Lemma 8.9. It follows that

$$n-\big|\chi(\mathcal{T})\big| \geq 3-(|\hat{I}|-2) = 0 = ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|.$$

Thus, by Lemma 8.9 and Corollary 8.7, $\mathcal{T}$ is not effective.
(2) Suppose $|\hat{I}|=4$. If $|H_{\hat{0}}\mathcal{T}|\geq 3$, $\mathcal{M}_{\mathcal{T}}(\mu)$ is empty by Lemma 8.9. Let $\hat{1}\in\hat{I}$ be as above. If $|H_{\hat{0}}\mathcal{T}|=2$,

$$n-\big|\chi(\mathcal{T})\big| \geq 3-(|\hat{I}|-1) = 0 = ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|.$$

If $|H_{\hat{0}}\mathcal{T}|=1$ and $d_{\hat{1}}>0$,

$$n-\big|\chi(\mathcal{T})\big| = 3-1 > 1 = ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|.$$

If $|H_{\hat{0}}\mathcal{T}|=1$, $d_{\hat{1}}=0$, and $|H_{\hat{1}}\mathcal{T}|=3$,

$$n-\big|\chi(\mathcal{T})\big| \geq 3-(|\hat{I}|-1) = 0 = ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|-(|H_{\hat{1}}\mathcal{T}|-2).$$

Finally, if $|H_{\hat{0}}\mathcal{T}|=1$, $d_{\hat{1}}=0$, and $|H_{\hat{1}}\mathcal{T}|=2$,

$$n-\big|\chi(\mathcal{T})\big| \geq 3-(|\hat{I}|-2) = 1 = ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|.$$

Thus, by Corollary 8.7 and Lemma 8.9, in all four cases, no compact subset of $\mathcal{M}_{\mathcal{T}}(\mu)$ is effective. Since $\bar{\mathcal{M}}_{\mathcal{T}}(\mu)-\mathcal{M}_{\mathcal{T}}(\mu)$ consists of five-bubble strata that are not effective by (1) above, it follows that $\mathcal{T}$ is not effective.
(3) Suppose $|\hat{I}|=3$. If $H_{\hat{0}}\mathcal{T}=\hat{I}$ and $j_l=\hat{0}$ for some $l\in[N]$,

$$ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|-|M_{\hat{0}}\mathcal{T}| = 6-3-3-1 < 0,$$

and thus $\mathcal{M}_{\mathcal{T}}(\mu)$ is empty by Lemma 8.9. If $|H_{\hat{0}}\mathcal{T}|=2$,

$$n-\big|\chi(\mathcal{T})\big| \geq 3-(|\hat{I}|-1) = 1 \geq ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|.$$

If $|H_{\hat{0}}\mathcal{T}|=1$ and $d_{\hat{1}}>0$,

$$n - |\chi(\mathcal{T})| = 2 = ng - |\hat{I}| - |H_{\hat{0}}\mathcal{T}|.$$

If $|H_{\hat{0}}\mathcal{T}|=1$ and $|H_{\hat{1}}\mathcal{T}|=1$,

$$n - |\chi(\mathcal{T})| = 2 = ng - |\hat{I}| - |H_{\hat{0}}\mathcal{T}|.$$

Thus, in all three cases, by Lemma 8.9 and Corollary 8.7, no compact subset of $\mathcal{M}_{\mathcal{T}}(\mu)$ is effective. Since $\bar{\mathcal{M}}_{\mathcal{T}}(\mu)-\mathcal{M}_{\mathcal{T}}(\mu)$ consists of four- and five-bubble strata that are not effective by (1) and (2) above, $\mathcal{T}$ is not effective in these three cases. On the other hand, if $|H_{\hat{0}}\mathcal{T}|=2$, $j_l=\hat{0}$ or $j_l=\hat{1}$ for some $l\in[N]$, and $d_{\hat{1}}=0$,

$$n - |\chi(\mathcal{T})| \geq 1 \geq ng - |\hat{I}| - |H_{\hat{0}}\mathcal{T}| - |M_{\hat{0}}\mathcal{T}| - \big(|H_{\hat{1}}\mathcal{T}| + |M_{\hat{1}}\mathcal{T}| - 2\big).$$

Thus, by Lemmas 8.8 and 8.9, no compact subset of $\mathcal{M}_{\mathcal{T}}(\mu)$ is effective. Similarly to the above, it follows that $\mathcal{T}$ is not effective.

(4) Suppose $|\hat{I}|=2$ and $j_l=\hat{0}$ for some $l\in[N]$. If $|H_{\hat{0}}\mathcal{T}|=2$,

$$n - |\chi(\mathcal{T})| \geq 1 \geq ng - |\hat{I}| - |H_{\hat{0}}\mathcal{T}| - |M_{\hat{0}}\mathcal{T}|.$$

If $|H_{\hat{0}}\mathcal{T}|=1$,

$$n - |\chi(\mathcal{T})| = 2 \geq ng - |\hat{I}| - |H_{\hat{0}}\mathcal{T}| - |M_{\hat{0}}\mathcal{T}|.$$

Thus, in either case, no compact subset of $\mathcal{M}_{\mathcal{T}}(\mu)$ is effective by Lemmas 8.8 and 8.9. Furthermore,

$$\bar{\mathcal{M}}_{\mathcal{T}}(\mu) - \mathcal{M}_{\mathcal{T}}(\mu) = \bigcup_{\mathcal{T}'<\mathcal{T}} \mathcal{M}_{\mathcal{T}'}(\mu),$$

where $\mathcal{T}'$ is either a four- or five-bubble strata, or a three bubble-strata $\mathcal{T}' = (\Sigma, [N], I'; j', \underline{d}')$ such that either $|H_{\hat{0}}\mathcal{T}|=1$, or $d'_{\hat{1}'}=0$ and $j'_l=\hat{0}$ or $\hat{1}'$. By (1)-(3) above, none of such bubble types is effective, and thus $\mathcal{T}$ is not effective.

8.4 Second-Order Estimate for $\psi^{\mu}_{\mathcal{T},t\nu}$, Case 1

We now refine the first-order estimate for $\psi^{\mu}_{\mathcal{T},t\nu}$ along the sets on which the section α^- defined above does not have full rank. These are precisely the sets on which the section $\bar{\mathcal{D}}_{\mathcal{T},|\chi(\mathcal{T})|}$ defined in the proof of Lemma 8.9 does not have full rank.

One set on which $\bar{\mathcal{D}}_{\mathcal{T},|\chi(\mathcal{T})|}$ fails to have full rank is the zero set of $\mathcal{D}_{\mathcal{T},h_1}$. If $n=2,3$, by Lemma 8.9, $\mathcal{D}_{\mathcal{T},h_1}$ does not vanish unless h_1 is the only element of the set $\chi(\mathcal{T})$. Thus, we assume that this is the case. We denote the zero-locus of $\mathcal{D}_{\mathcal{T},h_1}$ by $\mathcal{S}_{\mathcal{T},1}\subset\mathcal{M}_{\mathcal{T}}$, which will be abbreviated as $\mathcal{S}$ in this subsection. Since $\mathcal{D}_{\mathcal{T},h_1}$ is transversal to zero by Corollary C.3, $\mathcal{S}$ is a complex submanifold of $\mathcal{M}_{\mathcal{T}}$ of codimension n. Its normal bundle $\mathcal{NS}$ in $\mathcal{M}_{\mathcal{T}}$ is the restriction of $L^*_{h_1}\mathcal{T}\otimes\mathrm{ev}^*T\mathbb{P}^n$ to $\mathcal{S}_{\mathcal{T},1}$. Let $(\Phi_{\mathcal{S}},\Phi^{\mu}_{\mathcal{S}})$ be a regularization of $\mathcal{S}_{\mathcal{T},1}(\mu)\equiv\mathcal{S}\cap\mathcal{M}_{\mathcal{T}}(\mu)$. This regularization can be chosen so that

$$\mathcal{D}_{\mathcal{T},h_1}\tilde{\phi}_{\mathcal{S}}(b,X) = \Pi_{b,\tilde{\phi}_{\mathcal{S}}(b,X)}X \quad \forall\ (b,X)\in\mathcal{N}\tilde{\mathcal{S}} = \mathrm{ev}^*T\mathbb{P}^n, \tag{8.10}$$

where $\tilde{\phi}_{\mathcal{S}}$ is the lift of $\phi_{\mathcal{S}}$ to the preimage $\tilde{\mathcal{S}}$ of $\mathcal{S}$ and its normal bundle $\mathcal{N}\tilde{\mathcal{S}}$ in $\mathcal{M}_{\mathcal{T}}^{(0)}$; see Section 3.8. The bundle $\mathcal{N}\mathcal{S}$ carries a natural norm induced by the $g_{\mathbb{P}^n,\mathrm{ev}}$-metric on $\mathbb{P}^n$. Denote by $F\mathcal{S}$ and $F^{\emptyset}\mathcal{S}$ the bundles described in Section 3.8 corresponding to the submanifold $\mathcal{S}_{\mathcal{T},1}$. Let $\hat{1}\in H_{\hat{0}}\mathcal{T}$ be the unique element such that $h_1\in\bar{D}_{\hat{1}}\mathcal{T}$. If $[b;X,v]\in F\mathcal{S}=\mathcal{N}\mathcal{S}\oplus F\mathcal{T}$, put

$$^{(2)}\alpha_{\mathcal{T};1}(X,v)=X(b_v)s_{x_{\hat{1}}}\tilde{v}_{h_1}+\alpha^{(2)}_{\mathcal{T},h_1}(v).$$

Lemma 8.12 *There exist $\delta, C\in C^{\infty}(\mathcal{S};\mathbb{R}^+)$ such that for all $\varpi=[(b;X,v)]\in F^{\emptyset}\mathcal{S}_{\delta}$,*

$$\Big\|\pi^{0,1}_{\Phi_{\mathcal{S}}(\varpi),-}\bar{\partial}u_{\Phi_{\mathcal{S}}(\varpi)}+\tilde{R}_{\Phi_{\mathcal{S}}(\varpi)}\Pi_{b,\phi_{\mathcal{S}}(X)}{}^{(2)}\alpha_{\mathcal{T};1}(X,v)\Big\|_2\le C(b)|v|\big(|v|^2_{h_1}+|X||v|_{h_1}\big).$$

Proof: The proof is almost identical to the proof of Lemma 8.5. The only difference is that we use two terms of the power series of Proposition 7.6. We then make use of the assumption (8.10) on $\phi_{\mathcal{S}}$ and smooth dependence of $\mathcal{D}^{(2)}_{\mathcal{T},h_1}$ on X.

Lemma 8.13 *There exist $\delta, C\in C^{\infty}(\mathcal{S}_{\mathcal{T},1}(\mu);\mathbb{R}^+)$ such that for all $\varpi=[(b;X,v)]\in F^{\emptyset}\mathcal{S}_{\delta}$,*

$$\Big\|\psi^{\mu}_{\mathcal{S},t\nu}(\varpi)-\big(t\bar{\nu}_{ev(b)}+{}^{(2)}\alpha_{\mathcal{T};1}(X,v)\big)\Big\|_2\le C(b)\big(t+|v|^{\frac{1}{p}}\big)\big(t+|v|^2_{h_1}+|X||v|_{h_1}\big).$$

Proof: This claim follows from Lemmas 7.3 and 8.12 in a way analogous to the proof of Lemma 8.6. The only difference is that we need to improve the estimate on $\pi^{0,1}_{v,-}D_v\xi_{v,t\nu}$ made in the proof of Lemma 8.6. Let $\{\psi_j\}$ be an orthonormal basis for $\mathcal{H}^{0,1}_{\Sigma}$, such that $\psi_1\in\mathcal{H}^{+}_{\Sigma}\big(\tilde{x}_{h_1}(v)\big)$, and $\{X_i\}$ an orthonormal basis for $T_{\mathrm{ev}(\phi_{\mathcal{S}}(X))}\mathbb{P}^n$. By Theorem 7.2, with $v(X)=\Phi_{\mathcal{S}}(\varpi)$,

$$\begin{aligned}\Big|\big\langle\big\langle\pi^{0,1}_{v(X),-}D_{v(X)}\xi_{v(X),t\nu},R_{v(X)}X_i\psi_j\big\rangle\big\rangle\Big|&=\Big|\big\langle\big\langle\xi_{v(X),t\nu},D^*_{v(X)}R_{v(X)}X_i\psi_j\big\rangle\big\rangle\Big|\\&\le C(b)(t+|v|^{\frac{1}{p}})\|D^*_{v(X)}R_{v(X)}X_j\psi_j\|_{C^0}.\end{aligned}\tag{8.11}$$

Since $\xi\in\tilde{\Gamma}_+(v)$, by construction in Section 8.1,

$$\big\langle\big\langle\xi_{v(X),t\nu},D^*_{v(X)}R_{v(X)}X_i\psi_1\big\rangle\big\rangle=0.\tag{8.12}$$

On the other hand, since $\psi_2|_{\tilde{x}_{h_1}(v)}=0$ and $\|\nabla\psi_2\|_{g_{\phi_{\mathcal{S}}(X),\hat{0}},C^0}\le C(b)$, by equation (7.5)

$$\|D^*_{v(X)}R_{v(X)}X_i\psi_2\|_{C^0(\tilde{A}_{v(X),h_1})}\le C(b)|v|^2_{h_1},\tag{8.13}$$

where $\tilde{A}_{v(X),h_1}$ is the annulus defined in Lemma 7.1. By equations (8.11)-(8.13),

$$\Big|\pi^{0,1}_{v(X),-}D_v\xi_{v(X),t\nu}\Big|\le C(b)(t+|v|^{\frac{1}{p}})|v|^2_{h_1}.$$

The next step is to apply Lemma 5.2 or Corollary 5.6 whenever possible. Let

$$F^+=\mathrm{ev}^*T\mathbb{P}^n\otimes\bigotimes_{i\in\hat{I},h_1\in D_i\mathcal{T}}F_i\mathcal{T},\quad F^-=F\mathcal{T},\quad\tilde{F}^-=\Big(\bigotimes_{i\in\hat{I},h_1\in\bar{D}_i\mathcal{T}}F_i\mathcal{T}\Big)^{\otimes 2},\quad\mathcal{O}^{\pm}=\mathcal{H}^{\pm}_{\Sigma}\otimes\mathrm{ev}^*T\mathbb{P}^n;$$

$$\alpha^+\big([X,v]\big)=Xs_{x_{\hat{1}}}\tilde{v}_{h_1},\quad\rho\big([b,v_{\hat{I}}]\big)=\big[b,\tilde{v}_{h_1}\otimes\tilde{v}_{h_1}\big],\quad\alpha^-\big(\rho(v)\big)\equiv\pi^-_{x_{\hat{1}}(b_v)}\alpha^{(2)}_{\mathcal{T}}(v).$$

Note that $\alpha^+ \in \Gamma(\mathcal{S}; F^{+*} \otimes \mathcal{O}^+)$, since $\pi^- \circ X s = 0$. Since the map $(X, v) \longrightarrow (X \otimes \tilde{v}_{h_1}, v)$ is injective on $F^{\emptyset}\mathcal{T}$, we can view $\psi^{\mu}_{\mathcal{S},t\nu}$ as a map on an open subset of $F^- \oplus F^+$. Analogously to the first-order case of Section 8.2, subset $K \subset \mathcal{S}_{\mathcal{T},1}(\mu)$ will be called *second-order regular* if α^- has full rank over K.

Corollary 8.14 *Suppose d is a positive integer, $\mathcal{T} = (\Sigma, [N], I; j, \underline{d})$ is a simple bubble type, with $d_{\hat{0}} = 0$ and $\sum_{i \in I} d_i = d$, and μ is an N-tuple of constraints in general position such that*

$$codim_{\mathbb{C}} \mu = d(n+1) - n(g-1) + N.$$

Let $\nu \in \Gamma(\Sigma \times \mathbb{P}^n; \Lambda^{0,1} \pi_\Sigma^* T^* \Sigma \otimes \pi_{\mathbb{P}^n}^* T\mathbb{P}^n)$ be a generic section. If $|\hat{I}| > 1$, for every second-order regular compact subset K of $\mathcal{S}_{\mathcal{T},1}(\mu)$, there exist a neighborhood U_K of K in $\bar{C}^{\infty}_{(d;[N])}(\Sigma; \mu)$ and $\epsilon_K > 0$ such that for any $t \in (0, \epsilon_K)$, $U_K \cap \mathcal{M}_{\Sigma,d,t\nu}(\mu) = \emptyset$. If $|\hat{I}| = 1$, there exists a compact regular subset $K_{\mathcal{T},1}$ of $\mathcal{S}_{\mathcal{T},1}(\mu)$ with the following property. If K is a compact subset of $\mathcal{S}_{\mathcal{T},1}(\mu)$ containing $K_{\mathcal{T},1}$, there exist a neighborhood U_K of K in $\bar{C}^{\infty}_{(d;[N])}(\Sigma; \mu)$ and $\epsilon_K > 0$ such that for all $t \in (0, \epsilon_K)$, the signed cardinality of $U_K \cap \mathcal{M}_{\Sigma,d,t\nu}(\mu)$ equals to twice the signed number of zeros of the map

$$T\Sigma^{\otimes 2} \otimes L_{\hat{1}} \mathcal{T}^{\otimes 2} \big| \mathcal{S}_{\mathcal{T},1}(\mu) \longrightarrow \mathcal{H}_\Sigma^- \otimes ev^* T\mathbb{P}^n, \quad [b, v] \longrightarrow \bar{\nu}_b^- + \alpha^{(2,-)}\big([b, v]\big). \tag{8.14}$$

Proof: In either case, by Theorem 7.2, there exist a neighborhood U_K of K in $\bar{C}^{\infty}_{(d;[N])}(\Sigma; \mu)$ and $\delta_K, \epsilon_K > 0$ such that for any $t \in (0, \epsilon_K)$, there exists a sign-preserving bijection between $U_K \cap \mathcal{M}_{\Sigma,d,t\nu}(\mu)$ and the zeros of $\psi^{\mu}_{\mathcal{S},t\nu}$ on $F^{\emptyset}\mathcal{S}_{\delta_K} \big| \big(U_K \cap \mathcal{S}_{\mathcal{T},1}(\mu)\big)$, provided $U_K \cap \mathcal{S}_{\mathcal{T},1}(\mu)$ is precompact in $\mathcal{S}_{\mathcal{T},1}(\mu)$. If K is second-order regular, U_K can be chosen so that the closure of $U_K \cap \mathcal{S}_{\mathcal{T},1}(\mu)$ in $\mathcal{S}_{\mathcal{T},1}(\mu)$ is also second-order regular. Since K is regular and α^+ is injective on all fibers,

$$|v|_{h_1}^2 = |\phi(v)| \le C_K \big|\alpha^-(\phi(v))\big| \Longrightarrow |v|_{h_1}^2 + |X||v|_{h_1} !le C'_K \big|^{(2)}\alpha_{\mathcal{T};1}(X, v)\big| \quad \forall (X, v) \in F^{\emptyset}\mathcal{S}_{\delta_K} \big| K,$$

where $C_K, C'_K > 0$ depend only on K. Thus, by Lemma 8.13,

$$\Big\|\psi^{\mu}_{\mathcal{S},t\nu}(\varpi) - \big(t\bar{\nu}_{\mathrm{ev}(b_\varpi)} + {}^{(2)}\alpha_{\mathcal{T};1}(\varpi)\big)\Big\|_2 \le C_K\big(t + |\varpi|^{\frac{1}{p}}\big)\big(t + \big|^{(2)}\alpha_{\mathcal{T};1}(\varpi)\big|\big) \quad \forall \varpi \in F^{\emptyset}\mathcal{S}_{\delta_K} \big| K,$$

where $C_K > 0$ depends only on K. The first claim now follows from Lemma 5.2. The second follows from Corollary 5.6, provided that for a generic ν the set of zeros of the map in (8.14) is second-order regular and finite; see the last paragraph of Section 8.2.

8.5 Third-Order Estimate for $\psi^{\mu}_{\mathcal{T},t\nu}$, Case 1

We continue with the case of Section 8.4. Then

$$\alpha^-([b, \tilde{v}_{h_1}]) = (\mathcal{D}^{(2)}_{\mathcal{T},h_1} b) s^{(2,-)}_{b,x_{\hat{1}}}(\tilde{v}_{h_1}).$$

By Corollary C.3, for a generic choice of the constraints $\mu_1, \ldots, \mu_N$, $\mathcal{D}^{(2)}_{\mathcal{T},h_1}$ is transversal to zero along $\mathcal{S}_{\mathcal{T},1}(\mu)$ if $d_{h_1} \ge 2$. Since the zero set of $\mathcal{D}^{(2)}_{\mathcal{T},h_1}$ must have dimension at least $d_{\min}(\mathcal{T}) \ge 1$ by the same argument as in the proof of Lemma 8.9, $\mathcal{D}^{(2)}_{\mathcal{T},h_1}$ does not vanish along $\mathcal{S}_{\mathcal{T},1}(\mu)$ if $d_{h_1} \ge 2$. On the other hand, if $d_{h_1} = 1$, $\mathcal{S}_{\mathcal{T},1} = \emptyset$, since the differential of any degree-one holomorphic map from S^2 to $\mathbb{P}^n$ is nowhere zero. In fact, $\mathcal{S}_{\mathcal{T},1}(\mu)=\emptyset$ even for $d_{h_1}=2$, since the image of any degree-two map with a somewhere vanishing differential is a line, and no line intersects $\mu_1,\ldots,\mu_N$ if $n=2,3$. Thus, we can assume $d_{h_1}\geq 3$. It follows that the only way the above homomorphism α^- can fail to have full rank on $\tilde{F}^-$ is if $s^{(2,-)}_{b,x_{\hat{1}}}=0$. While $s^{(2)}_{b,x_{\hat{1}}}$ depends on the choice of the metric $g_{b,\hat{0}}$ on Σ, the section

$$s^{(2)} \equiv s^{(2,-)}_b \in \Gamma\big(\Sigma; T^*\Sigma^{\otimes 2}\otimes \mathcal{H}^-_{\Sigma}\big)$$

is independent of the metric and is globally defined on Σ. This can be seen by a direct computation. It has transverse zeros at the six branch points of the double cover $\Sigma\longrightarrow\mathbb{P}^1$ induced by s; see [GH, p246]. Denote by $z_1,\ldots,z_6$ these six points. Then the set on which α^- fails to have full rank is $\bigcup_{m\in[6]}\mathcal{S}^{(m)}_{\mathcal{T},1}(\mu)$, where

$$\mathcal{S}^{(m)}_{\mathcal{T},1} = \{b\in\mathcal{S}_{\mathcal{T},1}: x_{\hat{1}}(b)=z_m\}, \quad \mathcal{S}^{(m)}_{\mathcal{T},1}(\mu) = \mathcal{S}^{(m)}_{\mathcal{T},1}\cap\mathcal{M}_{\mathcal{T}}(\mu).$$

The sets $\mathcal{S}^{(m)}_{\mathcal{T},1}$ are obviously disjoint.

Since the normal bundle of $\mathcal{S}^{(m)}_{\mathcal{T},1}$ in $\mathcal{S}_{\mathcal{T},1}$ is $T_{z_m}\Sigma$, the normal bundle $\mathcal{NS}$ of $\mathcal{S}^{(m)}_{\mathcal{T},1}$ in $\mathcal{M}_{\mathcal{T}}(\mu)$ is $T_{z_m}\Sigma\oplus\mathcal{NS}_1$, where $\mathcal{NS}_1$ is the normal bundle of $\mathcal{S}_{\mathcal{T},1}$ in $\mathcal{M}_{\mathcal{T}}(\mu)$, as described in the previous section. Let $(\Phi_{\mathcal{S}},\Phi^{\mu}_{\mathcal{S}})$ be a regularization of $\mathcal{S}^{(m)}_{\mathcal{T},1}(\mu)$ induced by the regularization of $\mathcal{S}_{\mathcal{T},1}(\mu)$ described in section 8.4. In particular,

$$\mathcal{D}_{\mathcal{T},h_1}\tilde{\phi}_{\mathcal{S}}(b,w,X) = \Pi_{b,\tilde{\phi}_{\mathcal{S}}(b,w,X)}X \quad \forall (b,w,X)\in T_{z_m}\Sigma\oplus\mathcal{N}\tilde{\mathcal{S}}_1 = T_{z_m}\Sigma\oplus\mathrm{ev}^*T\mathbb{P}^n, \tag{8.15}$$

where $\tilde{\phi}_{\mathcal{S}}$ is the lift of $\phi_{\mathcal{S}}$ to $\mathcal{M}^{(0)}_{\mathcal{T}}$. We can also assume that $\Phi^{\mu}_{\mathcal{S}}$ is given by the $g_{\mathbb{P}^n,b}$-parallel transport on $\mathcal{N}_b\mathcal{S}_1$. The bundle $\mathcal{NS}$ carries a natural norm induced by the $g_{\mathbb{P}^n,\mathrm{ev}}$-metric on $\mathbb{P}^n$ and $g_{\cdot,\hat{0}}$-metric on Σ. Denote by $F\mathcal{S}$ and $F^{\emptyset}\mathcal{S}$ the bundles described in Section 3.8 corresponding to the submanifold $\mathcal{S}^{(m)}_{\mathcal{T},1}$. If $(b,w,X,v)\in F^{\emptyset}\mathcal{S}$ is sufficiently small, let

$$\tilde{x}_{\hat{1}}(w,v) = \tilde{x}_{\hat{1}}\big(\phi_{\mathcal{S}}(w,X,v)\big) = \tilde{x}_{\hat{1}}\big(\phi_{\mathcal{S}}(w,0,v)\big)\in\Sigma.$$

We identify a small neighborhood of z_m in Σ with a neighborhood of 0 in $T_{z_m}\Sigma$ via the $g_{b,\hat{0}}$-exponential map. Put

$$\tilde{\alpha}(w,X,v) = (Xb)s_{\tilde{x}_{\hat{1}}(w,v)}(\tilde{v}_{h_1}) + \Pi^{-1}_{b,\phi_{\mathcal{S}}(b,X)}\big(\mathcal{D}^{(2)}_{\mathcal{T},h_1}\phi_{\mathcal{S}}(b,X)\big)s^{(2)}_{b,\tilde{x}_{\hat{1}}(w,v)}(\tilde{v}_{h_1}) + \big(\mathcal{D}^{(3)}_{\mathcal{T},h_1}b\big)s^{(3)}_{b,z_m}(\tilde{v}_{h_1}).$$

If $(b,w,X,v)\in F^{\emptyset}\mathcal{S}|\mathcal{S}^{(m)}_{\mathcal{T},1}(\mu)$ is sufficiently small, let

$$\tilde{\alpha}^{\mu}(w,X,\mu) = (Xb)s_{\tilde{x}_{\hat{1}}(w,v)}(\tilde{v}_{h_1}) + \big(\mathcal{D}^{\mu,(2)}_{\mathcal{S},t\nu}(w,X,v)\big)s^{(2)}_{b,\tilde{x}_{\hat{1}}(w,v)}(\tilde{v}_{h_1}) + \big(\mathcal{D}^{(3)}_{\mathcal{T},h_1}b\big)s^{(3)}_{b,z_m}(\tilde{v}_{h_1}),$$

where, with $\varphi^{\mu}_{\mathcal{S},t\nu}$ as in Theorem 7.2,

$$\mathcal{D}^{\mu,(2)}_{\mathcal{S},t\nu}(w,X,v) = \Pi^{-1}_{\phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,v),\phi_{\mathcal{S}}\Phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,v)}\Pi^{-1}_{b,\phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,v)}\big(\mathcal{D}^{(2)}_{\mathcal{T},h_1}\phi_{\mathcal{S}}\Phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,v)\big).$$

Lemma 8.15 *There exist* $\delta, C\in C^\infty(\mathcal{S}^{(m)}_{\mathcal{T},1};\mathbb{R}^+)$ *such that for all* $\varpi=[(b,w,X,v)]\in F^\emptyset\mathcal{S}_\delta$,

$$\Big\|\pi^{0,1}_{\Phi_{\mathcal{S}}(\varpi),-}\bar{\partial}u_{\Phi_{\mathcal{S}}(\varpi)}+\tilde{R}_{\Phi_{\mathcal{S}}(\varpi)}\Pi_{b,\phi_{\mathcal{S}}(X)}\tilde{\alpha}(w,X,v)\Big\|_2\le C(b)|\varpi||v|^3_{h_1}.$$

Proof: The proof is the same as that of Lemma 8.12, except here we use the first three terms of the expansion of Proposition 7.6. Note that $|\tilde{x}_{\hat{1}}(w,v)|\le C(b)(|w|+|v|)$.

Lemma 8.16 *There exist* $\delta, C\in C^\infty(\mathcal{S}^{(m)}_{\mathcal{T},1}(\mu);\mathbb{R}^+)$ *such that for all* $\varpi=(b,w,X,v)\in F^\emptyset\mathcal{S}_\delta$

$$\Big\|\psi^\mu_{\mathcal{S},t\nu}(\varpi)-\big(t\bar{\nu}_{ev(b)}+\tilde{\alpha}^\mu(w,X,v)\big)\Big\|_2\le C(b)(t+|\varpi|^{\frac{1}{p}})\big(t+|v|^3_{h_1}+|\tilde{x}_1(w,v)||v|^2_{h_1}\big).$$

Proof: The proof is similar to the proofs of Lemmas 8.6 and 8.13, but we need to obtain an even stronger bound on

$$\|\pi^{0,1}_{\Phi_{\mathcal{S}}(\varpi),-}D_{\Phi_{\mathcal{S}}(\varpi)}\xi_{\Phi_{\mathcal{S}}(\varpi),t\nu}\|_2.$$

Let $\{\psi_j\}$ be an orthonormal basis for $\mathcal{H}^{0,1}_\Sigma$ such that $\psi_1\in\mathcal{H}^+_\Sigma(\tilde{x}_{h_1}(w,v))$, and $\{X_i\}$ an orthonormal basis for $T_{\mathrm{ev}(\phi_{\mathcal{S}}(X,v))}\mathbb{P}^n$. Then, as in the proof of Lemma 8.13, with $v(\varpi)=\Phi_{\mathcal{S}}(\varpi)$,

$$\langle\langle D_{\Phi_{\mathcal{S}}(\varpi)}\xi_{v(\varpi),t\nu},R_{v(\varpi)}X_i\psi_1\rangle\rangle=0; \tag{8.16}$$

$$\Big|\langle\langle\pi^{0,1}_{v(\varpi),-}D_{\Phi_{\mathcal{S}}(\varpi)}\xi_{v(\varpi),t\nu},R_{v(\varpi)}X\psi_2\rangle\rangle\Big|\le C(b)|t,v|^{\frac{1}{p}}\big\|D^*_{v(\varpi)}R_{v(\varpi)}X_i\psi_2\big\|_{v(\varpi),1}. \tag{8.17}$$

The one-form ψ_2 vanishes at $\tilde{x}_{h_1}(w,v)$ by definition and $\|\nabla\psi_2\|_{g_{b,\hat{0}},C^0}\le C|\tilde{x}_{h_1}(w,v)|$, since the derivative of the corresponding one-form for z_m vanishes. Thus, by equation (7.5)

$$\big\|D^*_{v(\varpi)}R_{v(\varpi)}X_i\psi_2\big\|_{g_{v(\varpi)},L^1(\tilde{A}_{v(\varpi),h})}\le C(b)(|\tilde{x}_{h_1}(w,v)||v|_{h_1}+|v|^2_{h_1})|v|_{h_1}, \tag{8.18}$$

as needed for our bound. Finally, we use our assumption that $\Phi^\mu_{\mathcal{S}}$ is given by the $g_{b,\hat{0}}$-parallel transport on $\mathcal{N}_b\mathcal{S}_1$.

For any $(w,X,v)\in F^\emptyset_b\mathcal{S}|\mathcal{S}^{(m)}_{\mathcal{T},1}(\mu)$ sufficiently small, let

$$Y(w,X,v)=(Xb)s_{\tilde{x}_1(w,v)}(\tilde{v}_{h_1})+\Big(\mathcal{D}^{\mu,(2)}_{\mathcal{S},t\nu}(w,X,v)\Big)s^{(2,+)}_{b,\tilde{x}_{\hat{1}}(w,v)}(\tilde{v}_{h_1},\tilde{v}_{h_1});$$
$${}^{(3)}\alpha^{(m),-}_{\mathcal{T};1}(w,v)=\big(\mathcal{D}^{(2)}_{\mathcal{T},h_1}b\big)s^{(3,-)}_{b,z_m}\big(\tilde{x}_{\hat{1}}(w,v),\tilde{v}_{h_1},\tilde{v}_{h_1}\big)+\big(\mathcal{D}^{(3)}_{\mathcal{T},h_1}b\big)s^{(3,-)}_{b,z_m}(\tilde{v}_{h_1});$$
$$r^+_{\mathcal{T};1}(v)=\big(\mathcal{D}^{(3)}_{\mathcal{T},h_1}b\big)s^{(3,+)}_{b,z_m}(\tilde{v}_{h_1}),\quad \bar{\nu}^\pm_b=\pi_{z_m}\bar{\nu}_b.$$

Corollary 8.17 *There exist* $\delta, C\in C^\infty(\mathcal{S}^{(m)}_{\mathcal{T},1}(\mu);\mathbb{R}^+)$ *such that for all* $\varpi=[(b,w,X,v)]\in F^\emptyset\mathcal{S}_\delta$

$$\Big\|\pi^+_{x_{\hat{1}}(w,v)}\psi^\mu_{\mathcal{S},t\nu}(w,X,v)-\big(t\pi^+_{x_{\hat{1}}(w,v)}\bar{\nu}_b+Y(w,X,v)+r^+_{\mathcal{T};1}(v)\big)\Big\|_2$$
$$\le C(b)(t+|\varpi|^{\frac{1}{p}})\big(t+|v|^3_{h_1}+|\tilde{x}_1(w,v)||v|^2_{h_1}\big);$$
$$\Big\|\pi^-_{x_{\hat{1}}(w,v)}\psi^\mu_{\mathcal{S},t\nu}(w,X,v)-\big(t\pi^-_{z_m}\bar{\nu}_b+{}^{(3)}\alpha^{(m),-}_{\mathcal{T};1}(w,v)\big)\Big\|_2$$
$$\le C(b)(t+|\varpi|^{\frac{1}{p}})\big(t+|v|^3_{h_1}+|\tilde{x}_1(w,v)||v|^2_{h_1}\big).$$

Proof: The first estimate is clear from Lemma 8.16. For the second, note that since $s^{(2,-)}_{b,z_m}=0$, $|\pi^-_{\tilde{x}_{\hat{1}}(w,v)}-\pi^-_{z_m}|\le C|\tilde{x}_{\hat{1}}(w,v)|^2$, and thus

$$\begin{gathered}\left|s^{(2,-)}_{b,\tilde{x}_{\hat{1}}(w,v)}(\tilde{v}_{h_1})-s^{(3,-)}_{b,z_m}(\tilde{x}_{\hat{1}}(w,v),\tilde{v}_{h_1},\tilde{v}_{h_1})\right|\le C|\tilde{x}_{\hat{1}}(w,v)|^2|\tilde{v}_{h_1}|^2\Longrightarrow\\ \left|\pi^-_{x_{\hat{1}}(w,v)}\tilde{\alpha}^{\mu}(w,X,v)-{}^{(3)}\alpha^-_{\mathcal{T};1}(w,v)\right|\le C(b)|(t,w,X,v)|^{\frac{1}{p}}\left(|x_{\hat{1}}(w,v)||\tilde{v}_{h_1}|^2+|\tilde{v}_{h_1}|^3\right)\end{gathered}$$

Furthermore, $|\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,v)|_b\le C(b)(t+|\varpi|^{\frac{1}{p}})|$.

The next step is to apply Lemma 5.2 and Corollary 5.6. Let

$$\begin{gathered}F^+=\mathcal{H}^+_{\Sigma}\otimes \mathrm{ev}^*T\mathbb{P}^n,\quad F^-=T_{z_m}\Sigma\oplus F\mathcal{T},\quad \mathcal{O}^{\pm}=\mathcal{H}^{\pm}_{\Sigma}\otimes\mathrm{ev}^*T\mathbb{P}^n;\\ \tilde{F}^-=T_{z_m}\Sigma\otimes\Big(\bigotimes_{i\in\hat{I},h_1\in\bar{D}_i\mathcal{T}}F_i\mathcal{T}\Big)^{\otimes 2}\oplus\Big(\bigotimes_{i\in\hat{I},h_1\in\bar{D}_i\mathcal{T}}F_i\mathcal{T}\Big)^{\otimes 3};\\ \rho([b;w,v_{\hat{I}}])=\big[b,x_{\hat{1}(w,v)}\otimes\tilde{v}_{h_1}\otimes\tilde{v}_{h_1},\tilde{v}_{h_1}\otimes\tilde{v}_{h_1}\otimes\tilde{v}_{h_1}\big];\\ \pi^+\alpha(w,v)=r^+_{\mathcal{T};1}(v),\quad \alpha^+(Y)=\pi^+_{z_m}Y,\quad \alpha^-(\rho(w,v))\equiv{}^{(3)}\alpha^{(m),-}_{\mathcal{T},1}(w,v).\end{gathered}$$

Note that $\alpha^-\in\Gamma(\mathcal{S};\tilde{F}^{-*}\otimes\mathcal{O}^-)$ is well-defined. Since the map

$$(w,X,v)\longrightarrow\big(Y(w,X,v),w,v\big)$$

is injective on $F^{\emptyset}\mathcal{S}$, we can view $\psi^{\mu}_{\mathcal{S},t\nu}$ as a map on an open subset of $F^-\oplus F^+$.

Corollary 8.18 *Suppose d is a positive integer, $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ is a simple bubble type, with $d_{\hat{0}}=0$ and $\sum_{i\in I}d_i=d$, and μ is an N-tuple of constraints in general position such that*

$$\mathit{codim}_{\mathbb{C}}\mu=d(n+1)-n(g-1)+N.$$

Let $\nu\in\Gamma(\Sigma\times\mathbb{P}^n;\Lambda^{0,1}\pi^*_{\Sigma}T^*\Sigma\otimes\pi^*_{\mathbb{P}^n}T\mathbb{P}^n)$ be a generic section. If $|\hat{I}|>1$, for every compact subset K of $\mathcal{S}^{(m)}_{\mathcal{T},1}(\mu)$, there exist a neighborhood U_K of K in $\bar{C}^{\infty}_{(d;[N])}(\Sigma;\mu)$ and $\epsilon_K>0$ such that for any $t\in(0,\epsilon_K)$, $U_K\cap\mathcal{M}_{\Sigma,d,t\nu}(\mu)=\emptyset$. If $|\hat{I}|=1$, there exists a compact subset $\tilde{K}^{(m)}_{\mathcal{T},1}$ of $\mathcal{S}_{\mathcal{T},1}(\mu)$ with the following property. If K is a compact subset of $\mathcal{S}^{(m)}_{\mathcal{T},1}(\mu)$ containing $\tilde{K}^{(m)}_{\mathcal{T},1}$, there exist a neighborhood U_K of K in $\bar{C}^{\infty}_{(d;[N])}(\Sigma;\mu)$ and $\epsilon_K>0$ such that for all $t\in(0,\epsilon_K)$, the signed cardinality of $U_K\cap\mathcal{M}_{\Sigma,d,t\nu}(\mu)$ equals to three times the signed number of zeros of the map

$$\begin{aligned}T_{z_m}\Sigma^{\otimes 3}\otimes\big(L_{\hat{1}}\mathcal{T}^{\otimes 2}\oplus L_{\hat{1}}\mathcal{T}^{\otimes 3}\big)|\mathcal{U}_{\bar{\mathcal{T}}}(\mu)&\longrightarrow\mathcal{H}^-_{\Sigma}\otimes ev^*T\mathbb{P}^n,\\ [b,w,v_{\hat{1}}]&\longrightarrow\bar{\nu}^-_b+\big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}b\big)s^{(3,-)}_{b,z_m}(w)+\big(\mathcal{D}^{(3)}_{\mathcal{T},\hat{1}}b\big)s^{(3,-)}_{b,z_m}(v).\end{aligned}\tag{8.19}$$

Proof: The proof is similar to the proofs of Corollaries 8.7 and 8.14, but two modifications are needed to be mentioned. First, we need to show that α^- always has full rank. Since we are assuming that $d_{h_1}\ge 3$, the sections $\mathcal{D}^{(1)}_{\mathcal{T},h_1}$, $\mathcal{D}^{(2)}_{\mathcal{T},h_1}$, and $\mathcal{D}^{(3)}_{\mathcal{T},h_1}$ over $\mathcal{M}_{\mathcal{T}}$ have transverse images in $T\mathbb{P}^n$. Thus, the sections of $\mathbb{P}(\mathrm{ev}^*T\mathbb{P}^n)\longrightarrow\mathcal{S}^{(m)}_{\mathcal{T},1}(\mu)$ induced by $\mathcal{D}^{(2)}_{\mathcal{T},h_1}$ and $\mathcal{D}^{(3)}_{\mathcal{T},h_1}$ are mutually transversal. However, the fiber dimension of $\mathbb{P}(\mathrm{ev}^*T\mathbb{P}^n)$ is $n-1$, while the dimension of $\mathcal{S}^{(m)}_{\mathcal{T},1}(\mu)$ is $n-2$. Thus, the two sections do not intersect and α^- has full rank on all fibers over $\mathcal{S}^{(m)}_{\mathcal{T},1}(\mu)$. The second difference with the proofs of Corollaries 8.7 and 8.14 is that we replace the section $\psi^{\mu}_{\mathcal{S},t\nu}$ by the map

$$(w,v,X)\longrightarrow \pi^{+}_{z_m}\pi^{+}_{x_{\hat{1}}(w,v)}\psi^{\mu}_{\mathcal{S},t\nu}(w,v,X)+\pi^{-}_{z_m}\pi^{-}_{x_{\hat{1}}(w,v)}\psi^{\mu}_{\mathcal{S},t\nu}(w,v,X),$$

which has exactly the same zeros provided w and v are sufficiently small (depending only on Σ).

8.6 Second-Order Estimate for $\psi^{\mu}_{\mathcal{T},t\nu}$, Case 2a

We now understand all cases except for (2a) and (3b) of Corollary 8.11. Let $\{h_1,h_2\}=\{\hat{1},\hat{2}\}$ in Case (2a) and $\{\hat{2},\hat{3}\}$ in (3b). By dimension count as in the proof of Lemma 8.9, $\mathcal{D}_{\mathcal{T},h_1}$ and $\mathcal{D}_{\mathcal{T},h_2}$ do not vanish on $\mathcal{M}_{\mathcal{T}}(\mu)$ in these two cases. By Corollary C.3, $\pi^{\perp}_b\circ\mathcal{D}_{\mathcal{T},h_2}$ is transversal to zero, where $\pi^{\perp}_b$ denotes the projection onto the orthogonal complement E_1 of the image of $\mathcal{D}_{\mathcal{T},h_1}$ in $\mathrm{ev}^*T\mathbb{P}^n$. Since

$$\alpha_{\mathcal{T}}(v)=\big(\mathcal{D}_{\mathcal{T},h_1}b_v\big)s_{x_{\tilde{h}_1(\mathcal{T})}}(\tilde{v}_{h_1})+\big(\mathcal{D}_{\mathcal{T},h_2}b_v\big)s_{x_{\tilde{h}_2(\mathcal{T})}}(\tilde{v}_{h_2}),$$

$\alpha_{\mathcal{T}}$ can fail to have the full rank only on the zero set of $\pi^{\perp}_b\circ\mathcal{D}_{\mathcal{T},h_2}$. Furthermore, $s_{x_{\tilde{h}_1}}$ and $s_{x_{\tilde{h}_2}}$ must have the same image in $\mathcal{H}^{0,1}_{\Sigma}$. This is automatic in Case (3b), since $\tilde{h}_1(\mathcal{T})=\tilde{h}_2(\mathcal{T})=\hat{1}$, but in Case (2a), this means that $x_{\hat{1}}$ and $x_{\hat{2}}$ differ by the nontrivial holomorphic automorphism of Σ; see [GH, p254].

We first treat Case (2a); so we can assume $h_1=\hat{1}$, $h_2=\hat{2}$. Let $\mathcal{S}\equiv\mathcal{S}_{\mathcal{T},2}$ denote the subset of $\mathcal{M}_{\mathcal{T}}$ on which the section $\alpha_{\mathcal{T}}$ has rank one. By Corollary C.3, this is a complex submanifold of $\mathcal{M}_{\mathcal{T}}$. Furthermore, $\mathcal{S}=\mathcal{S}_0\times\mathcal{S}_1$, where $\mathcal{S}_1$ is the subspace of $\mathcal{U}_{\bar{\mathcal{T}}}$ on which the operator $\bar{\mathcal{D}}_{\mathcal{T},2}$, defined as in the proof of Lemma 8.9, has rank one,

$$\mathcal{S}_0=\big\{(x_{\hat{1}},-x_{\hat{1}})\colon x_{\hat{1}}\in\Sigma^*\big\},$$

$-x_{\hat{1}}\in\Sigma$ denotes the image of $x_{\hat{1}}$ under the nontrivial automorphism of Σ, and Σ^* is the subset of Σ which is not fixed by this automorphism, i.e. the complement of the points $z_1,\ldots,z_6$ described in Section 8.5. By Corollary C.3, $\mathcal{S}_1$ is a complex submanifold of $\mathcal{U}_{\bar{\mathcal{T}}}$. The normal bundle of $\mathcal{S}$ in $\mathcal{M}_{\mathcal{T}}$ is

$$\mathcal{N}\mathcal{S}=\mathcal{N}\mathcal{S}_0\oplus\mathcal{N}\mathcal{S}_1,\quad\text{where}\quad \mathcal{N}\mathcal{S}_0=\pi^*_{\Sigma,\hat{2}}T\Sigma,\;\;\mathcal{N}\mathcal{S}_1=L^*_{\hat{2}}\mathcal{T}\otimes E_1,$$

and $\pi_{\Sigma,h}:\mathcal{S}_0\subset\Sigma\times\Sigma\longrightarrow\Sigma$ is the projection on the hth component. Let $(\Phi_{\mathcal{S}},\Phi^{\mu}_{\mathcal{S}})$ be a regularization of $\mathcal{S}_{\mathcal{T},2}(\mu)\equiv\mathcal{S}\cap\mathcal{M}_{\mathcal{T}}(\mu)$. This regularization can be chosen so that

$$\pi^{\perp}_{\phi_{\mathcal{S}}(b,X)}\mathcal{D}_{\mathcal{T},\hat{2}}\tilde{\phi}_{\mathcal{S}}(b,X)=\Pi_{b,\tilde{\phi}_{\mathcal{S}}(b,X)}X\quad\forall(b,X)\in\mathcal{N}\tilde{\mathcal{S}}_1=E_1,\tag{8.20}$$

where $\tilde{\phi}_{\mathcal{S}}$ is the lift of $\phi_{\mathcal{S}}$ to $\mathcal{M}^{(0)}_{\mathcal{T}}$. We also assume that $\Phi^{\mu}_{\mathcal{S}}$ is given by the $g_{\mathbb{P}^n,b}$-parallel transport on $\mathcal{N}_b\mathcal{S}_1$. Since the section s is invariant under the automorphism group of Σ, we identify $\pi^*_{\Sigma,\hat{2}}T\Sigma|\mathcal{S}_0$ with $\pi^*_{\Sigma,\hat{1}}T\Sigma|\mathcal{S}_0$. If $(b;w)\in\mathcal{N}\mathcal{S}_0$ is sufficiently small, let

$$x_{\hat{2}}(w)=\exp_{b,x_{\hat{2}}}w.$$

The bundle $\mathcal{NS}$ carries a natural norm induced by the $g_{\mathbb{P}^n,\mathrm{ev}}$-metric on $\mathbb{P}^n$ and $g_{\cdot,\hat{0}}$-metric on Σ. Denote by $F\mathcal{S}$ and $F^{\emptyset}\mathcal{S}$ the bundles described in Section 3.8 corresponding to the submanifold $\mathcal{S}_{\mathcal{T},2}$. If $(w,X,v)\in F\mathcal{S}=\mathcal{NS}\oplus F\mathcal{T}$, put

$$\tilde{\alpha}(w,X,v)=\Pi^{-1}_{b,\phi_{\mathcal{S}}(b,X)}\Big(\big(\mathcal{D}_{\mathcal{T},\hat{1}}\phi_{\mathcal{S}}(b,X)\big)s_{x_{\hat{1}}}(v_{\hat{1}})+\big(\mathcal{D}_{\mathcal{T},\hat{2}}\phi_{\mathcal{S}}(b,X)\big)s_{x_{\hat{2}}(w)}(v_{\hat{2}})\Big)$$
$$+\Big(\big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}b\big)s^{(2)}_{b,x_{\hat{1}}}(v_{\hat{1}})+\big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{2}}b\big)s^{(2)}_{b,x_{\hat{1}}}(v_{\hat{2}})\Big).$$

If $(w,X,v)\in F^{\emptyset}\mathcal{S}|\mathcal{S}_{\mathcal{T},2}(\mu)$ is sufficiently small, let

$$\tilde{\alpha}^{\mu}(w,X,v)=\Big(\big(\mathcal{D}^{\mu}_{\mathcal{S},t\nu,\hat{1}}(w,X,v)\big)s_{x_{\hat{1}}}(v_{\hat{1}})+\big(\mathcal{D}^{\mu}_{\mathcal{S},t\nu,\hat{2}}(w,X,v)\big)s_{x_{\hat{2}}(w)}(v_{\hat{2}})\Big)$$
$$+\Big(\big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}b\big)s^{(2)}_{b,x_{\hat{1}}}(v_{\hat{1}})+\big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{2}}b\big)s^{(2)}_{b,x_{\hat{1}}}(v_{\hat{2}})\Big),$$

where, with $\varphi^{\mu}_{\mathcal{S},t\nu}$ as in Theorem 7.2,

$$\mathcal{D}^{\mu}_{\mathcal{S},t\nu,h}(w,X,v)=\Pi^{-1}_{\phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,v),\phi_{\mathcal{S}}\Phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,v)}\Pi^{-1}_{b,\phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,v)}\big(\mathcal{D}_{\mathcal{T},h}\phi_{\mathcal{S}}\Phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,v)\big).$$

Lemma 8.19 *There exist $\delta,C\in C^{\infty}(\mathcal{S};\mathbb{R}^{+})$ such that for all $\varpi=[(b,w,X,v)]\in F^{\emptyset}\mathcal{S}_{\delta}$,*

$$\Big\|\pi^{0,1}_{\Phi_{\mathcal{S}}(\varpi),-}\bar{\partial}u_{\Phi_{\mathcal{S}}(\varpi)}+\tilde{R}_{\Phi_{\mathcal{S}}(\varpi)}\Pi_{b,\phi_{\mathcal{S}}(X)}\tilde{\alpha}(w,X,v)\Big\|_2\le C(b)|\varpi||v|^2.$$

Proof: The proof is analogous to the proof of Lemma 8.15; here we use Proposition 7.6 with two terms for $h=\hat{1}$ and two terms for $h=\hat{2}$.

Lemma 8.20 *There exist $\delta,C\in C^{\infty}(\mathcal{S}_{\mathcal{T},2}(\mu);\mathbb{R}^{+})$ such that for all $\varpi=[(b,w,X,v)]\in F^{\emptyset}\mathcal{S}_{\delta}$,*

$$\Big\|\psi^{\mu}_{\mathcal{S},t\nu}(\varpi)-\big(t\bar{\nu}_{ev(b)}+\tilde{\alpha}^{\mu}(w,X,v)\big)\Big\|_2\le C(b)(t+|\varpi|^{\frac{1}{p}})\big(t+|v|^2+|w||v_{\hat{2}}|\big).$$

Proof: As in the proof of Lemmas 8.13 and 8.16, we need to obtain an appropriate estimate on

$$\big\|D^{*}_{\Phi_{\mathcal{S}}(\varpi)}R_{\Phi_{\mathcal{S}}(\varpi)}X_i\psi_2\big\|_{L^1},$$

where ψ_2 is a $(0,1)$-form vanishing at $x_{\hat{1}}$ and with norm 1. From equation (7.4), we see that the L^1-norm over the small annulus centered at $x_{\hat{1}}$ is bounded by $C(b)|v_{\hat{1}}|^2$; see also the proof of Lemma 8.13. Furthermore, since $x_{\hat{2}}$ is "dual" to $x_{\hat{1}}$, ψ_2 also vanishes at $x_{\hat{2}}$. Thus, the L^1-norm over the small annulus centered at $x_{\hat{2}}(w)$ is bounded by $C(b)(|w|+|v_{\hat{2}}|)|v_{\hat{2}}|$ as can be seen from equation (7.4).

Let $\tilde{s}^{(2,+)}_{b,x}\in T^*_x\Sigma$ be given by $s^{(2,+)}_{b,x}(v,v)=\tilde{s}^{(2,+)}_{b,x}(v)s_x(v)$. For any $b\in\mathcal{S}_{\mathcal{T},2}(\mu)$, define

$$\kappa(b)\in L^{*}_{\hat{2}}\mathcal{T}\otimes L_{\hat{1}}\mathcal{T}-\{0\}\quad\text{and}\quad\mu(b)\in L^{*}_{\hat{2}}\mathcal{T}\otimes L_{\hat{1}}\mathcal{T}\quad\text{by}$$
$$\big(\mathcal{D}_{\mathcal{T},\hat{2}}b\big)=\kappa(b)\big(\mathcal{D}_{\mathcal{T},\hat{1}}b\big),\quad\pi_b\big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}b\big)=\mu(b)\big(\mathcal{D}^{(1)}_{\mathcal{T},\hat{1}}b\big),$$

where $\pi_b\colon \mathrm{ev}^*T\mathbb{P}^n\longrightarrow\mathrm{Im}(\mathcal{D}_{\mathcal{T},\hat{1}})$ is the orthogonal projection map. If $(w,X,v)\in F^{\emptyset}\mathcal{S}|\mathcal{S}_{\mathcal{T},2}(\mu)$ is sufficiently small, let $\tilde{\kappa}(w,X,v)\in\mathbb{C}^*$ be given by

$$\pi_{\phi_{\mathcal{S}}\Phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,v)}\big(\mathcal{D}_{\mathcal{T},\hat{2}}\phi_{\mathcal{S}}\Phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,v)\big)=\tilde{\kappa}(w,X,v)\big(\mathcal{D}_{\mathcal{T},\hat{1}}\phi_{\mathcal{S}}\Phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,v)\big).$$

Note that by Theorem 7.2, $|\tilde{\kappa}(w,X,v)-\kappa(b)|\le C(b)(t+|\varpi|^{\frac{1}{p}})$. Let

$$Y^t(w,X,v)=\big(\mathcal{D}^{\mu}_{\mathcal{S},t\nu,\hat{1}}(w,X,v)\big)s_{x_{\hat{1}}}\big(v_{\hat{1}}+\tilde{\kappa}(w,X,v)v_{\hat{2}}+\mu(b)\tilde{s}^{(2,+)}_{\Sigma,x_{\hat{1}}}(v_{\hat{1}})v_{\hat{1}}\big),$$
$$Y^{\perp}(X,v_{\hat{2}})=X s_{\Sigma,x_{\hat{1}}}(v_{\hat{2}});$$
$${}^{(2)}\alpha^-_{\mathcal{T};2}(w,v_{\hat{2}})=\big(\mathcal{D}_{\mathcal{T},\hat{1}}b\big)s^{(2,-)}_{b,x_{\hat{1}}}(w,v_{\hat{2}})+\big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}b\big)s^{(2,-)}_{b,x_{\hat{1}}}(\kappa(b)v_{\hat{2}})+\big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{2}}b\big)s^{(2,-)}_{b,x_{\hat{1}}}(v_{\hat{2}});$$
$$r^+_{\mathcal{T};2}(w,v)=\big(\mathcal{D}^{(1)}_{\mathcal{T},\hat{1}}(b)\big)s^{(2,+)}_{b,x_{\hat{1}}}(w,v_{\hat{2}})+\pi^{\perp}_b\big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}(b)\big)s^{(2,+)}_{b,x_{\hat{1}}}(\kappa(b)v_{\hat{2}})+\big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{2}}(b)\big)s^{(2,+)}_{b,x_{\hat{1}}}(v_{\hat{2}}).$$

Let $Y=Y^t+Y^{\perp}$ and $\bar{\nu}^{\pm}_b=\pi^{\pm}_{x_{\hat{1}}}\bar{\nu}_b$.

Corollary 8.21 *There exist $\delta,C\in C^{\infty}(\mathcal{S}_{\mathcal{T},2}(\mu);\mathbb{R}^+)$ such that for all $\varpi=[(b,w,X,v)]\in F^{\emptyset}\mathcal{S}_{\delta}$,*

$$\Big\|\pi^+_{x_{\hat{1}}}\psi^{\mu}_{\mathcal{S},t\nu}(\varpi)-\big(t\bar{\nu}^+_b+Y(w,X,v)+r^+_{\mathcal{T};2}(w,v)\big)\Big\|_2\le C(b)(t+|\varpi|^{\frac{1}{p}})\big(|v|^2+|w||v_{\hat{2}}|+|Y|\big);$$
$$\Big\|\pi^-_{x_{\hat{1}}}\psi^{\mu}_{\mathcal{S},t\nu}(\varpi)-\big(t\bar{\nu}^-_b+{}^{(2)}\alpha^-_{\mathcal{T};2}(w,v_{\hat{2}})\big)\Big\|_2\le C(b)(t+|\varpi|^{\frac{1}{p}})\big(|v|^2+|w||v_{\hat{2}}|+|Y|\big).$$

Proof: The proof is similar to that of Corollary 8.17, but we use

$$\Big|s_{\Sigma,x_{\hat{2}}(w)}(v_{\hat{2}})-\big(s_{\Sigma,x_{\hat{1}}}(v_{\hat{2}})+s^{(2)}_{b,x_{\hat{1}}}(w,v_{\hat{2}})\big)\Big|\le C(b)|w|^2|v_{\hat{2}}|.$$

We also use $\big|\big(\mathcal{D}^{\mu}_{\mathcal{S},t\nu,\hat{1}}(w,X,v)\big)\big|\ge C(b)^{-1}$.

The next step is to apply Corollary 5.6. Let $\mathcal{O}^{\pm}=\mathcal{H}^{\pm}_{\Sigma}\otimes\mathrm{ev}^*T\mathbb{P}^n$,

$$F^+=\mathcal{H}^+_{\Sigma}\otimes\mathrm{ev}^*T\mathbb{P}^n,\quad F^-=\pi^*_{\Sigma,\hat{1}}T\Sigma\oplus F_{\hat{2}}\mathcal{T},\quad \tilde{F}^-=\pi^*_{\Sigma,\hat{1}}T\Sigma\otimes F_{\hat{2}}\mathcal{T}\oplus F_{\hat{2}}\mathcal{T}^{\otimes 2};$$
$$\rho([b;w,v_{\hat{2}}])=[b,w\otimes v_{\hat{2}},v_{\hat{2}}\otimes v_{\hat{2}}],\quad \alpha^-\big(\rho(w,v_{\hat{2}})\big)\equiv{}^{(2)}\alpha^-_{\mathcal{T},2}(w,v_{\hat{2}}),\quad \pi^+r(w,v)=r^+_{\mathcal{T};1}(w,v).$$

Note that $\alpha^-\in\Gamma(\mathcal{S};\tilde{F}^{-*}\otimes\mathcal{O}^-)$ is well-defined. Since the map

$$(w,X,v)\longrightarrow\big(Y(w,X,v),w,v_{\hat{2}}\big)$$

is injective on $F^{\emptyset}\mathcal{S}$ as long as $\delta\in C^{\infty}(\mathcal{S}_{\mathcal{T},2}(\mu);\mathbb{R}^+)$ is sufficiently small, we can view $\psi^{\mu}_{\mathcal{S},t\nu}$ as a map on an open subset of $F^-\oplus F^+$.

Corollary 8.22 *Suppose d is a positive integer, $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ is a simple bubble type, with $\hat{I}=\{\hat{1},\hat{2}\}$, $M_{\hat{0}}\mathcal{T}=\emptyset$, $d_{\hat{0}}=0$, and $\sum_{i\in I}d_i=d$, and μ is an N-tuple of constraints in general position such that*

$$\mathrm{codim}_{\mathbb{C}}\mu=d(n+1)-n(g-1)+N.$$

Let $\nu\in\Gamma(\Sigma\times\mathbb{P}^n;\Lambda^{0,1}\pi^*_{\Sigma}T^*\Sigma\otimes\pi^*_{\mathbb{P}^n}T\mathbb{P}^n)$ be a generic section. Then there exists a compact subset $\tilde{K}_{\mathcal{T},2}$ of $\mathcal{S}_{\mathcal{T},2}(\mu)$ with the following property. If K is a compact subset of $\mathcal{S}_{\mathcal{T},2}(\mu)$ containing $\tilde{K}_{\mathcal{T},1}$, there exist a neighborhood U_K of K in $\bar{C}^{\infty}_{(d;[N])}(\Sigma;\mu)$ and $\epsilon_K>0$ such that for any $t\in(0,\epsilon_K)$, the signed cardinality of $U_K\cap\mathcal{M}_{\Sigma,d,t\nu}(\mu)$ equals to twice the signed number of zeros of the map

$$\pi^*_{\Sigma}T\Sigma^{\otimes 2}\otimes\big(L_{\hat{2}}\tilde{\mathcal{T}}\oplus L_{\hat{2}}\tilde{\mathcal{T}}^{\otimes 2}\big)|\Sigma^*\otimes\mathcal{U}_{\bar{\mathcal{T}}}(\mu)\longrightarrow\mathcal{H}^-_{\Sigma}\otimes ev^*T\mathbb{P}^n,$$
$$[(x,b);(w,v)]\longrightarrow\bar{\nu}^-_b+\big(\mathcal{D}_{\bar{\mathcal{T}},\hat{2}}b\big)s^{(2)}_x(w,v)+\big(\mathcal{D}^{(2)}_{\bar{\mathcal{T}},\hat{1}}b\big)s^{(2)}_x(\kappa(b)v)+\big(\mathcal{D}^{(2)}_{\bar{\mathcal{T}},\hat{2}}b\big)s^{(2)}_x(v).\quad(8.21)$$

Proof: The proof is similar to that of Corollary 8.14. We only need to see that the section α^- defined above has rank two. If $d_{\hat{1}} = d_{\hat{2}} = 1$, the space $\mathcal{S}_{\mathcal{T},2}(\mu) = \emptyset$, since any two tangent lines in $\mathbb{P}^n$ agree, and no line passes through all of the constraints $\mu_1, \ldots, \mu_N$ if $n=3$. Thus, it can be assumed that $d_{\hat{1}} \geq 2$. Note that $\mathcal{S}_{\mathcal{T},2}(\mu)$ is one-dimensional, with the only dimension coming from the singular point $x_{\hat{1}} \in \Sigma$. Thus, by Corollary C.3, if the constraints $\mu_1, \ldots, \mu_N$ are in general position, the image of $\mathcal{D}^{(2)}_{\mathcal{T};\hat{1}}$ does not lie in the linear span of $(\mathcal{D}_{\mathcal{T},\hat{2}} b)$ and $(\mathcal{D}^{(2)}_{\mathcal{T},\hat{2}} b)$. Furthermore, $(\mathcal{D}_{\mathcal{T},\hat{2}} b) \neq 0$.

8.7 Second-Order Estimate for $\psi^{\mu}_{\mathcal{T},t\nu}$, Case 2b

We now treat Case (3b) of Corollary 8.11; we can assume $h_1 = \hat{2}$, $h_2 = \hat{3}$. Let $\mathcal{S} \equiv \mathcal{S}_{\mathcal{T},2}$ denote the subset of $\mathcal{M}_{\mathcal{T}}$ on which the operator $\bar{\mathcal{D}}_{\mathcal{T},2}$ of Lemma 8.9 has rank one. Similarly to the case of Section 8.6, $\mathcal{S}$ is a regular submanifold of $\mathcal{M}_{\mathcal{T}}$ with normal bundle $\mathcal{N}\mathcal{S} = L^*_{\hat{3}}\mathcal{T} \otimes E_1$. As before, we can choose a regularization $(\Phi_{\mathcal{S}}, \Phi^{\mu}_{\mathcal{S}})$ of $\mathcal{S}_{\mathcal{T},2}(\mu) \equiv \mathcal{S} \cap \mathcal{M}_{\mathcal{T}}(\mu)$ such that

$$\pi^{\perp}_{\tilde{\phi}_{\mathcal{S}}(b,X)} \mathcal{D}_{\mathcal{T},\hat{3}} \tilde{\phi}_{\mathcal{S}}(b, X) = \Pi_{b,\tilde{\phi}_{\mathcal{S}}(b,X)} X \quad \forall (b, X) \in \mathcal{N}\tilde{\mathcal{S}}_1 = E_1, \tag{8.22}$$

where $\tilde{\phi}_{\mathcal{S}}$ is the lift of $\phi_{\mathcal{S}}$ to $\mathcal{M}^{(0)}_{\mathcal{T}}$, and $\Phi^{\mu}_{\mathcal{S}}$ is given by the $g_{\mathbb{P}^n,b}$-parallel transport on $\mathcal{N}_b\mathcal{S}$. Denote by $F\mathcal{S}$ and $F^{\emptyset}\mathcal{S}$ the bundles described in Section 3.8 corresponding to the submanifold $\mathcal{S}_{\mathcal{T},2}$. If $(X, v) \in F\mathcal{S} = \mathcal{N}\mathcal{S} \oplus F\mathcal{T}$ is sufficiently small, let

$$\tilde{\alpha}(X, v) = \big(\mathcal{D}_{\mathcal{T},\hat{2}} \phi_{\mathcal{S}}(b, X)\big) s_{\tilde{x}_{\hat{2}}(v)}(\tilde{v}_{\hat{2}}) + \big(\mathcal{D}_{\mathcal{T},\hat{3}} \phi_{\mathcal{S}}(b, X)\big) s_{\tilde{x}_{\hat{3}}(v)}(\tilde{v}_{\hat{3}});$$
$$\tilde{\alpha}^{\mu}(X, v) = \big(\mathcal{D}^{\mu}_{\mathcal{S},t\nu,\hat{2}}(X, v)\big) s_{\tilde{x}_{\hat{2}}(v)}(\tilde{v}_{\hat{2}}) + \big(\mathcal{D}^{\mu}_{\mathcal{S},t\nu,\hat{3}}(X, v)\big) s_{x_{\hat{3}}(v)}(v_{\hat{3}}),$$

where, with $\varphi^{\mu}_{\mathcal{S},t\nu}$ as in Theorem 7.2,

$$\mathcal{D}^{\mu}_{\mathcal{S},t\nu,h}(X, v) = \Pi^{-1}_{\phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(X,v), \phi_{\mathcal{S}}\Phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(X,v)} \Pi^{-1}_{b,\phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(X,v)} \big(\mathcal{D}_{\mathcal{T},h} \phi_{\mathcal{S}} \Phi^{\mu}_{\mathcal{S}} \varphi^{\mu}_{\mathcal{S},t\nu}(X, v)\big).$$

Lemma 8.23 *There exist $\delta, C \in C^{\infty}(\mathcal{S}_{\mathcal{T},2}; \mathbb{R}^+)$ such that for all $\varpi = [(b, X, v)] \in F^{\emptyset}\mathcal{S}_{\delta}$,*

$$\Big\| \pi^{0,1}_{\Phi_{\mathcal{S}}(\varpi),-} \bar{\partial} u_{\Phi_{\mathcal{S}}(\varpi)} + \tilde{R}_{\Phi_{\mathcal{S}}(\varpi)} \tilde{\alpha}(X, v) \Big\|_2 \leq C(b) \big(|\tilde{v}_{\hat{2}}|^2 + |\tilde{v}_{\hat{3}}|^2 \big).$$

Proof: This lemma is immediate from Proposition 7.6 applied with one term for each $h = \hat{2}, \hat{3}$.

Lemma 8.24 *There exist $\delta, C \in C^{\infty}(\mathcal{S}_{\mathcal{T},2}(\mu); \mathbb{R}^+)$ such that for all $\varpi = [(b, X, v)] \in F^{\emptyset}\mathcal{S}_{\delta}$,*

$$\Big\| \psi^{\mu}_{\mathcal{S},t\nu}(\varpi) - \big(t\bar{\nu}_b + \tilde{\alpha}^{\mu}(X, v)\big) \Big\|_2 \leq C(b)(t + |\varpi|^{\frac{1}{p}})\big(t + |v_{\hat{1}}|(|\tilde{v}_{\hat{2}}| + |\tilde{v}_{\hat{3}}|)\big).$$

Proof: As usually, we only need to obtain a good bound on

$$\| D^*_{\Phi_{\mathcal{S}}(\varpi)} R_{\Phi_{\mathcal{S}}(\varpi)} X_i \psi_2 \|_{L^1},$$

where the notation is as in the proof of Lemma 8.20. By equation (7.4), the L^1-norm on the small annulus centered at $\tilde{x}_{\hat{2}}(v)$ is bounded by $|\tilde{v}_{\hat{2}}|^2$. Since $g_{b,\hat{0}}$-distance between $\tilde{x}_{\hat{2}}(v)$ and $\tilde{x}_{\hat{3}}(v)$ is bounded by $C(b)|v_{\hat{1}}|$, the L_1-norm over the annulus centered at $\tilde{x}_{\hat{3}}(v)$ is bounded by $|v_{\hat{1}}||\tilde{v}_{\hat{3}}|$.

For any $b\in\mathcal{S}_{\mathcal{T},2}(\mu)$, let $\kappa(b)\in(L_{\hat{3}}^*\mathcal{T}\otimes L_{\hat{2}}\mathcal{T}-\{0\})$ be given by $(\mathcal{D}_{\mathcal{T},\hat{3}}b)=\kappa(b)(\mathcal{D}_{\mathcal{T},\hat{2}}b)$. Define $\tilde{\kappa}(X,v)\in(L_{\hat{3}}^*\mathcal{T}\otimes L_{\hat{2}}\mathcal{T}-\{0\})$ for $(X,v)\in F^{\emptyset}\mathcal{S}|\mathcal{S}_{\mathcal{T},2}(\mu)$ sufficiently small by

$$\pi_{\phi_{\mathcal{S}}\Phi_{\mathcal{S}}^{\mu}\varphi_{\mathcal{S},t\nu}^{\mu}(X,v)}\big(\mathcal{D}_{\mathcal{T},\hat{3}}\phi_{\mathcal{S}}\Phi_{\mathcal{S}}^{\mu}\varphi_{\mathcal{S},t\nu}^{\mu}(X,v)\big)=\tilde{\kappa}(X,v)\big(\mathcal{D}_{\mathcal{T},\hat{2}}\phi_{\mathcal{S}}\Phi_{\mathcal{S}}^{\mu}\varphi_{\mathcal{S},t\nu}^{\mu}(X,v)\big).$$

Note that by Theorem 7.2, $|\tilde{\kappa}(X,v)-\kappa(b)|\le C(b)(t+|\varpi|^{\frac{1}{p}})$. Let

$$Y^t(X,v)=\big(\mathcal{D}_{\mathcal{S},t\nu,\hat{2}}^{\mu}(X,v)\big)\Big(s_{x_{\hat{1}}}\big(\tilde{v}_{\hat{2}}+\tilde{\kappa}(X,v)\tilde{v}_{\hat{3}}\big)+s_{b,x_{\hat{1}}}^{(2,+)}\big(v_{\hat{1}},x_{\hat{2}}\tilde{v}_{\hat{2}}+x_{\hat{3}}\tilde{\kappa}(b)\tilde{v}_{\hat{3}}\big)\Big);$$
$$Y^{\perp}(X,v)=Xs_{x_{\hat{1}}}(\tilde{v}_{\hat{3}}),\quad {}^{(2)}\alpha_{\mathcal{T};2}^{-}(v)=\big(\mathcal{D}_{\mathcal{T},\hat{2}}b\big)s_{b,x_{\hat{1}}}^{(2,-)}\big(v_{\hat{1}},x_{\hat{2}}\tilde{v}_{\hat{2}}+x_{\hat{3}}\kappa(b)\tilde{v}_{\hat{3}}\big).$$

Let $Y=Y^t+Y^{\perp}$ and $\bar{\nu}_b^{\pm}=\pi_{x_{\hat{1}}}^{\pm}\bar{\nu}_b$.

Corollary 8.25 *There exist $\delta,C\in C^{\infty}(\mathcal{S}_{\mathcal{T},2}(\mu);\mathbb{R}^+)$ such that for all $\varpi=[(b,X,v)]\in F^{\emptyset}\mathcal{S}_{\delta}$,*

$$\Big\|\pi_{x_{\hat{1}}}\psi_{\mathcal{S},t\nu}^{\mu}(X,\varpi)-\big(t\bar{\nu}_b^{+}+Y(X,v)\big)\Big\|_2\le C(b)|t,\varpi|^{\frac{1}{p}}\big(t+|v_{\hat{1}}|(|\tilde{v}_{\hat{2}}|+|\tilde{v}_{\hat{3}}|)+|Y^{\perp}(X,v)|\big);$$
$$\Big\|\pi_{x_{\hat{1}}}^{-}\psi_{\mathcal{S},t\nu}^{\mu}(\varpi)-\big(t\bar{\nu}_b^{-}+{}^{(2)}\alpha_{\mathcal{T};2}^{-}(v)\big)\Big\|_2\le C(b)|t,\varpi|^{\frac{1}{p}}\big(t+|v_{\hat{1}}|(|\tilde{v}_{\hat{2}}|+|\tilde{v}_{\hat{3}}|)\big).$$

Proof: This claim is proved similarly to Corollary 8.21.

The next step is to apply Lemma 5.2. Let

$$F^{+}=\mathcal{H}_{\Sigma}^{+}\otimes E_1,\quad F^{-}=F\mathcal{T},\quad \mathcal{O}^{\pm}=\mathcal{H}_{\Sigma}^{\pm}\otimes\mathrm{ev}^*T\mathbb{P}^n,\quad \tilde{F}^{-}=\pi_{\Sigma}^*T\Sigma\otimes F_{\hat{2}}\mathcal{T};$$
$$\rho([b;v])=\big[b,v_{\hat{1}}\otimes(x_{\hat{2}}\tilde{v}_{\hat{2}}+x_{\hat{3}}\kappa(b)\tilde{v}_{\hat{3}})\big],\quad \alpha^{-}\big(\rho(v)\big)\equiv{}^{(2)}\alpha_{\mathcal{T},2}^{-}(v),\quad \alpha(X,v)=Y(X,v)+{}^{(2)}\alpha_{\mathcal{T},2}^{-}(v).$$

Note that $\alpha^{-}\in\Gamma(\mathcal{S};\tilde{F}^{-*}\otimes\mathcal{O}^{-})$ is well-defined. Since the map

$$(X,v)\longrightarrow\big(Y^{\perp}(X,v),v\big)$$

is injective on $F^{\emptyset}\mathcal{S}$, we can view $\psi_{\mathcal{S},t\nu}^{\mu}$ as a map on an open subset of $F^{+}\oplus F^{-}$.

Corollary 8.26 *Suppose d is a positive integer, $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ is a simple bubble type, with $\hat{I}=\{\hat{1},\hat{2},\hat{3}\}$, $H_{\hat{1}}\mathcal{T}=\{\hat{2},\hat{3}\}$, $d_{\hat{0}}=0$, and $\sum_{i\in I}d_i=d$, and μ is an N-tuple of constraints in general position such that*

$$\mathrm{codim}_{\mathbb{C}}\mu=d(n+1)-n(g-1)+N.$$

Let $\nu\in\Gamma(\Sigma\times\mathbb{P}^n;\Lambda^{0,1}\pi_{\Sigma}^*T^*\Sigma\otimes\pi_{\mathbb{P}^n}^*T\mathbb{P}^n)$ be a generic section. For every compact subset K of $\mathcal{S}_{\mathcal{T},2}(\mu)$, such that $x_{\hat{1}}(b)\in\Sigma^*$ for all $b\in K$, there exist a neighborhood U_K of K in $\bar{C}_{(d;[N])}^{\infty}(\Sigma;\mu)$, where $d=\sum d_h$, and $\epsilon_K>0$ such that for any $t\in(0,\epsilon_K)$, $U_K\cap\mathcal{M}_{\Sigma,d,t\nu}(\mu)=\emptyset$.

Proof: The set $\mathcal{S}_{\mathcal{T},2}^*(\mu)\equiv\{b\in\mathcal{S}_{\mathcal{T},2}(\mu):x_{\hat{1}}\in\Sigma^*\}$ is an open subset of $\mathcal{S}_{\mathcal{T},2}(\mu)$ on which the section α^{-} has full rank, since $\mathcal{D}_{\mathcal{T},\hat{2}}$ does not vanish on $\mathcal{S}_{\mathcal{T},2}(\mu)$. Note that the dimension of $\mathcal{S}_{\mathcal{T},2}(\mu)$ is 1, the rank of $\tilde{F}^{-}$ is also 1, while the rank $\mathcal{O}^{-}$ is 3. Thus, the claim follows from Theorem 7.2, Lemma 5.2, and Corollary 8.25, provided

$$|v_{\hat{1}}|\big(|\tilde{v}_{\hat{2}}|+|\tilde{v}_{\hat{3}}|\big)\le C(b)\big(|v_{\hat{1}}||x_{\hat{2}}\tilde{v}_{\hat{2}}+x_{\hat{3}}\kappa(b)\tilde{v}_{\hat{3}}|+|Y^t(X,v)|\big)$$

for some $C\in C^\infty(\mathcal{S}^*_{\mathcal{T},2}(\mu);\mathbb{R}^+)$. By definition of $Y^t(X,v)$,

$$|\tilde{v}_{\hat{2}}+\kappa(b)\tilde{v}_{\hat{3}}|\le|Y^t(X,v)|+C(b)|x_{\hat{2}}\tilde{v}_{\hat{2}}+x_{\hat{3}}\kappa(b)\tilde{v}_{\hat{3}}|.$$

Since $x_{\hat{2}}\neq x_{\hat{3}}$,

$$\begin{aligned}|v_{\hat{1}}|\big(|\tilde{v}_{\hat{2}}|+|\tilde{v}_{\hat{3}}|\big)&\le C(b)|v_{\hat{1}}|\big(|\tilde{v}_{\hat{2}}+\kappa(b)\tilde{v}_{\hat{3}}|+|x_{\hat{2}}\tilde{v}_{\hat{2}}+x_{\hat{3}}\kappa(b)\tilde{v}_{\hat{3}}|\big)\\&\le C'(b)|v_{\hat{1}}|\big(|x_{\hat{2}}\tilde{v}_{\hat{2}}+x_{\hat{3}}\kappa(b)\tilde{v}_{\hat{3}}|+|Y^t(X,v)|\big).\end{aligned}$$

8.8 Third-order Estimate for $\psi^\mu_{\mathcal{T},t\nu}$, Case 2b

It remains to consider gluing along the subset $\mathcal{S}^{(m)}_{\mathcal{T},2}(\mu)$ of $\mathcal{S}_{\mathcal{T},2}(\mu)$ consisting of bubble maps b such that $x_{\hat{1}}(b)=z_m$, one of the six distinguished points of Σ. Let

$$\mathcal{S}=\mathcal{S}^{(m)}_{\mathcal{T},2}=\{b\in\mathcal{S}_{\mathcal{T},2}\colon x_{\hat{1}}(b)=z_m\}.$$

The normal bundle of $\mathcal{S}^{(m)}_{\mathcal{T},2}$ in $\mathcal{M}_\mathcal{T}$ is $\mathcal{N}\mathcal{S}=T_{z_m}\Sigma\oplus\mathcal{N}\mathcal{S}_1$, where $\mathcal{N}\mathcal{S}_1$ is the normal bundle of $\mathcal{S}_{\mathcal{T},2}$ in $\mathcal{M}_\mathcal{T}$ described in the previous subsection. Let $(\Phi_\mathcal{S},\Phi^\mu_\mathcal{S})$ be a regularization of $\mathcal{S}^{(m)}_{\mathcal{T},2}(\mu)$ induced by the regularization of $\mathcal{S}_{\mathcal{T},2}(\mu)$ described in Section 8.7. In particular,

$$\pi^\perp_{\phi_\mathcal{S}(b,X)}\mathcal{D}_{\mathcal{T},\hat{3}}\tilde{\phi}_\mathcal{S}(b,w,X)=\Pi_{b,\tilde{\phi}_\mathcal{S}(b,w,X)}X\quad\forall(b,w,X)\in T_{z_m}\Sigma\oplus\mathcal{N}\tilde{\mathcal{S}}_1=T_{z_m}\Sigma\oplus E_1,$$

where $\tilde{\phi}_\mathcal{S}$ is the lift of $\phi_\mathcal{S}$ to $\mathcal{M}^{(0)}_\mathcal{T}$. We also assume that $\Phi^\mu_\mathcal{S}$ is given by the $g_{\mathbb{P}^n,b}$-parallel transport on $\mathcal{N}_b\mathcal{S}_1$. The bundle $\mathcal{N}\mathcal{S}$ carries a natural norm induced by the $g_{\mathbb{P}^n,\mathrm{ev}}$-metric on $\mathbb{P}^n$ and $g_{\cdot,\hat{0}}$-metric on Σ. Denote by $F\mathcal{S}$ and $F^\emptyset\mathcal{S}$ the bundles described in Section 3.8 corresponding to the submanifold $\mathcal{S}^{(m)}_{\mathcal{T},2}$. If $(b,w,X,v)\in F^\emptyset\mathcal{S}$ is sufficiently small, let

$$\tilde{x}_h(w,v)=\tilde{x}_h(\phi_\mathcal{S}(w,X,v))=\tilde{x}_h(\phi_\mathcal{S}(w,0,v))\in\Sigma,\quad h=\hat{2},\hat{3}.$$

We identify a small neighborhood of z_m in Σ with a neighborhood of 0 in $T_{z_m}\Sigma$ via the $g_{b,\hat{0}}$-exponential map. Put

$$\begin{aligned}\tilde{\alpha}(w,X,v)&=\Pi^{-1}_{b,\phi_\mathcal{S}(b,X)}\Big(\big(\mathcal{D}_{\mathcal{T},\hat{2}}\phi_\mathcal{S}(b,X)\big)s_{\tilde{x}_{\hat{2}}(w,v)}(\tilde{v}_{\hat{2}})+\big(\mathcal{D}_{\mathcal{T},\hat{3}}\phi_\mathcal{S}(b,X)\big)s_{\tilde{x}_{\hat{3}}(w,v)}(\tilde{v}_{\hat{3}})\\&\qquad+\big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{2}}\phi_\mathcal{S}(b,X)\big)s^{(2)}_{b,z_m}(\tilde{v}_{\hat{2}})+\big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{3}}\phi_\mathcal{S}(b,X)\big)s^{(2)}_{b,z_m}(\tilde{v}_{\hat{3}})\Big);\\ \tilde{\alpha}^\mu(w,X,v)&=\Big(\big(\mathcal{D}^\mu_{\mathcal{S},t\nu,\hat{2}}(w,X,v)\big)s_{\tilde{x}_{\hat{2}}(w,v)}(\tilde{v}_{\hat{2}})+\big(\mathcal{D}^\mu_{\mathcal{S},t\nu,\hat{3}}(w,X,v)\big)s_{x_{\hat{3}}(w,v)}(v_{\hat{3}})\Big)\\&\qquad+\Big(\big(\mathcal{D}^{\mu,(2)}_{\mathcal{S},t\nu,\hat{2}}b\big)s^{(2)}_{b,z_m}(\tilde{v}_{\hat{2}})+\big(\mathcal{D}^{\mu,(2)}_{\mathcal{S},t\nu,\hat{3}}b\big)s^{(2)}_{b,z_m}(\tilde{v}_{\hat{3}})\Big),\end{aligned}$$

where, with $\varphi^\mu_{\mathcal{S},t\nu}$ as in Theorem 7.2,

$$\mathcal{D}^{\mu,(k)}_{\mathcal{S},t\nu,h}(w,X,v)=\Pi^{-1}_{\phi^\mu_\mathcal{S}\varphi^\mu_{\mathcal{S},t\nu}(w,X,v),\phi_\mathcal{S}\Phi^\mu_\mathcal{S}\varphi^\mu_{\mathcal{S},t\nu}(w,X,v)}\Pi^{-1}_{b,\phi^\mu_\mathcal{S}\varphi^\mu_{\mathcal{S},t\nu}(w,X,v)}\big(\mathcal{D}^{(k)}_{\mathcal{T},h}\phi_\mathcal{S}\Phi^\mu_\mathcal{S}\varphi^\mu_{\mathcal{S},t\nu}(w,X,v)\big).$$

With $\kappa(b)$ as in the previous section, let

$$\alpha^+(v)=\big(\mathcal{D}_{\mathcal{T},\hat{2}}b\big)s_{z_m}\big(\tilde{v}_{\hat{2}}+\kappa(b)\tilde{v}_{\hat{3}}\big),\quad \alpha_{\hat{2}}^-(w,v)=\big(\mathcal{D}_{\mathcal{T},\hat{2}}b\big)s_{b,z_m}^{(3,-)}\big(\tilde{x}_{\hat{2}}(w,v),(x_{\hat{2}}-x_{\hat{3}})v_{\hat{1}},\tilde{v}_{\hat{2}}\big);$$
$$\alpha_{\hat{3}}^-(w,v)=\big(\mathcal{D}_{\mathcal{T},\hat{3}}b\big)s_{b,z_m}^{(3,-)}\big(\tilde{x}_{\hat{3}}(w,v),(x_{\hat{3}}-x_{\hat{2}})v_{\hat{1}},\tilde{v}_{\hat{3}}\big).$$

Lemma 8.27 *There exist $\delta,C\in C^\infty(\mathcal{S}_{\mathcal{T},2}^{(m)};\mathbb{R}^+)$ such that for all $\varpi=(b,w,X,v)\in F^\emptyset\mathcal{S}_\delta$,*

$$\Big\|\pi_{\Phi_{\mathcal{S}}(\varpi),-}^{0,1}\bar{\partial}u_{\Phi_{\mathcal{S}}(\varpi)}-\tilde{R}_{\Phi_{\mathcal{S}}(\varpi)}\tilde{\alpha}(w,X,v)\Big\|_2\le C(b)|\varpi|\big(|\tilde{v}_{\hat{2}}|^2+|\tilde{v}_{\hat{3}}|^2\big).$$

Proof: This lemma follows from Proposition 7.6 applied with first- and second-order terms.

Lemma 8.28 *There exist $\delta,C>0$ such that for all $\varpi=(b,w,X,v)\in F^\emptyset\mathcal{S}_\delta|\mathcal{S}_{\mathcal{T};2}^{(m)}(\mu)$,*

$$\Big\|\psi_{\mathcal{S},t\nu}^{\mu}(\varpi)-\big(t\bar{\nu}_b+\tilde{\alpha}^\mu(w,X,v)\big)\Big\|_2\le C(t+|\varpi|^{\frac{1}{p}})\big(t+(|v_{\hat{1}}|^2+|v_{\hat{1}}||w|)(|\tilde{v}_{\hat{2}}|+|\tilde{v}_{\hat{3}}|)\big).$$

Proof: Note that the space $\mathcal{S}_{\mathcal{T},2}^{(m)}(\mu)$ is zero-dimensional and compact if $n=3$. As before, we need to bound

$$\big\|D_{\Phi_{\mathcal{S}}(\varpi)}^{*}R_{\Phi_{\mathcal{S}}(\varpi)}X_i\psi_2\big\|_{L^1},$$

where the notation is as in the proof of Lemma 8.20. By equation (7.4), the L^1-norm on the annulus centered at $\tilde{x}_{\hat{2}}=\tilde{x}_{\hat{2}}(w,v)$ is bounded by $(|\tilde{x}_{\hat{2}}||\tilde{v}_{\hat{2}}|+|\tilde{v}_{\hat{2}}|^2)|\tilde{v}_{\hat{2}}|$, while the norm over the other annulus is bounded by $(|\tilde{x}_{\hat{2}}||v_{\hat{1}}|+|v_{\hat{1}}|^2)|\tilde{v}_{\hat{3}}|$, since the $g_{b,\hat{0}}$-distance between $\tilde{x}_{\hat{2}}$ and $\tilde{x}_{\hat{3}}$ is bounded by $C|v_{\hat{1}}|$. See the proof of Lemma 8.16 for more detail. The claim follows from $\tilde{x}_{\hat{2}}=w+x_{\hat{2}}v_{\hat{1}}$.

Lemma 8.29 *There exist $\delta,C>0$ such that for all $\varpi=(b,w,X,v)\in F^\emptyset\mathcal{S}_\delta|\mathcal{S}_{\mathcal{T};2}^{(m)}(\mu)$,*

$$\Big\|\tilde{\alpha}^\mu(w,X,v)-\alpha^+(w,v)\Big\|_2\le C(t+|\varpi|^{\frac{1}{p}})\big(|\tilde{v}_{\hat{2}}|+|\tilde{v}_{\hat{3}}|\big);$$
$$\Big\|\pi_{\tilde{x}_{\hat{2}}(w,v)}^{-}\tilde{\alpha}^\mu(w,X,v)-\alpha_{\hat{3}}^-(w,v)\Big\|\le C(t+|\varpi|^{\frac{1}{p}})\big(|v_{\hat{1}}|+|w|\big)|v_{\hat{1}}|\big(|\tilde{v}_{\hat{2}}|+|\tilde{v}_{\hat{3}}|\big);$$
$$\Big\|\pi_{\tilde{x}_{\hat{3}}(w,v)}^{-}\tilde{\alpha}^\mu(w,X,v)-\alpha_{\hat{2}}^-(w,v)\Big\|\le C(t+|\varpi|^{\frac{1}{p}})\big(|v_{\hat{1}}|+|w|\big)|v_{\hat{1}}|\big(|\tilde{v}_{\hat{2}}|+|\tilde{v}_{\hat{3}}|\big).$$

Proof: The first bound is clear from the definition of $\tilde{\alpha}^\mu$, since

$$\big(\mathcal{D}_{\mathcal{T},\hat{3}}b\big)=\kappa(b)\big(\mathcal{D}_{\mathcal{T},\hat{2}}b\big),\qquad |\varphi(w,X,v)|_b\le C(t+|\varpi|^{\frac{1}{p}}).$$

Since $s_{b,z_m}^{(2,-)}=0$,

$$\big|\pi_{\tilde{x}_{\hat{2}}}^{-}s_{\tilde{x}_h}^{(2)}(\tilde{v}_h)\big|\le C\big(|\tilde{x}_{\hat{2}}|+|v_{\hat{1}}|\big)|\tilde{v}_h|^2. \tag{8.23}$$

where $\tilde{x}_h=\tilde{x}_h(w,v)$. Since $\tilde{x}_{\hat{3}}-\tilde{x}_{\hat{2}}=(x_{\hat{3}}-x_{\hat{2}})v_{\hat{1}}$,

$$\Big|s_{b,\tilde{x}_{\hat{3}}}(\tilde{v}_{\hat{3}})-\big(s_{b,\tilde{x}_{\hat{2}}}(\tilde{v}_{\hat{3}})+s_{b,\tilde{x}_{\hat{2}}}^{(2)}\big((x_{\hat{3}}-x_{\hat{2}})v_{\hat{1}},\tilde{v}_{\hat{3}}\big)+s_{b,\tilde{x}_{\hat{2}}}^{(3)}\big((x_{\hat{3}}-x_{\hat{2}})v_{\hat{1}},(v_{\hat{3}}-v_{\hat{2}})v_{\hat{1}},\tilde{v}_{\hat{3}}\big)\big)\Big|\le C|v_{\hat{1}}|^3|\tilde{v}_{\hat{3}}|.$$

Since $\pi^-_{\tilde{x}_{\hat{2}}} s_{\tilde{x}_{\hat{2}}} = 0$ and $s^{(2,-)}_{b,z_m} = 0$,

$$\left|\pi^-_{\tilde{x}_{\hat{2}}} s^{(2)}_{b,\tilde{x}_{\hat{2}}}\big((x_{\hat{3}} - x_{\hat{2}})v_{\hat{1}}, \tilde{v}_{\hat{3}}\big) - s^{(3,-)}_{b,z_m}\big(\tilde{x}_{\hat{2}}, (x_{\hat{3}} - x_{\hat{2}})v_{\hat{1}}, \tilde{v}_{\hat{3}}\big)\right| \le C|\tilde{x}_{\hat{2}}|^2|v_{\hat{1}}||\tilde{v}_{\hat{3}}|;$$
$$\left|\pi^-_{\tilde{x}_{\hat{2}}} s^{(3)}_{b,\tilde{x}_{\hat{2}}}\big((x_{\hat{3}}-x_{\hat{2}})v_{\hat{1}}, (x_{\hat{3}}-x_{\hat{2}})v_{\hat{1}}, \tilde{v}_{\hat{3}}\big) - s^{(3,-)}_{b,z_m}\big((x_{\hat{3}}-x_{\hat{2}})v_{\hat{1}}, (x_{\hat{3}}-x_{\hat{2}})v_{\hat{1}}, \tilde{v}_{\hat{3}}\big)\right| \le C|\tilde{x}_{\hat{2}}||v_{\hat{1}}|^2|\tilde{v}_{\hat{3}}|.$$

Putting the last three equations together, we see that

$$\left|\pi^-_{\tilde{x}_{\hat{2}}} s_{b,\tilde{x}_{\hat{3}}}(\tilde{v}_{\hat{3}}) - s^{(3,-)}_{b,z_m}\big(\tilde{x}_{\hat{3}}, (x_{\hat{3}} - x_{\hat{2}})v_{\hat{1}}, \tilde{v}_{\hat{3}}\big)\right| \le C\big(|\tilde{x}_{\hat{2}}| + |v_{\hat{1}}|\big)\big(|\tilde{x}_{\hat{2}}||v_1| + |v_{\hat{1}}|^2\big)|\tilde{v}_{\hat{3}}|. \tag{8.24}$$

The second bound follows from equations (8.23) and (8.24). The last estimate is proved similarly.

Corollary 8.30 *There exist $\delta, C > 0$ such that for all $\varpi = (b, w, X, v) \in F^{\emptyset}\mathcal{S}_\delta|\mathcal{S}^{(m)}_{\mathcal{T};2}(\mu)$,*

$$\left\|\psi^\mu_{\mathcal{S},t\nu}(\varpi) - \big(t\bar{\nu}_b + \tilde{\alpha}^\mu(\varpi)\big)\right\|_2 \le C(t + |\varpi|^{\frac{1}{p}})\big(t + |\tilde{\alpha}^\mu(\varpi)|\big).$$

Proof: In light of Lemma 8.28, it is sufficient to show that

$$\big(|v_{\hat{1}}| + |w|\big)|v_{\hat{1}}|\big(|\tilde{v}_{\hat{2}}| + |\tilde{v}_{\hat{3}}|\big) \le C\big|\tilde{\alpha}^\mu(w, X, v)\big| \tag{8.25}$$

for some $C > 0$. Since $\big(\mathcal{D}_{\mathcal{T},\hat{2}}b\big)s_{z_m}$, $\big(\mathcal{D}_{\mathcal{T},\hat{2}}b\big)s^{(3,-)}_{b,z_m}$ and $\big(\mathcal{D}_{\mathcal{T},\hat{3}}b\big)s^{(3,-)}_{b,z_m}$ are nonzero, by Lemma 8.29

$$\big|\tilde{v}_{\hat{2}} + \kappa(b)\tilde{v}_{\hat{3}}\big| \le C\Big(\big|\tilde{\alpha}^\mu(w, X, v)\big| + (t + |\varpi|^{\frac{1}{p}})\big(|\tilde{v}_{\hat{2}}| + |\tilde{v}_{\hat{3}}|\big)\Big);$$
$$|\tilde{x}_h||v_{\hat{1}}||\tilde{v}_h| \le C\Big(\big|\tilde{\alpha}^\mu(w, X, v)\big| + (t + |\varpi|^{\frac{1}{p}})\big(|v_{\hat{1}}| + |w|\big)|v_{\hat{1}}|\big(|\tilde{v}_{\hat{2}}| + |\tilde{v}_{\hat{3}}|\big)\Big).$$

Since $\kappa(b) \ne 0$, $x_{\hat{2}} \ne x_{\hat{3}}$, and $\tilde{x}_h = w + x_h v_{\hat{1}}$, we obtain

$$\begin{aligned}\big(|v_{\hat{1}}| + |w|\big)|v_{\hat{1}}|\big(|\tilde{v}_{\hat{2}}| + |\tilde{v}_{\hat{3}}|\big) &\le C\big(|\tilde{x}_{\hat{2}}| + |\tilde{x}_{\hat{3}}|\big)|v_{\hat{1}}|\big(|\tilde{v}_{\hat{2}}| + |\tilde{v}_{\hat{3}}|\big)\\ &\le C'\Big(|\tilde{x}_{\hat{2}}||v_{\hat{1}}|\big(|\tilde{v}_{\hat{2}}| + |\tilde{v}_{\hat{2}} + \kappa(b)\tilde{v}_{\hat{3}}|\big) + |\tilde{x}_{\hat{3}}||v_{\hat{1}}|\big(|\tilde{v}_{\hat{3}}| + |\tilde{v}_{\hat{3}} + \kappa(b)\tilde{v}_{\hat{3}}|\big)\Big)\\ &\le C''\Big(\big|\tilde{\alpha}^\mu(w, X, v)\big| + (t + |\varpi|^{\frac{1}{p}})\big(|v_{\hat{1}}| + |w|\big)|v_{\hat{1}}|\big(|\tilde{v}_{\hat{2}}| + |\tilde{v}_{\hat{3}}|\big)\Big).\end{aligned} \tag{8.26}$$

If δ is sufficiently small, estimate (8.25) follows from (8.26).

The next step is to apply Lemma 5.2. Let

$$F^+ = L^*_{\hat{3}}\mathcal{T} \otimes E_1, \quad F^- = T_{z_m}\Sigma \oplus F\mathcal{T}, \quad \mathcal{O}^\pm = \mathcal{H}^\pm_\Sigma \otimes \mathrm{ev}^*T\mathbb{P}^n, \quad \tilde{F}^- = \pi^*_\Sigma T\Sigma^{\otimes 3} \otimes L_{\hat{2}}\mathcal{T}^{\otimes 3};$$
$$\phi\big([b, w, v]\big) = \big[b, (w + x_{\hat{2}}v_{\hat{1}}) \otimes ((x_{\hat{2}} - x_{\hat{2}})v_{\hat{1}}) \otimes \tilde{v}_{\hat{2}}\big];$$
$$\alpha^-\big(\phi(w, v)\big) \equiv \alpha^-_{\hat{2}}(w, v), \quad \alpha(X, w, v) = \alpha^\mu(X, w, v).$$

Note that $\alpha^- \in \Gamma(\mathcal{S}; \tilde{F}^{-*} \otimes \mathcal{O}^-)$ is well-defined.

Corollary 8.31 *Suppose d is a positive integer, $\mathcal{T} = (\Sigma, [N], I; j, \underline{d})$ is a simple bubble type, with $\hat{I} = \{\hat{1}, \hat{2}, \hat{3}\}$, $H_{\hat{1}}\mathcal{T} = \{\hat{2}, \hat{3}\}$, $d_{\hat{0}} = 0$, and $\sum_{i \in I} d_i = d$, and μ is an N-tuple of constraints in* *general position such that*

$$codim_{\mathbb{C}}\mu = d(n+1) - n(g-1) + N.$$

Let $\nu \in \Gamma(\Sigma \times \mathbb{P}^n; \Lambda^{0,1}\pi_\Sigma^* T^*\Sigma \otimes \pi_{\mathbb{P}^n}^* T\mathbb{P}^n)$ *be a generic section. There exist a neighborhood* U *of* $\mathcal{S}^{(m)}_{\mathcal{T},2}(\mu)$ *in* $\bar{C}^\infty_{(d;[N])}(\Sigma;\mu)$*, and* $\epsilon > 0$ *such that for any* $t \in (0,\epsilon)$*,* $U \cap \mathcal{M}_{\Sigma,d,t\nu}(\mu) = \emptyset$*.*

Proof: Analogously to the proof of Corollary 8.18, we apply Lemma 5.2 to the map

$$(w,v,X) \longrightarrow \pi^+_{z_m}\pi^+_{x_{\hat{3}}(w,v)}\psi^\mu_{\mathcal{S},t\nu}(w,v,X) + \pi^-_{z_m}\pi^-_{x_{\hat{3}}(w,v)}\psi^\mu_{\mathcal{S},t\nu}(w,v,X)$$

instead of $\psi^\mu_{\mathcal{S},t\nu}$. The claim then follows from Theorem 7.2, Lemma 5.2, and Corollary 8.30.

8.9 Summary of Chapter 8

We conclude Chapter 8 by reviewing the main results concerning the structure of $CR_2(\mu)$ so far. Throughout this section, $\mathcal{T} = (\Sigma, [N], I; j, \underline{d})$ is a simple bubble type, with $d = \sum d_h$ and $d_{\hat{0}} = 0$, and μ is an N-tuple of linear subspaces of $\mathbb{P}^n$ in general position such that $\text{codim}_{\mathbb{C}}\mu = d(n+1) - n + N$. We continue to assume that the genus of Σ is two and $n = 2, 3$.

If $|\hat{I}| \geq n$, by Corollaries 8.10 and 8.11, there exist a neighborhood $U_{\mathcal{T}}$ of $\bar{\mathcal{M}}_{\mathcal{T}}(\mu)$ in $\bar{C}^\infty_{(d;[N])}(\Sigma;\mu)$ and $\epsilon_{\mathcal{T}} > 0$ such that for all $t \in (0, \epsilon_{\mathcal{T}})$, $U_{\mathcal{T}} \cap \mathcal{M}_{\Sigma,d,t\nu}(\mu) = \emptyset$. This is also true if $H_{\hat{0}}\mathcal{T} \neq \hat{I}$ or $M_{\hat{0}}\mathcal{T} \neq \emptyset$. If $n = 2$, this statement is just Corollary 8.10. If $n = 3$, we only need to consider Cases (1), (2b), and (3b) of Corollary 8.11. Case (3b) follows from Corollaries 8.7, 8.26, and 8.31. The claim for Case (2b) is obtained from Corollaries 8.7, 8.14, 8.18 and the same claim for Case (3b). Finally, in Case (1), we use Corollaries 8.7, 8.26, and 8.31, the statement of Corollary 8.11 for $|\hat{I}| \geq 2$, and the just stated result for Case (2b).

If $|\hat{I}| \leq n$, $H_{\hat{0}}\mathcal{T} = \hat{I}$, and $M_{\hat{0}}\mathcal{T} = \emptyset$, i.e. $\mathcal{T}$ is a *primitive* bubble type, by the previous paragraph and Corollaries 8.7, 8.14, 8.18, and 8.22, there exist a neighborhood $U_{\mathcal{T}}$ of $\bar{\mathcal{M}}_{\mathcal{T}}(\mu)$ in $\bar{C}^\infty_{(d;[N])}(\Sigma;\mu)$ and $\epsilon_{\mathcal{T}} > 0$ such that for all $t \in (0, \epsilon_{\mathcal{T}})$, the signed cardinality $N_{\mathcal{T}}(\mu)$ of $U_{\mathcal{T}} \cap \mathcal{M}_{\Sigma,d,t\nu}(\mu)$ is the sum of the numbers given by these four corollaries applied to $\mathcal{T}$. If $|\hat{I}| = 1$,

$$n_1(\mu) \equiv N_{\mathcal{T}}(\mu) = n_1^{(1)}(\mu) + 2n_1^{(2)}(\mu) + 18n_1^{(3)}(\mu), \tag{8.27}$$

where the numbers $n_1^{(k)}(\mu)$ are described as follows. The number $n_1^{(1)}(\mu)$ is the signed number of zeros of the affine map

$$\psi_1^{(1)} : T\Sigma \otimes L_{\hat{1}}\bar{\mathcal{T}} \longrightarrow \mathcal{H}^{0,1}_\Sigma \otimes \text{ev}^* T\mathbb{P}^n, \qquad \psi_1^{(1)}(x, [b, v_{\hat{1}}]) = \bar{\nu}_b + (\mathcal{D}_{\mathcal{T},\hat{1}} b) s_x(v_{\hat{1}}), \tag{8.28}$$

where the bundles are considered over $\Sigma \times \bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu) = \bar{\mathcal{M}}_{\mathcal{T}}(\mu)$ and $\hat{1}$ is the unique element of $\hat{I}$. Note that this number is the same as the number of zeros of the map in (8.9), since $\Sigma \times \bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu) - \mathcal{M}_{\mathcal{T}}(\mu)$ is a finite union of smooth manifolds of dimension less than the dimension of $\mathcal{M}_{\mathcal{T}}(\mu)$. Thus, if ν is generic, $\psi_1^{(1)}$ has no zeros over $\Sigma \times \bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu) - \mathcal{M}_{\mathcal{T}}(\mu)$. The number $n_1^{(2)}(\mu)$ is the signed number of zeros of the affine map

$$\psi_1^{(2)} : T\Sigma^{\otimes 2} \otimes L_{\hat{1}}\bar{\mathcal{T}}^{\otimes 2} \longrightarrow \mathcal{H}^-_\Sigma \otimes \text{ev}^* T\mathbb{P}^n, \quad \psi_2^{(2)}(x, [b, v_{\hat{1}}]) = \bar{\nu}^-_b + (\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}} b) s^{(2)}_x(v_{\hat{1}}), \tag{8.29}$$

where the bundles are considered over $\Sigma\times\bar{\mathcal{S}}_1(\mu)$ and $\bar{\mathcal{S}}_1(\mu)$ is the closure in $\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)$ of the space

$$\mathcal{S}_1(\mu) = \big\{b\in\mathcal{U}_{\bar{\mathcal{T}}}(\mu) : \mathcal{D}_{\mathcal{T},\hat{1}}\big|_b = 0\big\}.$$

If $n=2$, $\mathcal{S}_1(\mu)$ is a finite set and thus $\bar{\mathcal{S}}_1(\mu)=\mathcal{S}_1(\mu)$. If $n=3$, $\mathcal{S}_1(\mu)$ is one-dimensional over $\mathbb{C}$. The boundary $\bar{\mathcal{S}}_1(\mu)-\mathcal{S}_1(\mu)$ is a finite set; see Lemma 6.8. Thus, in either case, the maps in (8.29) and (8.14) have the same zeros. Finally, the number $n_1^{(3)}(\mu)$ is the signed number of zeros of the affine map

$$\begin{gathered}\psi_1^{(3)} : T\Sigma^{\otimes 3} \otimes \big(L_{\hat{1}}\bar{\mathcal{T}}^{\otimes 2} \oplus L_{\hat{1}}\bar{\mathcal{T}}^{\otimes 3}\big) \longrightarrow \mathcal{H}_\Sigma^- \otimes \mathrm{ev}^*T\mathbb{P}^n, \\ \psi_1^{(3)}(x,[b,v_{\hat{1}},w_{\hat{1}}]) = \bar{\nu}_b^- + \big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}b\big)s^{(3,-)}_{b,z_m}(v_{\hat{1}}) + \big(\mathcal{D}^{(3)}_{\mathcal{T},\hat{1}}b\big)s^{(3,-)}_{b,z_m}(w_{\hat{1}}),\end{gathered} \tag{8.30}$$

where the bundles are considered over $\bar{\mathcal{S}}_1(\mu)$ and z_m is one of the six distinguished points of Σ. By the same argument as above, this number is precisely the number of zeros of the map in (8.19).

If $|\hat{I}|=2$ and $n=2$, $N_{\mathcal{T}}(\mu)=n_{\mathcal{T}}^{(1)}(\mu)$ is the signed number of zeros of the affine map

$$\begin{gathered}\psi_{\mathcal{T}}^{(1)} : T\Sigma_{\hat{1}} \otimes L_{\hat{1}}\bar{\mathcal{T}} \oplus T\Sigma_{\hat{2}} \otimes L_{\hat{2}}\bar{\mathcal{T}} \longrightarrow \mathcal{H}_\Sigma^{0,1} \otimes \mathrm{ev}^*T\mathbb{P}^n, \\ \psi_{\mathcal{T}}^{(1)}(x_{\hat{1}},x_{\hat{2}},[b,v_{\hat{1}},v_{\hat{2}}]) = \bar{\nu}_b + \big(\mathcal{D}_{\mathcal{T},\hat{1}}b\big)s_{x_{\hat{1}}}(v_{\hat{1}}) + \big(\mathcal{D}_{\mathcal{T},\hat{2}}b\big)s_{x_{\hat{2}}}(v_{\hat{2}}),\end{gathered} \tag{8.31}$$

where the bundles are considered over $\Sigma^2 \times \bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu) = \Sigma_{\hat{1}} \times \Sigma_{\hat{2}} \times \bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)$ and $\hat{1},\hat{2}$ are the two elements of $\hat{I}$. By the same argument as before, the number $n_{\mathcal{T}}^{(1)}(\mu)$ is the same as the number of zeros of the map (8.7). If $|\hat{I}|=2$ and $n=3$,

$$N_{\mathcal{T}}(\mu) = n_{\mathcal{T}}^{(1)}(\mu) + 2n_{\mathcal{T}}^{(2)}(\mu), \tag{8.32}$$

where $n_{\mathcal{T}}^{(1)}(\mu)$ is defined the same way as in the $n=2$ case, while $n_{\mathcal{T}}^{(2)}(\mu)$ is the signed number of zeros of the affine map

$$\begin{gathered}\psi_{\mathcal{T}}^{(2)} : T\Sigma^{\otimes 2} \otimes \big(L_{\hat{2}}\bar{\mathcal{T}} \oplus L_{\hat{2}}\bar{\mathcal{T}}^{\otimes 2}\big) \longrightarrow \mathcal{H}_\Sigma^- \otimes \mathrm{ev}^*T\mathbb{P}^n, \\ \psi_{\mathcal{T}}^{(2)}(x,[b,v_{\hat{2}},w_{\hat{2}}]) = \bar{\nu}_b^- + \big(\mathcal{D}_{\mathcal{T},\hat{2}}b\big)s_x^{(2)}(w_{\hat{2}}) + \big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}b\big)s_x^{(2)}(\kappa(b)v_{\hat{2}}) + \big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{2}}b\big)s_x^{(2)}(v_{\hat{2}}),\end{gathered} \tag{8.33}$$

where the bundles are viewed over $\Sigma \times \mathcal{S}_{\bar{\mathcal{T}};2}(\mu)$,

$$\mathcal{S}_{\bar{\mathcal{T}};2}(\mu) = \big\{b\in\mathcal{U}_{\bar{\mathcal{T}}}(\mu) : \pi^\perp_{[b]} \circ \mathcal{D}_{\mathcal{T},\hat{2}}\big|_{[b]} = 0\big\}, \tag{8.34}$$

E_1 is the quotient of $\mathrm{ev}^*T\mathbb{P}^n$ by $\mathrm{Im}(\mathcal{D}_{\mathcal{T},\hat{1}})$, $\pi^\perp$: $\mathrm{ev}^*T\mathbb{P}^n \longrightarrow E_1$ is the projection map, and $\kappa(b) \in L_{\hat{2}}^*\bar{\mathcal{T}} \otimes L_{\hat{1}}\bar{\mathcal{T}}$ is a nonzero homomorphism. Note that $\mathcal{S}_{\bar{\mathcal{T}}}(\mu)$ is a finite set with our choice of constraints. Finally, if $|\hat{I}|=3$ and $n=3$, $N_{\mathcal{T}}(\mu)=n_{\mathcal{T}}^{(1)}(\mu)$ is the signed number of zeros of the affine map

$$\begin{gathered}\psi_{\mathcal{T}}^{(1)} : T\Sigma_{\hat{1}} \otimes L_{\hat{1}}\bar{\mathcal{T}} \oplus T\Sigma_{\hat{2}} \otimes L_{\hat{2}}\bar{\mathcal{T}} \oplus T\Sigma_{\hat{3}} \otimes L_{\hat{3}}\bar{\mathcal{T}} \longrightarrow \mathcal{H}_\Sigma^{0,1} \otimes \mathrm{ev}^*T\mathbb{P}^n, \\ \psi_{\mathcal{T}}^{(1)}(x_{\hat{1}},x_{\hat{2}},x_{\hat{3}},[b,v_{\hat{1}},v_{\hat{2}},v_{\hat{3}}]) = \bar{\nu}_b + \big(\mathcal{D}_{\mathcal{T},\hat{1}}b\big)s_{x_{\hat{1}}}(v_{\hat{1}}) + \big(\mathcal{D}_{\mathcal{T},\hat{2}}b\big)s_{x_{\hat{2}}}(v_{\hat{2}}) + \big(\mathcal{D}_{\mathcal{T},\hat{3}}b\big)s_{x_{\hat{3}}}(v_{\hat{3}}),\end{gathered} \tag{8.35}$$

where the bundles are considered over $\Sigma^3 \times \bar{\mathcal{U}}_{\mathcal{T}}(\mu) = \Sigma_{\hat{1}} \times \Sigma_{\hat{2}} \times \Sigma_{\hat{3}} \times \bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ and $\hat{1}, \hat{2}, \hat{3}$ are the three elements of $\hat{I}$. As before, the number $n_{\mathcal{T}}^{(1)}(\mu)$ is precisely the number of zeros of the map (8.7). If $m \geq 2$ and $k \geq 1$, we denote by $n_m^{(k)}(\mu)$ the sum of the numbers $n_{\mathcal{T}}^{(k)}(\mu)$ taken over all equivalence classes of primitive bubble types $\mathcal{T}$ with $|\hat{I}| = m$.

Chapter 9

Computation of $CR_2(\mu)$ for $\mathbb{P}^2$ and $\mathbb{P}^3$

In this chapter, we compute the numbers $CR_2(\mu)$ for $\mathbb{P}^2$ and $\mathbb{P}^3$ and thus the genus-two fixed-complex-structure enumerative invariants for $\mathbb{P}^2$ and $\mathbb{P}^3$. We use the description of $CR_2(\mu)$ as the total number of zeros of explicit affine maps between vector bundles given in Section 8.9, the topological tools of Chapter 5, and the results about spaces of rational maps of Chapter 6.

9.1 The Numbers $n_2^{(1)}(\mu)$ for $\mathbb{P}^2$ and $n_3^{(1)}(\mu)$ for $\mathbb{P}^2$

We start with the easiest cases.

Lemma 9.1 *If $d\geq 1$ and μ is an N-tuple of $3d-2$ points in general position in $\mathbb{P}^2$,*

$$n_1^{(3)}(\mu) = \big|\mathcal{S}_1(\mu)\big|.$$

Proof: By equation 8.30, $n_1^{(3)}(\mu) = N(\alpha_{1,3})$, where $\alpha_{1,3}$ is an isomorphism on every fiber of a bundle over $\mathcal{S}_1(\mu)$. Thus, the claim is clear.

Lemma 9.2 *If $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ is a primitive bubble type with $|\hat{I}|=n$ and μ is an N-tuple of constraints in general position such that*

$$\sum d_h = d \quad \textit{and} \quad \textit{codim}_{\mathbb{C}}\mu = (n+1)\sum_{i\in M} d_i - n + N,$$

the set $\mathcal{U}_{\bar{\mathcal{T}}}(\mu)$ is finite and $n_{\mathcal{T}}^{(1)}(\mu) = 2^n|\mathcal{U}_{\bar{\mathcal{T}}}(\mu)|$.

Proof: The first statement is clear by dimension counting. By equations (8.31) and (8.35), $n_{\mathcal{T}}^{(1)} = N(\alpha_n)$, where

$$\alpha_n \in \Gamma\big(\Sigma^n \times \mathcal{U}_{\bar{\mathcal{T}}}(\mu); \mathrm{Hom}(\tilde{E}_n;\mathcal{O})\big), \quad \alpha_n\big((v_h\otimes v_h)_{h\in\hat{I}}\big) = \sum_{h\in\hat{I}} \big(\mathcal{D}_{\mathcal{T},h}v_h\big)s_{x_h}v_h,$$

$$\tilde{E}_n = \bigoplus_{h\in\hat{I}} T\Sigma_h \otimes L_h\mathcal{T} \approx \bigoplus_{h\in\hat{I}} T\Sigma_h, \quad \mathcal{O} = \mathcal{H}_{\Sigma}^{0,1}\otimes \mathrm{ev}^*T\mathbb{P}^n \approx \mathbb{C}^{2n}.$$

By Lemma 8.9, α_n has full rank on every fiber. Thus, by Lemma 5.14,

$$n_{\mathcal{T}}^{(1)}(\mu) = N(\alpha_n) = \big\langle -e(E_n), \big[\Sigma^n \times \mathcal{U}_{\bar{\mathcal{T}}}(\mu)\big]\big\rangle = 2^n\big|\mathcal{U}_{\bar{\mathcal{T}}}(\mu)\big|,$$

since the euler characteristic of Σ is -2.

Corollary 9.3 *If* $n=2$, $n_2^{(1)}(\mu) = 4\big|\mathcal{V}_2(\mu)\big|$. *If* $n=3$, $n_3^{(1)}(\mu) = 8\big|\mathcal{V}_3(\mu)\big|$.

9.2 The Numbers $n_1^{(2)}(\mu)$ for $\mathbb{P}^2$ and $n_2^{(2)}(\mu)$ for $\mathbb{P}^3$

The number $n_1^{(2)}(\mu)$ is the signed cardinality of the zero set of the affine map $\psi_{\mathcal{T}}^{(2)}$ in (8.29). By Section 8.4, the linear part α of the affine map $\psi_{\mathcal{T}}^{(2)}$ has full rank, except over the zero set of $s^{(2)}$. In order to simplify our computations, we replace $s^{(2)}$ by another section that has no zeros on Σ, but so that the corresponding affine map has the same number of zero as the map in (8.29). The section $s^{(2)} \in \Gamma(\Sigma; T^*\Sigma^{\otimes 2} \otimes \mathcal{H}_\Sigma^-)$ has transverse zeros at the points $z_1, \ldots, z_6 \in \Sigma$; see Section 8.5. Thus, it induces a nonvanishing section

$$\tilde{s}^{(2)} \in \Gamma\big(\Sigma; \mathrm{Hom}(\tilde{T}\Sigma^*, \mathcal{H}_\Sigma^-)\big), \quad \text{where} \quad \tilde{T}\Sigma = T\Sigma^{\otimes 2} \otimes \mathcal{O}(z_1) \otimes \ldots \otimes \mathcal{O}(z_6)$$

and $\mathcal{O}(z_m)$ denotes the holomorphic line bundle corresponding to the divisor z_m on Σ. The bundles $\tilde{T}\Sigma$ and $T\Sigma^{\otimes 2}$ can be identified on Σ^*, the complement of the six points, in such a way that $\tilde{s}^{(2)} = \eta s^{(2)}$ on Σ^* for some $\eta \in C^\infty(\Sigma^*; \mathbb{R}^+)$. Let $\tilde{\psi}_{\mathcal{T}}^{(2)}$ denote the affine map obtained by replacing $T\Sigma^{\otimes 2}$ and $s^{(2)}$ by $\tilde{T}\Sigma$ and $\tilde{s}^{(2)}$, respectively, in (8.29). Since $\psi_{\mathcal{T}}^{(2)}$ and $\tilde{\psi}_{\mathcal{T}}^{(2)}$ have no zeros over $\{z_m\}$ if ν is generic and $s^{(2)}$ and $\tilde{s}^{(2)}$ differ by a nonzero multiple on Σ^*, there is a sign-preserving bijection between the zeros of $\psi_{\mathcal{T}}^{(2)}$ and of $\tilde{\psi}_{\mathcal{T}}^{(2)}$. Furthermore, the linear part $\tilde{\alpha}_{1;1}$ of $\tilde{\psi}_{\mathcal{T}}^{(2)}$ has full rank on every fiber. By the same argument, we can replace $s^{(2)}$ by $\tilde{s}^{(2)}$ to obtain an affine map $\psi_{\mathcal{T}}^{(2)}$ in (8.33) with linear part $\tilde{\alpha}_{2;2}$ that has full rank on every fiber.

Lemma 9.4 *If* $n=2$, $n_1^{(2)}(\mu) = 2|\mathcal{S}_1(\mu)|$. *If* $n=3$, $n_2^{(2)}(\mu) = 2|\mathcal{S}_{2;2}(\mu)|$.

Proof: (1) Suppose $n=2$. By Section 8.9 and the above construction, $n_1^{(2)}(\mu) = N(\tilde{\alpha}_{1;1})$, where

$$E = \tilde{T}\Sigma \otimes L \approx \tilde{T}\Sigma, \quad \mathcal{O} = \mathcal{H}_\Sigma^- \otimes \mathrm{ev}^*T\mathbb{P}^2 \approx \mathcal{H}_\Sigma^- \otimes \mathbb{C}^2, \quad \text{and} \quad \tilde{\alpha}_1 \in \Gamma\big(\Sigma \times \bar{\mathcal{S}}_1(\mu); \mathrm{Hom}(E, \mathcal{O})\big)$$

has full rank on every fiber. Thus, by Lemma 5.14,

$$n_1^{(2)}(\mu) = \big\langle c_1(\mathcal{O}) - c_1(E), [\Sigma \times \bar{\mathcal{S}}_1(\mu)]\big\rangle = 2|\mathcal{S}_1(\mu)|.$$

(2) Suppose $n=3$ and $\mathcal{T} = (\Sigma, [N], I; j, \underline{d})$ is a bubble type that contributes to $n_2^{(2)}(\mu)$. Then, $n_{\mathcal{T}}^{(2)}(\mu) = N(\tilde{\alpha}_{2;2})$, where

$$E = \tilde{T}\Sigma \otimes \big(L_{\hat{1}}\mathcal{T} \oplus L_{\hat{2}}\mathcal{T}\big) \approx \tilde{T}\Sigma \oplus \tilde{T}\Sigma, \quad \mathcal{O} = \mathcal{H}_\Sigma^- \otimes \mathrm{ev}^*T\mathbb{P}^2 \approx \mathcal{H}_\Sigma^- \otimes \mathbb{C}^3,$$
$$\text{and} \qquad \tilde{\alpha}_{2;2} \in \Gamma\big(\Sigma \times \bar{\mathcal{S}}_{\mathcal{T};2}(\mu); \mathrm{Hom}(E, \mathcal{O})\big)$$

has full rank on every fiber. Thus, by Lemma 5.14,

$$n_{\mathcal{T}}^{(2)}(\mu) = \big\langle c_1(\mathcal{O}) - c_1(E), [\Sigma \times \bar{\mathcal{S}}_{\mathcal{T};2}(\mu)]\big\rangle = 2|\mathcal{S}_{\mathcal{T};2}(\mu)|.$$

Summing the last identity over all equivalence classes of bubble types $\mathcal{T}$ as above, we obtain the second claim.

9.3 The Numbers $n_1^{(1)}(\mu)$ for $\mathbb{P}^2$ and $n_2^{(1)}(\mu)$ for $\mathbb{P}^3$

We combine these two cases in the same section because both spaces of rational maps involved, i.e. $\bar{\mathcal{V}}_1(\mu)$ for $\mathbb{P}^2$ and $\bar{\mathcal{V}}_2(\mu)$ for $\mathbb{P}^3$, are two-dimensional over $\mathbb{C}$.

Lemma 9.5 *If* $n=2$, $n_1^{(1)}(\mu)=2\langle 6a^2+3ac_1(\mathcal{L}^*),[\bar{\mathcal{V}}_1(\mu)]\rangle$.

Proof: By Section 8.9, $n_1^{(1)}(\mu)=N(\alpha_1)$, where

$$\alpha_1\in\Gamma\big(\Sigma\times\bar{\mathcal{V}}_1(\mu);\mathrm{Hom}(T\Sigma\otimes L;\mathcal{H}_\Sigma^{0,1}\otimes\mathrm{ev}^*T\mathbb{P}^2)\big),\quad \alpha_1(v\otimes v)=\big(\mathcal{D}v\big)(s_\Sigma v);$$

see equation (8.28). By Corollary 5.17,

$$N(\alpha_1)=2\langle 6a^2+3ac_1(\mathcal{L}^*),[\bar{\mathcal{V}}_1(\mu)]\rangle-2\sum_{\mathcal{D}^{-1}(0)=\sqcup\bar{\mathcal{Z}}_i}\langle c_1(ev^*T\mathbb{P}^2),[\bar{\mathcal{Z}}_i]\rangle,\tag{9.1}$$

where the sum is taken over $\mathcal{D}$-regular subsets in a decomposition of $\mathcal{D}^{-1}(0)$ into $\mathcal{D}$-regular and $\mathcal{D}$-hollow subsets; see Section 5.2. In (9.1), we also used $ac_1(L^*)=ac_1(\mathcal{L}^*)$. Such a decomposition of $\mathcal{D}^{-1}(0)$ is described in the proof of Lemma 6.3. Furthermore, the restriction of $ev^*T\mathbb{P}^2$ to every spaces $\bar{\mathcal{Z}}_i$ in that decomposition is trivial. Thus, the last term in (9.1) is zero, and the claim follows.

Lemma 9.6 *If* $n=3$, $n_2^{(1)}(\mu)=4\langle 10a^2+4a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big)+c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*),[\bar{\mathcal{V}}_2(\mu)]\rangle$.

Proof: (1) We use the same notation as in the proof of Lemma 6.4. By Section 8.9 and equation (8.31), $n_2^{(1)}(\mu)=N(\alpha_2)$, where

$$\alpha_2\in\Gamma\big(\Sigma^2\times\bar{\mathcal{V}}_2(\mu);\mathrm{Hom}(\tilde{E};\mathcal{O})\big),\quad \tilde{E}=T\Sigma_1\otimes L_1\oplus T\Sigma_2\otimes L_2,\quad \mathcal{O}=\mathcal{H}_\Sigma^{0,1}\otimes\mathrm{ev}^*T\mathbb{P}^3,$$
$$\alpha_2\big(x_1,x_2,b;v_1\otimes v_1,v_2\otimes v_2\big)=\big(\mathcal{D}v_1\big)(s_{\Sigma,x_1}v_1)+\big(\mathcal{D}v_2\big)(s_{\Sigma,x_2}v_2).$$

Here the bundles $L_i\longrightarrow\bar{\mathcal{V}}_2(\mu)$ and the sections $\mathcal{D}_i\in\Gamma\big(\bar{\mathcal{V}}_2(\mu);L_i^*\otimes\mathrm{ev}^*T\mathbb{P}^2\big)$ are defined as follows. If $b\in\mathcal{U}_{\mathcal{T}^*}(\mu)\subset\mathcal{V}_2(\mu)$, $\mathcal{T}^*=(S^2,[N],I^*;j^*,\underline{d}^*)$, and $I^*=\{k_1,k_2\}$, we let $L_i|_b=L_{k_i}\mathcal{T}$ and $\mathcal{D}_i=\mathcal{D}_{\mathcal{T},k_i}$. These bundles and sections are well-defined once we fix a representative for each equivalence class of such bubble types $\mathcal{T}^*$ and order the elements of the corresponding set I^*.

(2) By Lemma 5.14,

$$n_1^{(1)}(\mu)=\sum_{k=0}^{k=5}\langle c_k(\mathcal{O})\lambda_{\tilde{E}}^{5-k},[\mathbb{P}\tilde{E}]\rangle-\mathcal{C}_{\alpha_{\tilde{E}}^{-1}(0)}(\alpha_{\tilde{E}}^{\perp}),\tag{9.2}$$
$$=4\langle 28a^2+16a\big(c_1(L_1^*)+c_1(L_2^*)\big)+3\big(c_1^2(L_1^*)+c_1^2(L_2^*)\big)+4c_1(L_1^*)c_1(L_2^*),[\bar{\mathcal{V}}_2(\mu)]\rangle-\mathcal{C}_{\tilde{\alpha}^{-1}(0)}(\tilde{\alpha}^{\perp}),$$

where $\tilde{\alpha}\in\Gamma(\mathbb{P}\tilde{E};\gamma_{\tilde{E}}^*\otimes\mathcal{O})$ is the section induced by α_2. Let

$$\Sigma^{(\pm)}=\big\{(x_1,x_2)\in\Sigma_1^*\times\Sigma_2^*:x_1=\pm x_2\big\},\quad \Sigma^{(0)}=\big\{(z_m,z_m):m\in[6]\big\};$$
$$\mathcal{S}_{2;2}^{(\pm)}=\Sigma^{(\pm)}\times\mathcal{S}_{2;2}(\mu),\quad \mathcal{S}_{2;2}^{(0)}=\Sigma^{(0)}\times\mathcal{S}_{2;2}(\mu),$$

where $+x_2\equiv x_2$ and $-x_2$ is the image of x_2 under the nontrivial automorphism of Σ. The zero set of $\tilde{\alpha}$ is the union of a section of $\mathbb{P}\tilde{E}$ over $\mathcal{S}_{2;2}^{(\pm)}$, $\mathcal{S}_{2;2}^{(0)}$, and $\Sigma^2\times\mathcal{U}_{\mathcal{T}}(\mu)$, where $\mathcal{T}$ is as

in the proof of Lemma 6.4.
(3) The above section over $\Sigma^2\times\mathcal{U}_{\mathcal{T}}(\mu)$ is given by

$$\mathcal{Z}_{\mathcal{T}}\equiv\tilde{\alpha}^{-1}(0)\cap\big(\mathbb{P}\tilde{E}|\Sigma^2\times\mathcal{U}_{\mathcal{T}}(\mu)\big)=\big\{(x_1,x_2,b,T_{x_1}\Sigma_1\otimes L_1|_b):(x_1,x_2,b)\in\Sigma^2\times\mathcal{U}_{\mathcal{T}}(\mu)\big\}.$$

The map $\gamma_{\mathcal{T}}^{\mu}$ of Theorem 6.2 induces identifications of neighborhoods of $\mathcal{Z}_{\mathcal{T}}$ in

$$\mathcal{FS}=\pi_{\tilde{E}}^*\big(\mathcal{FT}\oplus T^*\Sigma_1\otimes L_1^*\otimes T\Sigma_2\otimes L_2\big)$$

and in $\mathbb{P}\tilde{E}$ as well as of appropriate bundles such that

$$\begin{gathered}\Big|\tilde{\alpha}\big(\gamma_{\mathcal{T}}^{\mu}(v,u)\big)-\tilde{\alpha}_{\mathcal{T}}\big(\rho_{\mathcal{T}}(v),u\big)\Big|\le C(b_v)|v|^{\frac{1}{p}}|\rho_{\mathcal{T}}(v)|\quad\forall(v,u)\in\mathcal{FS}_\delta,\quad\text{where}\\ \tilde{\alpha}_{\mathcal{T}}\in\Gamma\big(\mathcal{Z}_{\mathcal{T}};\mathrm{Hom}(\tilde{\mathcal{F}}\mathcal{S};\gamma_{\tilde{E}}^*\otimes\mathcal{O})\big),\quad\tilde{\mathcal{F}}\mathcal{S}=\pi_{\tilde{E}}^*\big(\tilde{\mathcal{F}}\mathcal{T}\oplus T^*\Sigma_1\otimes L_1^*\otimes T\Sigma_2\otimes L_2\big),\\ \tilde{\alpha}_{\mathcal{T}}(x_1,x_2,b;\tilde{v},u)=\big\{\alpha_{\mathcal{T}}(\tilde{v})\big\}\otimes s_{x_1}+\big(\mathcal{D}_2\otimes s_{x_2}\big)\circ u.\end{gathered}$$

By the proof of Lemma 8.9, $\tilde{\alpha}_{\mathcal{T}}$ is nondegenerate. The same is true of $\tilde{\alpha}^{\perp}$ as long as $\bar{\nu}\in\Gamma(\mathbb{P}\tilde{E};\mathcal{O})$ is generic. Thus, if $\hat{I}\neq H_{\hat{0}}\mathcal{T}$, $\mathcal{Z}_{\mathcal{T}}$ is $\tilde{\alpha}^{\perp}$-hollow and $\mathcal{C}_{\mathcal{Z}_{\mathcal{T}}}(\tilde{\alpha}^{\perp})=0$ by Proposition 5.13. If $|H_{k_1}\mathcal{T}|=|\hat{I}|=1$, i.e. $\mathcal{T}=\mathcal{T}^*(l)$ for some $l\in[N]$, $\tilde{\alpha}_{\mathcal{T}}$ has full rank on $\bar{\mathcal{Z}}_{\mathcal{T}}\approx\Sigma^2\times\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$. Thus, by Proposition 5.13,

$$\mathcal{C}_{\mathcal{Z}_{\mathcal{T}}}(\alpha^{\perp})=\big\langle c(\gamma_{\tilde{E}}^*\otimes\mathcal{O}^{\perp})c(\mathcal{FS})^{-1},[\bar{\mathcal{Z}}_{\mathcal{T}}]\big\rangle=4\big\langle 16a+4c_1(L_2^*)+3c_1(L_{\hat{1}}^*\mathcal{T}),\big[\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\big]\big\rangle,\tag{9.3}$$

since $F\mathcal{S}\approx L_{\hat{1}}\mathcal{T}\oplus T^*\Sigma_1\otimes T\Sigma_2\otimes L_2$. If $|H_{k_1}\mathcal{T}|=|\hat{I}|=2$, we similarly obtain

$$\mathcal{C}_{\mathcal{Z}_{\mathcal{T}}}(\alpha_{\tilde{E}}^{\perp})=\big\langle c(\gamma_{\tilde{E}}^*\otimes\mathcal{O}^{\perp})c(\mathcal{FS})^{-1},[\bar{\mathcal{Z}}_{\mathcal{T}}]\big\rangle=12|\mathcal{U}_{\mathcal{T}}(\mu)|\tag{9.4}$$

Note that $F\mathcal{S}\approx\mathbb{C}^2\oplus T^*\Sigma_1\otimes T\Sigma_2$ in this case. Summing up equations (9.3) and (9.4) over all equivalence classes of bubble types $\mathcal{T}$ of the appropriate form and using (6.14) and (6.15), we obtain

$$\mathcal{C}_{\mathbb{P}\tilde{E}|\partial\mathcal{V}_2(\mu)}(\alpha_{\tilde{E}}^{\perp})=4\sum_{[\mathcal{T}^*]\in M_i^*,i\neq j}\sum\big\langle 16a+3c_1(\mathcal{L}_i^*)+4c_1(\mathcal{L}_j^*),\big[\mathcal{U}_{\mathcal{T}^*(l)}(\mu)\big]\big\rangle-36\big|\mathcal{V}_3(\mu)\big|,\tag{9.5}$$

where the outer sum is taken over equivalence classes of bubbles $\mathcal{T}^*$ as in (1) above.
(4) It remains to compute $\mathcal{C}_{\mathbb{P}\tilde{E}|\mathcal{S}_{2;2}^{\pm}}(\alpha^{\perp})$ and $\mathcal{C}_{\mathbb{P}\tilde{E}|\mathcal{S}_{2;2}^{(0)}}(\alpha^{\perp})$. Note that

$$\alpha^{-1}(0)\cap\big(\mathbb{P}\tilde{E}|\mathcal{S}_{2;2}^{(\pm)}\big)=\mathcal{Z}_{2;2}^{(\pm)}\equiv\big\{(x,\pm x,b;[v\otimes v_1,v\otimes v_2])\in\mathbb{P}\tilde{E}|_{\mathcal{S}_{2;2*}^{(\pm)}}:\mathcal{D}|_{(b;[v_1,v_2])}=0\big\},$$

where $\mathcal{D}$ is the section of $\gamma_{\tilde{E}}^*\otimes\mathrm{ev}^*T\mathbb{P}^3$ defined in the proof of Lemma 6.4. Identify neighborhoods of $\mathcal{Z}_{2;2}^{(\pm)}$ in

$$\mathcal{FS}\equiv T\Sigma\oplus\gamma_{\tilde{E}}^*\otimes\mathrm{ev}^*T\mathbb{P}^3\approx T\Sigma\oplus\mathbb{C}^3$$

and in $\mathbb{P}\tilde{E}$ via a map γ in such a way that

$$\begin{gathered}\big|\tilde{\alpha}\big(\gamma(w,X)\big)-\alpha_{\mathcal{S}}(w,X)\big|\le C(x,b)\big(|w|+|X|\big)|w\quad\forall(w,X)\in\mathcal{FS}_\delta,\\ \text{where}\qquad\alpha_{\mathcal{S}}\in\Gamma\big(\mathcal{Z}_{2;2};\mathrm{Hom}(\mathcal{FS};\gamma_{\tilde{E}}^*\otimes\mathcal{O})\big),\\ \big\{\alpha_{\mathcal{S}}(w,X)\big\}(v\otimes v)=(Xv)(s_xv)+\big(\mathcal{D}_2v_2\big)\big(s_{b,x}^{(2)}(w,v)\big)\in\mathcal{O},\quad\text{if }\ v\in T_x\Sigma,\ v=(v_1,v_2)\in\gamma_E.\end{gathered}$$

Since $s_x^{(2)}=\pi_x^-\circ s_{b,x}^{(2)}$ does not vanish on Σ^*, $\alpha_{\mathcal{S}}$ has full rank on $\mathcal{Z}_{2;2}^{(\pm)}$ and extends over $\bar{\mathcal{Z}}_{2;2}^{(\pm)}\approx\Sigma\times\mathcal{S}_{2;2}(\mu)$. This extension is a regular polynomial in the sense of Definition 5.9. Furthermore,

$$\pi_{\bar{\nu}}^{\perp}\alpha_{\mathcal{S}}\colon \gamma_{\tilde{E}}^*\otimes \mathrm{ev}^*T\mathbb{P}^3 \longrightarrow \gamma_{\tilde{E}}^*\otimes \pi_{\bar{\nu}}^{\perp}\big(\mathcal{H}_{\Sigma}^+\otimes \mathrm{ev}^*T\mathbb{P}^3\big)$$

Thus, by Proposition 5.13, $\mathcal{C}_{\mathcal{Z}_{2;2}}^{(\pm)}(\tilde{\alpha})=N(\alpha_{\mathcal{S}}^-)$, where

$$\alpha_{\mathcal{S}}^-\in\Gamma\big(\bar{\mathcal{Z}}_{2;2};\mathrm{Hom}(T\Sigma;\mathcal{O}_2)\big),\quad \mathcal{O}_2=\gamma_{\tilde{E}}^*\otimes\big(\mathcal{H}_{\Sigma}^-\otimes\mathrm{ev}^*T\mathbb{P}^3\big)^{\perp}\approx T^*\Sigma\otimes\big(\mathcal{H}_{\Sigma}^-\otimes\mathbb{C}^3\big)^{\perp},$$
$$\big\{\alpha_{\mathcal{S}}^-(w)\big\}(v\otimes v)=\pi_{\pi_x^-\bar{\nu}}^{\perp}\big((\mathcal{D}_2v_2)s_x^{(2)}(w,v)\big).$$

As in the previous section, we can replace $T\Sigma$ with

$$\tilde{T}'\Sigma\equiv T\Sigma\otimes\mathcal{O}(z_1)\otimes\ldots\otimes\mathcal{O}(z_6)$$

and $s^{(2)}$ with $\tilde{s}^{(2)}\in\Gamma(\Sigma;\tilde{T}'\Sigma^*\otimes\mathcal{H}_{\Sigma}^-)$ above to obtain a non-vanishing linear map $\tilde{\alpha}_{\mathcal{S}}^-$ such that $N(\alpha_{\mathcal{S}}^-)=N(\tilde{\alpha}_{\mathcal{S}}^-)$. Thus, by Lemma 5.14,

$$\mathcal{C}_{\mathcal{Z}_{2;2}^{(+)}\cup\mathcal{Z}_{2;2}^{(-)}}(\tilde{\alpha}^{\perp})=2\langle c_1(\mathcal{O}_2)-c_1(\tilde{T}'\Sigma),[\mathcal{Z}_{2;2}^{(+)}]\rangle=2(10-4)|\mathcal{S}_{2;2}(\mu)|=12|\mathcal{S}_{2;2}(\mu)|. \quad (9.6)$$

(5) We next show that $\mathcal{C}_{\mathbb{P}\tilde{E}|\mathcal{S}_{2;2}^{(0)}}(\tilde{\alpha}^{\perp})=0$. Similarly to (4),

$$\alpha^{-1}(0)\cap\big(\mathbb{P}\tilde{E}|\mathcal{S}_{2;2}^{(0)}\big)=\mathcal{Z}_{2;2}^{(0)}\equiv\big\{\big(z_m,z_m,b;[v\otimes v_1,v\otimes v_2]\big)\in\mathbb{P}\tilde{E}\big|_{\mathcal{S}_{2;2}^{(0)}}:\mathcal{D}|_{(b;[v_1,v_2])}=0\big\}.$$

We can identify neighborhoods of $\mathcal{Z}_{2;2}^{(0)}$ in

$$\mathcal{FS}\equiv T\Sigma_1\oplus T\Sigma_2\oplus\gamma_{\tilde{E}}^*\otimes\mathrm{ev}^*T\mathbb{P}^3\approx\mathbb{C}^2\oplus\mathbb{C}^3$$

and in $\mathbb{P}\tilde{E}$ via a map γ in such a way that

$$\big|\pi_{w_1}^-\circ\tilde{\alpha}\big(\gamma(w_1,w_2,X)-\alpha_{\mathcal{S}}^-(w_1,w_2,X)\big|\le C\big|X,w_1,w_2\big||w_1||w_1-w_2|\quad\forall(w_1,w_2,X)\in\mathcal{FS}_{\delta},$$
$$\text{where}\quad\big\{\alpha_{\mathcal{S}}^-(w_1,w_2)\big\}(v\otimes v_1,v\otimes v_2)=\big(\mathcal{D}_2v_2\big)\big(s_{z_m}^{(3)}(w_1,w_2-w_1,v)\in\mathcal{H}_{\Sigma}^-(z_m)\otimes\mathrm{ev}^*T\mathbb{P}^3.$$

Since the rank of $\big(\mathcal{H}_{\Sigma}^-(z_m)\otimes\mathrm{ev}^*T\mathbb{P}^3\big)/\mathbb{C}\pi_{z_m}^-\bar{\nu}$ is two, while the rank of $T_{z_m}\otimes T_{z_m}$ is one, it follows that $\mathcal{Z}_{2;2}^{(0)}$ is $\tilde{\alpha}$-hollow, and $\mathcal{C}_{\mathbb{P}\tilde{E}|\mathcal{S}_{2;2}^{(0)}}(\tilde{\alpha}^{\perp})=0$ by Proposition 5.13. The lemma is obtained by plugging (9.5) and (9.6) into (9.2), using (6.14) and (6.15), and applying Lemma 6.4.

9.4 The Numbers $n_1^{(2)}(\mu)$ and $n_1^{(3)}(\mu)$ for $\mathbb{P}^3$

In this section, we give topological expressions for the numbers $n_1^{(2)}(\mu)$ and $n_1^{(3)}(\mu)$ in the $n=3$ case.

Lemma 9.7 *If $n=3$, $n_1^{(2)}(\mu)=4\langle 2a+c_1(\mathcal{L}^*),[\bar{\mathcal{S}}_1(\mu)]\rangle-2\big|\mathcal{S}_{2;2}(\mu)\big|$.*

Proof: (1) By Section 8.9, (see equation (8.29)) and an argument as in Section 9.2, $n_1^{(2)}(\mu) = N(\tilde{\alpha}_{1,2})$, where

$$\tilde{\alpha}_{1,2} \in \Gamma\big(\Sigma \times \bar{\mathcal{S}}_1(\mu); \mathrm{Hom}(\tilde{T}\Sigma \otimes L^{\otimes 2}; \mathcal{O})\big), \quad \mathcal{O} = \mathcal{H}_\Sigma^- \otimes \mathrm{ev}^* T\mathbb{P}^3,$$
$$\tilde{\alpha}_{1,2}(x, b; w \otimes v) = (\mathcal{D}^{(2)} v)(s_x^{(2)} w).$$

Since $\tilde{\alpha}_{1,2}$ does not vanish on $\Sigma \times \mathcal{S}_1(\mu)$ (see Section 8.4), by Lemma 5.14,

$$\begin{aligned} n_1^{(2)}(\mu) &= \sum_{k=0}^{k=2} \langle c_1^{2-k}(\tilde{T}\Sigma^* \otimes L^{*\otimes 2}) c_k(\mathcal{O}), [\Sigma \times \bar{\mathcal{S}}_1(\mu)] \rangle - \mathcal{C}_{\Sigma \times \partial \bar{\mathcal{S}}_1}(\tilde{\alpha}_{1,2}^\perp) \\ &= 4\langle 2a + c_1(L^*), [\bar{\mathcal{S}}_1(\mu)] \rangle - \mathcal{C}_{\Sigma \times \partial \bar{\mathcal{S}}_1}(\tilde{\alpha}_{1,2}^\perp). \end{aligned} \tag{9.7}$$

(2) If $\mathcal{T} = (S^2, [N], I; j, \underline{d}) < \mathcal{T}^* \equiv (S^2, [N], \{\hat{0}\}; \hat{0}, d)$ and $\mathcal{S}_\mathcal{T}(\mu) \neq \emptyset$, $\mathcal{T}$ must have one of the three forms given by Lemma 6.8. Since $\mathcal{D}^{(2)}$ does not vanish on $\mathcal{S}_\mathcal{T}(\mu)$ in Case (1) of Lemma 6.8, $\mathcal{C}_{\Sigma \times \mathcal{S}_\mathcal{T}(\mu)}(\alpha^\perp) = 0$ in this case. In Case (2), i.e. $\mathcal{T} = \mathcal{T}^*(l)$ for some $l \in [N]$, $\mathcal{D}_{\mathcal{T},\hat{1}}^{(2)}$ does not vanish over $\mathcal{S}_\mathcal{T}(\mu)$; see Section 8.5. Thus, the first estimate of Lemma 6.8 and Proposition 5.13,

$$\mathcal{C}_{\Sigma \times \mathcal{S}_\mathcal{T}(\mu)}(\tilde{\alpha}_{2;1}^\perp) = 2\langle c_1(\tilde{T}\Sigma^* \otimes \mathcal{O}^\perp) - c_1(\mathcal{FS}), [\Sigma \times \mathcal{S}_\mathcal{T}(\mu)] \rangle = 4|\mathcal{S}_\mathcal{T}(\mu)|. \tag{9.8}$$

(3) Suppose $\mathcal{T}$ is as in Case (3) of Lemma 6.8. Since $x_{\hat{1}} \neq x_{\hat{2}}$, $\mathcal{D}_{\mathcal{T},\hat{1}}$ and $\mathcal{D}_{\mathcal{T},\hat{2}}$ do not vanish on $\mathcal{S}_\mathcal{T}(\mu) = \mathcal{U}_\mathcal{T}(\mu) \cap \mathcal{S}_2(\mu)$ (see Section 8.6), and $\mathcal{D}_{\mathcal{T},\hat{1}} + \mathcal{D}_{\mathcal{T},\hat{2}}$ vanishes on $\mathcal{FS}$, $x_{\hat{1}}\mathcal{D}_{\mathcal{T},\hat{1}} + x_{\hat{2}}\mathcal{D}_{\mathcal{T},\hat{2}}$ does not vanish on $\mathcal{FS}$. Thus, the third estimate of Lemma 6.8 and Proposition 5.13,

$$\mathcal{C}_{\Sigma \times \mathcal{S}_\mathcal{T}(\mu)}(\alpha^\perp) = \langle c_1(\tilde{T}\Sigma^* \otimes \mathcal{O}^\perp) - c_1(\mathcal{FS}), [\Sigma \times \mathcal{S}_\mathcal{T}(\mu)] \rangle = 2|\mathcal{S}_\mathcal{T}(\mu)|. \tag{9.9}$$

Summing up equations (9.8) and (9.9) over all appropriate bubble types $\mathcal{T} < \mathcal{T}^*$ and substituting the result into (9.7), we obtain the claim.

Lemma 9.8 *If* $n=3$, $n_1^{(3)}(\mu) = \langle 4a + 5c_1(\mathcal{L}^*), [\bar{\mathcal{S}}_1(\mu)] \rangle - 3|\mathcal{S}_{2;2}(\mu)|$.

Proof: (1) By equation (8.30), $n_1^{(3)}(\mu) = N(\alpha_{1,3})$, where

$$\alpha_{1,3} \in \Gamma\big(\bar{\mathcal{S}}_1(\mu); \mathrm{Hom}(E; \mathcal{O})\big), \quad E = L^{\otimes 2} \oplus L^{\otimes 3}, \quad \mathcal{O} = \mathrm{ev}^* T\mathbb{P}^3,$$
$$\alpha_{1,3}(v_2, v_3) = \mathcal{D}^{(2)} v_2 + \mathcal{D}^{(3)} v_3.$$

Since $\alpha_{1,3}$ has full rank on $\mathcal{S}_1(\mu)$ (see Section 8.5),

$$\begin{aligned} n_1^{(3)}(\mu) &= \sum_{k=0}^{k=2} \langle \lambda_E^{2-k} c_k(\mathcal{O}), [\mathbb{P}E] \rangle - \mathcal{C}_{\mathbb{P}E|\partial \bar{\mathcal{S}}_1}(\tilde{\alpha}^\perp) \\ &= \langle 4a + 5c_1(L^*), [\bar{\mathcal{S}}_1(\mu)] \rangle - \mathcal{C}_{\mathbb{P}E|\partial \bar{\mathcal{S}}_1}(\tilde{\alpha}^\perp), \end{aligned} \tag{9.10}$$

where $\tilde{\alpha} \in \Gamma(\mathbb{P}E; \gamma_E^* \otimes \mathcal{O})$ is the section induced by $\alpha_{1,3}$.
(2) As in the proof of Lemma 9.7, $\mathcal{C}_{\mathbb{P}E|\mathcal{S}_\mathcal{T}(\mu)}(\tilde{\alpha}) = 0$ for bubble types $\mathcal{T}$ of Case (1) of Lemma 6.8. Suppose $\mathcal{T} = \mathcal{T}^*(l)$ for some $l \in [N]$, i.e. we are in Case (2) of Lemma 6.8. The map γ_S of Lemma 6.6 indices an identification of neighborhoods of $\mathbb{P}E|\mathcal{S}_\mathcal{T}(\mu)$ in $\mathcal{FS} \approx \mathbb{C}$

and in $\mathbb{P}E$ such that

$$\Big|\tilde{\alpha}\big(\gamma_{\mathcal{S}}([v_2,v_3],v)\big)-\tilde{\alpha}_{\mathcal{T}}\big([v_2,v_3],v\big)\Big|\le C|v|^{2+\frac{1}{p}}\quad\forall\big([v_2,v_3],v\big)\in\pi_E^*\mathcal{FS}_\delta\quad\text{where}$$

$$\tilde{\alpha}_{\mathcal{T}}\in\Gamma\big(\mathbb{P}E|\mathcal{S}_{\mathcal{T}}(\mu);\mathrm{Hom}(\mathcal{FS}^{\otimes 2},\gamma_E^*\otimes\mathcal{O})\big),$$

$$\big\{\tilde{\alpha}_{\mathcal{T}}(v)\big\}(v_2,v_3)=\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}\big(v^{\otimes 2}\otimes v_2\big)+3\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}\big(x_{\hat{1}}v^{\otimes 2}\otimes v_3\big);$$

see first and second estimates on Lemma 6.8. In particular, $\tilde{\alpha}_{\mathcal{T}}$ vanishes only on the subset

$$\mathcal{Z}_{\mathcal{T}}\equiv\big\{[v_2,v_3]\in\mathbb{P}E|\mathcal{S}_{\mathcal{T}}(\mu):\,v_2+3x_{\hat{1}}v_3=0\big\}.$$

Thus, by Proposition 5.13 and Lemma 5.14,

$$\begin{aligned}\mathcal{C}_{\mathbb{P}E|(\mathcal{S}_{\mathcal{T}}(\mu)-\mathcal{Z}_{\mathcal{T}})}(\tilde{\alpha}^{\perp})&=2\Big(\big\langle c_1\big(\gamma_E^*\otimes\mathcal{O}^{\perp}\big)-c_1(\mathcal{FS}),\big[\mathbb{P}E|\mathcal{S}_{\mathcal{T}}(\mu)\big]\big\rangle-\mathcal{C}_{\mathcal{Z}_{\mathcal{T}}}\big((\pi_{\bar{\nu}}^{\perp}\circ\tilde{\alpha}_{\mathcal{T}})^{\perp}\big)\Big)\\&=4|\mathcal{S}_{\mathcal{T}}(\mu)|-2\mathcal{C}_{\mathcal{Z}_{\mathcal{T}}}\big((\pi_{\bar{\nu}}^{\perp}\circ\tilde{\alpha}_{\mathcal{T}})^{\perp}\big).\end{aligned}$$

By definition of $\tilde{\alpha}_{\mathcal{T}}$ and Proposition 5.13, $\mathcal{C}_{\mathcal{Z}_{\mathcal{T}}}\big((\pi_{\bar{\nu}}^{\perp}\circ\tilde{\alpha}_{\mathcal{T}})^{\perp}\big)=|\mathcal{Z}_{\mathcal{T}}|$. On the other hand, since the images of $\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}$ and $\mathcal{D}^{(3)}_{\mathcal{T},\hat{1}}$ are linearly independent in every fiber of $\mathrm{ev}^*T\mathbb{P}^3$ over $\mathcal{S}_{\mathcal{T}}(\mu)$, by the first two estimates of Lemma 6.8 and Proposition 5.13, $\mathcal{C}_{\mathcal{Z}_{\mathcal{T}}}(\tilde{\alpha}^{\perp})=3|\mathcal{Z}_{\mathcal{T}}|$. Thus,

$$\mathcal{C}_{\mathbb{P}E|\mathcal{S}_{\mathcal{T}}(\mu)}(\tilde{\alpha}^{\perp})=\big(4|\mathcal{S}_{\mathcal{T}}(\mu)|-2|\mathcal{S}_{\mathcal{T}}(\mu)|\big)+3|\mathcal{S}_{\mathcal{T}}(\mu)|=5|\mathcal{S}_{\mathcal{T}}(\mu)|.\tag{9.11}$$

(3) Suppose $\mathcal{T}$ is as in Case (3). The map $\gamma_{\mathcal{S}}$ of Lemma 6.6 indices an identification of neighborhoods of $\mathbb{P}E|\mathcal{S}_{\mathcal{T}}(\mu)$ in $\mathcal{FS}\approx\mathbb{C}$ and in $\mathbb{P}E$ such that

$$\Big|\tilde{\alpha}^{\perp}\big(\gamma_{\mathcal{S}}([v_2,v_3],v)\big)-\tilde{\alpha}_{\mathcal{T}}\big([v_2,v_3],v\big)\Big|\le C|v|^{1+\frac{1}{p}}\quad\forall\big([v_2,v_3],v\big)\in\pi_E^*\mathcal{FS}_\delta\quad\text{where}$$

$$\tilde{\alpha}_{\mathcal{T}}\in\Gamma\big(\mathbb{P}E|\mathcal{S}_{\mathcal{T}}(\mu);\mathrm{Hom}(\mathcal{FS},\gamma_E^*\otimes\mathcal{O})\big),$$

$$\begin{aligned}\big\{\tilde{\alpha}_{\mathcal{T}}(v_{\hat{1}},v_{\hat{2}})\big\}(v_2,v_3)=&\,2\Big(\mathcal{D}^{(1)}_{\mathcal{T},\hat{1}}\big(x_{\hat{1}}v_{\hat{1}}\otimes v_2\big)+\mathcal{D}^{(1)}_{\mathcal{T},\hat{2}}\big(x_{\hat{2}}v_{\hat{2}}\otimes v_2\big)\Big)\\&+6(x_{\hat{1}}+x_{\hat{2}})\Big(\mathcal{D}^{(1)}_{\mathcal{T},\hat{1}}\big(x_{\hat{1}}v_{\hat{1}}\otimes v_3\big)+\mathcal{D}^{(1)}_{\mathcal{T},\hat{2}}\big(x_{\hat{2}}v_{\hat{2}}\otimes v_3\big)\Big);\end{aligned}$$

see the last two estimates on Lemma 6.8. In particular, $\tilde{\alpha}_{\mathcal{T}}$ vanishes only on the subset

$$\mathcal{Z}_{\mathcal{T}}\equiv\big\{[v_2,v_3]\in\mathbb{P}E|\mathcal{S}_{\mathcal{T}}(\mu):\,v_2+3(x_{\hat{1}}+x_{\hat{2}})v_3=0\big\}.$$

Thus, by Proposition 5.13 and Lemma 5.14,

$$\begin{aligned}\mathcal{C}_{\mathbb{P}E|(\mathcal{S}_{\mathcal{T}}(\mu)-\mathcal{Z}_{\mathcal{T}})}(\tilde{\alpha}^{\perp})&=\big\langle c_1\big(\gamma_E^*\otimes\mathcal{O}^{\perp}\big)-c_1(\mathcal{FS}),\big[\mathbb{P}E|\mathcal{S}_{\mathcal{T}}(\mu)\big]\big\rangle-\mathcal{C}_{\mathcal{Z}_{\mathcal{T}}}\big((\pi_{\bar{\nu}}^{\perp}\circ\tilde{\alpha}_{\mathcal{T}})^{\perp}\big)\\&=2|\mathcal{S}_{\mathcal{T}}(\mu)|-\mathcal{C}_{\mathcal{Z}_{\mathcal{T}}}\big((\pi_{\bar{\nu}}^{\perp}\circ\tilde{\alpha}_{\mathcal{T}})^{\perp}\big).\end{aligned}$$

By definition of $\tilde{\alpha}_{\mathcal{T}}$ and Proposition 5.13, $\mathcal{C}_{\mathcal{Z}_{\mathcal{T}}}\big((\pi_{\bar{\nu}}^{\perp}\circ\tilde{\alpha}_{\mathcal{T}})^{\perp}\big)=|\mathcal{Z}_{\mathcal{T}}|$. On the other hand, by the last two estimates of Lemma 6.8 and Proposition 5.13, $\mathcal{C}_{\mathcal{Z}_{\mathcal{T}}}(\tilde{\alpha}^{\perp})=2|\mathcal{Z}_{\mathcal{T}}|$. Thus,

$$\mathcal{C}_{\mathbb{P}E|\mathcal{S}_{\mathcal{T}}(\mu)}(\alpha^{\perp})=\big(2|\mathcal{S}_{\mathcal{T}}(\mu)|-|\mathcal{S}_{\mathcal{T}}(\mu)|\big)+2|\mathcal{S}_{\mathcal{T}}(\mu)|=3|\mathcal{S}_{\mathcal{T}}(\mu)|.\tag{9.12}$$

The claim follows by summing up (9.11) and (9.12) over the appropriate equivalence classes of bubble types $\mathcal{T}<\mathcal{T}^*$, plugging the result back into (9.10), and using (6.14).

9.5 The Numbers $n_1^{(1)}(\mu)$ for $\mathbb{P}^3$

Lemma 9.9 *If* $n=3$,

$$n_1^{(1)}(\mu) = 4\big\langle 10a^3c_1(\mathcal{L}^*) + 3a^2c_1^2(\mathcal{L}^*), [\bar{\mathcal{V}}_1(\mu)]\big\rangle - 12\big\langle a^2, [\bar{\mathcal{V}}_2(\mu)]\big\rangle.$$

Proof: (1) We use the same notation as in the proof of Lemma 6.9. By equation (8.28), $n_1^{(1)}(\mu)=N(\alpha_1)$, where

$$\alpha_1 \in \Gamma\big(\Sigma\times\bar{\mathcal{V}}_1(\mu); \mathrm{Hom}(T\Sigma\otimes L; \mathcal{O})\big), \quad \mathcal{O}=\mathcal{H}_\Sigma^{0,1}\otimes \mathrm{ev}^*T\mathbb{P}^3, \quad \alpha_1(x,b;w,v_2) = \big(\mathcal{D}v\big)(s_x w).$$

Since s has no zeros on Σ, by Lemma 5.14,

$$\begin{aligned} n_1^{(1)}(\mu) &= \sum_{k=0}^{k=5}\big\langle c_k(\mathcal{O})c_1^{5-k}(T^*\Sigma\otimes L^*), [\Sigma\times\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)]\big\rangle - \mathcal{C}_{\Sigma\times\mathcal{D}^{-1}(0)}(\alpha_1^\perp), \\ &= 2\big\langle 112a^3c_1(L^*) + 84a^2c_1^2(L^*) + 32ac_1^3(L^*) + 5c_1^4(L^*), [\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)]\big\rangle - \mathcal{C}_{\Sigma\times\mathcal{D}^{-1}(0)}(\alpha_1^\perp). \end{aligned} \tag{9.13}$$

Let $\mathcal{O}_2 = T^*\Sigma\otimes L^*\otimes\mathcal{O}^\perp$.
(2) By Corollary C.3, we can identify neighborhoods of $\mathcal{S}_1(\mu)$ in $E_2=L^*\otimes\mathrm{ev}^*T\mathbb{P}^3$ and in $\bar{\mathcal{V}}_1(\mu)$ via a map γ in such a way that $\mathcal{D}\big(\gamma(X)\big)=X$ for all $X\in E_2$ sufficiently small. Then,

$$\begin{aligned} \alpha_1\big(\gamma(X)\big) &= \alpha_{\mathcal{S}}(X) \quad \forall X\in F_{2,\delta}, \qquad \text{where} \\ \alpha_{\mathcal{S}} \in \Gamma\big(\Sigma\times\mathcal{S}_1(\mu); &\mathrm{Hom}(F_2, T^*\Sigma\otimes L^*\otimes\mathcal{O})\big), \quad \alpha_{\mathcal{S}}(X) = X\otimes s. \end{aligned}$$

Since $\alpha_{\mathcal{S}}$ extends over $\Sigma\times\bar{\mathcal{S}}_1(\mu)$ as an everywhere injective linear map, by Proposition 5.13,

$$\begin{aligned} \mathcal{C}_{\Sigma\times\mathcal{S}_1(\mu)}(\alpha_1^\perp) &= \big\langle c(T^*\Sigma\otimes L^*\otimes\mathcal{O}^\perp)c(F_2)^{-1}, [\Sigma\times\bar{\mathcal{S}}_1(\mu)]\big\rangle \\ &= 2\big\langle 12a + 5c_1(L^*), [\bar{\mathcal{S}}_1(\mu)]\big\rangle. \end{aligned} \tag{9.14}$$

On the other hand, if $\mathcal{T}=(S^2,[N],I;j,\underline{d})<\mathcal{T}^*$, by Corollary 6.7, $\mathcal{S}_{\mathcal{T}}(\mu)$ is $\alpha_1^\perp$-hollow. Thus, $\mathcal{C}_{\Sigma\times\mathcal{S}_{\mathcal{T}}(\mu)}(\alpha^\perp)=0$ by Proposition 5.13.
(3) With $\mathcal{T}$ as above, if $H_{\hat{0}}\mathcal{T}\neq\hat{I}$ or $d_{\hat{0}}\neq 0$, $\mathcal{S}_{\mathcal{T}}(\mu)=\emptyset$. Thus, by Theorem 6.2, $\Sigma\times\mathcal{U}_{\mathcal{T}}(\mu)$ is $\alpha_1^\perp$-hollow, and $\mathcal{C}_{\Sigma\times\mathcal{U}_{\mathcal{T}}(\mu)}(\alpha_1^\perp)=0$ by Proposition 5.13. On the other hand, if $H_{\hat{0}}\mathcal{T}=\hat{I}$ or $d_{\hat{0}}=0$, by Theorem 6.2, Proposition 5.13, and the splitting (6.17),

$$\begin{aligned} \mathcal{C}_{\Sigma\times(\mathcal{U}_{\mathcal{T}}(\mu)}(\alpha_1^\perp) &= N\big(\pi_{\bar{\mathcal{V}}}^\perp\circ(s\otimes\alpha_{\mathcal{T}})\big), \qquad \text{where} \\ \pi_{\bar{\mathcal{V}}}^\perp\circ(s\otimes\alpha_{\mathcal{T}}) &\in \Gamma\big(\Sigma\times\bar{\mathcal{M}}_{0,H_{\hat{0}}\mathcal{T}+M_{\hat{0}}\mathcal{T}}\times\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu); \mathrm{Hom}(L^*\otimes E\bar{\mathcal{T}}, T^*\Sigma\otimes L^*\otimes\mathcal{O}^\perp)\big), \\ E\bar{\mathcal{T}} = \bigoplus_{h\in H_{\hat{0}}\mathcal{T}} L_h\bar{\mathcal{T}}, &\quad \{\pi_{\bar{\mathcal{V}}}^\perp\circ(s\otimes\alpha_{\mathcal{T}})\}(w\otimes v) = \pi_{\bar{\mathcal{V}}}^\perp\Big((s_x w)\sum_{h\in H_{\hat{0}}\mathcal{T}}\mathcal{D}_{\bar{\mathcal{T}},h}(v_h\otimes v)\Big). \end{aligned}$$

Thus, by Lemma 5.14,

$$\mathcal{C}_{\Sigma\times(\mathcal{U}_{\mathcal{T}}(\mu)-\mathcal{S}_{\mathcal{T}}(\mu))}(\alpha_1^\perp) = \sum_{k=0}^{k=4}\big\langle \lambda_{L^*\otimes E\bar{\mathcal{T}}}^{4-k}c_k(\mathcal{O}_2), [\mathbb{P}(L^*\otimes E\bar{\mathcal{T}})]\big\rangle - \mathcal{C}_{\tilde{\alpha}_{\mathcal{T}}^{-1}(0)}(\tilde{\alpha}_{\mathcal{T}}^\perp), \tag{9.15}$$

where $\tilde{\alpha}_{\mathcal{T}}\in\Gamma(\mathbb{P}(L^*\otimes E\bar{\mathcal{T}});\gamma^*_{L^*\otimes E\bar{\mathcal{T}}}\otimes\mathcal{O}_2)$ is the section induced by $\pi^{\perp}_{\bar{\nu}}\circ(s\otimes\alpha_{\mathcal{T}})$.
(4) If $\mathcal{T}=\mathcal{T}^*(l)$ for some $l\in[N]$, $\tilde{\mathcal{M}}_{0,H_{\hat{0}}\mathcal{T}+M_{\hat{0}}\mathcal{T}}$ is a point. Thus, by (9.15),

$$\mathcal{C}_{\Sigma\times(\mathcal{U}_{\mathcal{T}}(\mu)-\mathcal{S}_{\mathcal{T}}(\mu))}(\alpha_1)=2\big\langle 112a^3+84a^2c_1(L^*_{\hat{1}}\mathcal{T})+32ac_1^2(L^*_{\hat{1}}\mathcal{T})+5c_1^3(L^*_{\hat{1}}\mathcal{T}),\,[\bar{\mathcal{U}}_{\mathcal{T}}(\mu)]\big\rangle \\ -\mathcal{C}_{\Sigma\times\mathcal{D}^{-1}_{\bar{\mathcal{T}},\hat{1}}(0)}(\tilde{\alpha}^{\perp}_{\mathcal{T}}),$$

if $\hat{1}$ is the unique element of $H_{\hat{0}}\mathcal{T}$. By the same argument as in (3),

$$\mathcal{C}_{\Sigma\times\mathcal{S}_{\bar{\mathcal{T}}}(\mu)}(\tilde{\alpha}^{\perp}_{\bar{\mathcal{T}}})=\big\langle c(L^*_{\hat{1}}\bar{\mathcal{T}}\otimes\mathcal{O}^{\perp}_2)c(L^*_{\hat{1}}\bar{\mathcal{T}}\otimes\mathrm{ev}^*T\mathbb{P}^3)^{-1},\,[\Sigma\times\mathcal{S}_{\bar{\mathcal{T}}}(\mu)]\big\rangle=10\big|\mathcal{S}_{\mathcal{T}}(\mu)\big|.$$

On the other hand, if $\mathcal{T}'=(S^2,[N]-\{l\},I';j',\underline{d}')<\bar{\mathcal{T}}$, $\mathcal{S}_{\mathcal{T}'}(\mu)=\emptyset$ by dimension counting and Corollary C.3. Thus, by Theorem 6.2 to applied with $\mathcal{T}'<\bar{\mathcal{T}}$ and Proposition 5.13 used as in (3) above, $\mathcal{C}_{\Sigma\times\mathcal{U}_{\mathcal{T}'}}(\tilde{\alpha}^{\perp}_{\bar{\mathcal{T}}})=0$ unless $d_{\hat{1}}=0$ and $H_{\hat{1}}\mathcal{T}'=I'-I$, i.e.

$$H_{\hat{1}}\mathcal{T}'=\hat{I}'=\{\hat{2},\hat{3}\},\quad d'_{\hat{1}}=0,\ d'_{\hat{2}}\neq 0,\ d'_{\hat{3}}\neq 0.$$

In such a case, by Proposition 5.13,

$$\begin{aligned}\mathcal{C}_{\Sigma\times\mathcal{U}_{\mathcal{T}'}(\mu)}(\tilde{\alpha}^{\perp}_{\bar{\mathcal{T}}})&=\big\langle c(L^*_{\hat{1}}\bar{\mathcal{T}}\otimes\mathcal{O}_2)c(L_{\hat{2}}\mathcal{T}'\oplus L_{\hat{3}}\mathcal{T}')^{-1},\,[\Sigma\times\bar{\mathcal{U}}_{\mathcal{T}'}(\mu)]\big\rangle\\&=\big\langle 32a+5\big(c_1(L^*_{\hat{2}}\mathcal{T}')+c_1(L^*_{\hat{3}}\mathcal{T}')\big),\,[\bar{\mathcal{U}}_{\mathcal{T}'}(\mu)]\big\rangle.\end{aligned}$$

Summing equation (9.15) over $\mathcal{T}=\mathcal{T}^*(l)$ and using (6.15), we obtain

$$\sum_{l\in[N]}\mathcal{C}_{\Sigma\times(\mathcal{U}_{\mathcal{T}^*(l)}(\mu)-\mathcal{S}_{\mathcal{T}^*(l)}(\mu))}(\alpha^{\perp}_1)=-\big\langle 64a^2+10\big(c_1(\mathcal{L}^*_{\hat{1}})+c_1(\mathcal{L}^*_{\hat{2}})\big),\,[\bar{\mathcal{V}}_{2,1}(\mu)]\big\rangle \tag{9.16}$$
$$+2\sum_{l\in[N]}\Big(\big\langle 112a^3+84a^2c_1(\mathcal{L}*)+32ac_1^2(\mathcal{L}^*)+5c_1^3(\mathcal{L}^*),\,[\bar{\mathcal{U}}_{\mathcal{T}^*(l)}(\mu)]\big\rangle-5\big|\bar{\mathcal{U}}_{\mathcal{T}^*(l)}\cap\bar{\mathcal{S}}_1(\mu)\big|\Big),$$

where $\bar{\mathcal{V}}_{2,1}(\mu)=\bigcup_{l\in[N]}\bar{\mathcal{V}}_{2,1;l}(\mu)$, and $\bar{\mathcal{V}}_{2,1;l}(\mu)$ denotes the union of the spaces $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ taken over all equivalence classes of basic bubble types $\mathcal{T}=(S^2,[N]-\{l\},\{\hat{1},\hat{2}\};j,\underline{d})$ with $d_{\hat{1}},d_{\hat{2}}>0$ and $d_{\hat{1}}+d_{\hat{2}}=d$.
(5) If $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=2$ and $|M_{\hat{0}}\mathcal{T}|=0$, the space $\tilde{\mathcal{M}}_{0,H_{\hat{0}}\mathcal{T}+M_{\hat{0}}\mathcal{T}}$ is again a point. The zero set of $\tilde{\alpha}^{-1}_{\mathcal{T}}$ consists of a section $\mathcal{Z}_{\mathcal{T}}$ of $\mathbb{P}E\bar{\mathcal{T}}$ over $\Sigma\times\mathcal{S}_{\mathcal{T}}(\mu)$ and a section $\mathcal{Z}_{\mathcal{T}'}$ over the spaces $\Sigma\times\bar{\mathcal{U}}_{\mathcal{T}'}(\mu)$, with $\mathcal{T}'$ corresponding to the bubble types $\mathcal{T}$ described in (2) in the proof of Lemma 6.4. Explicitly,

$$\mathcal{Z}_{\mathcal{T}}=\big\{(x,b;[v_{\hat{1}},v_{\hat{2}}])\in\mathbb{P}E\bar{\mathcal{T}}:\mathcal{D}_{\bar{\mathcal{T}},\hat{1}}v_{\hat{1}}+\mathcal{D}_{\bar{\mathcal{T}},\hat{2}}v_{\hat{2}}=0\big\}\approx\Sigma\times\mathcal{S}_{\mathcal{T}}(\mu),$$

if $H_{\hat{0}}\mathcal{T}=\{\hat{1},\hat{2}\}$. Similarly to (2) and (4), Corollary C.3 and Proposition 5.13 give

$$\mathcal{C}_{\mathcal{Z}_{\mathcal{T}}(\mu)}(\tilde{\alpha}^{\perp}_{\mathcal{T}})=\big\langle c(\gamma^*_{E\bar{\mathcal{T}}}\otimes\mathcal{O}^{\perp}_2)c(\mathbb{C}^3)^{-1},\,[\mathcal{Z}_{\mathcal{T}}]\big\rangle=10\big|\mathcal{S}_{\mathcal{T}}(\mu)\big|.$$

On the other hand, if $\mathcal{T}'=(S^2,[N],I';j',\underline{d}')<\bar{\mathcal{T}}$ and $\mathcal{C}_{\mathbb{P}\mathcal{F}\mathcal{T}|(\Sigma\times\mathcal{U}'_{\mathcal{T}}(\mu))}(\tilde{\alpha}^{\perp}_{\bar{\mathcal{F}}\mathcal{T}})\neq 0$, similarly to (4),

$$|H_{\hat{1}}\mathcal{T}'|=|\hat{I}'|\in\{1,2\},\quad d'_{\hat{1}}=0,\ d'_i\neq 0 \text{ if } i\in I'-\{\hat{1}\},$$

up to interchange of $\hat{1}$ and $\hat{2}$. If $|H_{\hat{1}}\mathcal{T}'|=|\hat{I}'|=1$, by Proposition 5.13 and equation (6.15),

$$\begin{aligned}\mathcal{C}_{\mathcal{Z}_{\mathcal{T}'}}(\tilde{\alpha}_{\mathcal{T}}^{\perp}) &= \big\langle c(\mathcal{O}_2)c(L_{\hat{3}}\mathcal{T}'\oplus L_{\hat{1}}\mathcal{T}')^{-1}, [\Sigma\times\bar{\mathcal{U}}_{\mathcal{T}'}(\mu)]\big\rangle \\ &= 2\big\langle 32a+5\big(c_1(\mathcal{L}_{\hat{1}}^*\mathcal{T}')+c_1(\mathcal{L}_{\hat{2}}^*\mathcal{T}')\big), [\bar{\mathcal{U}}_{\mathcal{T}'}(\mu)]\big\rangle.\end{aligned}$$

see the proof of Lemma 6.9 for more details. If $|I'-I|=2$, by Proposition 5.13,

$$\mathcal{C}_{\mathcal{Z}_{\mathcal{T}'}}(\tilde{\alpha}_{\mathcal{T}}^{\perp}) = \big\langle c_1(\mathcal{O}_2)-c_1(\mathbb{C}^3), [\Sigma\times\bar{\mathcal{U}}_{\mathcal{T}'}(\mu)]\big\rangle = 10\big|\bar{\mathcal{U}}_{\mathcal{T}'}(\mu)\big|.$$

Thus, summing equation (9.15) over $\mathcal{T}<\mathcal{T}^*$ with $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=2$ and $|M_{\hat{0}}\mathcal{T}|=0$, we obtain

$$\begin{aligned}\sum_{[\mathcal{T}]}\mathcal{C}_{\Sigma\times(\mathcal{U}_{\mathcal{T}}(\mu)-\mathcal{S}_{\mathcal{T}}(\mu))}(\alpha_{\hat{1}}^{\perp}) &= -30\big|\mathcal{V}_3(\mu)\big|-10\big|\mathcal{S}_{2;2}(\mu)\big| \\ &+2\big\langle 84a^2+32a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big)+5\big(c_1^2(\mathcal{L}_1^*)+c_1^2(\mathcal{L}_2^*)\big)+5c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*), [\bar{\mathcal{V}}_2(\mu)]\big\rangle\end{aligned} \tag{9.17}$$

(6) If $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=2$ and $|M_{\hat{0}}\mathcal{T}|=1$, $\tilde{\mathcal{M}}_{0,H_{\hat{0}}\mathcal{T}+M_{\hat{0}}\mathcal{T}}\approx\mathbb{P}^1$ and $L\longrightarrow\tilde{\mathcal{M}}_{0,H_{\hat{0}}\mathcal{T}+M_{\hat{0}}\mathcal{T}}$ is the tautological line bundle; see Lemma 6.23. Since $\tilde{\alpha}_{\mathcal{T}}$ has full rank on every fiber over in this case, by Proposition 5.13,

$$\begin{aligned}\mathcal{C}_{\Sigma\times\mathcal{U}_{\mathcal{T}}(\mu)}(\alpha^{\perp}) &= \big\langle c(\mathcal{O}_2)c\big(L^*\otimes E\bar{\mathcal{T}}\big)^{-1}, [\Sigma\times\tilde{\mathcal{M}}_{0,H_{\hat{0}}\mathcal{T}+M_{\hat{0}}\mathcal{T}}\times\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)]\big\rangle \\ &= 2\big\langle 32a+5\big(c_1(L_{\mathcal{T},\hat{1}}^*)+c_1(L_{\mathcal{T},\hat{2}}^*)\big), [\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)]\big\rangle.\end{aligned}$$

Summing equation (9.15) over $\mathcal{T}<\mathcal{T}^*$ with $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=2$ and $|M_{\hat{0}}\mathcal{T}|=1$ and using (6.15), we obtain

$$\sum_{[\mathcal{T}]}\mathcal{C}_{\Sigma\times\mathcal{U}_{\mathcal{T}}(\mu)}(\alpha_{\hat{1}}^{\perp}) = \big\langle 64a^2+10\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*), [\bar{\mathcal{V}}_{2,1}(\mu)]\big\rangle \tag{9.18}$$

(7) Finally, if $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=3$, $L\longrightarrow\tilde{\mathcal{M}}_{0,H_{\hat{0}}\mathcal{T}+M_{\hat{0}}\mathcal{T}}$ is again the tautological line bundle, and $\tilde{\alpha}_{\mathcal{T}}$ has full rank on every fiber. Thus by Proposition 5.13,

$$\mathcal{C}_{\Sigma\times\mathcal{U}_{\mathcal{T}}(\mu)}(\alpha_{\hat{1}}^{\perp}) = \big\langle c(\mathcal{O}_2)c\big(L^*\otimes E\bar{\mathcal{T}}\big)^{-1}, [\Sigma\times\bar{\mathcal{U}}_{\mathcal{T}}(\mu)]\big\rangle = 10|\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)|.$$

Thus, summing equation (9.15) over $\mathcal{T}<\mathcal{T}^*$ with $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=3$, we obtain

$$\sum_{|H_{\hat{0}}\mathcal{T}|=3}\mathcal{C}_{\Sigma\times\mathcal{U}_{\mathcal{T}}(\mu)}(\alpha_{\hat{1}}^{\perp}) = 10\big|\mathcal{V}_3(\mu)\big|. \tag{9.19}$$

From equations (9.13), (9.14), and (9.16)-(9.19), we conclude that

$$\begin{aligned}n_1^{(1)}(\mu) =& 2\big\langle 112a^3c_1(\mathcal{L}^*)+84a^2c_1^2(\mathcal{L}^*)+32ac_1^3(\mathcal{L}^*)+5c_1^4(\mathcal{L}^*), [\bar{\mathcal{V}}_1(\mu)]\big\rangle \\ &-2\big\langle 84a^2+32a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big)+5\big(c_1^2(\mathcal{L}_1^*)+c_1^2(\mathcal{L}_2^*)\big)+5c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*), [\bar{\mathcal{V}}_2(\mu)]\big\rangle \\ &-2\big\langle 12a+5c_1(\mathcal{L}^*), [\bar{\mathcal{S}}_1(\mu)]\big\rangle+10\big|\mathcal{S}_2(\mu)\big|+20\big|\mathcal{V}_3(\mu)\big|.\end{aligned}$$

The claim follows by using Lemma 6.4 and 6.9.

9.6 The Final Formulas

We finally put everything together to arrive at formulas for the numbers $n_{2,d}(\mu)$ in $\mathbb{P}^2$ and $\mathbb{P}^3$. It is assumed that μ is a tuple of $(3d-2)$ points in the case of $\mathbb{P}^2$ and of p points and q lines, with $2p+q=4d-3$, in the case of $\mathbb{P}^3$. In the former case, we write $n_{2,d}$ for $n_{2,d}(\mu)$ and n_d for the number of rational plane degree d curves passing through $3d$–1 points.

If $\nu \in \Gamma(\Sigma \times \mathbb{P}^n; \Lambda^{0,1}\pi_\Sigma^* T^*\Sigma \otimes \pi_{\mathbb{P}^n}^* T^*\mathbb{P}^n)$ is generic, for all $t \in (0,1)$, the signed cardinality of the set $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$ is the symplectic invariant $RT_{2,d}(;\mu)$. If $t>0$ is sufficiently small, every element of $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$ lies either in a small neighborhood U of the set $\mathcal{H}_{\Sigma,d}(\mu)$ or in a small neighborhood W of the space of all bubble map with singular domains. Furthermore,

$$^{\pm}\big|\mathcal{M}_{\Sigma,t\nu,d}(\mu)\cap U\big| = \big|\mathcal{H}_{\Sigma,d}(\mu)\big| = 2n_{2,d}(\mu).$$

On the other hand, by Section 8.9, $\big|\mathcal{M}_{\Sigma,t\nu,d}(\mu)\cap W\big| = CR_2(\mu)$, where

$$CR_2(\mu)=\begin{cases} n_1^{(1)}(\mu)+2n_1^{(2)}(\mu)+18n_1^{(3)}(\mu)+n_2^{(2)}(\mu), & \text{if } n=2;\\ n_1^{(1)}(\mu)+2n_1^{(2)}(\mu)+18n_1^{(3)}(\mu)+n_2^{(2)}(\mu)+2n_2^{(2)}(\mu)+n_3^{(1)}(\mu), & \text{if } n=3. \end{cases} \tag{9.20}$$

Thus, $n_{2,d}(\mu)$ is one-half of the difference between $RT_{2,d}(;\mu)$ and the number $CR_2(\mu)$.

We now give a very explicit formula in the $n=2$ case. Abbreviate $\mathcal{M}_{(d_1,d_2)}(\mu)$ as $\mathcal{M}_{d_1,d_2}$. Let

$$\mathcal{Z}_{2;d} = \Big(\bigcup_{\substack{d_1,d_2>0\\ d_1+d_2=0}} \mathcal{Z}_{d_1,d_2}\Big)/\mathbb{Z}_2, \quad \text{where} \quad \mathcal{Z}_{d_1,d_2} = \bigcup_{j_l=1,2} \bar{\mathcal{U}}_{(S^2,[N],I;j,\{0,d_1,d_2\})}(\mu),$$

where $I=\{\hat{0},1,2\}$ with the partial ordering $0<1,2$. The set $\mathcal{Z}_{2;d}$ is the the zero-dimensional space of three-bubble maps passing through the (3d-2) points μ, such that the map is trivial on the principal component. Note that

$$\big|\mathcal{Z}_{d;2}\big| = \big|\mathcal{V}_2(\mu)\big| = \frac{1}{2}\sum_{d_1+d_2=d}\binom{3d-2}{3d_1-1} d_1 d_2 n_{d_1} n_{d_2}. \tag{9.21}$$

The binomial coefficient counts the number of possible ways of distributing the constraints between the two nontrivial bubbles. Without the factor d_1d_2, the above number would have been precisely the number of two-component rational curves passing through $(3d$–$2)$ generic points in $\mathbb{P}^2$. However, we have to account for the image of the evaluation map at $\hat{0}$, which must be one of the d_1d_2 points of intersection of two rational curves of degrees d_1 and d_2.

Lemma 9.10 *In the $n=2$ case, the total correction is given by*

$$CR_2(\mu) = \big\langle 78a^2 + 72ac_1(\mathcal{L}^*) + 22c_1^2(\mathcal{L}^*), [\bar{\mathcal{V}}_1(\mu)]\big\rangle - 18\big|\mathcal{V}_2(\mu)\big|.$$

Proof: The four numbers of (9.20) are given by Lemmas 9.5, 9.4, 9.1, and 9.3. The cardinality of $\mathcal{S}_1(\mu)$ is given by Lemma 6.3.

Lemma 9.11 *With notation as above,*

$$\langle ac_1(\mathcal{L}^*), [\bar{\mathcal{V}}_1(\mu)]\rangle = \frac{1}{d}\Big(-n_d + \frac{1}{2}\sum_{d_1+d_2=d} d_1^2 d_2^2 \binom{3d-2}{3d_1-1} n_{d_1} n_{d_2}\Big).$$

Proof: By Corollary 6.18,

$$ac_1(\mathcal{L}^*) = \frac{1}{d^2}a\Big(\mathcal{H} - 2da + \sum_{d_1+d_2=d}^{>} d_2^2 \mathcal{M}_{d_1,d_2}\Big). \tag{9.22}$$

Note that

$$\begin{aligned}\sum_{d_1+d_2=d} d_2^2 \langle a, [\mathcal{M}_{d_1,d_2}]\rangle &= \sum_{d_1+d_2=d} d_1(d_1 d_2) d_2^2 \binom{3d-2}{3d_1-1} n_{d_1} n_{d_2} \\ &= \frac{1}{2} d \sum_{d_1+d_2=d} d_1^2 d_2^2 \binom{3d-2}{3d_1-1} n_{d_1} n_{d_2}.\end{aligned} \tag{9.23}$$

The reason for the appearance of the factor $d_1 d_2$ in (9.23) is the same one as in (9.21). On the other hand, the factor d_1 appears because we need to count the number of times the first rational component intersects a line in $\mathbb{P}^2$. Since

$$\langle a\mathcal{H}, [\bar{\mathcal{V}}_1(\mu)]\rangle = dn_d \quad \text{and} \quad \langle a^2, [\bar{\mathcal{V}}_1(\mu)]\rangle = n_d,$$

the claim follows by plugging (9.23) into (9.22).

Lemma 9.12 *With notation as above,*

$$\langle c_1^2(\mathcal{L}^*), [\bar{\mathcal{V}}_1(\mu)]\rangle = -\frac{1}{2}\sum_{d_1+d_2=d} \binom{3d-2}{3d_1-1} d_1 d_2 n_{d_1} n_{d_2}.$$

Proof: By Corollary 6.18,

$$c_1^2(\mathcal{L}^*) = \frac{1}{d^2} c_1(\mathcal{L}^*)\Big(\mathcal{H} - 2da + \sum_{d_1+d_2=d} d_2^2 \mathcal{M}_{d_1,d_2}\Big). \tag{9.24}$$

Since there are no two-component rational curves of total degree d passing through $(3d-1)$ generic points in $\mathbb{P}^2$ and there are no three-component rational curves of total degree d passing through $(3d-2)$ generic points in $\mathbb{P}^2$, by Corollary 6.18

$$\langle \mathcal{H}c_1(\mathcal{L}^*), [\bar{\mathcal{V}}_1(\mu)]\rangle = \frac{1}{d^2}\langle -2da\mathcal{H}, [\bar{\mathcal{V}}_1(\mu)]\rangle = -2n_d. \tag{9.25}$$

Similarly by Corollary 6.18 and Lemma 6.19,

$$\begin{aligned}\langle c_1(\mathcal{L}^*), [\mathcal{M}_{d_1,d_2}]\rangle &= \frac{1}{d_1^2}\langle -2d_1 a\mathcal{H}, [\mathcal{M}_{d_1,d_2}]\rangle + |\mathcal{Z}_{d_1,d_2}| = -|\mathcal{Z}_{d_1,d_2}| \\ &= -d_1 d_2 \binom{3d-2}{3d_1-1} n_{d_1} n_{d_2}.\end{aligned} \tag{9.26}$$

Note that by symmetry

$$\sum_{d_1+d_2=d} d_1 d_2^3 \binom{3d-2}{3d_1-1} n_{d_1} n_{d_2} = \frac{1}{2} \sum_{d_1+d_2=d} d_1 d_2 \left(d^2 - 2d_1 d_2\right) \binom{3d-2}{3d_1-1} n_{d_1} n_{d_2}. \tag{9.27}$$

The claim now follows from equations (9.24)-(9.27) and Lemma 9.11.

Corollary 9.13 *The total correction term is given by*

$$CR_2(\mu) = 78n_d + 72\frac{1}{d}\Big(-n_d + \frac{1}{2}\sum_{d_1+d_2=d} d_1^2 d_2^2 \binom{3d-2}{3d_1-1} n_{d_1} n_{d_2}\Big) - 20\sum_{d_1+d_2=d} d_1 d_2 \binom{3d-2}{3d_1-1} n_{d_1} n_{d_2}.$$

Proof: This claim is immediate from Lemmas 9.10-9.12 and equation (9.21).

Lemma 9.14 *The genus-two RT-invariant in $\mathbb{P}^2$ is given by*

$$RT_{2,d}(;\mu) \equiv RT_{2,d}(;p_{[3d-2]}) = 6d^2 n_d + \sum_{d_1+d_2=d} d_1^3 d_2^3 \binom{3d-2}{3d_1-1} n_{d_1} n_{d_2}.$$

Proof: Applying the genus-reducing composition law of [RT] twice, we obtain

$$\begin{aligned} RT_{2,d}(;p_{[3d-2]}) &= 2RT_{1,d}(p,\mathbb{P}^2;p_{[3d-2]}) + RT_{1,d}(\ell,\ell;p_{[3d-2]}) \\ &= 4RT_{0,d}(p,\mathbb{P}^2,p,\mathbb{P}^2;p_{[3d-2]}) + 4RT_{0,d}(p,\mathbb{P}^2,\ell,\ell;p_{[3d-2]}) + RT_{0,d}(\ell,\ell,\ell,\ell;p_{[3d-2]}) \\ &= 0 + 4RT_{0,d}(p,\ell,\ell;p_{[3d-2]}) + RT_{0,d}(\ell,\ell,\ell,\ell;p_{[3d-2]}). \end{aligned} \tag{9.28}$$

Since the genus-zero three-point RT-invariant is the usual enumerative invariant, the middle term above is simply $4d^2 n_d$. On the other hand, by the component-splitting composition law of [RT],

$$\begin{aligned} RT_{0,d}(\ell,\ell,\ell,\ell;p_{[3d-2]}) &= 2RT_{0,0}(\ell,\ell,\mathbb{P}^2;)RT_{0,d}(\ell,\ell,p;p_{[3d-2]}) \\ &\quad + \sum_{d_1+d_2=d}\ \sum_{J_1+J_2=[3d-2]} RT_{0,d_1}(\ell,\ell,\ell;p_{J_1})RT_{0,d_2}(\ell,\ell,\ell;p_{J_2}) \\ &= 2d^2 n_d + \sum_{d_1+d_2=d} d_1^3 d_2^3 \binom{3d-2}{3d_1-1} n_{d_1} n_{d_2}. \end{aligned} \tag{9.29}$$

The lemma follows from equations (9.28) and (9.29).

Theorem 1.2 is nearly proved. We can simplify the expression in Corollary 9.13 by using a recursive relation for the numbers n_d; see Section 11.3. The expression of Theorem 1.2 is half of the difference between the quantity of Lemma 9.14 and Corollary 9.13. Note that the numbers n_d with $d=1,2,3$ have long been known to be zero; see [ACGH]. Strictly speaking, our computation does not apply to the cases $d=1,2$. However, these two cases do provide a consistency check.

The case of $\mathbb{P}^3$ is significantly harder than the $n=2$ case. An explicit recursive formula as in Theorem 1.2 would be rather long, so we do not provide one. Instead we express $n_{2,d}(\mu)$ in terms of the corresponding symplectic invariant and intersection numbers of the spaces $\bar{\mathcal{V}}_1(\mu)$, $\bar{\mathcal{V}}_2(\mu)$, and $\bar{\mathcal{V}}_3(\mu)$; see Theorem 1.3. The expression for $CR_2(\mu)$ of Theorem 1.3 is obtained by plugging the results of Lemmas 9.9, 9.7, 9.8, 9.6, 9.4, and 9.3 into (9.20). The numbers $\langle a, [\bar{S}_1(\mu)]\rangle$, $\langle c_1(\mathcal{L}^*), [\bar{S}_1(\mu)]\rangle$, and $|\bar{S}_{2;2}(\mu)|$ are given by Lemmas 6.4 and 6.9.

As in the case of $\mathbb{P}^2$, we recover the well-known fact that all degree-one, -two, and -three numbers are zero. The only degree-one number, the number of genus-two degree-one curves through a line, is zero because there are no holomorphic degree-one maps from a positive-genus curve into $\mathbb{P}^n$; see [ACGH]. The eight degree-two and -three numbers are zero because the image of any holomorphic map of degree two or three from a genus-two curve into $\mathbb{P}^n$ is a line, see [ACGH], while no line passes through the required constraints. The first three degree-four numbers given below have also been known to be zero, since the image of any holomorphic map of degree four from a genus-two curve into $\mathbb{P}^n$ must lie in a plane. Precisely for this reason, the fourth degree-four number for $\mathbb{P}^3$ must be the number $n_{2,4}$ for $\mathbb{P}^2$.

degree	4					5
(p,q)	(6,1)	(5,3)	(4,5)	(3,7)	(0,13)	(5,7)
$\mathrm{RT}_{2,d}(\cdot;\mu)$	7,872	64,960	548,608	4,906,304	5,130,826,752	290,439,680
$CR(\mu)$	7,872	64,960	548,608	4,877,504	4,998,465,792	258,287,360
$n_{2,d}(\mu)$	0	0	0	14,400	66,180,480	16,076,160

Chapter 10

Computation of $CR_3(\mu)$ for $\mathbb{P}^2$

We now compute the genus-three correction term $CR_3(\mu)$ for $\mathbb{P}^2$ and thus prove Theorem 1.4. Throughout this chapter, Σ denotes a genus-three Riemann surface with a fixed generic complex structure. In particular, Σ is not hyperelleptic. Via the canonical embedding, Σ can be viewed as a degree-four embedded curve in $\mathbb{P}^2$. If Σ is generic, it will have twenty-four flexes, i.e. points at which the tangent line has contact of order three, instead of two, with the tangent line. More generally, Σ may have n hyperflexes, i.e. points at which the tangent line has contact of order four, and $24-2n$ (simple) flexes. The number of genus-three curves with a fixed complex structure does depend on this number n. We indicate the necessary changes in the computation to obtain this number for any n. Of course, n cannot exceed twelve. Throughout this chapter, $N=3d-4$ and μ is an N-tuple of points in general position in $\mathbb{P}^2$.

10.1 Description of $CR_3(\mu)$

We start by describing the number of elements of $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$ that lie near each strata $\mathcal{M}_{\mathcal{T}}(\mu)$ of bubble maps of type (2c) in terms of the zeros of affine maps between vector bundles over closures of certain subspaces of $\mathcal{M}_{\mathcal{T}}(\mu)$. These results are analogous to Section 8.9 and are proved by arguments as in Chapters 8. The necessary correction is outlined briefly at the end of this section.

As in the genus-two case, the section

$$s = s_b \in \Gamma\big(\Sigma; T^*\Sigma \otimes \mathcal{H}_\Sigma^{0,1}\big)$$

is well-defined and nowhere zero. Thus, its image determines a line subbundle $\mathcal{H}_\Sigma^+$ of $\Sigma\times\mathcal{H}_\Sigma^{0,1}\longrightarrow\Sigma$. We denote its orthogonal complement by $\mathcal{H}_\Sigma^-$ as before. Let

$$\pi^- \in \Gamma\big(\Sigma; (\Sigma\times\mathcal{H}_\Sigma^{0,1})^* \otimes \mathcal{H}_\Sigma^-\big)$$

be the orthogonal projection map onto $\mathcal{H}_\Sigma^-$. As in the genus-two case, the section

$$s^{(2)} = \pi^- \circ s_b \in \Gamma\big(\Sigma; T^*\Sigma^{\otimes 2} \otimes \mathcal{H}_\Sigma^-\big)$$

is globally defined. Unlike the genus-two case, $s^{(2)}$ does not vanish on Σ, since Σ is not hyperelleptic. Thus, $s^{(2)}$ determines a line subbundle $\mathcal{H}_\Sigma^{-+}$ of $\mathcal{H}_\Sigma^-$. We denote its orthogonal complement by $\mathcal{H}_\Sigma^{--}$ and the corresponding orthogonal projection map by π^{--}. The composition $s^{(3)} \equiv \pi_x^{--} \circ s_{b,x}^{(3)}$ is again independent of the choice of the metric $g_{b,\hat{0}}$. If Σ is generic, the section

$$s^{(3)} \in \Gamma(\Sigma; T^*\Sigma^{\otimes 3} \otimes \mathcal{H}_\Sigma^{--})$$

vanishes transversally at 24 distinct points $z_1, \ldots, z_{24}$ of Σ. These points correspond to the flexes of Σ under the canonical embedding into $\mathbb{P}^2$.

Theorem 10.1 *Suppose d is a positive integer, $N = 3d-4$, μ is an N-tuple of points in general position in $\mathbb{P}^2$, $\mathcal{T} = (\Sigma, [N], I; j, \underline{d})$ is a simple bubble type such that $d_{\hat{0}} = 0$ and $\sum d_i = d$. If*

$$\nu \in \Gamma(\Sigma \times \mathbb{P}^2; \Lambda^{0,1}\pi_\Sigma^* T^*\Sigma \otimes \pi_{\mathbb{P}^2}^* T\mathbb{P}^2)$$

is a generic section, there exist a compact subset $K_{\mathcal{T},\nu}$ of $\mathcal{M}_\mathcal{T}(\mu)$ and integer $N(\mathcal{T})$ with the following property. If K is a compact subset of $\mathcal{M}_\mathcal{T}(\mu)$ containing $K_{\mathcal{T},\nu}$, there exist a neighborhood U_K of K in $\bar{C}^\infty_{(d;[N])}(\Sigma; \mu)$ and $\epsilon_K > 0$ such that for all $t \in (0, \epsilon_K)$,

$$^{\pm}\big|U_K \cap \mathcal{M}_{\Sigma,d,t\nu}(\mu)\big| = N(\mathcal{T}).$$

If $\mathcal{T}$ is not primitive, $U_K \cap \mathcal{M}_{\Sigma,d,t\nu}(\mu) = \emptyset$. If $\mathcal{T}$ is primitive, $N(\mathcal{T})$ is the numbers of zeros of the affine maps between vector bundles described below.

If $|\hat{I}| > 3$, $\mathcal{M}_\mathcal{T}(\mu) = \emptyset$. If $|\hat{I}| = 1, 2, 3$, define

$$\alpha_{|\hat{I}|} \in \Gamma\big(\Sigma^{\hat{I}} \times \bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu); \mathrm{Hom}(\bigoplus_{h \in \hat{I}} T\Sigma_h \otimes L_h\mathcal{T}, \mathcal{H}_\Sigma^{0,1} \otimes \mathrm{ev}^* T\mathbb{P}^2)\big),$$

$$\text{by} \quad \alpha_{|\hat{I}|}\big(b, (x_h)_{h \in \hat{I}}; (v_h \otimes v_h)_{h \in \hat{I}}\big) = \sum_{h \in \hat{I}} \big(\mathcal{D}_{\mathcal{T},h} v_h\big)\big(s_{x_h} v_h\big). \tag{10.1}$$

If $|\hat{I}| = 3$, $N(\mathcal{T}) = n^{(1)}(\mathcal{T}) = N(\alpha_3)$. If $|\hat{I}| = 2$, $N(\mathcal{T}) = n^{(1)}(\mathcal{T}) + 2n^{(2)}(\mathcal{T}) = N(\alpha_2) + 2N(\alpha_{2;1})$, with $\alpha_{2;1}$ defined as follows. If $\hat{I} = \{\hat{1}, \hat{2}\}$, $b \in \mathcal{S}_{\bar{\mathcal{T}};1}(\mu)$, and $\mathcal{D}_{\bar{\mathcal{T}};\hat{1}}(b) = 0$,

$$\begin{aligned} \alpha_{2;1}\big(b, (x_h)_{h \in \hat{I}}; (v_h \otimes v_h)_{h \in \hat{I}}\big) &= \big(\mathcal{D}_{\mathcal{T},\hat{1}}^{(2)} v_{\hat{1}}\big)\big(s_{x_{\hat{1}}}^{(2)} v_{\hat{1}}\big) + \big(\mathcal{D}_{\mathcal{T},\hat{2}} v_{\hat{2}}\big)\big(\pi_{x_{\hat{1}}}^{-} s_{x_{\hat{2}}} v_{\hat{2}}\big) \in \mathcal{H}_\Sigma^{-}(x_{\hat{1}}) \otimes T_{\mathrm{ev}(b)}\mathbb{P}^2 \\ \text{if} \quad v_{\hat{1}} \otimes v_{\hat{1}} &\in T\Sigma_{\hat{1}}^{\otimes 2} \otimes L_{\hat{1}}\mathcal{T}^{\otimes 2}\big|_{(b, x_{\hat{1}})}, \quad v_{\hat{2}} \otimes v_{\hat{2}} \in T\Sigma_{\hat{2}} \otimes L_{\hat{2}}\mathcal{T}\big|_{(b, x_{\hat{2}})}. \end{aligned} \tag{10.2}$$

If $|\hat{I}| = 1$,

$$\begin{aligned} N(\mathcal{T}) &= n^{(1)}(\mathcal{T}) + 2n^{(2)}(\mathcal{T}) + 3n_3(\mathcal{T}) + 4n_4(\mathcal{T}) \\ &= N(\alpha_1) + 2N(\alpha_{1;1}) + 3N(\alpha_{1;2}) + 96|\mathcal{S}_{1;2}(\mu)|, \end{aligned} \tag{10.3}$$

where

$$\begin{aligned} \alpha_{1;1} &\in \Gamma\big(\Sigma \times \bar{\mathcal{S}}_1(\mu); \mathrm{Hom}(T\Sigma^{\otimes 2} \otimes L_{\hat{1}}\mathcal{T}^{\otimes 2}, \mathcal{H}_\Sigma^{-} \otimes \mathrm{ev}^* T\mathbb{P}^2)\big) \quad \text{and} \\ \alpha_{1;2} &\in \Gamma\big(\Sigma \times \mathcal{S}_{1;2}(\mu); \mathrm{Hom}(T\Sigma^{\otimes 3} \otimes L_{\hat{1}}^{\otimes 3}\mathcal{T}, \mathcal{H}_\Sigma^{--} \otimes \mathrm{ev}^* T\mathbb{P}^2)\big) \end{aligned}$$

are defined by

$$\alpha_{1;k}\big(x, b, v \otimes v\big) = \big(\mathcal{D}_{\mathcal{T},\hat{1}}^{(k+1)} v\big)\big(s_x^{(k+1)} v\big). \tag{10.4}$$

Finally, for each $m = 1, 2, 3$, we denote by $n_m^{(k)}(\mu)$ the sum of the numbers $n^{(k)}(\mathcal{T})$ taken over all equivalence classes of primitive bubble types $\mathcal{T}$ with $|\hat{I}|=m$.

Corollary 10.2 *With notation as above,*

$$CR_3(\mu)=\big(n_1^{(1)}(\mu)+2n_1^{(2)}(\mu)+3n_1^{(3)}(\mu)+96|\mathcal{S}_{1;2}(\mu)|\big)+\big(n_2^{(1)}(\mu)+2n_2^{(2)}(\mu)\big)+n_3^{(1)}(\mu).$$

Corollary 10.2 follows immediately from the preceding paragraph, Theorem 10.1, and the definition of $CR_3(\mu)$ in Section 7.1.

Remark 1: The multiplicity k for $n_m^{(k)}(\mu)$ is the degree of a polynomial map between two vector spaces of small dimensions. In the cases under consideration, these degrees are clear. In the genus-two case, they are described in Chapter 8.

Remark 2: The number 96 in (10.3) arises because each element of $\mathcal{M}_{\mathcal{T}}(\mu)$ corresponding to a simple flex of Σ and a rational map in $\mathcal{S}_{2;2}(\mu)$ enters with a multiplicity of 4. A hyperflex of Σ would result in a multiplicity of 10, at least if $d\geq 4$. Thus, if Σ has n hyperflexes and 24–$2n$ simple flexes, the number 96 in equation (10.3) and in Corollary 10.2 should be replaced by $96+2n$. No other changes are needed. The analogue of the term $n_1^{(4)}(\mu)=24|\mathcal{S}_{1;2}(\mu)|$ in the genus-two case is $n_1^{(3)}(\mu)=6|\mathcal{S}_1(\mu)|$, as each of the six hyperelleptic points of Σ and a cuspidal map through the fixed $3d$–2 points enters with a multiplicity of 3; see Section 8.5.

Remark 3: It should be possible to adapt this approach to the case Σ is a hyperelleptic genus-three surface, but significant changes would be required. In particular, there will likely be a contribution to $CR_3(\mu)$ from an affine map over the space $\Sigma\times\bar{\mathcal{S}}_{2;2}(\mu)$, as is the case in the genus-two case in $\mathbb{P}^3$; see Section 8.6. Furthermore, higher-order contributions $n_m^{(k)}(\mu)$, $k\geq 3$, will have a very different description, which will involve the hyperelleptic and Weierstrass points of Σ.

Theorem 10.1 follows from Theorem 7.2 and analysis of leading-order terms of $\psi^{\mu}_{\mathcal{S},t\mu}$ as in Chapter 8. As before, we need to choose a tangent-bundle model in such a way that the last term in the definition of $\psi_{\mathcal{T},t\mu}$ in Theorem 7.2 is always very small. By an argument similar to Section 8.1, we can choose $\tilde{\Gamma}_+(v)$ so that its image under D_v is orthogonal to

$$\big(\mathcal{H}^+_\Sigma(\tilde{x}_{h^*}(v))\oplus\mathcal{H}^{-+}_\Sigma(\tilde{x}_{h^*}(v))\big)\otimes \mathrm{ev}^*T\mathbb{P}^2,$$

provided $d_{h^*}\geq 2$. Then in all cases, $\pi^{0,1}_-\circ D_v|\tilde{\Gamma}_+(v)$ will be sufficiently small for the purposes of extracting dominant terms from $\psi^{\mu}_{\mathcal{S},t\nu}$ as in Chapter 8.

10.2 The Numbers $n_1^{(3)}(\mu)$, $n_2^{(2)}(\mu)$, and $n_3^{(1)}(\mu)$

In this section, we express the numbers $n_1^{(3)}(\mu)$, $n_2^{(2)}(\mu)$, and $n_3^{(1)}(\mu)$ in terms of the cardinalities of the finite sets $\mathcal{S}_{1;2}(\mu)$, $\mathcal{S}_{2;2}(\mu)$, and $\mathcal{V}_3(\mu)$, respectively.

Lemma 10.3 $n_1^{(3)}(\mu)=12|\mathcal{S}_{1;2}(\mu)|$

Proof: By Section 10.1, $n_1^{(3)}(\mu)=N(\alpha_{1;2})$, where

$$\alpha_{1;2}\in\Gamma\big(\Sigma\times\mathcal{S}_{1;2}(\mu);\mathrm{Hom}(T\Sigma^{\otimes3}\otimes L^{\otimes3},\mathcal{H}_{\Sigma}^{--}\otimes\mathrm{ev}^*T\mathbb{P}^2)\big),$$
$$\alpha_{1;2}\big(x,b,v\otimes v\big)=\big(\mathcal{D}^{(3)}v\big)\big(s_x^{(3)}v\big). \tag{10.5}$$

The section $s^{(3)}\in\Gamma\big(\Sigma;\mathrm{Hom}(T\Sigma^{\otimes3},\mathcal{H}_{\Sigma}^{--})\big)$ has simple zeros at $z_1,\ldots,z_{24}$. Thus, $s^{(3)}$ induces a non-vanishing section

$$\tilde{s}^{(3)}\in\Gamma\big(\Sigma;\mathrm{Hom}(\tilde{T}\Sigma,\mathcal{H}_{\Sigma}^{--})\big),\quad \text{where}\quad \tilde{T}\Sigma=T\Sigma^{\otimes3}\otimes\mathcal{O}(z_1)\otimes\ldots\otimes\mathcal{O}(z_{24}).$$

Furthermore, $N(\tilde{\alpha}_{1;2})=N(\alpha_{1;2})$, where

$$\tilde{\alpha}_{1;2}\in\Gamma\big(\Sigma\times\mathcal{S}_{1;2}(\mu);\mathrm{Hom}(\tilde{T}\Sigma\otimes L^{\otimes3},\mathcal{H}_{\Sigma}^{--}\otimes\mathrm{ev}^*T\mathbb{P}^2)\big)$$

is the section obtained by replacing $s^{(3)}$ by $\tilde{s}^{(3)}$ in (10.5). See Section 9.2 for a similar argument in the genus-two case. Since $\mathcal{D}^{(3)}$ does not vanish on $\mathcal{S}_{1;2}(\mu)$ by Corollary C.3, $\tilde{\alpha}_{1;2}$ does not vanish on $\Sigma\times\mathcal{S}_{1;2}(\mu)$. Thus, by Lemma 5.14,

$$n_1^{(3)}(\mu)=N(\tilde{\alpha}_{1;2})=\big\langle c_1(\mathcal{H}_{\Sigma}^{--}\otimes\mathrm{ev}^*T\mathbb{P}^2)-c_1(\tilde{T}\Sigma),\big[\Sigma\times\mathcal{S}_{1;2}(\mu)\big]\big\rangle=12\big|\mathcal{S}_{1;2}(\mu)\big|.$$

since the euler characteristic of Σ is -4.

Remark: Note that this argument remains valid even if Σ has hyperflexes, i.e. the points $z_1,\ldots,z_{24}$ are not all distinct.

Lemma 10.4 $n_2^{(2)}(\mu)=36\big|\mathcal{S}_{2;1}(\mu)\big|$

Proof: (1) By Section 10.1, $n_2^{(2)}(\mu)=N(\alpha_{2;1})$, where

$$\alpha_{2;1}\in\Gamma\big(\Sigma_1\times\Sigma_2\times\mathcal{S}_{2;1}(\mu);\mathrm{Hom}(E,\mathcal{O})\big),\quad E=T\Sigma_1^{\otimes2}\otimes L_1^{\otimes2}\oplus T\Sigma_2\otimes L_2,\quad \mathcal{O}=\mathcal{H}_{\Sigma_1}^{-}\otimes\mathrm{ev}^*T\mathbb{P}^2,$$
$$\alpha_{2;1}\big(x_1,x_2,b;v_1\otimes v_1,v_2\otimes v_2\big)=\big(\mathcal{D}_1^{(2)}v_1\big)\big(s_{x_1}^{(2)}v_1\big)+\big(\mathcal{D}_2v_2\big)\big(\pi_{x_1}^{-}s_{x_2}v_2\big)\in\mathcal{H}_{\Sigma}^{-}(x_1)\otimes T_{\mathrm{ev}(b)}\mathbb{P}^2.$$

Here we define the line bundles $L_1,L_2\longrightarrow\mathcal{S}_{2;1}(\mu)$ and the sections $\mathcal{D}_1^{(2)}$ and $\mathcal{D}_2^{(1)}$ as follows. If $b\in\mathcal{U}_{\mathcal{T}^*}(\mu)\cap\mathcal{S}_{2;1}(\mu)$, $\mathcal{T}^*=(S^2,[N],I^*;j^*,\underline{d}^*)$, $I^*=\{k_1,k_2\}$, and $\mathcal{D}_{\mathcal{T}^*,k_1}b=0$, we take

$$L_1\big|_b=L_{k_1}\mathcal{T}^*\big|_b,\quad L_2\big|_b=L_{k_2}\mathcal{T}^*\big|_b,\quad \mathcal{D}_1^{(2)}\big|_b=\mathcal{D}_{\mathcal{T}^*,k_1}^{(2)}\big|_b,\quad \mathcal{D}_2^{(1)}\big|_b=\mathcal{D}_{\mathcal{T}^*,k_2}^{(1)}\big|_b.$$

(2) By Lemma 5.14,

$$\begin{aligned}N(\alpha_{2;1})&=\sum_{k=0}^{k=2}\big\langle\lambda_E^{3-k}c_k(\mathcal{H}_{\Sigma_1}^{-}\otimes\mathbb{C}^2),[\mathbb{P}E]\big\rangle-\mathcal{C}_{\alpha^E{}^{-1}(0)}\big(\alpha^{E\perp}\big)\\&=64\big|\mathcal{S}_{2;1}(\mu)\big|-\mathcal{C}_{\alpha^E{}^{-1}(0)}\big(\alpha^{E\perp}\big).\end{aligned}\tag{10.6}$$

Since Σ is not hyperelleptic by assumption, $s_{x_1}=\lambda s_{x_2}$ for some $\lambda\in C^*$ if and only if $x_1=x_2$. Thus, $\pi_{x_1}^{-}s_{x_2}=0$ if and only if $x_1=x_2$. Since the images of $\mathcal{D}_1^{(2)}\big|_b$ and $\mathcal{D}_2^{(1)}\big|_b$ in $T_{\mathrm{ev}(b)}\mathbb{P}^2$ are linearly independent for all $b\in\mathcal{S}_{2;1}(\mu)$ by Corollary C.3, it follows that

$$\alpha^{E\,-1}(0)=\mathcal{Z}\equiv\big\{\big(x,x,b;T\Sigma_2\otimes L_2\big):x\in\Sigma,\ b\in\mathcal{S}_{2;1}(\mu)\big\}.$$

The normal bundle of $\mathcal{Z}$ in $\mathbb{P}E_2$ is

$$\mathcal{N}\mathcal{Z} = T\Sigma_2 \oplus (T\Sigma_2 \otimes L_2)^* \otimes T\Sigma_1^{\otimes 2} \otimes L_1^{\otimes 2} \approx T\Sigma \oplus T\Sigma \longrightarrow \mathcal{Z}.$$

With appropriate identifications,

$$\big|\alpha^E(x,x,b;w,u) - \alpha_{\mathcal{Z}}(x,b;w,u)\big| \le C|w|^2 \qquad \forall (x,x,b;w,u)\in\mathcal{N}\mathcal{Z}, \tag{10.7}$$

where $\alpha_{\mathcal{Z}}\in\Gamma\big(\mathcal{Z};\mathrm{Hom}(\mathcal{N}\mathcal{Z};\gamma_E^*\otimes\mathcal{O})\big)$, $\alpha_{\mathcal{Z}}(x,b;w,u) = \big\{\mathcal{D}_1^{(2)}\otimes s_x^{(2)}\big\}\circ u + \mathcal{D}_2^{(1)}\otimes s_x^{(2)}(w,\cdot)$.

Since the images of $\mathcal{D}_1^{(2)}\big|_b$ and $\mathcal{D}_2^{(1)}\big|_b$ in $T_{\mathrm{ev}(b)}\mathbb{P}^2$ are linearly independent for all $b\in\mathcal{S}_{2;1}(\mu)$, $\alpha_{\mathcal{Z}}$ has full rank over all of $\mathcal{Z}$. If $\bar{\nu}$ is generic, $\pi_{\bar{\nu}}^{\perp}\circ\alpha_{\mathcal{Z}}$ also has full rank on every fiber, where $\pi_{\bar{\nu}}^{\perp}:\mathcal{O}\longrightarrow\mathcal{O}/\mathbb{C}\bar{\nu}$ is the quotient projection as before. Then by the analytic estimate (10.7) and Proposition 5.13,

$$\begin{aligned}\mathcal{C}_{\alpha^{E\,-1}(0)}\big(\alpha^{E\perp}\big) &= \big\langle c_1(T^*\Sigma\otimes\mathcal{O}^{\perp}) - c_1(\mathcal{N}\mathcal{Z}), \big[\Sigma\times\mathcal{S}_{2;1}(\mu)\big]\big\rangle \\ &= \big\langle \big(3c_1(T^*\Sigma)+2c_1(T^*\Sigma)\big) + 2c_1(T^*\Sigma), [\Sigma]\big\rangle\big|\mathcal{S}_{2;1}(\mu)\big| = 28\big|\mathcal{S}_{2;1}(\mu)\big|.\end{aligned} \tag{10.8}$$

The claim is obtained by plugging (10.8) into (10.6).

Lemma 10.5 $n_3^{(1)}(\mu) = 36\big|\mathcal{V}_3(\mu)\big|$

Proof: (1) By Section 10.1, $n_3^{(1)}(\mu) = N(\alpha_3)$, where

$$\alpha_3\in\Gamma\big(\Sigma_1\times\Sigma_2\times\Sigma_3\times\mathcal{V}_3(\mu);\mathrm{Hom}(E,\mathcal{O})\big), \quad E=\bigoplus_i T\Sigma_i\otimes L_i, \quad \mathcal{O}=\mathcal{H}_{\Sigma}^{0,1}\otimes\mathrm{ev}^*T\mathbb{P}^2,$$

$$\alpha_3\big(x_1,x_2,x_3,b;v_1\otimes v_1, v_2\otimes v_2, v_3\otimes v_3\big) = \sum_i \big(\mathcal{D}_i v_i\big)\big(s_{x_i}v_i\big) \in \mathcal{H}_{\Sigma}^{0,1}\otimes T_{\mathrm{ev}(b)}\mathbb{P}^2.$$

Here the bundles $L_i\longrightarrow\mathcal{V}_3(\mu)$ and the sections $\mathcal{D}_i\in\Gamma\big(\mathcal{V}_3(\mu);L_i^*\otimes\mathrm{ev}^*T\mathbb{P}^2\big)$ are defined as follows. If $b\in\mathcal{U}_{\mathcal{T}^*}(\mu)\subset\mathcal{V}_3(\mu)$, $\mathcal{T}^*=(S^2,[N],I^*;j^*,\underline{d}^*)$, and $I^*=\{k_1,k_2,k_3\}$, we let $L_i\big|_b = L_{k_i}\mathcal{T}^*$ and $\mathcal{D}_i=\mathcal{D}_{\mathcal{T},k_i}$. These bundles and sections are well-defined once we fix a representative for each equivalence class of such bubble types $\mathcal{T}^*$ and order the elements of the corresponding set I^*.
(2) By Lemma 5.14,

$$\begin{aligned} N(\alpha_3) &= \sum_{k=0}^{k=3}\big\langle\lambda_E^{5-k}c_k(\mathbb{C}^5),[\mathbb{P}E]\big\rangle - \mathcal{C}_{\alpha^{E\,-1}(0)}\big(\alpha^{E\perp}\big) \\ &= 64\big|\mathcal{V}_3(\mu)\big| - \mathcal{C}_{\alpha^{E\,-1}(0)}\big(\alpha^{E\perp}\big).\end{aligned} \tag{10.9}$$

Since Σ is not hyperelleptic, $s_{x_1}=\lambda s_{x_2}$ for some $\lambda\in C^*$ if and only if $x_1=x_2$. Since the images of $\mathcal{D}_i\big|_b$ and $\mathcal{D}_j\big|_b$ in $T_{\mathrm{ev}(b)}\mathbb{P}^2$ are linearly independent for all $b\in\mathcal{V}_3(\mu)$ and $i\neq j$ by Corollary C.3, it follows that

$$\alpha^{E\,-1}(0) = \mathcal{Z} \equiv \Big\{\big(x,x,x,b;[v\otimes v_1, v\otimes v_2, v\otimes v_3]\big)\in\mathbb{P}E : x\in\Sigma,\ b\in\mathcal{V}_3(\mu),\ \sum_i\mathcal{D}_i v_i = 0\Big\}.$$

The normal bundle of $\mathcal{Z}$ in $\mathbb{P}E_2$ is

$$\mathcal{N}\mathcal{Z} = T\Sigma_2\oplus T\Sigma_3\oplus\big(T\Sigma_1\otimes L_1\big)^*\otimes\big(T\Sigma_2\otimes L_2\oplus T\Sigma_3\otimes L_3\big)\approx T\Sigma\oplus T\Sigma\oplus\mathbb{C}^2\longrightarrow\mathcal{Z}.$$

With appropriate identifications,

$$\big|\alpha^E(x,x,x,b;w_2,w_3,u_2,u_3)-\alpha_{\mathcal{Z}}(x,b;w_2,w_3,u_2,u_3)\big|\le C\big|(w_2,w_3)\big|^2 \quad \forall (w_2,w_3,u_2,u_3)\in\mathcal{N}\mathcal{Z},$$
$$\text{where} \qquad \alpha_{\mathcal{Z}}\in\Gamma\big(\mathcal{Z};\mathrm{Hom}(\mathcal{N}\mathcal{Z};(T\Sigma_1\otimes L_1)^*\otimes\mathcal{O})\big), \qquad (10.10)$$
$$\alpha_{\mathcal{Z}}\big(x,b;w_2,w_3,u_2,u_3\big) = \big\{\mathcal{D}_2\otimes s_x\big\}\circ u_2+\big\{\mathcal{D}_3\otimes s_x\big\}\circ u_3+\mathcal{D}_2\otimes s^{(2)}_{g_x,x}(w_2,\cdot)+\mathcal{D}_3\otimes s^{(2)}_{g_x,x}(w_3,\cdot),$$

and $\{g_x : x\in\Sigma\}$ is a smooth family of metrics on Σ such that g_x is flat on a neighborhood of x. Since the images of $\mathcal{D}_2\big|_b$ and $\mathcal{D}_3\big|_b$ are linearly independent in $T_{\mathrm{ev}(b)}\mathbb{P}^2$ for all $b\in\mathcal{V}_3(\mu)$ and the section $s_x^{(2)}=\pi_x^-\circ s_{g_x,x}$ does not vanish on Σ, the linear map $\alpha_{\mathcal{Z}}$ is injective over $\mathcal{Z}$. Thus, by the analytic estimate (10.10) and Proposition 5.13,

$$\begin{aligned}\mathcal{C}_{\alpha^{E^{-1}}(0)}\big(\alpha^{E\perp}\big) &= \big\langle c_1\big(T^*\Sigma\otimes\mathcal{O}^\perp\big)-c_1\big(\mathcal{N}\mathcal{Z}\big),\big[\Sigma\times\mathcal{V}_3(\mu)\big]\big\rangle \\ &= \big\langle 5c_1(T^*\Sigma)+2c_1(T^*\Sigma),[\Sigma]\big\rangle\big|\mathcal{V}_3(\mu)\big| = 28\big|\mathcal{V}_3(\mu)\big|.\end{aligned} \qquad (10.11)$$

The claim is obtained by plugging (10.11) into (10.9).

10.3 The Numbers $n_2^{(1)}(\mu)$

We now use the topological tools of Chapter 5 along with the analytic estimates of Chapter 6 to give a topological formula for the number $n_2^{(1)}(\mu)$ of Corollary 10.2.

Lemma 10.6 *If d is a positive integer and μ is a tuple of $3d-4$ points in general position in $\mathbb{P}^2$,*

$$n_2^{(1)}(\mu) = 12\big\langle 10a^2+3a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_1^*)\big)+c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_1^*),\big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle.$$

Proof: (1) By Section 10.1, $n_2^{(1)}(\mu)=N(\alpha_2)$, where

$$\alpha_2\in\Gamma\big(\Sigma_1\times\Sigma_2\times\bar{\mathcal{V}}_2(\mu);\mathrm{Hom}(\tilde{E},\mathcal{O})\big), \quad \tilde{E}=T\Sigma_1\otimes L_1\oplus T\Sigma_2\oplus L_2, \quad \mathcal{O}=\mathcal{H}_\Sigma^{0,1}\otimes\mathrm{ev}^*T\mathbb{P}^2,$$
$$\alpha_2\big(x_1,x_2,b;v_1\otimes v_1,v_2\otimes v_2\big) = \big(\mathcal{D}_1v_1\big)\big(s_{x_1}v_1\big)+\big(\mathcal{D}_2v_2\big)\big(s_{x_2}v_2\big)\in\mathcal{H}_\Sigma^{0,1}\otimes T_{\mathrm{ev}(b)}\mathbb{P}^2,$$

with the bundles $L_i\longrightarrow\bar{\mathcal{V}}_2(\mu)$ and the sections $\mathcal{D}_i\in\Gamma\big(\bar{\mathcal{V}}_2(\mu);L_i^*\otimes\mathrm{ev}^*T\mathbb{P}^2\big)$ defined as in the proof of Lemma 10.5. By Lemma 5.14,

$$\begin{aligned}N(\alpha_2) &= \sum_{k=0}^{k=3}\big\langle\lambda_{\tilde{E}}^{5-k}c_k(\mathcal{O}),[\mathbb{P}\tilde{E}]\big\rangle-\mathcal{C}_{\alpha^{\tilde{E}^{-1}}(0)}\big(\alpha^{\tilde{E}\perp}\big) \\ &= 16\big\langle 36a^2+18a\big(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)\big)+3\big(c_1^2(L_1^*)+c_1^2(L_2^*)\big)+4c_1(L_1^*)c_1(L_2^*),\big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle \\ &\qquad -\mathcal{C}_{\alpha^{\tilde{E}^{-1}}(0)}\big(\alpha^{\tilde{E}\perp}\big).\end{aligned} \qquad (10.12)$$

Since $s_{x_1}=\lambda s_{x_2}$ for some $\lambda\in\mathbb{C}$ if and only if $x_1=x_2$,

$$\alpha^{\tilde{E}^{-1}}(0) = \big\{\big(x,x,b;[v_1,v_2]\big):\mathcal{D}_1v_1+\mathcal{D}_2v_2=0\big\}\cup\bigcup_{i=1,2}\big\{\big(x_1,x_2,b;T_{x_i}\Sigma_i\otimes L_i|_b\big):\mathcal{D}_ib=0\big\}.$$

We now partition these sets further and apply the topological tools of Chapter 5. As usually, we denote by $\bar{\nu}\in\Gamma\big(\mathbb{P}\tilde{E};\mathcal{O}\big)$ a generic non-vanishing section.

(2) We start with the spaces $(\Sigma^2-\triangle)\times\mathcal{S}_{2;1}(\mu)$ and $\triangle\times\mathcal{S}_{2;1}(\mu)$. For notational simplicity, we assume that $\mathcal{D}_1 b=0$ for $b\in\mathcal{S}_{2;1}(\mu)$. The normal bundle of the subspace

$$\mathcal{Z}_{2;1}(\mu)=\big\{(x_1,x_2,b;T_{x_1}\Sigma_1\otimes L_1|_b): x_1\neq x_2,\ b\in\mathcal{S}_{2;1}(\mu)\big\}$$

in $\mathbb{P}\tilde{E}$ is given by

$$\mathcal{N}\mathcal{Z}_{2;1}=\pi_{\tilde{E}}^*\big(L_1^*\otimes \mathrm{ev}^*T\mathbb{P}^2\oplus T\Sigma_1^*\otimes L_1^*\otimes T\Sigma_2\otimes L_2\big)\approx\mathbb{C}^2\otimes T\Sigma_1^*\otimes T\Sigma_2.$$

With appropriate identifications, $\mathcal{D}_1(b,X)=X$ for all $X\in(L_1^*\otimes\mathrm{ev}^*T\mathbb{P}^2)_b$ sufficiently small. Thus,

$$\alpha^{\tilde{E}}(x_1,x_2,b;X,u)=\alpha_{2;1}(x_1,x_2,b;X,u)\equiv X\otimes s_{\Sigma,x_1}+\big\{\mathcal{D}_2\otimes s_{\Sigma,x_2}\big\}\circ u.$$

Since $\alpha_{2;1}$ has full rank on $\mathcal{Z}_{2;1}(\mu)\approx(\Sigma^2-\triangle)\times\mathcal{S}_{2;1}(\mu)$ and extends over $\Sigma^2\times\mathcal{S}_{2;1}(\mu)$,

$$\mathcal{C}_{(\Sigma^2-\triangle)\times\mathcal{S}_{2;1}(\mu)}(\alpha^{\tilde{E}\perp})=N(\pi_{\bar{\nu}}^{\perp}\circ\alpha_{2;1}),$$

as long as $\bar{\nu}$ is generic. In fact, it can be assumed the image of $\bar{\nu}$ is disjoint from $\mathcal{H}_{\Sigma}^{+}(x_1)\otimes\mathrm{ev}^*T\mathbb{P}^2$. Then $\pi_{\bar{\nu}}^{\perp}\big(\mathcal{H}_{\Sigma}^{+}(x_1)\otimes\mathrm{ev}^*T\mathbb{P}^2\big)$ is a rank-two subbundle of $\mathcal{O}^{\perp}$, and

$$\pi_{\bar{\nu}}^{\perp}\circ\alpha_{2;1}:\mathbb{C}^2\longrightarrow\gamma_{\tilde{E}}^*\otimes\pi_{\bar{\nu}}^{\perp}\big(\mathcal{H}_{\Sigma}^{+}(x_1)\otimes\mathrm{ev}^*T\mathbb{P}^2\big)$$

is an isomorphism. It follows that $N(\pi_{\bar{\nu}}^{\perp}\circ\alpha_{2;1})=N(\tilde{\alpha}_{2;1})$, where

$$\tilde{\alpha}_{2;1}\in\Gamma(\Sigma^2\times\mathcal{S}_{2;1}(\mu);\mathrm{Hom}(F_2;\mathcal{O}_2)),\quad \tilde{\alpha}_{2;1}(x_1,x_2,b;u)=\pi^{\perp}_{\pi_{x_1}^{-}\bar{\nu}}\circ\big\{\mathcal{D}_2\otimes\pi_{x_1}^{-}s_{x_2}\big\}\circ u,$$

$$F_2=T\Sigma_1^*\otimes L_1^*\otimes T\Sigma_2\otimes L_2\approx T\Sigma_1^*\otimes T\Sigma_2,\ \ \mathcal{O}_2=T\Sigma_1^*\otimes L_1^*\otimes\big(\mathcal{H}_{\Sigma_1}^{-}\otimes\mathrm{ev}^*T\mathbb{P}^2\big)^{\perp}\approx T\Sigma_1^*\otimes\big(\mathcal{H}_{\Sigma_1}^{-}\otimes\mathbb{C}^2\big)^{\perp};$$

By Lemma 5.14,

$$\begin{aligned}N(\tilde{\alpha}_{2;1})&=\big\langle c_1^2(F_2^*)+c_1(F_2^*)c_1(\mathcal{O}_2),[\Sigma^2\times\mathcal{S}_{2;1}(\mu)]\big\rangle-\mathcal{C}_{\tilde{\alpha}_{2;1}^{-1}(0)}(\tilde{\alpha}_{2;1}^{\perp})\\&=48|\mathcal{S}_{2;1}(\mu)|-\mathcal{C}_{\tilde{\alpha}_{2;1}^{-1}(0)}(\tilde{\alpha}_{2;1}^{\perp}).\end{aligned}$$

The zero set of $\tilde{\alpha}_{2;1}$ is $\triangle\times\mathcal{S}_{2;1}(\mu)$; see the proof of Lemma 10.4. Its normal bundle is $T\Sigma_2\approx T\Sigma$. If $\bar{\nu}$ and $\bar{\nu}_2$ are generic, as in the proof of Lemma 10.4, we obtain

$$\big|\tilde{\alpha}_{2;1}^{\perp}(x,x,b;w)-\tilde{\alpha}_{2;1;\triangle}(x,b;w)\big|\le C|w|^2\quad\forall w\in T\Sigma_\delta,$$

where $\tilde{\alpha}_{2;1;\triangle}:T\Sigma\longrightarrow F_2^*\otimes\mathcal{O}_2^{\perp}$ is an injection on every fiber. Thus, by Proposition 5.13,

$$\mathcal{C}_{\tilde{\alpha}_{2;1}^{-1}(0)}(\tilde{\alpha}_{2;1}^{\perp})=\big\langle c_1(F_2^*\otimes\mathcal{O}_2^{\perp})-c_1(T\Sigma),[\Sigma\times\mathcal{S}_{2;1}(\mu)]\big\rangle=24|\mathcal{S}_{2;1}(\mu)|.$$

We conclude that

$$\mathcal{C}_{(\Sigma^2-\triangle)\times\mathcal{S}_{2;1}(\mu)}(\alpha^{\tilde{E}\perp})=24|\mathcal{S}_{2;1}(\mu)|.\tag{10.13}$$

On the other hand, the space $\bar{\mathcal{Z}}_{2;1}-\mathcal{Z}_{2;1}\approx\triangle\times\mathcal{S}_{2;1}(\mu)$ is $\alpha^{\tilde{E}\perp}$-hollow, and thus

$$\mathcal{C}_{\triangle\times\mathcal{S}_{2;1}(\mu)}(\alpha^{\tilde{E}\perp})=0.$$

Indeed, its normal bundle in $\mathbb{P}\tilde{E}_2$ is given by

$$\mathcal{N}\mathcal{Z} = \pi_{\tilde{E}}^*\big(T\Sigma_2 \oplus L_{\hat{1}}^* \otimes \mathrm{ev}^*T\mathbb{P}^2 \oplus T\Sigma_{\hat{1}}^* \otimes L_{\hat{1}}^* \otimes T\Sigma_2 \otimes L_2\big).$$

With appropriate identifications,

$$\begin{gathered}\big|\alpha^{\tilde{E}}(x,x,b;w,X,u) - \tilde{\alpha}(x,b;w,X,u)\big| \le C|w|^2|u| \quad \forall (w,X,u)\in\mathcal{N}\mathcal{Z}_\delta, \quad \text{where} \\ \tilde{\alpha}(x,b;w,X,u) = X\otimes s_x + \big\{\mathcal{D}_2\otimes s_x\big\}\circ u + \big\{\mathcal{D}_2\otimes s^{(2)}_{g_x,x}(w,\cdot)\big\}\circ u.\end{gathered}$$

Since $\pi_x^- s^{(2)}_{g_x,x}$ does not vanish, $\tilde{\alpha}$ is a dominant term for $\alpha^{\tilde{E}}$; the same holds for composites with projection maps. Since

$$\mathrm{rk}\big(\mathcal{H}_\Sigma^-\otimes \mathrm{ev}^*T\mathbb{P}^2\big)^\perp > \mathrm{rk}\, T\Sigma_2\otimes\big(T\Sigma_{\hat{1}}^*\otimes L_{\hat{1}}^*\otimes T\Sigma_2\otimes L_2\big) + \frac{1}{2}\dim\big(\triangle\times\mathcal{S}_{2;1}(\mu)\big),$$

$\triangle\times\mathcal{S}_{2;1}(\mu)$ is $\alpha^{\tilde{E}\perp}$-hollow.
(3) Suppose $\mathcal{T}=(S^2,[N],I;j,\underline{d})$ is a non-basic bubble type and $\mathcal{D}_1 b=0$ for some element $b\in\mathcal{U}_\mathcal{T}(\mu)\subset\bar{\mathcal{V}}_2(\mu)$. Let I_1 and I_2 be the corresponding rooted trees and $k_1\in I_1$ and $k_2\in I_2$ the minimal elements. Then $d_{k_1}=0$, $d_{k_2}\neq 0$, and $\big|H_{k_1}\mathcal{T}\big|\in\{1,2\}$. Let

$$\mathcal{Z}_\mathcal{T} = \big\{(x_1,x_2,b;T_{x_1}\Sigma_1\otimes L_1|_b)\in\mathbb{P}\tilde{E} : b\in\mathcal{U}_\mathcal{T}(\mu)\big\}.$$

By Theorem 6.2, the normal bundle of $\mathcal{Z}_\mathcal{T}$ in $\mathbb{P}\tilde{E}$ is

$$\mathcal{N}\mathcal{Z}_\mathcal{T} = \pi_{\tilde{E}}^*\big(\mathcal{F}\mathcal{T}\oplus T\Sigma_{\hat{1}}^*\otimes L_{\hat{1}}^*\otimes T\Sigma_2\otimes L_2\big) \approx \mathcal{F}\mathcal{T}\oplus T\Sigma_{\hat{1}}^*\otimes T\Sigma_2\otimes L_2.$$

First suppose $H_{k_1}\mathcal{T}=\{\hat{1}\}$ is a one-element set. Then, with appropriate identifications,

$$\begin{gathered}\big|\alpha^{\tilde{E}}(x_1,x_2,b;v,u) - \alpha_{\mathcal{Z}_\mathcal{T}}(x_1,x_2,b;v_{\hat{1}},u)\big| \le C(b)|v|^{\frac{1}{p}}\big(|v_{\hat{1}}|+|u|\big) \quad \forall (v,u)\in\mathcal{N}_b\mathcal{Z}_{\mathcal{T},\delta(b)}, \\ \text{where} \quad \alpha_{\mathcal{Z}_\mathcal{T}}(x_1,x_2,b;v_{\hat{1}},u) = \big(\mathcal{D}_{\mathcal{T},\hat{1}}v_{\hat{1}}\big)\otimes s_{x_1} + \big\{\mathcal{D}_2\otimes s_{x_2}\big\}\circ u. \qquad (10.14)\end{gathered}$$

If $H_{k_1}\mathcal{T}\neq\hat{I}$, the images of $\mathcal{D}_{\mathcal{T},\hat{1}}\big|_b$ and of $\mathcal{D}_2\big|_b$ in $T_{\mathrm{ev}(b)}\mathbb{P}^2$ are linearly independent for all $b\in\mathcal{U}_\mathcal{T}(\mu)$. Thus, $\alpha_{\mathcal{Z}_\mathcal{T}}$ is injective on every fiber and $\mathcal{Z}_\mathcal{T}$ is $\alpha^{\tilde{E}\perp}$-hollow by (10.14), provided $\bar{\nu}$ is generic. Then, by Proposition 5.13,

$$\mathcal{C}_{\mathcal{Z}_\mathcal{T}}\big(\alpha^{\tilde{E}\perp}\big) = 0 \qquad \text{if} \quad \big|\mathcal{H}_{k_1}\mathcal{T}\big| = 1 < |\hat{I}|.$$

If $\mathcal{H}_{k_1}\mathcal{T}=\hat{I}$, $\alpha_{\mathcal{Z}_\mathcal{T}}$ has full rank outside of the set

$$\tilde{\mathcal{S}}_{\mathcal{T};2}(\mu) = \big\{(x,x,b;T_x\Sigma_1\otimes L_1|_b)\in\mathcal{Z}_\mathcal{T} : \mathcal{D}_{\mathcal{T},\hat{1}}v_{\hat{1}}+\mathcal{D}_2v_2=0 \text{ for some } (v_{\hat{1}},v_2)\neq 0\big\} \approx \Sigma\times\mathcal{S}_{\mathcal{T};2}(\mu).$$

Since $\alpha_{\mathcal{Z}_\mathcal{T}}$ extends naturally over $\bar{\mathcal{Z}}_\mathcal{T}\subset\mathbb{P}\tilde{E}$, by Proposition 5.13,

$$\begin{gathered}\mathcal{C}_{\mathcal{Z}_\mathcal{T}-\tilde{\mathcal{S}}_{\mathcal{T};2}(\mu)}\big(\alpha^{\tilde{E}\perp}\big) = N(\tilde{\alpha}_\mathcal{T}), \qquad \text{where} \quad \tilde{\alpha}_\mathcal{T} = \pi_{\bar{\nu}}^\perp\circ\alpha_{\mathcal{Z}_\mathcal{T}}\in\Gamma\big(\bar{\mathcal{Z}}_\mathcal{T};\mathrm{Hom}(F_2;\mathcal{O}_2)\big), \\ \tilde{\alpha}_\mathcal{T} = \pi_{\bar{\nu}}^\perp\circ\big((\mathcal{D}_{\mathcal{T},\hat{1}}v_{\hat{1}})\otimes s_{x_1} + \big\{\mathcal{D}_2\otimes s_{x_2}\big\}\circ u\big); \\ F_2 = L_{\hat{1}}^*\otimes L_{\hat{1}}\mathcal{T}\oplus T\Sigma_{\hat{1}}^*\otimes L_{\hat{1}}^*\otimes T\Sigma_2\otimes L_2 \approx L_{\hat{1}}\mathcal{T}\oplus T\Sigma_{\hat{1}}^*\otimes T\Sigma_2\otimes L_2, \quad \mathcal{O}_2 = \gamma_{\tilde{E}}^*\otimes\mathcal{O}^\perp \approx T\Sigma_{\hat{1}}^*\otimes\mathbb{C}^5.\end{gathered}$$

By Lemma 5.14,

$$N(\tilde{\alpha}_{\mathcal{T}}) = \sum_{k=0}^{k=4} \langle \lambda_{F_2}^{4-k} c_k(\mathcal{O}_2), [\mathbb{P}F_2] \rangle - \mathcal{C}_{\tilde{\alpha}_{\mathcal{T}}^{F_2\,-1}(0)}(\tilde{\alpha}_{\mathcal{T}}^{F_2\perp})$$
$$= 16\langle 3c_1(\mathcal{L}_{\hat{1}}^*) + 4c_1(\mathcal{L}_{\hat{2}}^*), [\bar{\mathcal{U}}_{\mathcal{T}}(\mu)] \rangle - \mathcal{C}_{\tilde{\alpha}_{\mathcal{T}}^{F_2\,-1}(0)}(\tilde{\alpha}_{\mathcal{T}}^{F_2\perp}),$$

since $a=0$, $c_1(L_{\mathcal{T},\hat{1}}^*) = c_1(\mathcal{L}_{\hat{1}}^*)$, and $c_1(L_{\hat{2}}^*) = c_1(\mathcal{L}_{\hat{2}}^*)$ in $H^*(\bar{\mathcal{U}}_{\mathcal{T}}(\mu))$. Furthermore,

$$\tilde{\alpha}_{\mathcal{T}}^{F_2\,-1}(0) = \{(x,x,b;[v\otimes v_{\hat{1}}, v\otimes v_2]) \in \mathbb{P}F_2 : \mathcal{D}_{\mathcal{T},\hat{1}} v_{\hat{1}} + \mathcal{D}_2 v_2 = 0\} \approx \Sigma \times \mathcal{S}_{\mathcal{T};2}(\mu),$$
$$F_3 \equiv \mathcal{N}\tilde{\alpha}_{\mathcal{T}}^{F_2\,-1}(0) = T\Sigma_2 \oplus \gamma_{E\mathcal{T}} \approx T\Sigma \oplus \mathbb{C}^2,$$
$$\big|\tilde{\alpha}_{\mathcal{T}}^{F_2\perp}(x,x,b;w,X) - \tilde{\alpha}_{\mathcal{T},\triangle}(x,b;w,X)\big| \le C|w|^2 \quad \forall (w,X) \in F_{3,\delta}, \qquad \text{where}$$
$$\tilde{\alpha}_{\mathcal{T},\triangle} \in \Gamma(\tilde{\alpha}_{\mathcal{T}}^{F_2\,-1}(0); \mathrm{Hom}(F_3, \mathcal{O}_3)), \quad \mathcal{O}_3 = \gamma_{F_2}^* \otimes \mathcal{O}_2^{\perp} \approx (T^*\Sigma \otimes \mathbb{C}^5)^{\perp},$$
$$\tilde{\alpha}_{\mathcal{T},\triangle}(w,X) = \pi_{\bar{\nu}_2}^{\perp} \circ \big(\pi_{\bar{\nu}}^{\perp} \circ \big(X \otimes s_x + \mathcal{D}_2 \otimes s_{g_x,x}^{(2)}(w,\cdot)\big)\big).$$

Since $\tilde{\alpha}_{\mathcal{T},\triangle}$ has full rank on every fiber, by Proposition 5.13,

$$\mathcal{C}_{\tilde{\alpha}_{\mathcal{T}}^{F_2\,-1}(0)}(\tilde{\alpha}_{\mathcal{T}}^{F_2\perp}) = \langle c_1(\mathcal{O}_3) - c_1(F_3), [\tilde{\alpha}_{\mathcal{T}}^{F_2\,-1}(0)] \rangle = 24|\mathcal{S}_{\mathcal{T};2}(\mu)|$$
$$= 24\langle c_1(\mathcal{L}_{\hat{1}}^*\mathcal{T}) + c_1(\mathcal{L}_{\hat{2}}^*), [\bar{\mathcal{U}}_{\mathcal{T}}(\mu)] \rangle.$$

On the other hand, $\tilde{\mathcal{S}}_{\mathcal{T};2}(\mu)$ is $\alpha^{\tilde{E}\perp}$-hollow, and thus

$$\mathcal{C}_{\tilde{\mathcal{S}}_{\mathcal{T};2}(\mu)}(\alpha^{\tilde{E}\perp}) = 0.$$

Indeed, by Theorem 6.2,

$$\mathcal{N}\tilde{\mathcal{S}}_{\mathcal{T};2} = T\Sigma_2 \oplus L_{\hat{2}}^* \otimes (\mathrm{Im}\,\mathcal{D}_{\mathcal{T},\hat{1}})^{\perp} \oplus L_{\hat{1}}^* \otimes L_{\hat{1}}\mathcal{T} \oplus T\Sigma_{\hat{1}}^* \otimes L_{\hat{1}}^* \otimes T\Sigma_2 \otimes L_2,$$
$$\big|\alpha^{\tilde{E}}(x,x,b;w,X,v_{\hat{1}},u) - \tilde{\alpha}(x,b;w,X,v_{\hat{1}},u)\big| \le C|(w,X)||w||u|, \quad \forall (w,X,v_{\hat{1}},u) \in \mathcal{N}\tilde{\mathcal{S}}_{\mathcal{T};2;\delta},$$
$$\text{where} \qquad \pi_x^{-1}\tilde{\alpha}(x,b;w,X,v_{\hat{1}},u) = \{\mathcal{D}_2 \otimes s_x^{(2)}(w,\cdot)\} \circ u.$$

The claim that $\tilde{\mathcal{S}}_{\mathcal{T};2}(\mu)$ is $\alpha^{\tilde{E}\perp}$-hollow then follows as in (2). Summing over bubble types as above, we conclude that

$$\sum_{|\chi(\mathcal{T})|=1} \mathcal{C}_{\Sigma^2 \times \mathcal{U}_{\mathcal{T}}(\mu)}(\alpha^{\tilde{E}\perp}) = 16 \sum_{\mathcal{T}^*} \sum_{\{i,j\}=\{1,2\}} \sum_{l \in M_i^*} \langle 3c_1(\mathcal{L}_i^*) + 4c_1(\mathcal{L}_j^*), [\bar{\mathcal{U}}_{\mathcal{T}^*(l)}] \rangle$$
$$- 24 \sum_{\mathcal{T}^*} \sum_{l \in [N]} \langle c_1(\mathcal{L}_1^*) + c_1(\mathcal{L}_2^*), [\bar{\mathcal{U}}_{\mathcal{T}^*(l)}] \rangle, \tag{10.15}$$

where the outer sums are taken over all equivalence classes of basic bubble types $\mathcal{T}^*$ such that $\mathcal{U}_{\mathcal{T}^*}(\mu) \subset \mathcal{V}_2(\mu)$.

(4) We next consider the case $H_{k_1}\mathcal{T} = \{\hat{1},\hat{2}\}$ is a two-element set. Then $\hat{I} = H_{k_1}\mathcal{T}$,

$$\big|\alpha^{\tilde{E}}(x_1,x_2,b;v,u) - \alpha_{\mathcal{Z}_{\mathcal{T}}}(x_1,x_2,b;v,u)\big| \le C(b)|v|^{\frac{1}{p}}(|v|+|u|) \quad \forall (v,u) \in \mathcal{N}_b\mathcal{Z}_{\mathcal{T},\delta(b)},$$
$$\text{where} \qquad \alpha_{\mathcal{Z}_{\mathcal{T}}}(x_1,x_2,b;v,u) = (\mathcal{D}_{\mathcal{T},\hat{1}} v_{\hat{1}}) \otimes s_{x_1} + (\mathcal{D}_{\mathcal{T},\hat{2}} v_{\hat{2}}) \otimes s_{x_1} + \{\mathcal{D}_2 \otimes s_{x_2}\} \circ u.$$

The map $\alpha_{\mathcal{Z}_\mathcal{T}}$ has full rank outside of the set

$$\tilde{\mathcal{S}}_{\mathcal{T};2}(\mu)=\big\{(x,x,b;T_x\Sigma_1\otimes L_1|_b)\in\mathcal{Z}_\mathcal{T}\big\}\approx\Sigma\times\mathcal{U}_\mathcal{T}(\mu).$$

Thus, by Proposition 5.13, $\mathcal{C}_{\mathcal{Z}_\mathcal{T}-\tilde{\mathcal{S}}_{\mathcal{T};2}(\mu)}(\alpha^{\tilde{E}\perp})=N(\pi_{\bar{\mathcal{V}}}^{\perp}\circ\alpha_{\mathcal{Z}_\mathcal{T}})$. By the same argument as in (2) above, $N(\pi_{\bar{\mathcal{V}}}^{\perp}\circ\alpha_{\mathcal{Z}_\mathcal{T}})=N(\tilde{\alpha}_\mathcal{T})$, where

$$\tilde{\alpha}_\mathcal{T}\in\Gamma(\mathcal{Z}_\mathcal{T};\mathrm{Hom}(F_2;\mathcal{O}_2)),\quad \tilde{\alpha}_\mathcal{T}(u)=\pi_{\bar{\mathcal{V}}}^{\perp}\circ\big(\{\mathcal{D}_2\otimes\pi_{x_1}^{-}\circ s_{x_2}\}\circ u\big);$$
$$F_2=T\Sigma_1^*\otimes L_1^*\otimes T\Sigma_2\otimes L_2\approx T\Sigma_1^*\otimes T\Sigma_2,\quad \mathcal{O}_2=\gamma_{\tilde{E}}^*\otimes\big(\mathcal{H}_{\Sigma_1}^{-}\otimes\mathrm{ev}^*T\mathbb{P}^2\big)^{\perp}\approx T\Sigma_1^*\otimes\big(\mathcal{H}_{\Sigma_1}^{-}\otimes\mathbb{C}^2\big)^{\perp}.$$

Thus, applying Lemma 5.14 and again Proposition 5.13, similarly to (2) we obtain

$$N(\tilde{\alpha}_\mathcal{T})=\big\langle c_1(F_2^*)c_1(\mathcal{O}_2)+c_2(\mathcal{O}_2),[\mathcal{Z}_\mathcal{T}]\big\rangle-\mathcal{C}_{\tilde{\alpha}_\mathcal{T}^{-1}(0)}(\tilde{\alpha}_\mathcal{T}^{\perp})=48\big|\mathcal{U}_\mathcal{T}(\mu)\big|-\mathcal{C}_{\tilde{\alpha}_\mathcal{T}^{-1}(0)}(\tilde{\alpha}_\mathcal{T}^{\perp});$$
$$\mathcal{C}_{\tilde{\alpha}_\mathcal{T}^{-1}(0)}(\tilde{\alpha}_\mathcal{T}^{\perp})=\big\langle c_1(T^*\Sigma)+c_1(F_2^*\otimes\mathcal{O}_2^{\perp}),[\tilde{\mathcal{S}}_{\mathcal{T};2}(\mu)]\big\rangle=24\big|\mathcal{U}_\mathcal{T}(\mu)\big|.$$

On the other hand, by an argument similar to (3) above, $\tilde{\mathcal{S}}_{\mathcal{T};2}(\mu)$ is $\alpha^{\tilde{E}\perp}$-hollow. We conclude that

$$\sum_{|\chi(\mathcal{T})|=2}\mathcal{C}_{\Sigma^2\times\mathcal{U}_\mathcal{T}(\mu)}(\alpha^{\tilde{E}\perp})=24\cdot3\big|\mathcal{V}_3(\mu)\big|=72\big|\mathcal{V}_3(\mu)\big|.\tag{10.16}$$

(5) We finally compute the $\alpha^{\tilde{E}\perp}$-contribution to $e(\gamma_{\tilde{E}}^*\otimes\mathcal{O}^{\perp})$ from the space

$$\mathcal{Z}_{2;2}(\mu)=\big\{(x,x,b;[v\otimes v_1,v\otimes v_2])\in\mathbb{P}\tilde{E}:\mathcal{D}_1v_1+\mathcal{D}_1v_2=0,\ v_1,v_2\neq0\big\}\approx\Sigma\times\mathcal{S}_{2;2}(\mu).$$

Its normal bundle in $\mathbb{P}\tilde{E}$ is

$$\mathcal{N}\mathcal{Z}_{2;2}=T\Sigma\oplus\gamma_{E_2}^*\otimes\mathrm{ev}^*T\mathbb{P}^2,\quad\text{where}\quad E_2=L_1\oplus L_2\longrightarrow\bar{\mathcal{V}}_2(\mu).$$

With appropriate identifications, $\mathcal{D}X=X$ for all $X\in\gamma_{E_2}^*\otimes\mathrm{ev}^*T\mathbb{P}^2$ sufficiently small, where $\mathcal{D}\in\Gamma(\mathbb{P}E_2;\gamma_{E_2}^*\otimes\mathrm{ev}^*T\mathbb{P}^2)$ is the section defined in the proof of Lemma 6.13. Then,

$$\big|\alpha^{\tilde{E}}(x,x,b;w,X)-\tilde{\alpha}_{2;2}(x,b;w,X)\big|\le C|w|^2\quad\forall(w,X)\in\mathcal{N}\mathcal{Z}_{2;2;\delta},\qquad\text{where}$$
$$\tilde{\alpha}_{2;2}(x,b;w,X)=X\otimes s_x+\mathcal{D}_{2;2}\otimes s_{g_x,x}^{(2)}(w,\cdot),$$

and $\mathcal{D}_{2;2}\in\Gamma(\bar{\mathcal{S}}_{2;2}(\mu);\gamma_{E_2}^*\otimes\mathrm{ev}^*T\mathbb{P}^2)$ is the map defined in Section 6.7. With the identification of small neighborhoods of Δ in $T\Sigma\longrightarrow\Delta$ and in Σ^2 used above, the coefficients defining $\mathcal{D}_{2;2}$ are $c_1=0$ and $c_2=1$. By the same argument as in (2) and (4), $\mathcal{C}_{\mathcal{Z}_{2;2}}(\alpha^{\tilde{E}\perp})=N(\tilde{\alpha}_{2;2}^{-})$, where

$$\tilde{\alpha}_{2;2}^{-}\in\Gamma(\Sigma\times\bar{\mathcal{S}}_{2;2}(\mu);\mathrm{Hom}(F_2;\mathcal{O}_2)),\quad\tilde{\alpha}_{2;2}^{-}(w)=\pi_{\bar{\mathcal{V}}}^{\perp}\circ\big\{\mathcal{D}_{2;2}\otimes s_x^{(2)}(w,\cdot)\big\};$$
$$F_2=T\Sigma,\quad\mathcal{O}_2=\gamma_{\tilde{E}}^*\otimes\big(\mathcal{H}_\Sigma^{-}\otimes\mathrm{ev}^*T\mathbb{P}^2\big)^{\perp}\approx T^*\Sigma\otimes\gamma_{E_2}^*\otimes\big(\mathcal{H}_\Sigma^{-}\otimes\mathrm{ev}^*T\mathbb{P}^2\big)^{\perp}.$$

By Lemmas 5.14 and 6.13,

$$N(\tilde{\alpha}_{2;2}^{-}) = \langle c_1(F_2^*)c_1(\mathcal{O}_2) + c_2(\mathcal{O}_2), [\Sigma\times\bar{\mathcal{S}}_{2;2}(\mu)]\rangle - \mathcal{C}_{\tilde{\alpha}_{2;2}^{-1}(0)}(\tilde{\alpha}_{2;2}^{-\perp})$$
$$= 4\langle 120a^2+66a(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*))+13(c_1^2(\mathcal{L}_1^*)+c_1^2(\mathcal{L}_2^*))+13c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*), [\bar{\mathcal{V}}_2(\mu)]\rangle - \mathcal{C}_{\tilde{\alpha}_{2;2}^{-1}(0)}(\tilde{\alpha}_{2;2}^{-\perp}).$$

By Proposition 5.13 and an argument similar to the proof of Lemma 6.16,

$$\mathcal{C}_{\tilde{\alpha}_{2;2}^{-1}(0)} = \langle c_1(F_2^*\otimes\mathcal{O}_2^{\perp}), [\Sigma\times\partial\bar{\mathcal{S}}_{2;2}(\mu)]\rangle = 28|\partial\bar{\mathcal{S}}_{2;2}(\mu)|$$
$$= 28|\mathcal{S}_{2;1}(\mu)| + 84|\mathcal{V}_3(\mu)| + 28\sum_{[\mathcal{T}^*]}\sum_{l\in[N]}\langle c(\mathcal{L}_1^*)+c(\mathcal{L}_2^*), [\bar{\mathcal{U}}_{\mathcal{T}^*(l)}(\mu)]\rangle.$$

It follows that

$$\mathcal{C}_{\mathcal{Z}_{2;2}}(\alpha^{\tilde{E}\perp}) = 4\langle 120a^2+66a(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*))+7(c_1^2(\mathcal{L}_1^*)+c_1^2(\mathcal{L}_2^*)) \tag{10.17}$$
$$+7c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*) + 6(c_1^2(L_1^*)+c_1^2(L_2^*)) + 6c_1(L_1^*)c_1(L_2^*), [\bar{\mathcal{V}}_2(\mu)]\rangle - 28|\mathcal{S}_{2;1}(\mu)| - 84|\mathcal{V}_3(\mu)|.$$

Combining equations (10.12), (10.13), (10.15), (10.16), and (10.17), we obtain

$$n_2^{(1)}(\mu) = 4\langle 24a^2 + 6a(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_1^*)) + 3c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_1^*) - (c_1^2(\mathcal{L}_1^*)+c_1^2(\mathcal{L}_1^*)), [\bar{\mathcal{V}}_2(\mu)]\rangle$$
$$+4|\mathcal{S}_{2;1}(\mu)| + 12|\mathcal{V}_3(\mu)|.$$

The claim then follows by using Lemma 6.5.

10.4 The Numbers $n_1^{(2)}(\mu)$ and $n_1^{(1)}(\mu)$

We now determine the two remaining numbers of Corollary 10.2.

Lemma 10.7 *If d is a positive integer and μ is a tuple of $3d-4$ points in general position in $\mathbb{P}^2$,*

$$n_1^{(2)}(\mu) = 12\langle 7a^2+6ac_1(\mathcal{L}^*), [\bar{\mathcal{S}}_1(\mu)]\rangle - 12\langle 9a^2+3a(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)), [\bar{\mathcal{V}}_2(\mu)]\rangle.$$

Proof: (1) By Section 10.1, $n_1^{(2)}(\mu)=N(\alpha_{1;1})$, where

$$\alpha_{1;1}\in\Gamma(\Sigma\times\bar{\mathcal{S}}_1(\mu);\mathrm{Hom}(T\Sigma^{\otimes 2}\otimes L^{\otimes 2},\mathcal{O})),\quad \mathcal{O}=\mathcal{H}_\Sigma^-\otimes\mathrm{ev}^*T\mathbb{P}^2,$$
$$\alpha_{1;1}(x,b;v\otimes v) = (\mathcal{D}^{(2)}v)(s_x^{(2)}v)\in\mathcal{H}_\Sigma^-(x)\otimes T_{\mathrm{ev}(b)}\mathbb{P}^2.$$

Thus, we can apply Corollary 5.17 with

$$\bar{\mathcal{M}}=\bar{\mathcal{S}}_1(\mu),\quad L_\Sigma=T\Sigma^{\otimes 2},\quad V_\Sigma=\mathcal{H}_\Sigma^-,\quad L_{\mathcal{M}}=L^{\otimes 2},\quad V_{\mathcal{M}}=\mathrm{ev}^*T\mathbb{P}^2,\quad s=s^{(2)},\quad \alpha=\mathcal{D}^{(2)}.$$

The first term of Corollary 5.17 gives the intersection number on $\bar{\mathcal{S}}_1(\mu)$ in the statement of the lemma, since $ac_1(L^*)=ac_1(\mathcal{L}^*)$. A decomposition of the zero set of $\mathcal{D}^{(2)}$ is given in the proof of Lemma 6.15. The only stratum of $\alpha^{-1}(0)$ contributing to the second term of Corollary 5.17 is $\mathcal{S}_{2;2}(\mu)$. Lemma 6.13 reduces this contribution to the intersection number on $\bar{\mathcal{V}}_2(\mu)$ of the lemma.

Lemma 10.8 $n_1^{(1)}(\mu) = 0$

Proof: By Section 10.1, $n_1^{(1)}(\mu) = N(\alpha_1)$, where

$$\alpha_1 \in \Gamma\big(\Sigma \times \bar{\mathcal{V}}_1(\mu); \mathrm{Hom}(T\Sigma \otimes L, \mathcal{O})\big), \qquad \mathcal{O} = \mathcal{H}_\Sigma^{0,1} \otimes \mathrm{ev}^* T\mathbb{P}^2,$$
$$\alpha_1\big(x, b; v \otimes v\big) = \big(\mathcal{D}v\big)\big(s_x v\big) \in \mathcal{H}_\Sigma^{0,1} \otimes T_{\mathrm{ev}(b)}\mathbb{P}^2.$$

It will be shown that there exists $\bar{\nu} \in \Gamma\big(\mathbb{P}^2; \mathcal{H}_\Sigma^{0,1} \otimes T\mathbb{P}^2\big)$ such that the affine map

$$\psi_{\alpha_1, \mathrm{ev}^*\bar{\nu}} : T\Sigma \otimes L \longrightarrow \mathcal{O}, \qquad \big(x, b; v \otimes v\big) \longrightarrow \bar{\nu}_{\mathrm{ev}(b)} + \alpha_1\big(x, b; v \otimes v\big), \tag{10.18}$$

has no zeros over $\Sigma \times \bar{\mathcal{V}}_1(\mu)$. The map

$$\tilde{\alpha} : \Sigma \times \mathbb{P}T\mathbb{P}^2 \longrightarrow \mathbb{P}\big(\mathcal{H}_\Sigma^{0,1} \otimes T\mathbb{P}^2\big) \approx \mathbb{P}\big(\mathbb{C}^3 \otimes T\mathbb{P}^2\big), \qquad (x, \ell) \longrightarrow (\mathrm{Im}\ s_x) \otimes \ell,$$

is an embedding, since Σ is not hyperelleptic. Let $\mathcal{W}$ denote the image of $\gamma_{T\mathbb{P}^2}\big|\big(\mathrm{Im}\ \tilde{\alpha}\big)$ under the projection map

$$\gamma_{T\mathbb{P}^2} \longrightarrow T\mathbb{P}^2, \qquad (q, \ell, v) \longrightarrow (q, v).$$

Then $\mathcal{W}$ is a closed subspace of $\mathcal{H}_\Sigma^{0,1} \otimes T\mathbb{P}^2$ and $\mathcal{W} \longrightarrow \mathbb{P}^2$ is a bundle of affine varieties of dimension three. Thus, by transversality and dimension-counting, we can choose $\bar{\nu} \in \Gamma\big(\mathbb{P}^2; \mathcal{H}_\Sigma^{0,1} \otimes T\mathbb{P}^2\big)$ such that the image $\bar{\nu}$ does not intersect $\mathcal{W}$. Then the map $\psi_{\alpha_1, \mathrm{ev}^*\bar{\nu}}$ of (10.18) does not vanish.

We now plug in the results of Lemmas 10.8, 10.7, 10.3, 10.6, 10.4, and 10.5 into Corollary 10.2. Using Lemmas 6.5, 6.14, and 6.15, we then obtain the expression for $CR_3(\mu)$ given in Theorem 1.4.

Chapter 11

A Computation via Degeneration

Genus-one and -two fixed-complex-structure enumerative numbers for $\mathbb{P}^2$ can also be computed by degenerating the complex structure to a singular one and studying what happens in the limit. It is shown in [P1] that the count does not change in the limit in the genus-one case. In the genus-two case, the same claim for a specific degeneration was made in [KQR]. Theorem 1.2 contradicts the resulting formula of [KQR]. In this chapter, we fix the argument in [KQR] to recover Theorem 1.2.

11.1 Summary

Since out goal here is to correct the argument in [KQR], we follow the notation of [KQR]. In particular, we write N_d and $N_{2,d}$ for the numbers n_d and $n_{2,d}$. The relation between the formulas of Theorem 1.2 and of [KQR] is

$$N^Z_{2,d} = 6\big(N^{KQR}_{2,d} + T_d\big), \tag{11.1}$$

where T_d is the number of degree-d tacnodal rational plane curves passing through $(3d-2)$ points. There are three errors in [KQR], one of which is significant. They are described briefly in the next paragraph and in more detail in Section 11.3. Once these errors are corrected, the formula of Theorem 1.2 is recovered via the relation (11.1).

The first step in the proof of Theorem 1.2 via the recipe of [KQR] is Lemma 11.1, which allows one to reduce the computation to a very degenerate genus-two curve. The relevant intersection number is then computed by Propositions 11.2-11.5. Propositions 11.2 and 11.4 are proved in [KQR]. Proposition 11.5 is implied by Remark 3.12 in [KQR]. However, this remark is stated without a proof and contradicts Proposition 11.3. This is the significant error in [KQR]. A minor error is the statement about boundary relations at the beginning of the proof of Lemma 2.18. A posteriori, it turns out that this statement is in fact correct, at least in the relevant cases, but it does not follow from the argument given. The remaining error is dividing by an extra factor of six when computing contributions to the intersection number.

The outline of this chapter is as follows. We first review the notation and setup used in [KQR]. In Section 11.3, four propositions are stated which imply Theorem 1.2. The last two sections prove the two propositions not proved in [KQR].

11.2 Review of Notation and Setup

Denote by $\overline{\mathfrak{M}}_2$ the Deligne-Mumford moduli space of stable genus-two curves. If $d \geq 3$, let

$$\overline{\mathfrak{M}}_2(d) \equiv \overline{\mathfrak{M}}_{2,3d-2}(\mathbb{P}^2, d\ell)$$

be Kontsevich's moduli space of stable maps from $(3d-2)$-pointed genus-two curves to $\mathbb{P}^2$ of degree d, where $\ell \in H_2(\mathbb{P}^2;\mathbb{Z})$ is the homology class of a line. Let $\pi: \overline{\mathfrak{M}}_2(d) \longrightarrow \overline{\mathfrak{M}}_2$ be the forgetful map. Denote by $W_2(d) \subset \overline{\mathfrak{M}}_2(d)$ the subset of stable maps with irreducible domains and by $\overline{W}_2(d)$ the closure of $W_2(d)$ in $\overline{\mathfrak{M}}_2(d)$.

Every element of $\overline{\mathfrak{M}}_2(d)$ can be written as $[\mu\colon (D, p_1, \ldots, p_{3d-2})]$, where D is a prestable genus-two curve, $\mu\colon D \longrightarrow \mathbb{P}^2$ is a (holomorphic) map, and $p_1, \ldots, p_{3d-2} \in D$ are the marked points. There are natural evaluation maps

$$e_i\colon \overline{\mathfrak{M}}_2(d) \longrightarrow \mathbb{P}^2, \quad e_i\big([\mu\colon (D, p_1, \ldots, p_{3d-2})]\big) = \mu(p_i), \qquad i = 1, \ldots, 3d-2.$$

Let $\mathcal{L}_i = e_i^*\big(\mathcal{O}_{\mathbb{P}^2}(1)\big)$ and

$$Z = \big[\overline{W}_2(d)\big] \cap c_1^2(\mathcal{L}_1) \cap \ldots \cap c_1^2(\mathcal{L}_{3d-2}) \in H_6\big(\overline{W}_2(d)\big).$$

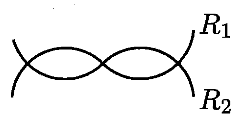
If $q_1, \ldots, q_{3d-2}$ are points in $\mathbb{P}^2$ in general position, then $\{e_1 \times \ldots \times e_{3d-2}\}^{-1}(q_1 \times \ldots \times q_{3d-2})$ is a representative for Z; see [KQR] for details.

Lemma 11.1 *For every* $[C] \in \overline{\mathfrak{M}}_2$,

$$N_{2,d} = [\pi^{-1}(C)] \cdot Z,$$

where $[\pi^{-1}(C)] \cdot Z$ *is the intersection pairing of* $\pi^{-1}([C])$ *and* Z *in* $\overline{W}_2(d)$.

This is a special case of Lemma 2.5 in [KQR]. In particular, if C_0 consists of two rational components identified at 3 pairs of points, i.e.

$C_0 =$ 

then $N_{2,d} = [\pi^{-1}(C_0)] \cdot Z$. The space $\pi^{-1}(C_0) \subset \overline{\mathfrak{M}}_2(d)$ can be written as the disjoint union $\bigsqcup W_T$, where W_T is the space of stable maps $[\mu\colon (D, p_1, \ldots, p_{3d-2})]$, such that the domain D is the union of R_1, R_2, and trees $T_1, \ldots, T_s$ of $\mathbb{P}^1$ in a way encoded by T. The stable reduction of D must be C_0. See Figure 1 below for some examples.

In order to compute $[\pi^{-1}(C_0)] \cdot Z$, [KQR] consider the intersection of Z with every nonempty space W_T. It is fairly easy to show that $Z \cap W_T$ is empty for all but a small number of trees T, independent of d. If $[\mu\colon (D, p_1, \ldots, p_{3d-2})] \in Z \cap W_T$, the map $\mu\colon D \longrightarrow \mathbb{P}^2$ has degree d and passes through $3d-2$ points in $\mathbb{P}^2$ in general position. Thus, if $D_1, \ldots, D_m$ are the irreducible components of D to which μ restricts non-trivially, $m=1$ or $m=2$. Then D can have at most two components, other than R_1, R_2, on which the map μ is constant.

The complete list of possibilities for D, up to equivalence, is given in Figure 1. Denote by C_{ij} the curve as in the ith row and jth column of Figure 1. Similarly, denote by W_{ij} be

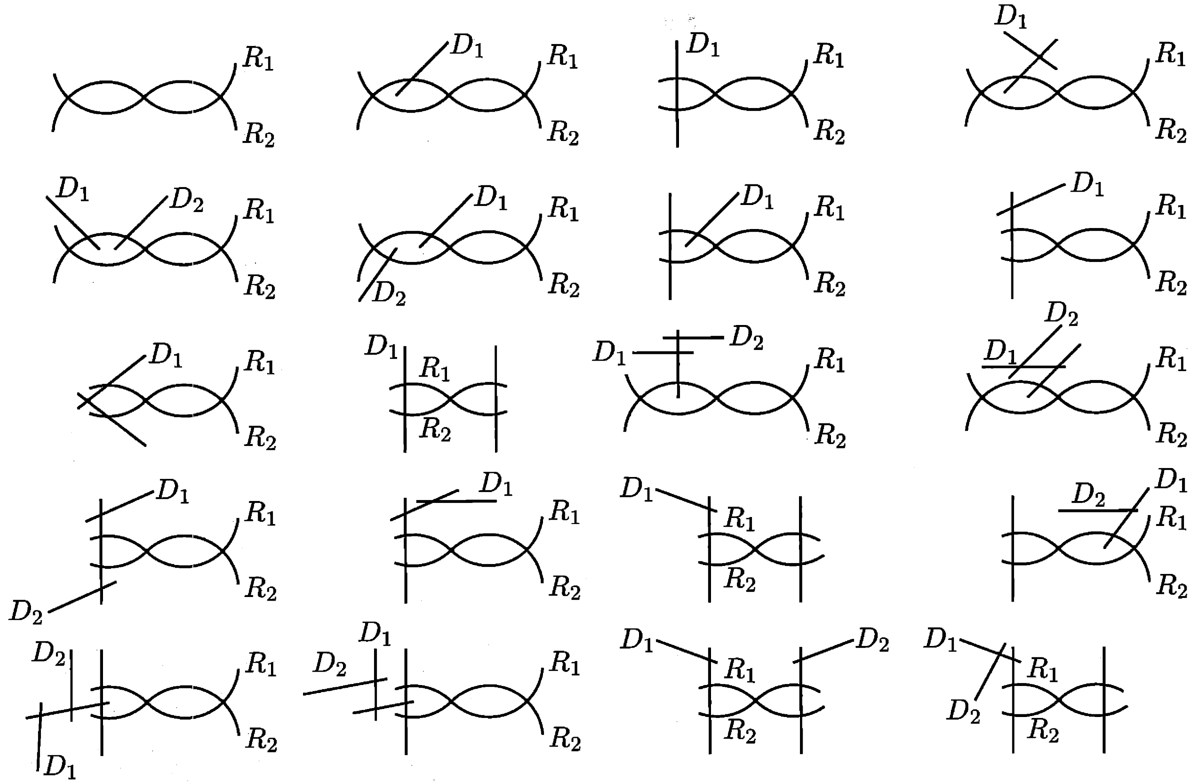


Figure 1

the space of stable maps with domain C_{ij} and a distribution of the degree d between the components of C_{ij} such that the image of some stable map in W_{ij} passes through $(3d-2)$ points. We clarify this statement in the relevant cases:
(1) if $[\mu\colon (D, p_1, \ldots, p_{3d-2})]$ lies in W_{13}, W_{32}, W_{41}, W_{43}, or W_{5j}, the degree of $\mu|D_i$ is $d_i \neq 0$, and the restriction of μ to all other components is constant;
(2) if $[\mu\colon (D, p_1, \ldots, p_{3d-2})]$ lies in W_{24}, W_{31}, or W_{42}, the degree of $\mu|D_1$ is $d_1 \neq 0$, $\mu|R_i$ is constant, and in the case of W_{42} the restriction of μ to the vertical component (in the diagram) is constant.
Furthermore, for stability reasons, every component of C_{ij}, on which μ is constant and which does not contain three singular point of C_{ij}, must contain one of the marked points p_i.

11.3 Computation of the Intersection Number

Proposition 11.2 *The contribution to* $[\pi^{-1}(C_0)] \cdot Z$ *from* W_{11} *is*

$$\frac{3(d-1)(d-2)(d-3)}{d} N_d + \frac{1}{2} \sum_{d_1+d_2=d} \left(d_1^2 d_2^2 - 6d_1d_2 - 4 + 18\frac{d_1d_2}{d}\right) \binom{3d-2}{3d_1-1} d_1d_2N_{d_1}N_{d_2}.$$

This proposition is essentially proved in [KQR]; see equation (2.9) and Lemmas 2.12, 2.16, and 3.2 in [KQR]. The above number is six times the number given by Theorem 1.1 of [KQR]. It is easy to see that the authors divide by six an extra time. For example, in Lemma 2.12, one should take *ordered* triplets of nodes, i.e. $\binom{d_1d_2}{3}$ should be replaced by

$$d_1d_2(d_1d_2-1)(d_1d_2-2),$$

since they are dividing by the order of $\mathrm{Aut}(C_0)$. Similarly, the number in Lemma 2.16

should be replaced by six times itself.

Proposition 11.3 *The contribution to* $[\pi^{-1}(C_0)]\cdot Z$ *from* W_{13} *is*

$$\frac{6(3d^2-12d+9)n_d}{d}+3\sum_{d_1+d_2=d}\Big(d_1d_2+4-9\frac{d_1d_2}{d}\Big)\binom{3d-2}{3d_1-1}d_1d_2N_{d_1}N_{d_2}.$$

We prove this proposition in Section 11.5. What we show is that $\overline{W}_2(d)\cap W_{13}$ is the space of all stable maps $[\mu\colon (D,p_1,\ldots,p_{3d-2})]$ such that $\mu(D)$ is a tacnodal curve in $\mathbb{P}^2$, and μ maps the two nodes of D to the same tacnode of $\mu(D)$. The number of Proposition 11.3 is $6T_d$. Note that the number T_d is known; see equation (1.2) in [DH] and Subsection 3.2 in [V1].

Proposition 11.4 *If* $(i,j)\in\{(1,2),(1,4),(2,1),(2,2),(2,3),(3,3),(3,4),(4,4)\}$, $Z\cap W_{ij}=\emptyset$. *Thus,* W_{ij} *does not contribute to* $[\pi^{-1}(C_0)]\cdot Z$.

Most of this proposition is proved by Lemmas 2.18 and 3.7 of [KQR]. The cases $(i,j)=(3,3)$ and $(i,j)=(3,4)$ can be deduced from the proofs of these two lemmas. The modification required is similar to the extension of the main part of the proof of Lemma 1 in [P1] to cases of multiple blowups; see also the proof of Lemma 11.9 below. Note that since Lemma 3.7 of [KQR] does not apply to the remaining possibilities for (i,j), neither does Lemma 2.18 of [KQR].

Proposition 11.5 *If* $(i,j)\in\{(2,4),(3,1),(3,2),(4,1),(4,2),(4,3),(5,1),(5,2),(5,3),(5,4)\}$, $Z\cap W_{ij}=\emptyset$. *Thus,* W_{ij} *does not contribute to* $[\pi^{-1}(C_0)]\cdot Z$.

We prove this proposition in the next section. The number in Theorem 1.2 is the sum of the numbers in Propositions 11.2 and 11.3. However, one has to make use of Kontsevich's recursion to obtain the formula in Theorem 1.2:

$$N_d=\frac{1}{6(d-1)}\sum_{d_1+d_2=d}\Big(d_1d_2-2\frac{(d_1-d_2)^2}{3d-2}\Big)\binom{3d-2}{3d_1-1}d_1d_2N_{d_1}N_{d_2}.$$

11.4 Proof of Proposition 11.5

11.4.1 The Semi-Standard Cases

We prove Proposition 11.5 by exhibiting conditions that stable maps in $\overline{W}_2(d)\cap W_{ij}$ must satisfy. This approach is analogous to methods in [P1] and [KQR], but we make no use of the spaces X and Y of these two papers. It should be possible to describe the space $\overline{W}_2(d)\cap\pi^{-1}(C_0)\subset\overline{\mathfrak{M}}_2(d)$ explicitly by using arguments as in this section to obtain necessary conditions for an element of $\pi^{-1}(C_0)$ to be in $\overline{W}_2(d)$ and by applying methods similar to the next section to show that these conditions are sufficient. However, much less is needed to prove Theorem 1.2.

Suppose $[\mu\colon (D,p_1,\ldots,p_{3d-2})]\in\overline{W}_2(g)\cap W_{ij}$. Then by definition of stable-map convergence, there exist
(T1) a one-parameter family of curves $\tilde{\eta}:\tilde{\mathcal{F}}\longrightarrow\Delta$ such that Δ is a neighborhood of 0 in $\mathbb{C}$, $\tilde{\mathcal{F}}$ is a smooth space, $\tilde{\eta}^{-1}(0)=D$, and $C_t\equiv\tilde{\eta}^{-1}(t)$ is a smooth genus-two curve for all

$t \in \Delta^* \equiv \Delta - \{0\}$;
(T2) a map $\tilde{\mu}: \tilde{\mathcal{F}} \longrightarrow \mathbb{P}^2$ such that $\tilde{\mu}|\eta^{-1}(0) = \mu$.
In many cases, $\tilde{\mathcal{F}}$ can be obtained by a sequence of blowups from another smooth bundle $\eta: \mathcal{F} \longrightarrow \Delta$ of curves. This observation is used often in the proofs of the lemmas that follow.

Lemma 11.6 *If* $[\mu: (D, p_1, \ldots, p_{3d-2})] \in \overline{W}_2(d) \cap W_{24}$ *and the degree of* $\mu|D_1$ *is* d, $\mu(D)$ *has a cusp at the image of the node of* D_1.

Proof: (1) Let $\tilde{\eta}: \tilde{\mathcal{F}} \longrightarrow \Delta^*$ be a family as in (T1) above with central fiber $\tilde{C}_0 = D$, and $\tilde{\mu}: \tilde{\mathcal{F}} \longrightarrow \mathbb{P}^2$ a map as in (T2). Then there exists another family $\eta: \mathcal{F} \longrightarrow \Delta$ as in (T1) such that the central fiber is C_{13} and $\tilde{\mathcal{F}}$ is the blowup of $\mathcal{F}$ at a smooth point $p \in D_1 \subset C_{13}$.
(2) Let $\psi \in H^0(C_{13}; \omega_{C_{13}})$ be an element such that $\psi|D_1 \neq 0$. From the point of view of complex geometry, $H^0(C_{13}; \omega_{C_{13}})$ is the space harmonic $(1,0)$-forms on the three components of C_{13}, which have simple poles at the singular points with residues that add up to zero at each node. Thus, such an element exists. Let (t, w) be coordinates near $p \in \mathcal{F}$ such that w is the vertical coordinate, i.e. $d\eta|\frac{\partial}{\partial w} = 0$. Then ψ extends to a family of elements $\psi_t \in H^0(C_t; \omega_{C_t})$ such that

$$\psi_t\big|_w = a\big(1 + o(1_{(t,w)})\big)dw, \tag{11.2}$$

for some $a \in \mathbb{C}^*$.
(3) On a neighborhood of $D_1^* \subset D = C_{24}$, the complement of the node in D_1, we have local coordinates $(t, z) \longrightarrow (t, w = tz, [1, z])$. Note that in these coordinates, (11.2) becomes

$$\psi_t\big|_z = at\big(1 + o(1_t)\big)dz. \tag{11.3}$$

Let L_1 and L_2 be any two lines in general position in $\mathbb{P}^2$. In particular, we assume that they miss the image under μ of the node of D_1. Then for all $t \in \Delta$, sufficiently small,

$$\mu_t^{-1}(L_i) = \{z_1^{(i)}(t), \ldots, z_d^{(i)}(t)\} \subset C_t \quad \text{and} \quad z_j^{(i)}(t) = z_j^{(i)}(0) + o(1_t), \tag{11.4}$$

where $\mu_t = \tilde{\mu}|C_t$. Since $\sum z_j^{(1)}(t)$ and $\sum z_j^{(2)}(t)$ are linearly equivalent divisors in C_t,

$$\sum_{j=1}^{j=d} \int_{z_j^{(1)}(t)}^{z_j^{(2)}(t)} \psi_t = 0 \qquad \forall t \in \Delta^*, \tag{11.5}$$

where the line integrals are taken inside of the coordinate chart. Plugging (11.3) and (11.4) into (11.5) gives

$$at \sum_{j=1}^{j=d} \big(z_j^{(2)}(0) - z_j^{(1)}(0) + o(1_t)\big) = 0 \qquad \forall t \in \Delta^*. \tag{11.6}$$

Dividing this equation by at and then taking the limit as $t \longrightarrow 0$, we conclude that

$$\sum_{j=1}^{j=d} z_j^{(1)}(0) = \sum_{j=1}^{j=d} z_j^{(2)}(0). \tag{11.7}$$

Condition (11.7) can be explicitly interpreted as follows. Let $[u, v]$ be homogeneous coordinates on D_1 such that $z = \frac{v}{u}$. Then a map $D_1 \longrightarrow \mathbb{P}^2$ of degree-d corresponds to three homogeneous polynomials

$$p_i = \sum_{j=0}^{j=d} p_{ij} u^j v^{d-j}.$$

Since equality (11.7) holds for a dense subset of lines in $\mathbb{P}^2$, there exists $K = K(\mu) \in \mathbb{P}^1$ such that

$$\frac{c_0 p_{0,d-1} + c_1 p_{1,d-1} + c_2 p_{2,d-1}}{c_0 p_{0,d} + c_1 p_{1,d} + c_2 p_{2,d}} = K \quad \forall (c_0, c_1, c_2) \in \mathbb{C}^3 - \{0\} \Longrightarrow$$
$$(p_{0,d-1}, p_{1,d-1}, p_{2,d-1}) = K(p_{0,d}, p_{1,d}, p_{2,d}). \tag{11.8}$$

Equation (11.8) imposes two linearly independent conditions on the map $\mu|D_1$ whenever $\mu \in \overline{W}_2(d) \cap W_{24}$. Geometrically, they mean that $\mu(D)$ has a cusp at the image of the node of D_1.

Corollary 11.7 *If $[\mu\colon (D, p_1, \ldots, p_{3d-2})] \in Z \cap W_{24}$, the degree of $\mu|D_1$ is less than d.*

Proof: Suppose the degree of $\mu|D_1$ is d. Then by Lemma 11.6, $\mu(D_1)$ has a cusp at the image of the node of D_1. Since the points $q_1, \ldots, q_{3d-2}$ are in general position, $\mu(D_1)$ has one simple cusp and $\binom{d-1}{2} - 1$ simple nodes. Let $\tilde{\mathcal{F}}$ and $\tilde{\mu}$ be as in the proof of Lemma 11.6. Then $\tilde{\mu}(C_t)$ converges to $\mu(D_1)$. By Lemma 2.4.1 or Example 3.2.2 in [V2], D_1 must have an elliptic tail, i.e. the map $\tilde{\mu}\colon \tilde{\mathcal{F}} \longrightarrow \mathbb{P}^2$ cannot exist. In the given case, this can also be seen directly as follows. The image under μ_t of the intersection of C_t with the coordinate chart described in (3) of the proof of Lemma 11.6 has $\binom{d-1}{2} - 1$ simple nodes, close to the simple nodes of $\mu(D_1)$. The complement of the coordinate chart in C_t is a genus two curve with a small coordinate neighborhood removed. Thus, it contributes at least 2 to the arithmetic genus of $\mu(C_t)$. This means that the arithmetic genus of $\mu(C_t)$ is at least $\binom{d-1}{2} + 1$, instead of $\binom{d-1}{2}$.

Lemma 11.8 *The image of every element $[\mu\colon (D, p_1, \ldots, p_{3d-2})] \in \overline{W}_2(d) \cap W_{43}$ has a cusp at $\mu(p_i)$ for some $i = 1, \ldots, 3d-2$. The same is true for every element of $\overline{W}_2(d) \cap W_{42}$ such that the degree of $\mu|D_1$ is d. Thus, $Z \cap W_{43} = \emptyset$, while for every element*

$$[\mu\colon (D, p_1, \ldots, p_{3d-2})] \in Z \cap W_{42},$$

the degree of $\mu|D_1$ is less than d.

Proof: (1) The proof of the first statement is nearly the same as the proof of Lemma 11.6. The only difference is that the central fiber of $\mathcal{F}$ will be C_{32}.
(2) The family $\tilde{\mathcal{F}}$ of the second claim of this lemma is obtained from $\tilde{\mathcal{F}}$ of Lemma 11.6 by blowing up a smooth point of the exceptional divisor $D_1 \subset C_{24}$. Thus, nearly the same argument as in Lemma 11.6 applies if the degree of $\mu|D_1$ is d; see [P1] for an extension in an analogous situation.

Lemma 11.9 *If $(i, j) \in \{(5, 2), (5, 4)\}$, the image of every element of $\overline{W}_2(d) \cap W_{ij}$ is a two-component rational cuspidal curve. The same is true for all*

$$[\mu\colon (D, p_1, \ldots, p_{3d-2})] \in \overline{W}_2(d) \cap W_{42},$$

such that the degree of $\mu|D_1$ is less than d. Thus, $Z \cap W_{ij} = \emptyset$ in all three cases.

Proof: (1) We first consider the case $[\mu\colon (D,p_1,\ldots,p_{3d-2})]\in\overline{W}_2(d)\cap W_{42}$ and the degree of $\mu|D_1$ is $d_1<d$. The case $d_1=d$ is considered in Lemma 11.8. The family $\tilde{\mathcal{F}}\longrightarrow\Delta$ corresponding to this case can be obtained as follows. We start with a family $\mathcal{F}\longrightarrow\Delta$ as in (2) of the proof of Lemma 11.6, blow it up at a smooth point $p\in D_1\subset C_{13}$, and then blow up the resulting space at a smooth point p_1 of the new exceptional divisor $E\equiv D_1\subset C_{24}$. Denote the last exceptional divisor by E_1. We use coordinates (t,z) near E^* as before and coordinates $(t,z_1)\longrightarrow\big(t,z=p_1+tz_1,[1,z_1]\big)$ near E_1^*. Then,

$$\begin{gathered}
\psi_t\big|_z=at\big(1+o(1_t)\big)dz,\qquad \psi_t\big|_{z_1}=at^2\big(1+o(1_t)\big)dz_1;\\
\mu_t^{-1}(L_i)=\big\{z_{1,1}^{(i)}(t),\ldots,z_{1,d_1}^{(i)}(t),z_{d_1+1}^{(i)}(t),\ldots,z_d^{(i)}(t)\big\}\subset C_t,\qquad\text{with}\\
z_{1,j}^{(i)}(t)=z_{1,j}^{(i)}(0)+o(1_t),\quad z_j^{(i)}(t)=z_j^{(i)}(0)+o(1_t);\\
\sum_{j=1}^{j=d_1}\int_{z_{1,j}^{(1)}(t)}^{z_{1,j}^{(2)}(t)}\psi_t+\sum_{j=d_1+1}^{j=d}\int_{z_j^{(1)}(t)}^{z_j^{(2)}(t)}\psi_t=0\qquad\forall t\in\Delta^*.
\end{gathered}$$

Each line integral is taken inside the corresponding coordinate chart. Proceeding as in the proof of Lemma 11.6, we obtain

$$\begin{gathered}
at^2\sum_{j=1}^{j=d_1}\big(z_{1,j}^{(1)}(0)-z_{1,j}^{(2)}(0)+o(1_t)\big)+at\sum_{j=d_1+1}^{j=d}\big(z_j^{(2)}(0)-z_j^{(1)}(0)+o(1_t)\big)=0\qquad\forall t\in\Delta^*\\
\Longrightarrow\sum_{j=d_1+1}^{j=d}z_j^{(1)}(0)=\sum_{j=d_1+1}^{j=d}z_j^{(2)}(0).
\end{gathered}$$

As before, the last identity implies that $\mu|E$ maps $z=\infty\in E$ to a cusp of $\mu(E)$.
(2) The argument in the case of W_{54} is the same, except we replace the family $\mathcal{F}$ of Lemma 11.6 with the family $\mathcal{F}$ of (1) of Lemma 11.8. Finally, the case of W_{52} simply involves an extra blowup at a smooth point as compared to the case of W_{42}.

Lemma 11.10 *If $(i,j)\in\{(4,1),(5,1)\}$, the image of every element of $\overline{W}_2(d)\cap W_{ij}$ is a two-component rational curve that has a tacnode. Thus, $Z\cap W_{ij}=\emptyset$.*

Proof: (1) The family $\tilde{\mathcal{F}}$ corresponding to the case of W_{41} is obtained by blowing up the family $\mathcal{F}$ of Lemma 11.6 at two smooth points, p_1 and p_2, of $D_1\subset C_{13}$. On a neighborhood of $D_i^*\subset C_{41}$, we use local coordinates $(t,z_i)\longrightarrow\big(t,p_i+tz_i,[1,z_i]\big)$. Then,

$$\begin{gathered}
\psi_t\big|_{z_i}=a_it\big(1+o(1_t)\big)dz_i;\\
\mu_t^{-1}(L_i)=\big\{z_{1,1}^{(i)}(t),\ldots,z_{1,d_1}^{(i)}(t),z_{2,d_1+1}^{(i)}(t),\ldots,z_{2,d}^{(i)}(t)\big\}\subset C_t,\quad z_{\iota,j}^{(i)}(t)=z_{\iota,j}^{(i)}(0)+o(1_t);\\
\sum_{j=1}^{j=d_1}\int_{z_{1,j}^{(1)}(t)}^{z_{1,j}^{(2)}(t)}\psi_t+\sum_{j=d_1+1}^{j=d}\int_{z_{2,j}^{(1)}(t)}^{z_{2,j}^{(2)}(t)}\psi_t=0\qquad\forall t\in\Delta^*.
\end{gathered}$$

for some $a_1, a_2 \in \mathbb{C}^*$, which depend on D, but not on $\mu|D_i$. Proceeding as before, we obtain

$$a_1 t \sum_{j=1}^{j=d_1} \big(z_{1,j}^{(1)}(0) - z_{1,j}^{(2)}(0) + o(1_t)\big) + a_2 t \sum_{j=d_1+1}^{j=d} \big(z_{2,j}^{(2)}(0) - z_{2,j}^{(1)}(0) + o(1_t)\big) = 0 \quad \forall t \in \Delta^* \Longrightarrow$$

$$a_1 \sum_{j=1}^{j=d_1} z_{1,j}^{(1)}(0) + a_2 \sum_{j=d_1+1}^{j=d} z_{2,j}^{(1)}(0) = a_1 \sum_{j=1}^{j=d_1} z_{1,j}^{(2)}(0) + a_2 \sum_{j=d_1+1}^{j=d} z_{2,j}^{(2)}(0). \tag{11.9}$$

Let $p_i^{(1)}$ and $p_i^{(2)}$ be the homogeneous polynomials corresponding to $\mu|D_1$ and $\mu|D_2$, respectively. Since (11.9) holds for a dense subset of lines, there exist $K = K(\mu) \in \mathbb{C}$ such that

$$a_1 \frac{c_0 p_{0,d_1-1}^{(1)} + c_1 p_{1,d_1-1}^{(1)} + c_2 p_{2,d_1-1}^{(1)}}{c_0 p_{0,d_1}^{(1)} + c_1 p_{1,d_1}^{(1)} + c_2 p_{2,d_1}^{(1)}} + a_2 \frac{c_0 p_{0,d_2-1}^{(2)} + c_1 p_{1,d_2-1}^{(2)} + c_2 p_{2,d_2-1}^{(2)}}{c_0 p_{0,d_2}^{(2)} + c_1 p_{1,d_2}^{(2)} + c_2 p_{2,d_2}^{(2)}} = K, \tag{11.10}$$

for all $(c_0, c_1, c_2) \in \mathbb{C}^3 - \{0\}$. Since μ maps the nodes of D_1 and D_2 to the same point,

$$\big(p_{0,d_1}^{(1)}, p_{1,d_1}^{(1)}, p_{2,d_1}^{(1)}\big) = \kappa \big(p_{0,d_2}^{(2)}, p_{1,d_2}^{(2)}, p_{2,d_2}^{(2)}\big)$$

for some $\kappa \in \mathbb{C}$. Using this equation, it is easy to see that condition (11.10) is equivalent to saying that μ maps the singular points of D_1 and D_2 into a tacnode of its image. Thus, the image of very element of $\overline{W}_2(d) \cap W_{41}$ is a two-component curve with a tacnode.
(2) Nearly the same argument applies to W_{51}. In this case, an extra blowup is required, and we will have $a_1 = a_2 = a$.

Lemma 11.11 *The image of every element of $\overline{W}_2(d) \cap W_{53}$ is a two-component rational curve such that both components have a cusp at one of the nodes of the image curve. Thus, $Z \cap W_{53} = \emptyset$.*

Proof: The proof is a minor modification of the proof of Lemma 11.6. The central fiber of $\mathcal{F}$ in this case is C_{32}. We can then choose $\psi \in H^0(C_{32}; \omega_{C_{32}})$ such that the restriction of ψ to the right vertical component (in the diagram) is zero. In terms of coordinates (t, w_1) and (t, w_2) near the smooth points p_1 and p_2 of the two vertical components, we will have

$$\psi_t\big|_{w_1} = a\big(1 + o(1_{(t,w)})\big) dw_1 \quad \text{and} \quad \psi_t\big|_{w_2} = o(1_t) dw_2,$$

for some $a \in \mathbb{C}^*$. Proceeding as above, we then conclude that μ maps the node of $D_1 \subset C_{53}$ to a cusp of $\mu(D_1)$. The same argument applies to $\mu|D_2$.

11.4.2 The Remaining Cases

The arguments in the previous subsection look very much like the arguments in [P1] and [KQR]. However, some differences appear in this subsection.

Lemma 11.12 *If $[\mu : (D, p_1, \ldots, p_{3d-2})] \in \overline{W}_2(d) \cap W_{13}$, the image of μ is a tacnodal rational curve and μ maps the nodes of D to a tacnode of $\mu(D)$.*

Proof: (1) We use coordinates (t,w) near $D_1^* \subset C_{13}$ such that the two nodes of D_1 correspond to $w=0$ and $w=\infty$. Let $\psi_t \in H^0(C_t;\omega_{C_t})$ be such that

$$\psi_t\big|_w = \big(1+o(1_t)\big)\frac{dw}{w}.$$

Proceeding as above, we obtain

$$\begin{aligned} \mu_t^{-1}(L_i) &= \{w_1^{(i)}(t),\ldots,w_d^{(i)}(t)\} \subset C_t, \quad w_j^{(i)}(t) = w_j^{(i)}(0)+o(1_t);\\ &\sum_{j=1}^{j=d}\int_{w_j^{(1)}(t)}^{w_j^{(2)}(t)} \psi_t = 0 \in \mathbb{C}/2\pi i\mathbb{Z} \qquad \forall t\in\Delta^*;\\ &\prod_{j=1}^{j=d} w_j^{(1)}(0) = \prod_{j=1}^{j=d} w_j^{(2)}(0) \equiv K; \qquad (11.11)\\ &\big(p_{0,0},p_{1,0},p_{2,0}\big) = K\big(p_{0,d},p_{1,d},p_{2,d}\big). \qquad (11.12)\end{aligned}$$

for some $K=K(\mu)\in\mathbb{C}$. Condition (11.12) on the coefficients of the homogeneous polynomials corresponding to $\mu|D_1$ follows from the fact that (11.11) holds for a dense subset of lines in $\mathbb{P}^2$. However, (11.12) by itself tells us nothing new about $\mu|D_1$, since we already know that μ maps the nodes of D_1 to the same point.

(2) We instead consider the limit of the left-hand side of (11.11) as L_1 approaches the line tangent to the branch $w=0$ of $\mu(D)$. If the node $\mu(0)$ of $\mu(D)$ is simple, two of the numbers $w_j^{(1)}(0)$ tend to 0 and one to ∞, all at comparable rates. Thus, we must have $K=0$. By the same argument, $K=\infty$. This means

$$p_{0,0}=p_{1,0}=p_{2,0}=p_{0,d}=p_{1,d}=p_{2,d}=0.$$

If $[u,v]$ are homogeneous coordinates on $E^{(1)}$ with $w=\frac{v}{u}$, it follows that uv divides all three homogeneous polynomials p_0,p_1,p_2, i.e. $\mu|D$ has degree at most $d-2$, not d, contrary to the assumption. Thus, $\mu(0)=\mu(\infty)$ has to be a tacnode of $\mu(D)$ if

$$\big[\mu\colon (D,p_1,\ldots,p_{3d-2})\big]\in\overline{W}_2(d)\cap W_{13}.$$

Lemma 11.13 *The image of every element* $\big[\mu : (D,p_1,\ldots,p_{3d-2})\big]\in\overline{W}_2(d)\cap W_{32}$ *has a tacnode at* $\mu(p_i)$ *for some* $i=1,\ldots,3d-2$. *If* $\big[\mu\colon (D,p_1,\ldots,p_{3d-2})\big]\in\overline{W}_2(d)\cap W_{24}$ *and the degree of* $\mu|D_1$ *is less than d, then* $\mu(D)$ *is a two-component rational tacnodal curve. Thus,* $Z\cap W_{ij}=\emptyset$ *in both cases.*

Proof: Since the proof of Lemma 11.12 carries over to the case of W_{32} with no change, the first claim is clear. For the second claim, we use coordinates (t,w) and (t,z) as in the proofs of Lemmas 11.6 and 11.12. Then,

$$\psi_t\big|_w = \big(1+o(1_t)\big)\frac{dw}{w}, \qquad \psi_t\big|_z = o(1_t);$$

$$\mu_t^{-1}(L_i) = \big\{z_1^{(i)}(t),\ldots,z_{d_1}^{(i)}(t), w_{d_1+1}^{(i)}(t),\ldots,w_d^{(i)}(t)\big\}\subset C_t, \qquad \text{with}$$

$$z_j^{(i)}(t) = z_j^{(i)}(0)+o(1_t), \quad w_j^{(i)}(t) = w_j^{(i)}(0)+o(1_t);$$

$$\sum_{j=1}^{j=d_1}\int_{z_j^{(1)}(t)}^{z_j^{(2)}(t)}\psi_t + \sum_{j=d_1+1}^{j=d}\int_{w_j^{(1)}(t)}^{w_j^{(2)}(t)}\psi_t = 0\in\mathbb{C}/2\pi i\mathbb{Z} \qquad \forall t\in\Delta^*;$$

$$\prod_{j=d_1+1}^{j=d} w_j^{(1)}(0) = \prod_{j=d_1+1}^{j=d} w_j^{(2)}(0).$$

The last identity implies that $\mu|D_2$ has a tacnode. The remaining claim of the lemma follows from the first two and Corollary 11.7.

Lemma 11.14 *If* $[\mu\colon(D,p_1,\ldots,p_{3d-2})]\in\overline{W}_2(d)\cap W_{31}$ *and the degree of* $\mu|D_1$ *is* d, $\mu(D)$ *has a tacnode at* $\mu(p_i)$ *for some* $i=1,\ldots,3d-2$. *If the degree of* $\mu|D_1$ *is less than* d, $\mu(D)$ *is a two-component tacnodal rational curve. Thus,* $Z\cap W_{31}=\emptyset$.

Proof: The proof of Lemma 11.12 applies to the first case with no change. For the second case, we use coordinate $(t,w_1)=(t,w)$ and (t,w_2) analogous to (t,w), such that $w_1=\infty$ and $w_2=\infty$ are identified in C_{31}. Since the residues of $\psi\in H^0(\tilde{C}_0;\omega_{\tilde{C}_0})$ at $w_1=\infty$ and $w_2=\infty$ add up to zero, $\psi|D_2=-\frac{dw_2}{w_2}$. Thus, proceeding as in the proof of Lemma 11.12, we obtain

$$\prod_{j=1}^{j=d_1} w_{1,j}^{(1)}(0)\cdot\Big(\prod_{j=d_1+1}^{j=d} w_{2,j}^{(1)}(0)\Big)^{-1} = \prod_{j=1}^{j=d_1} w_{1,j}^{(2)}(0)\cdot\Big(\prod_{j=d_1+1}^{j=d} w_{2,j}^{(2)}(0)\Big)^{-1} \equiv K;$$

$$\frac{c_0p_{0,0}^{(1)}+c_1p_{1,0}^{(1)}+c_2p_{2,0}^{(1)}}{c_0p_{0,d_1}^{(1)}+c_1p_{1,d_1}^{(1)}+c_2p_{2,d_1}^{(1)}}\cdot\frac{c_0p_{0,d_2}^{(2)}+c_1p_{1,d_2}^{(2)}+c_2p_{2,d_2}^{(2)}}{c_0p_{0,0}^{(2)}+c_1p_{1,0}^{(2)}+c_2p_{2,0}^{(2)}} = K \quad \forall(c_0,c_1,c_2)\in\mathbb{C}^3-\{0\}, \tag{11.13}$$

for some $K\in\mathbb{C}$. Since $\mu(w_2=\infty)=\mu(w_1=\infty)$,

$$\big(p_{0,d_1}^{(1)},p_{1,d_1}^{(1)},p_{2,d_1}^{(1)}\big) = \kappa\big(p_{0,d_2}^{(2)},p_{1,d_2}^{(2)},p_{2,d_2}^{(2)}\big)$$

for some $\kappa\in\mathbb{C}^*$. Thus, as a condition on μ, (11.13) is equivalent to

$$\big(p_{0,0}^{(1)},p_{1,0}^{(1)},p_{2,0}^{(1)}\big) = K\big(p_{0,0}^{(2)},p_{1,0}^{(2)},p_{2,0}^{(2)}\big)$$

for some $K\in\mathbb{C}$. Suppose $\mu(w_2=\infty)=\mu(w_1=0)$ is not a tacnode of $\mu(D)$. Then as in (2) of the proof of Lemma 11.12, we conclude that

$$p_{0,0}^{(1)}=p_{1,0}^{(1)}=p_{2,0}^{(1)} = p_{0,0}^{(2)}=p_{1,0}^{(2)}=p_{2,0}^{(2)}.$$

This means $\mu|D_1$ and $\mu|D_2$ have degrees at most d_1-1 and d_2-1, respectively, contrary to the assumption.

11.5 Proof of Proposition 11.3

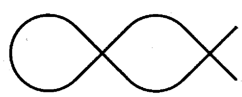
By Lemma 11.12, if $\big[\mu : (D, p_1, \ldots, p_{3d-2})\big] \in Z \cap W_{13}$, μ maps the nodes of D into the tacnode of $\mu(D)$. We now prove the converse and determine the multiplicity with which the number T_d enters into $[\pi^{-1}(C_0)] \cdot Z$.

Lemma 11.15 *Suppose C_0' is a tacnodal rational curve and $\eta\colon \mathcal{W} \longrightarrow \mathcal{B}$ is a local deformation space for C_0. Let $q_1, \ldots, q_{3d-2}$ be points in general position in $\mathbb{P}^2$ and $f\colon C_0' \longrightarrow \mathbb{P}^2$ be a map of degree d passing through the $(3d-2)$ points. Then there exists a map $\tilde{f}\colon \mathcal{W} \longrightarrow \mathbb{P}^2$, perhaps after shrinking $\mathcal{B}$, such that $\tilde{f}|C_0' = f$ and $\tilde{f}|\eta^{-1}(t)$ passes through the $(3d-2)$ points.*

Proof: Since $T_d = 0$ for $d \leq 3$, we can assume $d \geq 3$. Then $H^1\big(C_0'; f^*\mathcal{O}_{\mathbb{P}^2}(1)\big) = 0$. Thus, there is no obstruction to extending f to a neighborhood of C_0' in $\mathcal{W}$.

Corollary 11.16 *Suppose $\big[\mu\colon (D, p_1, \ldots, p_{3d-2})\big] \in W_{13}$, $\mu(p_i) = q_i$ for all $i = 1, \ldots, 3d-2$, and μ maps the nodes of D_1 to the tacnode of $\mu(D)$. Then $\big[\mu\colon (D, p_1, \ldots, p_{3d-2})\big] \in \overline{W}_2(d)$.*

Proof: We apply Lemma 11.15 to the normalization $f\colon C_0' \longrightarrow \mu(D)$ of $\mu(D)$ at the simple nodes. Let C_t be a family of rational curves identified at two pairs of points, i.e.

$C_t =$ 

As the nodes of C_t come together, C_t approaches C_0' in $\mathcal{B}$. For all $t \neq 0$ sufficiently small, let $f_t\colon C_t \longrightarrow \mathbb{P}^2$ be the maps provided by Lemma 11.15. Then $f_t(C_t)$ converges to $f(C_0')$. Furthermore, C_t converges to C_0 in $\overline{\mathfrak{M}}_2$. Thus, if

$$\lim_{t \longrightarrow 0} \big[f_t\colon (C_t, f_t^{-1}(q_1), \ldots, f_t^{-1}(q_{3d-2}))\big] = \big[\mu'\colon (D', p_1', \ldots, p_{3d-2}')\big] \in \overline{\mathfrak{M}}_2(d),$$

D' must be one of the curves C_{ij} of Figure 1, and $\mu'(D')$ is a tacnodal rational curve. By Propositions 11.4 and 11.5, we conclude that

$$\big[\mu\colon (D, p_1, \ldots, p_{3d-2})\big] = \big[\mu' : (D', p_1', \ldots, p_{3d-2}')\big] \in \overline{\mathfrak{M}}_2(d).$$

Lemma 11.17 *The contribution of W_{13} to $[\pi^{-1}(C_0)] \cdot Z$ is $6T_d$.*

Proof: Suppose $\big[\mu : (D, p_1, \ldots, p_{3d-2})\big] \in Z \cap W_{13}$. Given a fixed complex structure j on Σ such that (Σ, j) is very close to $[C_0]$ in $\overline{\mathfrak{M}}_2$, we need to determine the number of maps $\mu_j\colon \Sigma \longrightarrow \mathbb{P}^2$ close to μ. By Corollary 11.16, there exists a family of curves $\tilde{\eta}\colon \tilde{\mathcal{F}} \longrightarrow \Delta$ and of maps $\tilde{\mu}\colon \tilde{\mathcal{F}} \longrightarrow \mathbb{P}^2$ restricting to μ on the central fiber D. There are six automorphisms of C_0 that preserve its components. Corresponding to these automorphisms and $(\tilde{\mathcal{F}}, \tilde{\eta})$, we obtain six maps $\mu_j : \Sigma \longrightarrow \mathbb{P}^2$. None of these maps are equivalent, since we did not switch the two components of C_0.

Appendix A

Fundamental Analytic Estimates

In this appendix, we collect the basic Sobolev and elliptic estimates necessary for gluing pseudoholomorphic curves in symplectic geometry via the method of [LT]. Of crucial importance in applications of such estimates in [LT] is the dependence of the constants involved on the domain. Proofs of the needed properties of these constants are omitted in [LT]. The estimates described here are actually sharper than required for the purposes of [LT] or even of Appendix B. However, these sharpened estimates may help globalize the gluing construction of Chapter 3.

Section A.1 collects various simple facts from Riemannian geometry to obtain a Poincare lemma for vector fields along closed curves in Proposition A.5 and an expansion for the $\bar{\partial}$-operator in Proposition A.11. In Section A.2, we refine, in the $n=2$ case, the proofs of Sobolev Embedding Theorems given in [Mr] to obtain a C^0-estimate in Proposition A.18 and elliptic estimates for the $\bar{\partial}$-operator in Propositions A.21 and A.23 for vector fields along smooth maps into a compact manifold.

As in the rest of this thesis, if $u\colon \mathcal{D}\longrightarrow V$ is a smooth map, we write $\Gamma(u)$ and $\Gamma^1(u)$ for $\Gamma(\mathcal{D};u^*TV)$ and $\Gamma(\mathcal{D};T^*\mathcal{D}\otimes u^*TV)$, respectively. We denote the subspace of compactly supported sections in $\Gamma(u)$ by $\Gamma_c(u)$.

A.1 Riemannian Geometry Estimates

A.1.1 Parallel Transport

Let (V,g,∇) be a compact Riemannian manifold, where g is a metric on V and ∇ is a connection in TV, which is metric-compatible, but not necessarily torsion-free. Let T_∇ denote the torsion of ∇. If $p\in V$ and $X\in T_pV$, denote by $\exp_p X$ the exponential of X defined with respect to the connection ∇ and by r_V the injectivity radius of V defined with respect to the connection ∇ and metric g; see [C]. If $\alpha\colon (a,b)\longrightarrow V$ is a piecewise smooth curve, let $\Pi_\alpha\colon T_{\alpha(a)}V\longrightarrow T_{\alpha(b)}V$ denote the parallel-transport map along α defined with respect to the connection ∇. If $R=[a,b]\times[c,d]$ is a rectangle in $\mathbb{R}^2$ and $u\colon R\longrightarrow V$ is a smooth map, let

$$\Pi_{\partial u}\colon T_{u(a,c)}V\longrightarrow T_{u(a,c)}V$$

be the parallel transport along u restricted to the boundary of R traversed in the positive direction. Denote by Π_X the parallel-transport map along the geodesic $\gamma_X\colon s\longrightarrow \exp_p sX$, where $s \in [0,1]$. If α is a smooth curve and ξ is a smooth vector field along α, we write $\frac{D}{dt}\xi$ for the covariant derivative of ξ along α if the variable parameterizing α is t. More generally, if $u\colon \mathcal{D} \longrightarrow V$ is any smooth map, we define $\nabla\xi \in \Gamma^1(u)$ as follows. If $\alpha\colon (-\epsilon,\epsilon) \longrightarrow \mathcal{D}$ is a smooth curve, let

$$\{\nabla\xi\}_{\alpha(0)}\big(\alpha'(0)\big) = \frac{D}{dt}(\xi\circ\alpha)\Big|_{t=0}.$$

Lemma A.1 *There exists a constant $C > 0$ such that for any smooth map $u\colon R \longrightarrow V$,*

$$\big|\Pi_{\partial u} - I\big| \le C \int_R |u_s||u_t| ds dt,$$

where the norm of $(\Pi_{\partial u} - I) \in End(T_{u(a,c)}V)$ is computed with respect to the metric g.

Proof: (1) Let $R = [a,b] \times [c,d]$ as before. Choose an orthonormal frame $\{X_i\}$ for $T_{u(a,c)}V$. Extend each X_i to a vector field along $t \longrightarrow u(a,t)$, where $t \in [b,d]$, by parallel transport. Then extend each X_i to a vector field along $u\colon R \longrightarrow V$ by parallel-transporting the vector $X_i(a,t)$ along the curves $s \longrightarrow u(s,t)$. By definition, $\frac{D}{ds}X_i = 0$ on R. Let A be the matrix-valued function on R such that

$$\frac{D}{dt}X_i\Big|_{(s,t)} = A_{ik}(s,t)X_k(s,t);$$

here and below we use the generalized Einstein summation convention. Note that $A_{ij}(a,t) = 0$, while

$$\big\langle \mathcal{R}(u_s,u_t)X_i, X_j\big\rangle = \Big\langle \frac{D}{ds}\frac{D}{dt}X_i - \frac{D}{dt}\frac{D}{ds}X_i, X_j\Big\rangle = \Big\langle\Big(\frac{\partial}{\partial s}A_{ik}\Big)X_k, X_j\Big\rangle = \frac{\partial}{\partial s}A_{ij}. \tag{A.1}$$

where $\mathcal{R}$ denotes the Riemann curvature tensor of the connection of ∇. Since V is compact, it follows that

$$|A_{ij}(b,t)| \le C \int_a^b |u_s||u_t| ds. \tag{A.2}$$

(2) The parallel transport of X_i along the curves

$$\tau \longrightarrow u(\tau,c), \quad \tau \longrightarrow u(\tau,d), \quad \tau \longrightarrow u(a,\tau)$$

is X_i itself. Thus, it remains to estimate the parallel transport of each X_i along the curve $\tau \longrightarrow u(b,\tau)$. Let h_{ij} be the SO_N-valued function on $[c,d]$ such that

$$h(c) = I \quad\text{and}\quad \frac{D}{dt}h_{ij}X_j\Big|_{(b,t)} = 0 \;\; \forall i,t.$$

The second equation is equivalent to

$$h'_{ij}(t)X_j(b,t) + h_{ij}(t)A_{jk}(b,t)X_k(b,t) = 0 \Longleftrightarrow h' = -hA(b,\cdot). \tag{A.3}$$

Since A_{ij} is always traceless by (A.1), equation (A.3) has a unique solution in SO_n such that $h(0) = I$, where n is the dimension of V. Furthermore, by (A.2)

$$\big|h(d) - I\big| \le \int_c^d |h'(t)| dt \le \int_c^d |h||A| dt \le n^2 \int_c^d\!\!\int_a^b C|u_s||u_t| ds dt. \tag{A.4}$$

Since $\Pi_{\partial\alpha}X_i = h_{ij}(d)X_j$ by the above, the lemma follows from equation (A.4).

Corollary A.2 *There exists $C>0$ such that for any closed curve $\alpha\colon [a,b]\longrightarrow V$,*

$$\big|\Pi_\alpha - I\big| \le C\min\big(\|d\alpha\|_1, (b-a)\|d\alpha\|_2^2\big).$$

Proof: Since the group SO_N is compact and $\|d\alpha\|_1^2 \le (b-a)\|d\alpha\|_2^2$ by Holder's inequality, it is enough to assume that $\|d\alpha\|_1 \le \frac{1}{2}r_V$. Then we can write

$$\alpha(t) = \exp_{\alpha(0)}\tilde{\alpha}(t), \quad \text{where } |\tilde{\alpha}(t)|_{\alpha(0)} < r_V.$$

Define $u\colon [0,1]\times[a,b]\longrightarrow V$ by

$$u(s,t) = \exp_{\alpha(0)} s\tilde{\alpha}(t).$$

Note that

$$u_s(s,t) = d\exp_{\alpha(0)}|_{s\tilde{\alpha}(t)}\tilde{\alpha}(t), \quad u_t(s,t) = s\,d\exp_{\alpha(0)}|_{s\tilde{\alpha}(t)}\big(d\exp_{\alpha(0)}|_{\tilde{\alpha}(t)}\big)^{-1}\alpha'(t).$$

Thus, $|u_s|_{(s,t)} \le C\|d\alpha\|_1$, while $|u_t|_{(s,t)} \le C|d\alpha|_s$. From the above lemma, we then obtain

$$\big|\Pi_\alpha - I\big| = \big|\Pi_{\partial u} - I\big| \le C\int_a^b\!\!\int_0^1 |u_s||u_t|\,ds\,dt \le C\|d\alpha\|_1^2 \le C(b-a)\|d\alpha\|_2^2.$$

Since $\|d\alpha\|_1 \le \frac{1}{2}r_V$, it follows that $|\Pi_\alpha - I| \le C'\|d\alpha\|_1$.

Corollary A.3 *There exist $C, C' > 0$ such that for all smooth maps $\alpha, \xi\colon(-\epsilon,\epsilon)\longrightarrow T_pV$ with $|\alpha(0)| \le \frac{1}{2}r_V$,*

$$\left|\frac{D}{dt}\Big(\Pi_{\alpha(t)}\xi(t)\Big)\Big|_{t=0} - \Pi_{\alpha(0)}\xi'(0)\right| \le C|\alpha(0)||\alpha'(0)||\xi(0)| \le C'|\alpha'(0)||\xi(0)|.$$

Proof: Let $R = [0,1]\times\big[0,\frac{1}{2}\epsilon\big]$ and define $u\colon R\longrightarrow V$ by $u(s,t) = \exp_p s\alpha(t)$. Let $\{X_i\}$ be an orthonormal basis for T_pV. Extend each $\{X_i\}$ to a vector field along u by parallel transport along the geodesics $s\longrightarrow u(s,t)$. Write $\xi(t) = f(t)X_i$; then $\Pi_{\alpha(t)}\xi(t) = f(t)X_i(1,t)$, and thus

$$\frac{D}{dt}\Big(\Pi_{\alpha(t)}\xi(t)\Big)\Big|_{t=0} = f'(t)X_i(1,t) + f(t)\frac{D}{dt}X_i(1,t)\Big|_{t=0} = \Pi_{\alpha(0)}\xi'(0) + f(t)\frac{D}{dt}X_i(1,t)\Big|_{t=0}.$$

By the proof of Lemma A.1,

$$\left|\frac{D}{dt}X_i(1,t)\right|_{t=0} \le C\int_0^1 |u_s|_{(s,0)}|u_t|_{(s,0)}\,ds \le C'|\alpha(0)||\alpha'(0)| \le C''\big|\alpha'(0)\big|,$$

since $|\alpha(0)| \le \frac{1}{2}r_V$ by assumption.

A.1.2 Poincare Lemmas

Lemma A.4 *If $f\colon S^1\longrightarrow\mathbb{R}^N$ is a smooth function such that $\int_0^{2\pi} f(\theta)d\theta = 0$,*

$$\int_0^{2\pi}|f(\theta)|^2 d\theta \le \int_0^{2\pi}|f'(\theta)|^2 d\theta.$$

Proof: We can write $f(\theta)=\sum\limits_{n>-\infty}^{n<\infty} a_n e^{in\theta}$. Since $\int_0^{2\pi} f(\theta)d\theta=0$, $a_0=0$. Thus,

$$\int_0^{2\pi}|f(\theta)|^2d\theta=\sum_{n>-\infty}^{n<\infty}|a_n|^2\le\sum_{n>-\infty}^{n<\infty}|na_n|^2=\int_0^{2\pi}|f'(\theta)|^2d\theta.$$

Proposition A.5 *Let (V,g,∇) be a compact Riemannian manifold. There exists $C>0$ such that for any curve $\alpha\colon S^1\longrightarrow V$ and vector fields ξ_1 and ξ_2 along α,*

$$\big|\langle\langle\nabla_\theta\xi_1,\xi_2\rangle\rangle\big|\le\|\nabla_\theta\xi_1\|_2\|\nabla_\theta\xi_2\|_2+C\min\big(\|d\alpha\|_1,\|d\alpha\|_2^2\big)\|\xi_1\|_{2,1}\|\xi_2\|_2,$$

where all the norms are computed with respect to the standard metric on S^1.

Proof: (1) Identify $T_{\alpha(0)}V$ with $\mathbb{R}^N$, preserving the metric. Given $v\in T_{\alpha(0)}V$, let $v(\theta)$ denote the parallel transport of v along the curve $t\longrightarrow\alpha(t)$ with $0\le t\le\theta$. By Corollary A.2, there exists $X\in so(T_{\alpha(0)}V)=so_N$ such that

$$|X|\le C\min\big(\|d\alpha\|_1,\|d\alpha\|_2^2\big)\quad\text{and}\quad v(2\pi)=\{\exp(X)\}\cdot v(0)=\{\exp(X)\}\cdot v$$

for all $v\in T_{\alpha(0)}V$, where $\exp(X)$ is taken in $SO_N=SO(T_{\alpha(0)}V)$.
(2) For any $v\in T_{\alpha(0)}V$, let $\Psi_\theta v$ denote the parallel transport of $e^{-\theta X/2\pi}v$ along the curve $t\longrightarrow\alpha(t)$ with $0\le t\le\theta$. Then $\Psi\colon S^1\times\mathbb{R}^N\longrightarrow\alpha^*TV$ is a smooth isometry. Put

$$\bar{\xi}_2=\frac{1}{2\pi}\int_0^{2\pi}\Psi^{-1}\xi_2(\theta).$$

Holder's inequality and the previous lemma then imply that

$$\begin{aligned}\big|\langle\langle\nabla_\theta\xi_1,\xi_2-\Psi\bar{\xi}_2\rangle\rangle\big|&\le\|\nabla_\theta\xi_1\|_2\|\xi_2-\Psi\bar{\xi}_2\|_2\\&=\|\nabla_\theta\xi_1\|_2\|\Psi^{-1}\xi_2-\bar{\xi}_2\|_2\le\|\nabla_\theta\xi_1\|_2\|d\Psi^{-1}\xi_2\|_2.\end{aligned}\tag{A.5}$$

Note that

$$\|d\Psi^{-1}\xi_2\|_2\le\|\nabla_\theta\xi_2\|_2+|X/2\pi|\|\xi_2\|_2\le\|\nabla_\theta\xi_2\|_2+C\min\big(\|d\alpha\|_1,\|d\alpha\|_2^2\big)\|\xi_2\|_2.\tag{A.6}$$

On the other hand, by integration by parts, we obtain

$$\langle\langle\nabla_\theta\xi_1,\xi_2-\Psi\bar{\xi}_2\rangle\rangle=\langle\langle\nabla_\theta\xi_1,\xi_2\rangle\rangle+\langle\langle\xi_1,\nabla_\theta\Psi\bar{\xi}_2\rangle\rangle.\tag{A.7}$$

Since $\Psi\bar{\xi}_2$ is the parallel transport of $e^{\theta X/2\pi}\bar{\xi}_2$,

$$\begin{aligned}\big|\langle\langle\xi_1,\nabla_\theta\Psi\bar{\xi}_2\rangle\rangle\big|&\le\|\xi_1\|_2\|\nabla_\theta\Psi\bar{\xi}_2\|_2=\|\xi_1\|_2|X/2\pi|\|\Psi\bar{\xi}_2\|_2\\&\le C\min\big(\|d\alpha\|_1,\|d\alpha\|_2^2\big)\|\xi_1\|_2\|\xi_2\|_2.\end{aligned}\tag{A.8}$$

The corollary follows from equations (A.5)-(A.8).

If $R>r\ge0$, let

$$B_{R,r}=\big\{x\in\mathbb{R}^2: r<|x|<R\big\}\quad\text{and}\quad\tilde{B}_{R,r}=\big\{x\in\mathbb{R}^2: r\le|x|<R\big\}.$$

Lemma A.6 *For all $p \geq 1$, there exists $C_p > 0$ such that for all $R > 0$, $\epsilon < \frac{1}{2}$, and $f \in C^\infty(B_{R,\epsilon R}; \mathbb{R}^n)$,*

$$\int_{B_{R,\epsilon R}} f = 0 \Longrightarrow \|f\|_{C^0} \leq C_p R^{\frac{p-2}{p}} \|f\|_p.$$

Proof: (1) We first assume $R=1$. Suppose $\epsilon_k \longrightarrow \epsilon \in [0, \frac{1}{2}]$, $f_k \in C^\infty(B_{1,\epsilon_k}; \mathbb{C}^n)$,

$$\int_{B_{1,\epsilon_k}} f_k = 0, \quad \|f_k\|_{C^0} = 1, \quad \text{and} \quad \|df_k\|_p \longrightarrow 0.$$

By the Sobolev Embedding Theorem, f_k converges to some $f \in L^p(B_{1,\epsilon}; \mathbb{R}^n)$ with respect to the L^p-norm on compact subsets of $B_{1,\epsilon}$. Since $\|df_k\|_p \longrightarrow 0$, $df=0$ and $f \in L^p_1(B_{1,\epsilon}; \mathbb{R}^n)$. Since d is elliptic, it follows that f is smooth and constant, and thus zero. However, this contradicts the fact that $\|f\|_{C^0}=1$.
(2) Suppose $R > 0$, $f \in C^\infty(B_{R,\epsilon R}; \mathbb{R}^n)$, and $\int f = 0$. Let $\tilde{f} \in C^\infty(B_{1,\epsilon}; \mathbb{C}^n)$ be given by $\tilde{f}(z) = f(Rz)$. Then $\int_{B_{1,\epsilon}} \tilde{f} = 0$, and thus by (1),

$$\|f\|_{C^0} = \|\tilde{f}\|_{C^0} \leq C_p \|d\tilde{f}\|_p \leq C_p R^{\frac{p-2}{p}} \|df\|_p.$$

A.1.3 The Exponential Map and Differentiation

With notation as in Subsection A.1.1, let

$$\tilde{T}V = \bigcup_{p \in V} \tilde{T}_p V, \quad \text{where} \quad \tilde{T}_p V = \{X \in T_p V : |X| \leq \frac{1}{2} r_V\}.$$

If $\alpha: (-\epsilon, \epsilon) \longrightarrow V$ is a smooth curve and ξ is a vector fields along α such that $\xi(0) \in \tilde{T}V$, put

$$\Phi_{\alpha(0)}\Big(\xi(0); \alpha'(0), \frac{D}{ds}\xi\Big|_{s=0}\Big) = \Pi^{-1}_{\xi(0)}\Big(\frac{d}{ds} \exp_{\alpha(s)} \xi(s)\Big|_{s=0}\Big).$$

Note that Φ is well-defined, i.e. its definition is dependent only on the vectors $\alpha'(0)$, $\xi(0)$, and $\frac{D}{ds}\xi(0)\big|_{s=0}$, since Φ involves only first-order differentiation.

Lemma A.7 *There exists $C>0$ such that for all $p \in V$, $X \in \tilde{T}_p V$, and $Y, Z \in T_p V$,*

$$\Big|\Phi_p(X; Y, Z) - \big(Y + Z + T_\nabla(Y, Z)\big)\Big| \leq C|X|^2\big(|Y| + |Z|\big).$$

Proof: Let $\alpha: (-\epsilon, \epsilon) \longrightarrow V$ be a smooth curve and ξ a smooth vector field along α such that

$$\alpha(0) = p, \quad \alpha'(0) = Y, \quad \xi(0) = X, \quad \frac{D}{ds}\xi(s)\Big|_{s=0} = Z.$$

Put

$$F_{X,Y,Z}(t) = \frac{d}{ds} \exp_{\alpha(s)} t\xi(s)\Big|_{s=0} \quad \text{and} \quad H_{X,Y,Z}(t) = \Pi_{t\xi(0)}\big(Y + tZ + tT_\nabla(Y, X)\big).$$

Then,

$$F_{X,Y,Z}(0) = \frac{d}{ds}\alpha(s)\Big|_{s=0} = H_{X,Y,Z}(0);$$
$$\frac{D}{dt}F_{X,Y,Z}(t)\Big|_{t=0} = \frac{D}{ds}\frac{d}{dt}\exp_{\alpha(s)} t\xi(s)\Big|_{t=0}\Big|_{s=0} + T_\nabla\big(\alpha'(0),\xi(0)\big) = \frac{D}{dt}H_{X,Y,Z}(t)\Big|_{t=0}.$$

Combining the last two equations, we obtain

$$\big|F_{X,Y,Z}(t) - H_{X,Y,Z}(t)\big| \le C_p(X,Y,Z)t^2 \quad \forall\, t\in[-1,1],\ |X|,|Y|,|Z| \le 1+r_V, \tag{A.9}$$

where C is a smooth function on $TV\oplus TV\oplus TV$. Since $F_{X,Y,Z}$ and $H_{X,Y,Z}$ are linear in (Y,Z), $F_{X,Y,Z}(t)=F_{tX,Y,Z}(1)$, $H_{X,Y,Z}(t)=H_{tX,Y,Z}(1)$, and the space $\{X\in TV\colon |X|=1\}$ is compact, we conclude that there exists $C'>0$ such that

$$C_p(X,Y,Z) \le C'|X|^2\big(|Y|+|Z|\big) \quad \forall\, X\in\tilde{T}_pV,\ Y,Z\in T_pV. \tag{A.10}$$

The claim follows from equations (A.9) and (A.10).

For any $X,Y,Z\in T_pV$, let $\tilde{\Phi}_p(X;Y,Z) = \Phi_p(X;Y,Z) - \big(Y+Z+T_\nabla(Y,Z)\big)$.

Corollary A.8 *There exists $C>0$ such that for all $p\in V$, $X_1,X_2\in\tilde{T}_pV$, and $Y,Z_1,Z_2\in T_pV$,*

$$\begin{aligned}\Big|\tilde{\Phi}_p(X_1;Y,Z_1) - \tilde{\Phi}_p(X_2;Y,Z_2)\Big| \\ \le C\Big(\big(|X_1|+|X_2|\big)\big(|Y|+|Z_1|+|Z_2|\big)|X_1-X_2| + \big(|X_1|^2+|X_2|^2\big)|Z_1-Z_2|\Big).\end{aligned}$$

Proof: We can view $\tilde{\Phi}_p$ as a smooth section of

$$\pi_{\tilde{T}V}^*\big((T^*V\oplus T^*V)\otimes TV\big) \longrightarrow \tilde{T}V.$$

By Lemma A.7, $|\tilde{\Phi}_p(X;)|\le C|X|^2$. Since $\tilde{T}V$ is compact, there exists $C''>0$ such that

$$\big|\tilde{\Phi}_p(X_1;) - \tilde{\Phi}_p(X_2;)\big| \le C'\big(|X_1|+|X_2|\big)|X_1-X_2| \quad \forall\, X_1,X_2\in\tilde{T}_pV.$$

Since $\tilde{\Phi}_p$ is linear in the last two inputs, we conclude that

$$\begin{aligned}\Big|\tilde{\Phi}_p(X_1;Y,Z_1) - \tilde{\Phi}_p(X_2;Y,Z_2)\Big| &\le \Big|\tilde{\Phi}_p(X_1;0,Z_1-Z_2)\Big| + \Big|\big\{\tilde{\Phi}_p(X_2;) - \tilde{\Phi}_p(X_1;)\big\}(Y,Z_2)\Big| \\ &\le \tilde{C}\Big(|X_1|^2|Z_1-Z_2| + \big(|Y|+|Z_2|\big)\big(|X_1|+|X_2|\big)|X_1-X_2|\Big).\end{aligned}$$

A.1.4 Expansion of the $\bar{\partial}$-Operator

Let (V,g,J) be a compact almost-complex manifold. Here J is a complex structure in TV, which is orthogonal with respect to g. Then there exists a canonical connection ∇ in TV that commutes with J. Explicitly, if ∇^{LC} is the Levi-Civita connection of the metric g, for any $X\in T_pV$ and $\xi\in\Gamma(V;TV)$, let

$$\nabla_X\xi = \frac{1}{2}\Big(\nabla_X^{LC}\xi - J\nabla_X^{LC}(JX)\Big).$$

We will always associate this canonical connection with any given triple (V,g,J). This connection agrees with the Levi-Civita connection if and only if (V,g,J) is Kahler.

Let $(\mathcal{D},j)$ be any one-dimensional complex manifold. If $u:\mathcal{D}\longrightarrow V$ is a smooth map, denote by $\Lambda^{0,1}_{J,j}T^*\mathcal{D}\otimes u^*TV\longrightarrow\mathcal{D}$ the bundle of (J,j)-antilinear homomorphisms from $T\mathcal{D}$ to u^*TV. Let

$$\bar{\partial}u=\frac{1}{2}\Big(du+J\circ du\circ j\Big)\in\Gamma^{0,1}_{J,j}(u)\equiv\Gamma\big(\mathcal{D};\Lambda^{0,1}_{J,j}T^*\mathcal{D}\otimes u^*TV\big).$$

If $\xi\in\Gamma(u)$, we define $\bar{\partial}_u\xi, D_u\xi, L_u\xi, \tilde{D}_u\xi, N_u\xi\in\Gamma^{0,1}_{J,j}(u)$ by

$$D_u\xi=\nabla\xi+J\circ\nabla\xi\circ j,\quad \big\{L_u\xi\big\}_z(v)=T_\nabla\big(du\big|_zv,\xi(z)\big)+JT_\nabla\big(du\big|_zjv,\xi(z)\big),$$
$$\tilde{D}_u\xi=D_u\xi+L_u\xi,\quad \big\{\bar{\partial}_u\xi\big\}_z(v)=\Pi^{-1}_{\xi(z)}\Big(\bar{\partial}\{\exp_u\xi\}\big|_zv\Big)=\bar{\partial}u+\tilde{D}_u\xi+N_u\xi.$$

The operator $\tilde{D}_u:\Gamma(u)\longrightarrow\Gamma^{0,1}_{J,j}(u)$ is the linearization of the $\bar{\partial}$-operation at u with respect to the connection ∇; see [MS]. What this means is described in detail by Lemma A.9 and Proposition A.11 below.

Lemma A.9 *There exists a constant $C>0$ dependent only on (V,g,J) such that for any smooth map $u\colon(\mathcal{D},j)\longrightarrow V$, $\xi_1,\xi_2\in\Gamma(u)$, $z\in\mathcal{D}$, and $v\in T_z\mathcal{D}$, with $|\xi_1|_z,|\xi_2|_z\le\frac{1}{2}r_V$,*

$$\Big|\big\{N_u\xi_1\big\}_zv-\big\{N_u\xi_2\big\}_zv\Big|\le C\Big(\big|du|_zv\big|\big(|\xi_1|_z+|\xi_2|_z\big)\big|\xi_1-\xi_2\big|+\big(|\xi_1|_z+|\xi_2|_z\big)\big|\nabla\xi_1(v)-\nabla\xi_2(v)\big|\Big).$$

Furthermore, $N_u0=0$.

Proof: Since the connection ∇ commutes with J, so does the parallel transport Π. Thus, with notation as in the previous subsection,

$$\{N\xi\}_zv=\tilde{\Phi}\big(\xi(z);du|_zv,\nabla\xi|_zv\big)+J\big(u(z)\big)\tilde{\Phi}\big(\xi(z);du|_zjv,\nabla\xi|_zjv\big).$$

The claim now follows from Corollary A.8.

Definition A.10 *If $C_0>0$ and $u:\mathcal{D}\longrightarrow V$ is a smooth map, norms $\|\cdot\|_{p,1}$ and $\|\cdot\|_p$ on $\Gamma(u)$ and $\Gamma^1(u)$, respectively, are <u>C_0-admissible</u> if for all $\xi\in\Gamma(u)$, $\eta\in\Gamma^1(u)$, and any continuous function $f:\mathcal{D}\longrightarrow\mathbb{R}$,*

$$\|\nabla\xi\|_p\le\|\xi\|_{p,1},\quad \|f\eta\|_p\le\|f\|_{C^0}\|\eta\|_p,\quad \|\xi\|_{C^0}\le C_0\|\xi\|_{p,1}.$$

Proposition A.11 *For every compact almost-complex manifold (V,g,J), there exists $C_V\in C^\infty(\mathbb{R};\mathbb{R})$ with the following property. Suppose $(\mathcal{D},j)$ is a one-dimensional complex manifold, $u\colon(\mathcal{D},j)\longrightarrow V$ is a smooth map, and $\|\cdot\|_{p,1}$ and $\|\cdot\|_p$ are C_0-admissible norms on $\Gamma(u)$ and $\Gamma^1(u)$, respectively. Then for all $\xi_1,\xi_2\in\Gamma(u)$ such that $\|\xi_1\|_{p,1},\|\xi_2\|_{p,1}<\frac{r_V}{2C_0}$,*

$$\big\|N_u\xi_1-N_u\xi_2\big\|_p\le C_V\big(C_0+\|du\|_p\big)\big(\|\xi_1\|_{p,1}+\|\xi_2\|_{p,1}\big)\|\xi_1-\xi_2\|_{p,1}.$$

Furthermore, $N_u0=0$. If the geodesic ball of radius δ about $u(z)$ in (V,g,J) is isomorphic to an open subset of $\mathbb{C}^n$ and $|\xi|_z<\delta$, then $\{N_u\xi\}_z=0$.

Proof: The first two statements follow from Lemma A.9 and Definition A.10. The last claim is clear from the construction of N_u.

Remark: As the notation suggests, one possibility for the norms $\|\cdot\|_{p,1}$ and $\|\cdot\|_p$ is the usual Sobolev L^p_1 and L^p-norms with respect to some Riemannian metric on $\mathcal{D}$, where $p>2$. Another possibility is the modified Sobolev norms of [LT] described in Section 3.3, which seem to be ideally suited for gluing pseudoholomorphic maps. Proposition B.11, which is a key ingredient in the approach of [LT] to gluing pseudoholomorphic maps, is valid for the modified Sobolev norms, but not for the usual ones. In Proposition A.18 below, we show that for both types of Sobolev norms the constant C_0 itself is a function of $\|du\|_p$ only.

A.2 Sobolev and Elliptic Inequalities

A.2.1 Eucledian Case

Proofs of the next four statements are slight refinements of the proofs of similar statements in [Mr] in the $n=2$ case.

Lemma A.12 *For any bounded convex domain $\mathcal{D}\subset\mathbb{R}^2$, $f\in C^\infty(\mathcal{D})$, and $x\in\mathcal{D}$,*

$$\big|f_{\mathcal{D}}-f(x)\big|\le\frac{2r_0^2}{|\mathcal{D}|}\int_{\mathcal{D}}|df|_y|y-x|^{-1}dy,$$

where $2r_0$ is the diameter of $\mathcal{D}$, $|\mathcal{D}|$ is the area of $\mathcal{D}$, and

$$f_{\mathcal{D}}=\frac{1}{|\mathcal{D}|}\Big(\int_{\mathcal{D}}f(y)dy\Big)$$

is the average value of f on D.

Proof: For any $y\in\mathcal{D}$,

$$f(y)-f(x)=\int_0^1\frac{d}{dt}f\big(x+t(y-x)\big)dt=\int_0^1 df|_{x+t(y-x)}(y-x)dt.$$

Put $g(z)=|df|_z$ if $z\in\mathcal{D}$ and $g(z)=0$ otherwise. Then,

$$\big|f_{\mathcal{D}}-f(x)\big|\le\frac{1}{|\mathcal{D}|}\int_{y\in\mathcal{D}}|f(y)-f(x)|dy\le\frac{1}{|\mathcal{D}|}\int_{y\in\mathcal{D}}\int_0^\infty g\big(x+t(y-x)\big)|y-x|dtdy.$$

Rewriting the last integral in polar coordinates (r,θ) centered at x, we obtain

$$\begin{aligned}\big|f_{\mathcal{D}}-f(x)\big|&\le\frac{1}{|\mathcal{D}|}\int_0^{2\pi}\int_0^{2r_0}\int_0^\infty g(tr,\theta)r^2dtdrd\theta\\&=\frac{1}{|\mathcal{D}|}\int_0^{2\pi}\int_0^{2r_0}\int_0^\infty g(t,\theta)rdtdrd\theta=\frac{2r_0^2}{|\mathcal{D}|}\int_0^{2\pi}\int_0^\infty g(t,\theta)dtd\theta\\&=\frac{2r_0^2}{|\mathcal{D}|}\int_D|df|_y|y-x|^{-1}dy.\end{aligned}$$

Corollary A.13 *For any $R>0$ and $p>2$, there exists $C_p \in C^\infty(\mathbb{R};\mathbb{R})$ such that*

$$r\in\big[0,\frac{1}{2}R\big],\quad f\in C^\infty(B_{R,r}) \Longrightarrow \|f\|_{C^0}\le C_p(R)\|f\|_{p,1}.$$

Proof: For any $x^*\in B_{R,r}$, put

$$\mathcal{D}_{x^*}=\big\{x\in B_{R,r}:\big\langle x^*,|x^*|x-rx^*\big\rangle\ge 0\big\}.$$

If $x^*\neq 0$, $\mathcal{D}_{x^*}$ is the part of the annulus on the same side of the line $\big\langle x^*,x-rx^*/|x^*|\big\rangle=0$ as x^*. In particular,

$$\text{diam}(\mathcal{D}_{x^*})\le 2R \quad\text{and}\quad \big|\mathcal{D}_{x^*}\big|\ge\Big(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\Big)R^2.$$

Thus, by the above lemma and Holder's inequality,

$$\begin{aligned}|f(x^*)| &\le \frac{1}{|\mathcal{D}_{x^*}|}\|f\|_1+8\int_{y\in\mathcal{D}_{x^*}}|df|_y|y-x|^{-1}dy\\ &\le |\mathcal{D}|^{-\frac{1}{p}}\|f\|_p+8\Big(\int_{y\in B_{2R}(x)}|y-x|^{-\frac{p}{p-1}}\Big)^{\frac{p-1}{p}}\|df\|_p\le C_p(R)\|f\|_{p,1},\end{aligned} \tag{A.11}$$

since $\frac{p}{p-1}<2$.

Lemma A.14 *For any $R>0$ and $r\in[0,R)$,*

$$f\in C^\infty(B_{R,r}),\quad supp(f)\subset\tilde{B}_{R,r}\Longrightarrow\|f\|_2\le\|df\|_1.$$

Proof: Such a function f can be viewed as a function on the complement of the ball $\bar{B}(0,r)$ in $\mathbb{R}^2$. Since f vanishes at infinity, for any $(x,y)\in B_{R,r}$

$$f(x,y)=\begin{cases}\int_{-\infty}^{x}f_s(s,y)ds, & \text{if } x\le 0;\\ -\int_x^\infty f_s(s,y)ds, & \text{if } x\ge 0;\end{cases}\quad\text{and}\quad f(x,y)=\begin{cases}\int_{-\infty}^{y}f_t(x,t)dt, & \text{if } y\le 0;\\ -\int_y^\infty f_t(x,t)dt, & \text{if } y\ge 0.\end{cases}$$

Taking the absolute value in these equations, we obtain

$$|f|_{(x,y)}\le\int_{-\infty}^{\infty}|df|_{(s,y)}ds\quad\text{and}\quad|f|_{(x,y)}\le\int_{-\infty}^{\infty}|df|_{(x,t)}dt, \tag{A.12}$$

where we formally set f and df to be zero on the smaller disk. Multiplying the two inequalities in (A.12) and integrating with respect to x and y, we conclude

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f|^2_{(x,y)}dxdy\le\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|df|_{(x,y)}dxdy\Big)^2,$$

as claimed.

Corollary A.15 *For any $R>0$, $p,q\ge 1$ with $1-\frac{2}{p}\ge-\frac{2}{q}$, there exists $C_{p,q}\in C^\infty(\mathbb{R};\mathbb{R})$ such that*

$$r\in[0,R),\quad f\in C^\infty(B_{R,r}),\quad supp(f)\subset\tilde{B}_{R,r}\Longrightarrow\|f\|_q\le C_{p,q}(R)\|df\|_p.$$

Proof: For $\epsilon>0$, let $h_\epsilon=(f^2+\epsilon)^{\frac{q}{4}}-\epsilon^{\frac{q}{4}}$. By the above lemma and Holder's inequality,

$$\begin{aligned}\|f\|_q^q &\le \|h_\epsilon+\epsilon^{\frac{q}{4}}\|_2^2 \le 2\|dh_\epsilon\|_1^2+2\epsilon^{\frac{q}{2}}\pi R^2 = 2\Big\|\frac{q}{2}(f^2+\epsilon)^{\frac{q}{4}-1}f df\Big\|_1^2+2\epsilon^{\frac{q}{2}}\pi R^2 \\ &\le q^2\big\|(f^2+\epsilon)^{\frac{q}{4}-\frac{1}{2}}df\big\|_1^2+2\epsilon^{\frac{q}{2}}\pi R^2 \le q^2\|df\|_p^2\|(f^2+\epsilon)^{\frac{q-2}{4}}\|_{\frac{p}{p-1}}^2+2\epsilon^{\frac{q}{2}}\pi R^2.\end{aligned} \tag{A.13}$$

Note that

$$1-\frac{2}{p}=-\frac{2}{q} \Longrightarrow \frac{q-2}{4}\frac{p}{p-1}=\frac{q-2}{4}\frac{2q}{q-2}=\frac{q}{2}.$$

Thus, letting ϵ go to zero in (A.13), we obtain

$$\|f\|_q^q \le q^2\|df\|_p^2\|f\|_p^{q-2} \Longrightarrow \|f\|_q \le q\|df\|_p.$$

The case $1-\frac{2}{p}>-\frac{2}{q}$ follows by Holder's inequality.

A.2.2 Vector Fields along Smooth Maps into Compact Manifolds

Let (V,g,∇) be a compact Riemannian manifold.

Lemma A.16 *For any $R>0$ and $p,q\ge 1$ with $1-\frac{2}{p}\ge -\frac{2}{q}$, there exists $C_{p,q}\in C^\infty(\mathbb{R};\mathbb{R})$ such that for any $r\in[0,R)$, $u\in C^\infty(\tilde{B}_{R,r};V)$, and $\xi\in\Gamma_c(u)$,*

$$\|\xi\|_q \le C_{p,q}(R)\big(\|\xi\|_{p,1}+\|\xi du\|_p\big).$$

Proof: (1) Let $\{U_i : i\in[N]\}$ be a finite open cover of V such that the diameter of each set U_i is at most $\frac{1}{2}r_V$. Let $\{W_i : i\in[N]\}$ be an open cover of V such that $\bar{W}_i\subset U_i$. Choose smooth functions $\eta_i\colon V\longrightarrow[0,1]$ such that $\eta_i=1$ on W_i and $\eta_i=0$ outside of U_i. For each $i\in[N]$, pick $x_i\in W_i$. If $z\in\tilde{B}_{R,r}$ and $u(z)\in U_i$, define $\tilde{u}_i(z),\xi_i(z)\in T_{x_i}V$ by

$$\exp_{x_i}\tilde{u}_i(z)=u(z),\quad |\tilde{u}_i(z)|<r_V;\quad \Pi_{\tilde{u}_i(z)}\xi_i(z)=\xi(z).$$

For any $z\in B_{R,r}$, put $\tilde{\xi}_i(z)=\eta_i(u(z))\xi_i(z)$. Then $\tilde{\xi}_i\in C_c^\infty(\tilde{B}_{R,r};T_{x_i}V)$.
(2) By Corollary A.15, there exists $C_{p,q}(R)>0$ such that

$$\|\xi\|_{L^q(u^{-1}(W_i))} \le \|\tilde{\xi}_i\|_q \le C_{p,q}(R)\|\tilde{\xi}_i\|_{p,1} \le C_{p,q}(R)\big(\|\xi\|_p+\|d\tilde{\xi}_i\|_p\big). \tag{A.14}$$

Since $d\tilde{\xi}_i=(d\eta_i\circ du)\xi_i+(\eta\circ u)d\xi_i$ on $u^{-1}(U_i)$ and vanishes outside of $u^{-1}(U_i)$,

$$\|d\tilde{\xi}_i\|_p \le \|d\xi_i\|_{L^p(u^{-1}(U_i))}+C\|\xi_i du\|_p. \tag{A.15}$$

On the other hand, by Corollary A.3, if $u(z)\in U_i$

$$\Big|\nabla\xi|_z-\Pi_{\tilde{u}(z)}\circ d\xi_i|_z\Big|_z \le C|du|_z|\xi|_z. \tag{A.16}$$

Combining equations (A.14)-(A.16), we obtain

$$\|\xi\|_{L^q(u^{-1}(W_i))} \le C_{p,q}(R)\big(\|\xi\|_{p,1}+\|\xi du\|_p\big).$$

The claim follows by summing the last inequality over all i.

Lemma A.17 *For any $R>0$ and $p>2$, there exists $C_p\in C^\infty(\mathbb{R};\mathbb{R})$ such that for any $r\in[0,\frac{1}{2}R]$, $u\in C^\infty(B_{R,r};V)$, and $\xi\in\Gamma(u)$,*

$$\|\xi\|_{C^0}\le C_p(R)\big(\|\xi\|_{p,1}+\|\xi du\|_p\big).$$

Proof: With notation as above, by Corollary A.13, there exists $C_p(R)$ such that

$$\|\xi\|_{C^0(u^{-1}(W_i))}\le\|\tilde{\xi}_i\|_{C^0}\le C_p(R)\|\tilde{\xi}_i\|_{p,1}\le C_p(R)\big(\|\xi\|_{L^p(u^{-1}(U_i))}+\|d\tilde{\xi}_i\|_p\big).$$

As above, we obtain

$$\|d\tilde{\xi}_i\|_p\le C\big(\|\xi\|_{p,1}+\|\xi du\|_p\big),$$

and the claim follows.

Proposition A.18 *If (V,g,∇) is a compact Riemannian manifold and $p>2$, there exists $C_p\in C^\infty(\mathbb{R}\times\mathbb{R};\mathbb{R})$ such that for all $R>0$, $r\in[0,\frac{1}{2}R]$, $u\in C^\infty(B_{R,r};V)$, and $\xi\in\Gamma_c(u)$,*

$$\|\xi\|_{C^0}\le C_p\big(R,\|du\|_p\big)\|\xi\|_{p,1}.$$

The same statement holds if $B_{R,r}$ is replaced by a fixed compact Riemann surface (Σ,g_Σ).

Proof: By Lemma A.17 applied with $\tilde{p}=\frac{p+2}{2}$ and Holder's inequality,

$$\|\xi\|_{C^0}\le C_{\tilde{p}}(R)\big(\|\xi\|_{\tilde{p},1}+\|\xi du\|_{\tilde{p}}\big)\le C'_{\tilde{p}}(R)\big(\|\xi\|_{p,1}+\|du\|_p\|\xi\|_{q_1}\big),\tag{A.17}$$

where $q_1=\frac{2p}{p-2}$. If $q_1\le p$, then the proof is complete. Otherwise, apply Lemma A.16 with $p_1=\frac{2q_1}{q_1+2}$ and Holder's inequality:

$$\|\xi\|_{q_1}\le C_{p_1,q_1}(R)\big(\|\xi\|_{p_1,1}+\|\xi du\|_{p_1}\big)\le C'_{p_1,q_1}(R)\big(\|\xi\|_{p,1}+\|du\|_p\|\xi\|_{q_2}\big),\tag{A.18}$$

where $q_2=\frac{pp_1}{p-p_1}$. If $q_2\le p$, then the claim follows from equations (A.17) and (A.18). Otherwise, we can continue and construct sequences $\{p_i\},\{q_i\},\{C_i\}$ such that

$$p_i=\frac{2q_i}{q_i+2},\quad q_{i+1}=\frac{pp_i}{p-p_i};\tag{A.19}$$

$$\|\xi\|_{q_i}\le C_i(R)\big(\|\xi\|_{p,1}+\|du\|_p\|\xi\|_{q_{i+1}}\big).\tag{A.20}$$

Equation (A.19) implies that

$$q_{i+1}=\frac{2pq_i}{pq_i-2(q_i-p)}\Longrightarrow \text{if } q_i>0,\ \text{then } q_{i+1}<q_i.$$

Thus, if $q_i>2$ for all i, then the sequence $\{q_i\}$ must have a limit $q\ge 2$. The limiting value must satisfy

$$q=\frac{2pq}{pq-2(q-p)}\Longrightarrow(p-2)q=0\Longrightarrow q=0,$$

since $p>2$ by assumption. Thus, for N sufficiently large $q_N\le p$ and the first claim follows from (A.17) and equations (A.20) with i running from 1 to $N+1$, where N is the smallest integer such that $q_N\le p$. The second claim is easily obtained from the first.

A.2.3 Elliptic Estimates

If $A_1 = B_{R_1,r_1}$ and $A_2 = \bar{B}_{R_2,r_2}$ are two annuli in $\mathbb{R}^2$, we write $A_2 \Subset_\delta A_1$ if $R_1 - R_2 > \delta$ and $r_2 - r_1 \geq \delta$.

Lemma A.19 *For any $\delta > 0$, $p \geq 1$, and open annulus A_1, there exists $C_{\delta,p}(A_1) > 0$ such that for any annulus $A_2 \Subset_\delta A_1$ and $f \in C^\infty(A_1; \mathbb{C}^n)$,*

$$\|f\|_{L^p_1(A_2)} \leq C_p(A_1)\big(\|\bar{\partial} f\|_p + \|df\|_2 + \|f\|_1\big),$$

where the norms are taken with respect to the standard metric on $\mathbb{R}^2$.

Proof: We can assume that A_2 is the maximal annulus such that $A_2 \Subset_\delta A_1$. Let $\eta \colon A_1 \longrightarrow [0,1]$ be a compactly supported smooth function such that $\eta|A_2 = 1$. By the usual elliptic inequalities for S^2,

$$\begin{aligned} \|df\|_{L^p(A_2)} &\leq \|d(\eta f)\|_p \leq C_p(A_1)\big(\|\bar{\partial}(\eta f)\|_p + \|\eta f\|_p\big) \\ &\leq C_p(A_1)\big(\|\bar{\partial} f\|_p + \|(d\eta) f\|_p + \|\eta f\|_p\big). \end{aligned} \tag{A.21}$$

By Corollary A.15,

$$\begin{aligned} \|\eta f\|_p &\leq C_p(A_1)\big(\|df\|_2 + \|(d\eta) f\|_2 + \|\eta f\|_2\big) \\ &\leq C'_p(A_1)\big(\|df\|_2 + \|df\|_1 + \|(\nabla^2 \eta) f\|_1 + \|(d\eta) f\|_1 + \|\eta f\|_1\big) \\ &\leq C_{\delta,p}(A_1)\big(\|df\|_2 + \|f\|_1\big). \end{aligned} \tag{A.22}$$

Similarly,

$$\|(d\eta) f\|_p \leq C_{\delta,p}(A_1)\big(\|df\|_2 + \|f\|_1\big). \tag{A.23}$$

The claim follows by plugging (A.22) and (A.23) into (A.21).

Corollary A.20 *For any $\delta > 0$, $p \geq 1$, and open annulus A_1, there exists $C_{\delta,p}(A_1) > 0$ such that for any annulus $A_2 \Subset_\delta A_1$, and $f \in C^\infty(A_1; \mathbb{C}^n)$,*

$$\|df\|_{L^p(A_2)} \leq C_p(A_1)\big(\|\bar{\partial} f\|_p + \|df\|_2\big).$$

Proof: Let $\bar{f} = \frac{1}{|A_1|}\int_{A_1} f$, where $|A_1|$ is the area of A_1. By Lemma A.19,

$$\begin{aligned} \|df\|_{L^p(A_2)} = \|d(f - \bar{f})\|_{L^p(A_2)} &\leq C_p(A_1)\big(\|\bar{\partial}(f - \bar{f})\|_p + \|d(f - \bar{f})\|_2 + \|f - \bar{f}\|_1\big) \\ &= C_p(A_1)\big(\|\bar{\partial} f\|_p + \|df\|_2 + \|f - \bar{f}\|_1\big). \end{aligned} \tag{A.24}$$

The claim follows by applying Lemma A.6 or its proof to A_1, depending on A_1, and $f - \bar{f}$ with $p = 2$ to the last term in (A.24).

Proposition A.21 *If (V, g, J) is a compact almost complex manifold, for any $\delta > 0$, $p \geq 1$, and open annulus A_1, there exists $C_{\delta,p}(A_1)$ such that for any annulus $A_2 \Subset_\delta A_1$, smooth function $u \colon A_1 \longrightarrow V$, and $\xi \in \Gamma(u)$,*

$$\|\nabla \xi\|_{L^p(A_2)} \leq C_{\delta,p}(A_1)\big(\|D_u \xi\|_p + \|\nabla \xi\|_2 + \|\xi du\|_p\big),$$

where the norms are taken with respect to the standard metric on $\mathbb{R}^2$.

Proof: We continue with the notation of the proof of Lemma A.19. By Corollary A.20,

$$\begin{aligned}\|d\tilde{\xi}_i\|_{L^p(A_2)} &\le C_{\delta,p}(A_1)\big(\|\bar\partial\tilde{\xi}_i\|_p + \|d\tilde{\xi}_i\|_2\big)\\ &\le C'_{\delta,p}(A_1)\big(\|\bar\partial\xi_i\|_{L^p(u_i^{-1}(U_i))} + \|d\xi_i\|_{L^2(u_i^{-1}(U_i))} + \|\xi du\|_p\big)\end{aligned} \tag{A.25}$$

If $u(z)\in U_i$, by Corollary A.3,

$$\big|\nabla\xi - \Pi_{\tilde{u}_i(z)}d\xi_i\big|_z \le C|du|_z|\xi|_z. \tag{A.26}$$

Since $\nabla J=0$ and $\bar\partial_j\tilde{\xi}_i = (\bar\partial_j(\eta_i\circ u))\xi_i + \eta_i\bar\partial_j\xi_i$ on $u^{-1}(U_i)$, it follows from (A.25) and (A.26) that

$$\begin{aligned}\|\nabla\xi\|_{L^p(A_2\cap u_i^{-1}(W_i))} &\le \|d\tilde{\xi}_i\|_p + C\|\xi du\|_p\\ &\le C_{p,j}(A_1)\big(\|D_{u,j}\xi\|_p + \|\nabla\xi\|_2 + \|\xi du\|_p\big).\end{aligned} \tag{A.27}$$

The claim is obtained by summing the last equation over all i.

Lemma A.22 *For any $p\ge 1$ and open ball $B\subset\mathbb{R}^2$, there exists $C_{B,p}\in C^\infty(\mathbb{R};\mathbb{R})$ such that for any $u\in C^\infty(B;V)$ and $\xi\in\Gamma_c(u)$,*

$$\|\xi\|_{p,1} \le C_{B,p}\big(\|du\|_p\big)\big(\|D_u\xi\|_p + \|\xi\|_p\big),$$

where the norms are taken with respect to the standard metric on $\mathbb{R}^2$.

Proof: By an argument nearly identical to the proof of Proposition A.21, for all $p\ge 1$,

$$\|\xi\|_{p,1} \le C_{B,p}\big(\|D_u\xi\|_p + \|\xi\|_p + \|\xi du\|_p\big). \tag{A.28}$$

On the other hand, by Proposition A.18, for all $p>2$,

$$\|\xi\|_{C^0} \le C_{B,p}\big(\|du\|_p\big)\|\xi\|_p. \tag{A.29}$$

The claim is obtained from (A.28) and (A.29) by taking a sequence (p_i,q_i) as in the proof of Proposition A.18.

Proposition A.23 *If (V,g,J) is a compact almost complex manifold, (Σ,g_Σ) is a compact Riemann surface, and $p>2$, there exists $C_p\in C^\infty(\mathbb{R};\mathbb{R})$ such that for all $u\in C^\infty(B_{R,r};V)$ and $\xi\in\Gamma(u)$,*

$$\|\xi\|_{p,1} \le C_p\big(\|du\|_p\big)\big(\|D_u\xi\|_p + \|\xi\|_p\big).$$

Proof: This statement is immediate from Lemma A.22.

Appendix B

Fiber-Uniform Analytic Bounds

This appendix contains some of the more technical aspects of the gluing procedure of Chapter 3. In particular, we prove statements (3) and (4) of Lemma 3.5 and statement (1) of Lemma 3.16. Slightly weaker versions of these statements, but for arbitrary almost complex manifolds, are proved in [LT]. Only minor, essentially notational, changes are necessary to adapt the proofs of results below to such manifolds. We also describe certain properties of smooth families of metrics on a Riemann surface that are used in our gluing procedure. These properties are needed to prove the surjectivity of the gluing map.

B.1 Properties of Smooth Families of Metrics on Σ

Let m be a positive integer and

$$\aleph = \big\{x = x_{[m]} : x_h \in \Sigma,\ x_h \neq x_l \text{ if } h \neq l\big\}.$$

Suppose $\{g_x : x \in \aleph\}$ is a smooth family of metrics on Σ such that for any $x = x_{[m]} \in \aleph$ the metric g_x is flat on a neighborhood of x_h in Σ for all $h \in [m]$. If $x = x_{[m]} \in \aleph$ and $v \in T_y\Sigma$, let

$$T_x\aleph = \bigoplus_{h\in[m]} T_{x_h}\Sigma, \quad |v|_x = |v|_{g_x,y}.$$

If $w = w_{[m]} \in T_x\aleph$, let $|w|$ denote $\sum\limits_{h\in[m]} |w_h|_x$. Define $x(w) \in \Sigma^m$ by

$$x(w) = \big(x_1(w), \ldots, x_m(w)\big) = \big(\exp_{g_x,x_1} w_1, \ldots, \exp_{g_x,x_m} w_m\big).$$

We denote by $\phi_{x,y}$ the map $\phi_{g_x,y}$ and by $B_x(y,\delta)$ the set $B_{g_x}(y,\delta)$ described in Section 2.1. If $\delta\colon \aleph \longrightarrow \mathbb{R}$, let

$$T\aleph_\delta = \big\{(x,w) : x \in \aleph;\ w \in T_x\aleph,\ |w|_x < \delta(x)\big\}.$$

Lemma B.1 *There exist $\delta \in C^\infty(\aleph;\mathbb{R}^+)$ and a smooth families of holomorphic maps*

$$\big\{\tilde{p}_{h,(x,w)} : \{z \in B_x(x_h,\delta(x))\} \longrightarrow \Sigma \mid (x,w) \in T\aleph_\delta\big\},$$

such that each map $\tilde{p}_{h,(x,w)}$ is a $(g_x, g_{x(w)})$-isometry,

$$\begin{aligned} d\phi_{x,x_h}\big|_{x_h(w)}\phi_{x(w),x_h(w)}\tilde{p}_{h,(x,w)}(z) &= \phi_{x,x_h}(z), \\ \text{and} \qquad d_{g_x}\big(z, \tilde{p}_{h,(x,w)}(z)\big) &\le 2|w|_x \qquad \forall z \in B_x\big(x_h, \delta(x)\big). \end{aligned} \tag{B.1}$$

In particular, both sides of (B.1) are defined.

Proof: We choose δ such that if $w \in T_x\aleph$ and $|w| \le 4\delta(x)$, then $x(w) \in \aleph$ and the metric $g_{x(w)}$ is flat on $B_x\big(x_h, 2\delta(x)\big)$. This choice of δ insures that both sides of (B.1) are defined. Equation (B.1) is equivalent to

$$\phi_{x(w),x_h(w)}\tilde{p}_{h,(x,w)}(z) = d\exp_{g_x,x_h}\big|_{w_h}\phi_{x,x_h}(z) = \phi_{x,x_h(w)}z + d\exp_{g_x,x_h}\big|_{w_h}w_h, \tag{B.2}$$

since the metric g_x is flat on $B_x\big(x_h, 2\delta(x)\big)$. This equation defines the required map $\tilde{p}_{h,(x,w)}$. Since the metrics g_x and $g_{x(w)}$ are flat on $B_x\big(x_h, 2\delta(x)\big)$, the maps $\phi_{x,x_h(w)}z$ and $\phi_{x(w),x_h(w)}$ are holomorphic, and thus $\tilde{p}_{h,(x,w)}$ is holomorphic. Taking the differential of (B.2), we obtain

$$d\phi_{x(w),x_h(w)}\big|_{\tilde{p}_{h,(x,w)}(z)} \circ d\tilde{p}_{h,(x,w)}\big|_z = d\phi_{x,x_h(w)}\big|_z. \tag{B.3}$$

Since $\phi_{x(w),x_h(w)}$ and $\phi_{x,x_h(w)}$ are $(g_{x(w)}, g_{x(w)})$- and (g_x, g_x)-isometries, respectively, on $B_x\big(x_h, 2\delta(x)\big)$, it follows that $\tilde{p}_{h,(x,w)}$ is a $(g_x, g_{x(w)})$-isometry on $B_x\big(x_h, 2\delta(x)\big)$. By (B.2),

$$\begin{aligned} d_{g_x}\big(z, \tilde{p}_{h,(x,w)}(z)\big) &\le |w_h|_x + \big|\big(\phi_{x(w),x_h(w)} - \phi_{x,x_h(w)}\big)\tilde{p}_{h,(x,w)}(z)\big|_x \\ &\le |w_h|_x + C(x)|w|\delta(x), \end{aligned} \tag{B.4}$$

since the family of metrics is smooth. If $C(x)\delta(x) < 1$, the remaining claim of the lemma follows from (B.4).

Lemma B.2 *There exist $\delta, C_k \in C^\infty(\aleph; \mathbb{R}^+)$, where k is a positive integer, $\alpha_h \in C^\infty(T\aleph_\delta; \mathbb{C})$, and smooth families of maps*

$$\big\{\Theta_{w,h}\colon \{v \in T_{x_h}\Sigma\colon |v|_x < \delta(x)\} \longrightarrow T_{x_h}\Sigma \;\big|\; (x,w) \in T\aleph_\delta\big\}$$

such that each map $\Theta_{w,h}$ is holomorphic, $\Theta_{w,h}(0) = 0$, $\Theta'_{w,h}(0) = 0$, $\|\Theta^{(k)}_{w,h}\|_{C^0} \le C_k(x)|w|$, $|\alpha_h(w)| \le C_0(x)|w|$, and

$$d\phi_{x,x_h}\big|_{x_h(w)}d\phi_{x(w),x_h(w)}\big|_{x_h}\big(\phi_{x(w),x_h}z\big) = \big(1 + \alpha_h(w)\big)\phi_{x,x_h}z + \Theta_{w,h}\big(\phi_{x,x_h}z\big). \tag{B.5}$$

for all $z \in B_x\big(x_h, \delta(x)\big)$. In particular, both sides of (B.5) are defined.

Proof: We choose δ as in the proof of Lemma B.1. This choice of δ insures that both side of (B.5) are defined. If w and z are as in the statement of the lemma, by the flatness of the metric $g_{x(w)}$ near x_h, $\mathbb{C}$-linearity of the differential of the exponential map near zero, and the smoothness of the family of the metrics

$$d\phi_{x,x_h}\big|_{x_h(w)}d\phi_{x(w),x_h(w)}\big|_{x_h}\big(\phi_{x(w),x_h}z\big) = \big(1 + a_h(w)\big)\big(\phi_{x(w),x_h}z\big), \tag{B.6}$$

for some $a_h \in C^\infty(T\aleph_\delta;\mathbb{C})$ such that $a_h(0)=0$. Note that if $g_{x(w)}=g_x$, $a_h(w)=0$, since the metric g_x is flat on $B_x(x_h,|w|)$. The map

$$\{v\in T_{x_h}\Sigma\colon |v|_x<2\delta(x)\} \longrightarrow T_{x_h}\Sigma, \quad v\longrightarrow \phi_{x(w),x_h}\phi^{-1}_{x,x_h}v-v$$

is holomorphic, since $\phi_{x(w),x_h}$ and ϕ_{x,x_h} are, and vanishes at 0. Thus,

$$\phi_{x(w),x_h}\phi^{-1}_{x,x_h}v=\big(1+b_h(w)\big)v+\Theta_{w,h}(v), \tag{B.7}$$

for some $b_h(w)\in\mathbb{C}$ and holomorphic function $\Theta_{w,h}$ such that $\Theta_{w,h}(0),\Theta'_{w,h}(0)=0$. Equation (B.5) follows from (B.6) and (B.7). Smoothness of b_h and $\Theta_{\cdot,h}$ in w follows from the smoothness of the family of the metrics. The bounds on α_h and the derivatives of $\Theta_{w,h}$ follow from their smoothness and compactness of the fibers of

$$\big\{w\in T_x\aleph\colon |w|\le\delta(x)\big\}\longrightarrow\aleph.$$

Lemma B.3 *There exist $\delta, C\in C^\infty(\aleph;\mathbb{R}^+)$ and smooth families of maps*

$$N_h\colon\{(x,w)\colon x\in\aleph;\ (x,w)\in T\aleph_\delta\}\longrightarrow T\aleph$$

such that $|N_h(w,w_h)|_x\le C(x)|w||w_h|$ and

$$d\phi_{x,x_h}\big|_{x_h(w)}\big(\phi_{x(w),x_h(w)}x_h\big)=-w_h+N_h(w,w_h). \tag{B.8}$$

In particular, the left-hand side of (B.8) is defined.

Proof: We take δ as in Lemma B.2. Then, the left-hand side of (B.8) is defined and

$$\begin{aligned}
&d\phi_{x,x_h}\big|_{x_h(w)}\big(\phi_{x(w),x_h(w)}x_h\big)\\
&\quad=d\phi_{x,x_h}\big|_{x_h(w)}\big(\phi_{x,x_h(w)}x_h\big)+d\phi_{x,x_h}\big|_{x_h(w)}\big\{\big(\phi_{x(w),x_h(w)}x_h\big)-\big(\phi_{x,x_h(w)}x_h\big)\big\}\\
&\quad=-w_h+N_h(w,w_h),
\end{aligned}\tag{B.9}$$

where $N(\cdot,\cdot)$ is some smooth function of both variables, that vanishes if either input is zero. Equation (B.8) is thus proved, while the bound on N_h is obtained from its smoothness and compactness of the fibers as in the proof of Lemma B.2.

Lemma B.4 *There exists $\delta\in C^\infty(\aleph;\mathbb{R}^+)$ such that for all $x\in\aleph$, $v\in T_x\aleph$ with $|v|<\delta(x)$, and $c=c_{[m]}\in\mathbb{C}^m$ with $|c||v|<\delta(x)$, there exists $w\in T\aleph$ with $|w_h|_x<2|c_h||v_h|_x$ such that for $z\in B_x\big(x_h,4\delta(x)^{\frac{1}{2}}\big)$*

$$d\phi_{x,x_h}\big|_{x_h(w)}\big(\phi_{x(w),x_h(w)}z\big)=\big(1+\alpha_h(w)\big)\big(c_hv_h+\phi_{x,x_h}z\big)+\Theta_{w,h}\big(\phi_{x,x_h}z\big), \tag{B.10}$$

where $\alpha_h(w)$ and $\Theta_{w,h}$ are as in Lemma B.2. In particular, both sides of (B.10) are defined.

Proof: We start by choosing δ so that $8\delta^{\frac{1}{2}}$ is smaller that the function δ of Lemmas B.2 and B.3. By flatness of the metric $g_{x(w)}$ on $B\big(x_h,8\delta(x)^{\frac{1}{2}}\big)$ for $w\in T_x\aleph$ with $|w|<\delta(x)$

$$\phi_{x(w),x_h(w)}z=d\phi_{x(w),x_h(w)}\big|_{x_h}\phi_{x(w),x_h}z+\phi_{x(w),x_h(w)}x_h \tag{B.11}$$

for any $z \in B(x_h, 4\delta(x)^{\frac{1}{2}})$. Taking $d\phi_{x,x_h}\big|_{x_h(w)}$ of both sides of (B.11) and applying Lemmas B.2 and B.3, we obtain

$$d\phi_{x,x_h}\big|_{x_h(w)}\big(\phi_{x(w),x_h(w)}z\big) = \big(1+\alpha_h(w)\big)\phi_{x,x_h}z+\Theta_{w,h}\big(\phi_{x,x_h}z\big)-w_h+N_h(w,w_h). \quad \text{(B.12)}$$

Thus, we need to solve the equations

$$-w_h + N_h(w, w_h) = \big(1+\alpha_h(w)\big)c_h v_h. \quad \text{(B.13)}$$

Let $\Psi_h(w) = N_h(w,w_h) - \big(1+\alpha_h(w)\big)c_h v_h$. If $|w| \le 2|c_h||v_h|$, then by Lemmas B.2 and B.3,

$$\begin{aligned} |\Psi(w)| &\le C(x)|c||v|\big(2|c||v|+1\big) \le 2|c||v|, \\ |\Psi(w)-\Psi(w')| &\le C(x)|c||v||w-w'| \le \frac{1}{2}|w-w'|, \end{aligned} \quad \text{(B.14)}$$

provided $4C(x)\delta(x) < 1$. In such a case, Ψ is a contracting operator, and thus (B.13) has a unique solution $w \in T_x\aleph$ with $|w| < 2|c||v|$. The estimate $|w_h| < 2|c_h||v_h|$ follows directly from (B.13) if $\delta(x)$ is sufficiently small.

Corollary B.5 *There exist $\delta, C_k \in C^\infty(\aleph;\mathbb{R}^+)$, where k is a positive integer, such that for all $x \in \aleph$, $v \in T_x\aleph$ with $|v| < \delta(x)$, $c = c_{[m]} \in \mathbb{C}^m$ with $|c| < \delta(x)$, and $r = r_{[m]} \in \mathbb{R}^m$ with $|r| < \frac{1}{2}$, there exists $\tilde{x} \in \aleph$ and $\tilde{v} \in T_{\tilde{x}}\aleph$ such that*
(1) $\tilde{x}_h \in B_x(x_h, 2|c_h||v_h|)$, $\left|\frac{g_{\tilde{x}}}{g_x} - 1\right| \le C_1(x)|c||v|$, $\big||\tilde{v}_h|_{\tilde{x}} - |v_h|_x\big| \le C_1(x)\big(|c||v| + |r_h|\big)|v_h|$;
(2) for any $z \in B_x(x_h, 4\delta(x)^{1/2})$,

$$\frac{\phi_{\tilde{x},\tilde{x}_h}z}{\tilde{v}_h} = (1+r_h)\Big\{c_h + \frac{\phi_{x,x_h}z}{v_h} + \Theta_{v,c,r,h}\Big(\frac{\phi_{x,x_h}z}{v_h}\Big)\Big\}, \quad \text{(B.15)}$$

where $\Theta_{v,c,r,h}$ is a holomorphic function, varying smoothly with the parameters, such that

$$\Theta_{v,c,r,h}(0) = 0, \quad \Theta'_{v,c,r,h}(0) = 0, \quad \big\|\Theta^{\langle k\rangle}_{v,c,r,h}\big\|_{C^0} \le C_k(x)|c||v||v_h|^{k-1}.$$

Proof: Let δ be as in Lemma B.4. Given v and c as in the statement of the lemma, let $w \in T_x\aleph$ be the element provided by Lemma B.4. Take

$$\tilde{x}_h = x_h(w), \qquad \tilde{v}_h = (1+r_h)^{-1}\big(1+a_h(w)\big)d\phi^{-1}_{x,x_h}|_{w_h}v_h.$$

The estimates in (1) are immediate from Lemma B.4, provided δ is sufficiently small. The inequalities in (2) arise from the smooth dependence of w on x, v, and c in Lemma B.4, and the fact that w is zero if either $v=0$ or $c=0$.

B.2 Sobolev Inequalities for the Metrics g_v

In this section, we prove (3) of Lemma 3.5. The reason this estimate holds is that (Σ_v, g_v) can be written as a union of the surfaces $(\Sigma_{b_v,i}, g_v)$ with small disks missing and annuli $(\tilde{A}^{\pm}_{v,h}, g_v)$ that are uniformly equivalent to annuli in $\mathbb{R}^2$ with the smaller radius less than half of the larger.

Suppose $\mathcal{T}=(S,M,I;j,\underline{\lambda})$ is a bubble type and

$$v=(b,v_{\hat{I}})=\big((S,M,I;x,(j,y),u),v_{\hat{I}}\big)\in F^{(0)}\mathcal{T}_\delta.$$

For any $h\in\hat{I}$ and $i\in I$, put

$$\tilde{A}^-_{v,h}=q^{-1}_{v,i_h}\Big(\big\{(\iota_h,z)\in\Sigma_{b_v,\iota_h}:(2\delta_{\mathcal{T}}(b_v))^{-1}|v_h|_b\le r_{b,h}(z)\le|v_h|_b^{\frac{1}{2}}\big\}\Big);$$
$$\tilde{A}^+_{v,h}=q^{-1}_{v,i_h}\Big(\big\{(\iota_h,z)\in\Sigma_{b_v,\iota_h}:|v_h|_b^{\frac{1}{2}}\le r_{b,h}(z)\le 2\delta_{\mathcal{T}}(b_v)\big\}\Big);$$
$$S_{v,i}=q^{-1}_{v,i}\Big(\big\{(i,z)\in S_{b_v,i}:r_{b,h}(z)\ge\delta_{\mathcal{T}}(b_v)\text{ if }\iota_h=i;\ |q_S^{-1}(z)|\ge\delta_{\mathcal{T}}(b_v)\text{ if }i>0\big\}\Big).$$

Let $\tilde{A}_{v,h}$ denote $\tilde{A}^-_{v,h}\cup\tilde{A}^+_{v,h}$.

Lemma B.6 *For any $p>2$, there exists $C_p\in C^\infty(\mathcal{M}^{(0)}_{\mathcal{T}};\mathbb{R})$ such that for any $v\in F^{(0)}\mathcal{T}_\delta$ and $h\in\hat{I}$,*

$$\xi\in\tilde{\Gamma}_c(\tilde{A}_{v,h};u_v^*TV)\Longrightarrow\|\xi\|_{C^0}\le C_p(b_v)\|\xi\|_{g_v,p,1}.$$

Proof: By construction of the metric g_v, $g_v|\tilde{A}_{v,h}$ is the pullback of the metric g_{v,ι_h} on $q_{v,\iota_h}(\tilde{A}_{v,h})$ by the map q_{v,ι_h}. Furthermore, the metric g_{v,ι_h} on $q_v(\tilde{A}^\pm_{v,h})$ differs from the standard metric on the annulus $B_{2\delta_{\mathcal{T}}(b_v),|v_h|^{\frac{1}{2}}}\subset\mathbb{R}^2$ by factors bounded by $C(b_v)$. Since $\|du_v\|_{g_v,p}\le C_p(b_v)$ by (1) of Lemma 3.5, the claim follows from Proposition A.18.

Proposition B.7 *For any $p>2$, there exists $C_p\in C^\infty(\mathcal{M}^{(0)}_{\mathcal{T}};\mathbb{R})$ such that for all $v\in F^{(0)}\mathcal{T}_\delta$,*

$$\|\xi\|_{C^0}\le C_p(b_v)\|\xi\|_{g_v,p,1}\qquad\forall\xi\in\Gamma(u_v).$$

Proof: (1) Note that $g_v|S_{v,i}$ is the pull-back of the metric $g_{b_v,i}$ on $q_{v,i}(S_{v,i})$ by the map $q_{v,i}$. Thus, by Proposition A.18, if $\xi\in\Gamma_c(S_{v,i};u_v^*TV)$,

$$\|\xi\|_{C^0}=\|\xi\circ q_{v,i}\|_{C^0}\le C_p(\|du_v\circ q_{v,i}\|_{g_{b_v,i},p})\|\xi\circ q_{v,i}\|_{g_{b_v,i},p,1}=C_p(b_v)\|\xi\|_{g_v,p,1},$$

since ξ vanishes outside of $S_{v,i}$.
(2) We now define a partition of unity subordinate to $\{S_{v,i},\tilde{A}_{v,h}:i\in I,h\in\hat{I}\}$. Put

$$\eta^+_{v,h}(z)=\begin{cases}1-\beta_{\delta^2_{\mathcal{T}}(b_v)}\big(r_{b_v,h}(q_{v,\iota_h}(z))\big), & \text{if } q_{v,\iota_h}(z)\in\Sigma_{b_v,\iota_h};\\ 1, & \text{otherwise};\end{cases}$$

$$\eta^-_{v,h}(z)=\begin{cases}1-\beta_{\delta^2_{\mathcal{T}}(b_v)}\big(|q_S^{-1}q_v(z)|\big), & \text{if } q_{v,h}(z)\in\Sigma_{b_v,h};\\ 1, & \text{otherwise};\end{cases}\qquad \tilde{\eta}_v(z)=1-\prod_{h\in\hat{I}}\eta^-_{v,h}(z)\eta^+_{v,h}(z).$$

Note that $d\eta^\pm_{v,h}$ is supported in $\tilde{A}^\pm_{v,h}$. It follows from the definition of g_v that

$$\|d\eta^\pm_{v,h}\|_{g_v,C^1}=\|d(\eta^\pm_{v,h}\circ q^{-1}_{v,i_h})\|_{g_{v,i_h},C^1}\le C(b_v).$$

Thus, if $\xi \in \Gamma(u_v)$ by (1) and Lemma B.6,

$$\|\xi\|_{C^0} \le \sum_{i\in I} \|\tilde{\eta}\xi\|_{C^0(S_{v,i})} + \sum_{h\in\hat{I}} \|\eta_{v,h}^- \eta_{v,h}^+ \xi\|_{C^0} \le C_p(b_v)\Big(\|\tilde{\eta}\xi\|_{g_v,p,1} + \sum_{h\in\hat{I}} \|\eta_{v,h}^- \eta_{v,h}^+ \xi\|_{g_v,p,1}\Big)$$
$$\le C_p(b_v)\Big(|I|\|\xi\|_{g_v,p,1} + 2\sum_{h\in\hat{I}} \|\eta_{v,h}^- \eta_{v,h}^+\|_{g_v,C^1}\|\xi\|_{g_v,p}\Big) \le C_p'(b_v)\|\xi\|_{g_v,p,1}.$$

B.3 Elliptic Estimates for the Metrics g_v

This subsection contains the proof of (4) of Lemma 3.5, the main elliptic estimate for the operators D_v and the modified Sobolev norms. This estimate does *not* hold for the standard Sobolev norms. The argument is essentially the same as in [LT], but we do include all of the details, based on Appendix A, and state a sharper estimate.

Let $\mathcal{T}$, v, $\tilde{A}_{v,h} = \tilde{A}_{v,h}^- \cup \tilde{A}_{v,h}^+$, and $S_{v,i}$ be as in Subsection B.2. If $\iota_h = \hat{0}$, the metric $g_{b_v,\hat{0}}$ is flat on $B_{b_v,h}(\delta_{\mathcal{T}}(b_v)^{\frac{1}{2}})$. Thus, for any $h \in \hat{I}$, we can choose conformal polar coordinates (r,θ) on $\tilde{A}_{v,h}$ such that $r(z) = r_{b,v}(q_{v,\iota_h}(z))$. Since $g_v|\tilde{A}_{v,h}$ is the pullback of the metric g_{v,ι_h} on $q_{v,\iota_h}(\tilde{A}_{v,h})$ by the map q_{v,ι_h},

$$g_v = \Big\{\big(1-\beta_{|v_h|}(2r)\big)\frac{2C(b_v)}{|v_h| + |v_h|^{-1}r^2} + \beta_{|v_h|}(r)\Big\}\big(dr^2 + r^2 d\theta^2\big) \qquad \text{on } \tilde{A}_{v,h}. \tag{B.16}$$

Similarly, since $\rho_v = \rho_{v,\iota_h} \circ q_{v,\iota_h}$

$$\rho_v = r^2 + \frac{|v_h|^2}{r^2} \qquad \text{on } \tilde{A}_{v,h}. \tag{B.17}$$

Lemma B.8 *For all $p>1$, there exists $C_p \in C^\infty(\mathcal{M}_{\mathcal{T}}^{(0)};\mathbb{R})$ such that for all $v \in F^{(0)}\mathcal{T}_\delta$ and $h \in \hat{I}$,*

$$\xi \in \Gamma_c(\tilde{A}_{v,h}; u_v^* TV) \Longrightarrow \Big(\int_{\tilde{A}_{v,h}} \rho_v^{-\frac{p-2}{p}} |\nabla^{b_v}\xi|^2\Big)^{\frac{1}{2}} \le C_p(b_v)\big(\|D_v\xi\|_{v,p} + \|\xi\|_{v,p}\big).$$

Proof: (1) Let ϵ_1 and ϵ_2 denote $\big(2\delta_{\mathcal{T}}(b_v)\big)^{-1}|v_h|$ and $2\delta_{\mathcal{T}}(b_v)$, respectively. Note that the integral on the left-hand side in the statement of the lemma is conformally invariant. With respect to the metric $dr^2 + r^2 d\theta^2$,

$$\big|D_v\xi\big|_{(r,\theta)} = \Big|\frac{D}{dr}\xi + Jr^{-1}\frac{D}{d\theta}\xi\Big|_{(r,\theta)},$$

where $\frac{D}{dr}$ and $\frac{D}{d\theta}$ denote covariant differentiation with respect to the connection ∇^{b_v} and the norms are taken with respect to the metric $g_{V,b}$ on V. Thus,

$$a_h^2 \equiv \int_{\tilde{A}_{v,h}} \rho_v^{-\frac{p-2}{p}} |\nabla^{b_v}\xi|^2 \le \|D_v\xi\|_{v,p}^2 - 2\int_0^{2\pi}\int_{\epsilon_1}^{\epsilon_2} \rho_v^{-\frac{p-2}{p}} \Big\langle \frac{D}{dr}\xi, J\frac{D}{d\theta}\xi\Big\rangle dr d\theta. \tag{B.18}$$

Since $\nabla^{b_v}J=0$, using integration by parts twice, we obtain

$$\begin{aligned}\int_0^{2\pi}\int_{\epsilon_1}^{\epsilon_2}\rho_v^{-\frac{p-2}{p}}\Big\langle\frac{D}{dr}\xi,J\frac{D}{d\theta}\xi\Big\rangle drd\theta &= -\int_0^{2\pi}\int_{\epsilon_1}^{\epsilon_2}\rho_v^{-\frac{p-2}{p}}\Big\langle\frac{D}{d\theta}\frac{D}{dr}\xi,J\xi\Big\rangle drd\theta\\ &= -\int_0^{2\pi}\int_{\epsilon_1}^{\epsilon_2}\rho_v^{-\frac{p-2}{p}}\Big(\Big\langle\frac{D}{dr}\frac{D}{d\theta}\xi,J\xi\Big\rangle-\langle\mathcal{R}(u_r,u_\theta)\xi,J\xi\rangle\Big)drd\theta \qquad \text{(B.19)}\\ = \int_0^{2\pi}\int_{\epsilon_1}^{\epsilon_2}\rho_v^{-\frac{p-2}{p}}\Big(\Big\langle\frac{D}{d\theta}\xi,J\frac{D}{dr}\xi\Big\rangle &- \frac{(p-2)}{p}\frac{\rho_v'(r)}{\rho_v(r)}\Big\langle\frac{D}{d\theta}\xi,J\xi\Big\rangle+\langle\mathcal{R}_{b_v}(u_r,u_\theta)\xi,J\xi\rangle\Big)drd\theta,\end{aligned}$$

where u_r and u_θ denote $\frac{d}{dr}u_v$ and $\frac{d}{d\theta}u_v$, respectively, and $\mathcal{R}_{b_v}$ is the curvature tensor of the connection ∇^{b_v}. Since

$$\Big\langle\frac{D}{d\theta}\xi,J\frac{D}{dr}\xi\Big\rangle=-\Big\langle\frac{D}{dr}\xi,J\frac{D}{d\theta}\xi\Big\rangle,$$

by (B.19) and (1) of Lemma 3.5,

$$\begin{aligned}&\Big|\int_0^{2\pi}\int_{\epsilon_1}^{\epsilon_2}\rho_v^{-\frac{p-2}{p}}\Big\langle\frac{D}{dr}\xi,J\frac{D}{d\theta}\xi\Big\rangle drdr\theta\Big|\\ &\qquad\le\frac{|p-2|}{2p}\Big|\int_{\epsilon_1}^{\epsilon_2}\rho_v^{-\frac{p-2}{p}}\frac{\rho_v'(r)}{\rho_v(r)}\int_0^{2\pi}\Big\langle\frac{D}{d\theta}\xi,J\xi\Big\rangle d\theta dr\Big|+C(b_v)\|\xi\|_{v,p}^2.\end{aligned}\qquad \text{(B.20)}$$

(2) By Poincare Lemma, see Proposition A.5, for every circle with r fixed,

$$\begin{aligned}\Big|\int_0^{2\pi}\Big\langle\frac{D}{d\theta}\xi,J\xi\Big\rangle d\theta\Big|\le\int_0^{2\pi}\Big|\frac{D}{d\theta}\xi\Big|^2d\theta+C(g_{b_v})\Big\{\Big(\int_0^{2\pi}|u_\theta|^2d\theta\Big)\Big(\int_0^{2\pi}|\xi|^2d\theta\Big)\qquad \text{(B.21)}\\ +\Big(\int_0^{2\pi}|u_\theta|d\theta\Big)\Big(\int_0^{2\pi}|\xi|^2d\theta\Big)^{\frac12}\Big(\int_0^{2\pi}\Big|\frac{D}{d\theta}\xi\Big|^2d\theta\Big)^{\frac12}\Big\}.\end{aligned}$$

Since $\big|\frac{\rho_v'(r)}{\rho_v(r)}\big|\le 2r^{-1}$ on $\tilde{A}_{v,h}$, by Holder's inequality and the first part of Lemma 3.5,

$$\begin{aligned}&\frac12\int_{\epsilon_1}^{\epsilon_2}\rho_v^{-\frac{p-2}{p}}\Big|\frac{r\rho_v'(r)}{\rho_v(r)}\Big|\Big(\int_0^{2\pi}r^{-1}|u_\theta|d\theta\Big)\Big(\int_0^{2\pi}|\xi|^2d\theta\Big)^{\frac12}\Big(\int_0^{2\pi}r^{-2}\Big|\frac{D}{d\theta}\xi\Big|^2d\theta\Big)^{\frac12}rdr\\ &\qquad\le C\|\xi\|_{v,p}\Big(\int_{\tilde{A}_{v,h}}\rho_v^{-\frac{p-2}{p}}r^{-2}\Big|\frac{D}{d\theta}\xi\Big|^2\Big)^{\frac12}.\end{aligned}\qquad \text{(B.22)}$$

Similarly,

$$\frac12\int_{\epsilon_1}^{\epsilon_2}\rho_v^{-\frac{p-2}{p}}\Big|\frac{r\rho_v'(r)}{\rho_v(r)}\Big|\Big(\int_0^{2\pi}r^{-2}|u_\theta|^2d\theta\Big)\Big(\int_0^{2\pi}|\xi|^2d\theta\Big)rdrd\theta\le C(b_v)\|\xi\|_{v,p}^2.\qquad \text{(B.23)}$$

Combining equations (B.21)-(B.23), we obtain

$$\begin{aligned}&\frac12\Big|\int_{\epsilon_1}^{\epsilon_2}\rho_v^{-\frac{p-2}{p}}\frac{\rho_v'(r)}{\rho_v(r)}\int_0^{2\pi}\Big\langle\frac{D}{d\theta}\xi,J\xi\Big\rangle d\theta dr\Big|\\ &\qquad\le\int_0^{2\pi}\int_{\epsilon_1}^{\epsilon_2}\rho_v^{-\frac{p-2}{p}}r^{-2}\Big|\frac{D}{d\theta}\xi\Big|^2rdrd\theta+C(b_v)\big(\|\xi\|_{v,p}^2+\|\xi\|_pa_h\big).\end{aligned}\qquad \text{(B.24)}$$

Note that

$$\int_0^{2\pi}\int_{\epsilon_1}^{\epsilon_2}\rho_v^{-\frac{p-2}{p}}r^{-2}\Big|\frac{D}{d\theta}\xi\Big|^2 rdrd\theta \tag{B.25}$$
$$=\frac{1}{2}\int_{\tilde{A}_{v,h}}\rho_v^{-\frac{p-2}{p}}\Big(r^{-2}\Big|\frac{D}{d\theta}\xi\Big|^2+\Big|\Big(\frac{D}{dr}\xi+Jr^{-1}\frac{D}{d\theta}\xi\Big)-\frac{D}{dr}\xi\Big|^2\Big)\le\frac{1+\epsilon}{2}a_h^2+C_\epsilon\|D_v\xi\|_{v,p}^2$$

for any $\epsilon>0$. Combining equations (B.18),(B.20), (B.24) and (B.25), we obtain

$$a_h^2\le\frac{|p-2|}{p}(1+\epsilon)a_h^2+\big(C(b_v)+C_\epsilon\big)\big(\|D_v\xi\|_{v,p}^2+\|\xi\|_{v,p}^2+\|\xi\|_{v,p}a_h\big).$$

Since $\frac{|p-2|}{p}<1$, the claim follows by choosing ϵ sufficiently small.

Lemma B.9 *For all $p\ge1$, there exists $C_p\in C^\infty(\mathcal{M}_{\mathcal{T}}^{(0)};\mathbb{R})$ such that for all $v\in F^{(0)}\mathcal{T}_\delta$ and $h\in\hat{I}$,*

$$\xi\in\Gamma_c(\tilde{A}_{v,h}|u_v^*TV)\Longrightarrow\|\nabla^{b_v}\xi\|_{g_v,p}\le C_p(b_v)\Big(\|D_v\xi\|_{v,p}+\|\xi\|_{v,p}+\Big(\int_{\tilde{A}_{v,h}}\rho_v^{-\frac{p-2}{2p}}|\nabla\xi|^2\Big)^{\frac{1}{2}}\Big).$$

Proof: Choose a sequence

$$\delta_0>\ldots>\delta_{N+1}>0\quad\text{such that}\quad\delta_0=\epsilon_2,\ \delta_{N+1}=\epsilon_1,\ \frac{1}{3}\le\frac{\delta_{l+1}}{\delta_l}\le\frac{1}{2}.$$

For each $l=1,\ldots,N-1$, let g_l denote the metric

$$g_l=\Big(\delta_l^2+\frac{|v_h|^2}{\delta_l^2}\Big)^{-1}g_v\quad\text{on}\quad\tilde{A}_l=\big\{(r,\theta)\in\tilde{A}_{v,h}:\delta_{l+2}\le r\le\delta_{l-1}\big\}.$$

Let $\rho_l=\delta_l^2+|v_h|^2\delta_l^{-2}$ and denote by A_l the annulus $\{(r,\theta)\in\tilde{A}_{v,h}:\delta_{l+1}\le r\le\delta_l\}$. The pullback of the metric g_l on $\tilde{A}_l$ to the annulus $(\frac{\delta_{l+2}}{\delta_l},\frac{\delta_{l-1}}{\delta_l})\times S^1\subset\mathbb{R}^2$ by the map $(r,\theta)\longrightarrow(\delta_l r,\theta)$ differs from the Eucledian metric by a conformal factor bounded by $C(b_v)$, since

$$\frac{1}{100}\le\Big\{\big(1-\beta_{|v_h|}(\delta_l r)\big)\frac{2}{|v_h|+|v_h|^{-1}\delta_l^2r^2}+\beta_{|v_h|}(\delta_l r)\Big\}\Big(\delta_l^2+\frac{|v_h|^2}{\delta_l^2}\Big)^{-\frac{1}{2}}\delta_l\le200,$$

whenever $r\in(\frac{1}{9},3)$ and $\delta_l\in(|v_h|,1)$. Thus, by Proposition A.21,

$$\|\nabla^{b_v}\xi\|_{g_l,L^p(A_l)}\le C\Big(\|D_v\xi\|_{g_l,L^p(\tilde{A}_l)}+\|\nabla^{b_v}\xi\|_{g_l,L^p(\tilde{A}_l)}+\|\xi du\|_{g_l,L^p(\tilde{A}_l)}\Big),\tag{B.26}$$

or equivalently

$$\|\nabla^{b_v}\xi\|_{g_v,L^p(A_l)}\le C\Big(\|D_v\xi\|_{g_v,L^p(\tilde{A}_l)}+\|\rho_l^{-\frac{p-2}{2p}}\nabla^{b_v}\xi\|_{L^2(\tilde{A}_l)}+\|\xi du\|_{g_v,L^p(\tilde{A}_l)}\Big).\tag{B.27}$$

Since $\frac{\rho_v(r)}{\rho_l}\in[\frac{1}{81},81]$ when $r\in[\delta_{l+2},\delta_{l-1}]$, (B.27) is equivalent to

$$\Big(\int_{A_l}|\nabla^{b_v}\xi|^p\Big)^{\frac{1}{p}}\le C_p(b_v)\Big(\Big(\int_{\tilde{A}_l}|D_v\xi|^p\Big)^{\frac{1}{p}}+\Big(\int_{\tilde{A}_l}\rho_v^{-\frac{p-2}{p}}|\nabla^{b_v}\xi|^2\Big)^{\frac{1}{2}}+\|\xi du_v\|_{g_v,L^p(\tilde{A}_l)}\Big).$$

The claim follows by summing up the last inequality over all l and using (1) of Lemma 3.5.

Remark: The above proof does not quite apply to the two outermost annuli A_1 and A_N. However, since ξ is compactly supported in $\tilde{A}_{v,h}$, the proof of Proposition A.21 can be applied to A_1 with $A_1 \cup A_2$ replacing $\tilde{A}_1$ to (B.26), and similarly for A_N. Alternatively, for the purposes of proving Proposition B.11 below, it is sufficient to prove Lemma B.9 and Corollary B.10 for ξ that vanish on A_1 and A_N.

Corollary B.10 *For all $p>1$, there exists $C_p \in C^\infty(\mathcal{M}_{\mathcal{T}}^{(0)};\mathbb{R})$ such that for all $v\in F^{(0)}\mathcal{T}_\delta$ and $h\in\hat{I}$*

$$\xi\in\Gamma_c(\tilde{A}_{v,h};u^*TV) \Longrightarrow \|\xi\|_{v,p,1} \le C_p(b_v)\big(\|D_v\xi\|_{v,p}+\|\xi\|_{v,p}\big).$$

Proof: This corollary follows immediately from Lemmas B.10 and B.10.

Proposition B.11 *For all $p>1$, there exists $C_p\in C^\infty(\mathcal{M}_{\mathcal{T}}^{(0)};\mathbb{R})$ such that for all $v\in F^{(0)}\mathcal{T}_\delta$,*

$$\|\xi\|_{v,p,1} \le C_p(b_v)\big(\|D_v\xi\|_{v,p}+\|\xi\|_{v,p}\big) \qquad \forall\ \ \xi\in\Gamma(u_v).$$

Proof: This proposition follows from Corollary B.10 and Proposition A.23 by taking a partition of unity as in the proof of Proposition B.7.

B.4 Right Inverse for the Operator D_v

Lemma B.12 *Let $\{v_k\}$ be a sequence in $F^{(0)}\mathcal{T}_\delta$ that converges to $b^*\in\mathcal{M}_{\mathcal{T}}^{(0)}$. Suppose $\xi_k\in\Gamma(u_{v_k})$ is such that $\|\xi_k\|_{v_k,p,1}\le 1$ for all k, while $\|D_{v_k}\xi_k\|_{v_k,p}\longrightarrow 0$ as $k\longrightarrow\infty$ for some $p>2$. Then a subsequence of $\{\xi_k\}$ C^0-converges $\xi^*\in\Gamma_-(b^*)$. Furthermore, $\|\xi_k\|_{v_k,C^0}$ converges to $\|\xi^*\|_{b^*,C^0}$.*

Proof: (1) Write $b^* = \big(S,M,I;x^*,(j,y^*),u\big)$ and

$$v_k=(b_k,v_k)=\big((S,M,I;x_k,(j,y_k),u_k),(v_k)_{\hat{I}}\big).$$

For each $i\in I$ and $\delta>0$, put

$$S^*_{i,\delta}=\{z\in\Sigma_{b,i}: r_{b^*,h}(z)\ge\delta\ \forall h\in\hat{I};\ |q_S^{-1}(z)|\ge\delta \text{ if } i>0\}.$$

For $i\in I$ and all k sufficiently large (depending on δ), define $\zeta_{k,i},\xi'_{k,i}\in\Gamma(u_i^*|S^*_{i,\delta})$ by

$$\exp_{b^*,u_i^*(z)}\zeta_{k,i}(z)=u_{v_k}(q_{v_k}^{-1}(z)),\ \ \|\zeta_{k,i}\|_{b^*,C^0}<\text{inj }g_{V,b^*};\quad \Pi_{b^*,\zeta_{k,i}(z)}\xi'_{k,i}(z)=\xi_k(q_{v_k}^{-1}(z)).$$

Since $\|\nabla^{b^*}\zeta_{k,i}\|_{b^*,C^0}\le C$ for k sufficiently large, (1) of Lemma 3.5 and by Corollary A.3,

$$\begin{aligned}\|\xi'_{k,i}\|_{b^*,p,1}&\le(1+\epsilon_k)\|\xi_k\|_{v_k,p,1}+\epsilon_k\|\xi_k\|_{v_k,C^0},\\ \|D_{b,u_i}\xi'_{k,i}\|_{b^*,p}&\le(1+\epsilon_k)\|D_{v_k}\xi_k\|_{v_k,p}+\epsilon_k\|\xi_k\|_{v_k,C^0},\end{aligned}\tag{B.28}$$

where $\epsilon_k\longrightarrow 0$ as $k\longrightarrow\infty$. Since $\|\xi_k\|_{v_k,p,1}\le 1$, (2) of Lemma 3.5 applied to (B.28),

$$\begin{aligned}\|\xi'_{k,i}\|_{b^*,p,1}&\le(1+\tilde{\epsilon}_k)\|\xi_k\|_{v_k,p,1}+\tilde{\epsilon}_k,\\ \|D_{b,u_i}\xi'_{k,i}\|_{b^*,p}&\le(1+\tilde{\epsilon}_k)\|D_{v_k}\xi_k\|_{v_k,p}+\tilde{\epsilon}_k,\end{aligned}\tag{B.29}$$

where $\tilde{\epsilon}_k \longrightarrow 0$ as $k \longrightarrow \infty$. Sobolev's embedding theorem then implies that $\xi'_{k,i}$ converges to a vector field $\xi_i^* \in \Gamma(u_i|\Sigma^*_{b^*,i})$ in the C^0-norm on the compact subsets of $\Sigma^*_{b^*,i}$. Furthermore, $\|\xi_i^*\|_{b^*,C^0} < \infty$, since

$$\|\xi'_{k,i}\|_{b^*,C^0} \le (1+\epsilon_k)\|\xi_k\|_{v_k,C^0} \le C.$$

(2) We will now show that $D_{b^*,u_i^*}\xi_i^* = 0$ weakly, i.e. $\langle\langle \xi_i^*, D^*_{b^*,u_i^*}\eta\rangle\rangle_{b^*} = 0$ for all $\eta \in \Gamma^{0,1}(u_i^*)$. We have

$$\langle\langle \xi_i^*, D^*_{b^*,u_i^*}\eta\rangle\rangle_{b^*} = \lim_{\delta\longrightarrow 0}\int_{S^*_{i,\delta}} \langle \xi_i^*, D^*_{b^*,u_i^*}\eta\rangle_{b^*} = \lim_{\delta\longrightarrow 0}\lim_{k\longrightarrow\infty}\int_{S^*_{i,\delta}} \langle \xi'_{k,i}, D^*_{b^*,u_i^*}\eta\rangle_{b^*}, \tag{B.30}$$

since $\xi'_{k,i} \longrightarrow \xi_i^*$ in the C^0-norm on $S_{i,\delta}$. By integration by parts,

$$\begin{aligned} \Big|\int_{S^*_{i,\delta}} \langle \xi'_{k,i}, D^*_{b^*,u_i^*}\eta\rangle_{b^*} - \int_{S^*_{i,\delta}} \langle D_{b^*,u_i^*}\xi'_{k,i}, \eta\rangle_{b^*}\Big| &\le C\int_{\partial S^*_{i,\delta}} |\xi'_{k,i}||\eta| \\ &\le C'\|\xi'_{k,i}\|_{b^*,C^0}\|\eta\|_{b^*,C^0}\delta. \end{aligned} \tag{B.31}$$

Since $\|D_{b^*,u_i*}\xi'_{k,i}\|_{b^*,p} \longrightarrow 0$ as $k \longrightarrow \infty$ on $S^*_{i,\delta}$ and $\|\xi'_{k,i}\|_{b^*,C^0} \le C$, by (B.30) and (B.31),

$$\langle\langle \xi_i^*, D^*_{b^*,u_i^*}\eta\rangle\rangle_{b^*} = 0 \qquad \forall \eta \in \Gamma^{0,1}(u_i^*).$$

(3) Since $D_{b^*,u_i^*}\xi_i^* = 0$ weakly on S_i and D_{b^*,u_i^*} is an elliptic operator, it follows that ξ_i^* is smooth and $D_{b^*,u_i^*}\xi_i^* = 0$. It will now be shown that $\xi^*_{\iota_h}(x_h^*) = \xi_h^*(\infty)$ for all $h \in \hat{I}$, i.e. $\xi^* \equiv \xi_I^* \in \Gamma(b^*)$. For each $h \in \hat{I}$, let $A_{h,\delta,k} \subset S$ denote the small cylinder connecting $q_{v_k}^{-1}(S^*_{h,\delta})$ and $q_{v_k}^{-1}(S^*_{\iota_h,\delta})$. Let $\epsilon > 0$ be any small number. Choose small $\delta > 0$ such that $u_h(B_{b^*,h}(\infty, 2\delta))$ and $u^*_{\iota_h}(B_{b^*,\iota_h^*}(x_h^*, 2\delta))$ lie in $B_{b^*}(u_h^*(\infty), \epsilon)$. Then we can write

$$u^*_{b^*}(z) = \exp_{b^*,u^*_{b^*}(x_h^*)}\bar{u}_{b^*}(z), \quad |\bar{u}_{b^*}(z)| < \text{inj } g_{V,b^*}; \quad \bar{\xi}'_k(z) \equiv \Pi^{-1}_{b^*,\bar{u}_b(z)}\xi'_k(z)$$

for $z \in B_{b^*,h}(\infty, \delta) \cup B_{b^*,\iota_h^*}(x_h^*, \delta)$. Similarly, put

$$\bar{\xi}_h^*(z) = \Pi^{-1}_{b^*,\bar{u}_{b^*}(z)}\xi_h^*(z) \quad \text{and} \quad \bar{\xi}^*_{\iota_h}(z) = \Pi^{-1}_{b^*,\bar{u}_{b^*}(z)}\xi^*_{\iota_h}(z)$$

for z in $B_{b^*,h}(\infty, \delta)$ and in $B_{b^*,i_h}(x_h^*, \delta)$, respectively. We can also assume that δ is so small that $\big|\bar{\xi}_h^* - \xi_h^*(\infty)\big|_{b^*}$ and $\big|\bar{\xi}^*_{\iota_h} - \xi^*_{i_h}(x_h^*)\big|_{b^*}$ do not exceed ϵ on $B_{b^*,h}(\infty, \delta)$ and on $B_{b^*,\iota_h}(x_h^*, \delta)$, respectively. Choose large k^* such that all $k > k^*$,

$$\|\xi^* - \xi'_k\|_{C^0(S^*_{h,\delta}\cup S^*_{\iota_h,\delta})} \le \epsilon.$$

It can be assumed that $u_k(A_{h,2\delta,k})$ lies in $B_{b^*}(u^*(x_h^*); 2\epsilon)$ for $k > k^*$. Thus, we can write

$$u_k(z) = \exp_{b^*,u(x_h^*)}\bar{u}_k(z), \quad |\bar{u}_k(z)|_{b^*} < \text{inj } g_{V,b^*}; \quad \bar{\xi}_k(z) \equiv \Pi^{-1}_{b^*,\bar{u}_k(z)}\xi_k(z)$$

if $z \in A_{h,\delta,k}$. Pick points z_1 and z_2, one on each component of the boundary of $A_{h,\delta,k}$. Then

$$\begin{aligned} \big|\xi_h^*(\infty) - \xi^*_{\iota_h}(x_h^*)\big|_{b^*} &\le 2\big(\epsilon + \big|\bar{\xi}_h^*(q_{v_k}(z_1)) - \bar{\xi}^*_{\iota_h}(q_{v_k}(z_2))\big|_{b^*}\big) \\ &\le 4\big(\epsilon + \big|\bar{\xi}'_{k,h}(q_{v_k}(z_1)) - \bar{\xi}'_{k,i_h}(q_{v_k}(z_2))\big|_{b^*}\big) \\ &\le C\big(\epsilon + \big|\bar{\xi}_k(z_1) - \bar{\xi}_k(z_2)\big|_{b^*} + \|\zeta_k\|_{b^*,C^0(S^*_{h,\delta}\cup S^*_{i_h,\delta})}\|\bar{\xi}_k\|_{b^*,C^0(A_{h,\delta,k})}\big). \end{aligned} \tag{B.32}$$

Since $A_{h,\delta,k}$ is uniformly equivalent to the union of two annuli with the larger radius bounded above by δ and the smaller radius less than half of the larger, by Lemma A.12 and Holder's inequality,

$$\big|\bar{\xi}_k(z_1)-\bar{\xi}_k(z_2)\big|_{b^*}\leq C\big|\bar{\xi}_k(z_1)-\bar{\xi}_k(z_2)\big|_{b_k}\leq C'\delta^{\frac{2(p-2)}{p}}\|d\bar{\xi}_k\|_{\upsilon_k,L^p(A_{h,\delta,k})}. \tag{B.33}$$

By Corollary A.3 and Proposition B.11,

$$\|d\bar{\xi}_k\|_{\upsilon_k,L^p(A_{h,\delta,k})}\leq\|\xi_k\|_{\upsilon_k,p,1}+\|du_{\upsilon_k}\|_{\upsilon_k,p}\|\xi_k\|_{\upsilon_k,C^0}\leq C. \tag{B.34}$$

Combining equations (B.32)-(B.34), we obtain

$$\big|\xi_h^*(\infty)-\xi_{\iota_h}^*(x_h^*)\big|_{b^*}\leq C\big(\epsilon+\delta^{\frac{2(p-2)}{p}}+\|\zeta_k\|_{b^*,C^0(S^*_{h,\delta}\cup S^*_{\iota_h,\delta})}\big). \tag{B.35}$$

Since the last term in (B.35) tends to zero as $k\longrightarrow\infty$ and ϵ and δ can be chosen to be arbitrarily small, it follows $\xi_h^*(\infty)=\xi_{\iota_h}^*(x_h^*)$.

Proposition B.13 *For any simple bubble type $\mathcal{T}$, there exist $C,\delta\in C^\infty(\mathcal{M}_{\mathcal{T}}^{(0)};\mathbb{R})$ such that for all $\upsilon\in F^{(0)}\mathcal{T}_\delta$ if $\mathcal{T}$ is regular and $\upsilon\in F^{(\emptyset)}\mathcal{T}_\delta$ if $\mathcal{T}$ is semiregular,*

$$\|\xi\|_{\upsilon,p,1}\leq C_p(b_\upsilon)\|D_\upsilon\xi\|_{\upsilon,p}\qquad\forall\xi\in\Gamma_+(\upsilon)\quad\textit{and}\quad\forall\xi\in\tilde{\Gamma}_+(\upsilon).$$

Proof: If not, we can choose a sequence $\upsilon_k\in F^{(0)}\mathcal{T}_\delta$, converging to some $b\in\mathcal{M}_{\mathcal{T}}^{(0)}$ and vector fields $\xi_k\in\Gamma_+(\upsilon_k)$ (or $\xi_k\in\tilde{\Gamma}_+(\upsilon_k)$) such that $\|\xi_k\|_{\upsilon_k,p,1}=1$, while $\|D_{\upsilon_k}\xi\|_{\upsilon_k,p}\longrightarrow0$. If $\xi_k\in\Gamma_+(\upsilon_k)$, note that $\{\Gamma_-(\upsilon_k)\}$ C^0-converges to $V\equiv\Gamma_-(b)$. If $\xi_k\in\tilde{\Gamma}_+(\upsilon_k)$, by Definition 3.11, a subsequence of $\{\tilde{\Gamma}_-(\upsilon_k)\}$ C^0-converges to a subspace $V\subset L_1^p(b)$ such that $\pi_{b,-}:V\longrightarrow\Gamma_-(b)$ is an isomorphism. In either case, by the first statement of Lemma B.12, a subsequence of $\{\xi_k\}$ C^0-converges to a vector field $\xi^*\in\Gamma_-(b)$. By the second statement of Lemma B.12, ξ^* must be orthogonal V, since $\xi_k\in\Gamma_+(\upsilon_k)$ (or $\xi_k\in\tilde{\Gamma}_+(\upsilon_k)$). Thus, $\xi^*=0$. On the other hand, by Proposition B.11, there exists $\epsilon>0$ such that $\|\xi_k\|_{\upsilon_k,p}\geq\epsilon$ for all k sufficiently large. However, by Lemma B.12, $\|\xi_k\|_{\upsilon_k,C^0}\longrightarrow0$, which is a contradiction.

Appendix C

Regularity Lemmas

In this appendix, we use sheaf cohomology to prove the surjectivity of certain operators. These regularity results are used throughout this thesis to conclude that various spaces of stable maps have the expected dimension. We conclude by proving that the spaces of maps of type (2a) and (2b) of Section 7 are empty, as claimed.

C.1 A Short Exact Sequence on $\mathbb{P}^n$

If M is a Kahler manifold and $E \longrightarrow M$ is a holomorphic vector bundle, let $\mathcal{O}(E)$ denote the sheaf of holomorphic sections of E. If $E \longrightarrow M$ is the trivial holomorphic line bundle, we write $\mathcal{O}$ for $\mathcal{O}(E)$. Let $H \longrightarrow \mathbb{P}^n$ be the hyperplane bundle.

Lemma C.1 *There is an exact sequence of sheaves over $\mathbb{P}^n$:*

$$0 \longrightarrow \mathcal{O} \longrightarrow (n+1)\mathcal{O}(H) \longrightarrow \mathcal{O}(T\mathbb{P}^n) \longrightarrow 0.$$

Proof: (1) Let $[X_0 : \ldots : X_n]$ denote the homogeneous coordinates on $\mathbb{P}^n$. Denote by $\bar{X}_i$ the section of the hyperplane bundle given by

$$\bar{X}_i|_{[X_0:\ldots:X_n]}(X_0, \ldots, X_n) = X_i \in \mathbb{C}.$$

Then we define a sheaf map $\mathcal{O} \longrightarrow (n+1)\mathcal{O}(H)$ by

$$f \longrightarrow (f\bar{X}_0, \ldots, f\bar{X}_n).$$

Let $U_i = \{[X_0 : \ldots : X_n] : X_i \neq 0\}$. On U_i, we can use the complex coordinates

$$z_{i,k} = \frac{X_k}{X_i}, \qquad k \in \{0, \ldots, n\} - \{i\}.$$

Using these coordinates, we define a sheaf map $(n+1)\mathcal{O}(H) \longrightarrow \mathcal{O}(T\mathbb{P}^n)$ by

$$(p_0, \ldots, p_n) \longrightarrow \sum_{k \neq i} \big(p_k(z_{i,0}, \ldots, z_{i,n}) - z_{i,k}p_i(z_{i,0}, \ldots, z_{i,n})\big)\frac{\partial}{\partial z_{i,k}}, \tag{C.1}$$

where $z_{i,i}=1$. We need to see that this map is well-defined. Suppose $j\neq i$. Then,

$$z_{j,l}=z_{i,j}^{-1}z_{i,l} \Longrightarrow \frac{\partial}{\partial z_{i,k}}=\sum_{l\neq j}\frac{\partial z_{j,l}}{\partial z_{i,k}}\frac{\partial}{\partial z_{j,l}}=\begin{cases} z_{i,j}^{-1}\frac{\partial}{\partial z_{j,k}}, & \text{if } k\neq j;\\ -z_{i,j}^{-2}\Big(\frac{\partial}{\partial z_{j,i}}+\sum_{l\neq i,j} z_{i,l}\frac{\partial}{\partial z_{j,l}}\Big), & \text{if } k=j.\end{cases} \tag{C.2}$$

Since each p_l is a linear functional, if $k\neq i,j$, we can write the kth summand in (C.1) as

$$\begin{aligned}&\big(z_{j,i}^{-1}p_k(z_{j,0},\dots,z_{j,n})-z_{j,i}^{-2}z_{j,k}p_i(z_{j,0},\dots,z_{j,n})\big)z_{i,j}^{-1}\frac{\partial}{\partial z_{j,k}}\\ &\qquad=\big(p_k(z_{j,0},\dots,z_{j,n})-z_{j,i}^{-1}z_{j,k}p_i(z_{j,0},\dots,z_{j,n})\big)\frac{\partial}{\partial z_{j,k}}.\end{aligned} \tag{C.3}$$

The remaining, $k=j$, summand in (C.1) equals

$$\begin{aligned}&\big(z_{j,i}^{-1}p_j(z_{j,0},\dots,z_{j,n})-z_{j,i}^{-2}p_i(z_{j,0},\dots,z_{j,n})\big)(-z_{i,j}^{-2})\Big(\frac{\partial}{\partial z_{j,i}}+\sum_{k\neq i,j}z_{i,k}\frac{\partial}{\partial z_{j,k}}\Big)\\ &\qquad=\big(p_i(z_{j,0},\dots,z_{j,n})-z_{j,i}p_j(z_{j,0},\dots,z_{j,n})\big)\Big(\frac{\partial}{\partial z_{j,i}}+\sum_{k\neq i,j}z_{i,k}\frac{\partial}{\partial z_{j,k}}\Big).\end{aligned} \tag{C.4}$$

Since $z_{j,i}z_{i,k}=z_{j,k}$, collecting similar terms in (C.3) and (C.4), we obtain equation (C.1) with i replaced by j.
(2) It is clear that the first map is injective, the second is surjective, and the composite is zero. Finally, if $(p_0,\dots,p_n)$ is mapped to zero by the second map, then (C.1) implies that $\bar{X}_jp_i=\bar{X}_ip_j$ for all i and j. Thus, the function f, given by

$$f([X_0:\dots:X_n])=\frac{p_i(X_0,\dots,X_n)}{X_i},$$

is well-defined and holomorphic wherever $(p_0,\dots,p_n)$ is.

C.2 On Regularity of Kernel of D_b

Lemma C.2 *If $u\colon S^2\longrightarrow\mathbb{P}^n$ is a holomorphic map, there is a surjection*

$$(n+1)H^1\big(S^2;\mathcal{O}(u^*H\otimes(-(k+1)p))\big)\longrightarrow H^1\big(S^2;\mathcal{O}(u^*T\mathbb{P}^n\otimes(-(k+1)p))\big),$$

where p denotes the divisor corresponding to a point $p\in S^2$. If the degree of u is at least k, then both cohomology groups are trivial.

Proof: Pulling back the short exact sequence of sheaves of Lemma C.1 by u, tensoring it with $-(k+1)p$, and taking the corresponding long exact sequence, we obtain:

$$\begin{aligned}&\longrightarrow(n+1)H^1\big(S^2;\mathcal{O}(u^*H\otimes(-(k+1)p))\big)\longrightarrow H^1\big(S^2;\mathcal{O}(u^*T\mathbb{P}^n\otimes(-(k+1)p))\big)\\ &\longrightarrow H^2\big(S^2;\mathcal{O}(-(k+1)p)\big)\longrightarrow\dots\end{aligned} \tag{C.5}$$

Since S^2 is a one-dimensional complex manifold, the last cohomology group in (C.5) must vanish, and the first statement of the lemma follows. On the other hand, by Kodaira-Serre duality,

$$\begin{aligned} H^1\big(S^2;\mathcal{O}(u^*H\otimes(-(k+1)p))\big) &= H^1\big(S^2;\Omega^1(u^*H\otimes(-(k-1)p)\big) \\ &\approx H^0\big(S^2;\mathcal{O}((u^*H\otimes(-(k-1)p))^*)\big)^*. \end{aligned} \tag{C.6}$$

The last group in (C.6) is trivial if $\mathcal{O}\big(u^*H\otimes(-(k-1)p)\big)$ is positive, i.e. if

$$\big\langle c_1\big(u^*H\otimes(-(k-1)p\big),[S^2]\big\rangle = d-(k-1)>0,$$

where d is the degree of u.

Corollary C.3 *If $u\colon S^2\longrightarrow\mathbb{P}^n$ is holomorphic map of degree d, for any $p\in S^2$ and nonzero $v\in T_pS^2$, the map*

$$\phi_{p,v}^{(k)}\colon \ker D_u \longrightarrow \bigoplus_{m\in\langle k\rangle} T_{u(p)}\mathbb{P}^n, \quad \phi_{p,v}^{(k)}\xi = \big(\xi_p, D\xi|_{p,v},\ldots,D^{(k)}\xi|_{p,v}\big),$$

where $D\xi|_{p,v}$ denotes the covariant derivative of ξ along u in the direction of v, is surjective provided $d\geq k$.

Remark: If one defines $D^{(k)}\xi$ with respect to the metric $g_{\mathbb{P}^n,u(p)}$ on $\mathbb{P}^n$, $D^{(k)}\in T_{u(p)}\mathbb{P}^n\otimes T^*S^{2\otimes k}$, where T^*S^2 is viewed as a complex line bundle. However, the statement is independent of the choice of metric on $\mathbb{P}^n$.

Proof: Since ξ is holomorphic, if $\phi_{p,v}^{(k)}\xi$ is zero, ξ has a zero of order $k+1$ at p. Thus, $\phi_{p,v}^{(k)}$ induces a short exact sequence of sheaves on S^2:

$$0\longrightarrow \mathcal{O}\big(u^*T\mathbb{P}^n\otimes(-(k+1)p)\big)\longrightarrow \mathcal{O}(u^*T\mathbb{P}^n)\xrightarrow{\phi_{p,v}^{(k)}}(k+1)\mathcal{O}\big((u^*T\mathbb{P}^n)_p\big)\longrightarrow 0,$$

where we view $\mathcal{O}\big((u^*T\mathbb{P}^n)_p\big)$ as a sheaf on S^2 via extension by 0; see [GH, p38]. Taking the corresponding long exact sequence in cohomology, we obtain

$$\begin{aligned} \ldots\longrightarrow H^0\big(S^2;\mathcal{O}(u^*T\mathbb{P}^n)\big)&\xrightarrow{\phi_{p,v}^{(k)}}(k+1)H^0\big(S^2;\mathcal{O}((u^*T\mathbb{P}^n)_p)\big) \\ \longrightarrow H^1\big(S^2;\mathcal{O}(u^*T\mathbb{P}^n\otimes(-(k+1)p))\big)\ldots \end{aligned} \tag{C.7}$$

By Lemma C.2, the last cohomology group in (C.7) is zero if $d\geq k$. It follows that the map $\phi_{p,v}^{(k)}$ is surjective.

C.3 Dimension Counts

Lemma C.4 *Let Σ be a compact Riemann surface. If $u\colon\Sigma\longrightarrow\mathbb{P}^n$ is a holomorphic map, there exists a surjection*

$$(n+1)H^1\big(\Sigma;\mathcal{O}(u^*H)\big)\longrightarrow H^1\big(\Sigma;\mathcal{O}(u^*T\mathbb{P}^n)\big).$$

Proof: Pulling back the short exact sequence of Lemma C.1 by u gives a long exact sequence in sheaf cohomology:

$$\ldots(n+1)H^1\big(\Sigma;\mathcal{O}(u^*H)\big)\longrightarrow H^1\big(\Sigma;\mathcal{O}(u^*T\mathbb{P}^n)\big)\longrightarrow H^2(\Sigma;\mathcal{O})\ldots \tag{C.8}$$

Since Σ is one-complex dimensional, the last group vanishes, and the claim follows.

Corollary C.5 *Let Σ be a compact Riemann surface. If $u:\Sigma \longrightarrow \mathbb{P}^n$ is a holomorphic map, the $\bar{\partial}$-operator for the bundle $u^*T\mathbb{P}^n$,*

$$D_u\colon \Gamma(\Sigma; u^*T\mathbb{P}^n) \longrightarrow \Gamma(\Sigma; \Lambda^{0,1}T^*\Sigma \otimes u^*T\mathbb{P}^n)$$

is surjective, provided $d+\chi(\Sigma)>0$, where d is the degree of u.

Proof: The cokernel of D_u is $H^1_{\bar{\partial}}(\Sigma; u^*T\mathbb{P}^n)$. By Dolbeault Theorem,

$$H^1_{\bar{\partial}}(\Sigma; u^*T\mathbb{P}^n) = H^1\big(\Sigma; \mathcal{O}(u^*T\mathbb{P}^n)\big) = (n+1)H^1\big(\Sigma; \mathcal{O}(u^*H)\big). \tag{C.9}$$

On the other hand, by Kodaira-Serre duality (see [GH, p153]),

$$\begin{aligned} H^1\big(\Sigma; \mathcal{O}(u^*H)\big) &= H^1\big(\Sigma; \Omega^1(T\Sigma \otimes u^*H)\big) \\ &= H^0\big(\Sigma; \mathcal{O}((T\Sigma \otimes u^*H)^*)\big)^* = H^0_{\bar{\partial}}\big(\Sigma; (T\Sigma \otimes u^*H)^*\big)^*. \end{aligned} \tag{C.10}$$

The bundle $(T\Sigma \otimes u^*H)^*$ does not admit any holomorphic section if it is negative, i.e. if

$$\big\langle c_1\big((T\Sigma \otimes u^*H)\big), [\Sigma]\big\rangle = \big\langle c_1(T\Sigma) + c_1(u^*H), [\Sigma]\big\rangle = \chi(\Sigma) + d > 0.$$

Thus, the claim follows from equations (C.9) and (C.10) and Lemma C.4,

Proposition C.6 *Let Σ be a Riemann surface of genus 2 and let d and n be positive integers with $n\leq 4$. If $n=4$, assume that $d\neq 2$. Suppose $\mu=(\mu_1,\ldots,\mu_N)$ is an N-tuple of proper complex submanifolds of $\mathbb{P}^n$ of total complex codimension $d(n+1)-n+N$ in general position. If $\mathcal{T}=\big(\Sigma,[N],I;j,\underline{d}\big)$ is a bubble type such that*

$$\mathcal{T} < \big(\Sigma, [N], \{\hat{0}\}; \hat{0}, d\big) \equiv \mathcal{T}^*$$

and $d_{\hat{0}}>0$, then $\mathcal{H}_{\mathcal{T}}(\mu)=\emptyset$. Furthermore, if

$$b = \big(\Sigma, [N], \{\hat{0}\};, (\hat{0}, y), u\big) \in \mathcal{H}_{\mathcal{T}^*}(\mu),$$

then the map u is not multiply-covered.

Proof: (1) If $d_{\hat{0}}\geq 3$, by Corollaries C.3 and C.5 and standard arguments such as in [MS], the space $\mathcal{H}_{\mathcal{T}}$ is a smooth manifold and the maps ev_l are smooth. If $b\in\mathcal{H}_{\mathcal{T}}$, a neighborhood of b in $\mathcal{H}_{\mathcal{T}}$ can be modeled on $\ker D_b \oplus \bigoplus_{l=1}^{l=n} T_{y_l}\Sigma_{b,j_l}$. In particular, by the Index Theorem,

$$\dim_{\mathbb{C}} \mathcal{H}_{\mathcal{T}} = \sum_{i\in I}\big(d_i'(n+1) + n(1-g(\Sigma_{b,i}))\big) - (n-1)|\hat{I}| + N = d(n+1) - n + |\hat{I}| + N.$$

Thus, if the map

$$\mathrm{ev}_{[N]} \equiv \mathrm{ev}_1 \times \ldots \times \mathrm{ev}_N \colon \mathcal{H}_{\mathcal{T}} \longrightarrow \mathbb{P}^n \times \ldots \times \mathbb{P}^n,$$

is smooth and transversal to $\mu_1 \times \ldots \times \mu_N$, $\mathcal{H}_{\mathcal{T}}(\mu)$ is a smooth manifold of (complex) dimension $|\hat{I}|$. Since the map $\mathrm{ev}_{[N]}$ is invariant under the action of $2|\hat{I}|$-dimensional group

$$\mathcal{G}_{\mathcal{T}} \equiv \{g \in PSL_2 \colon g(\infty) = \infty\}^{\hat{I}},$$

$\mathcal{G}_\mathcal{T}$ acts smoothly on $\mathcal{H}_\mathcal{T}(\mu)$. Furthermore, the stabilizer at each point is finite. Thus, $\mathcal{H}_\mathcal{T}(\mu)=\emptyset$.

(2) Suppose $d_{\hat{0}}=2$. If $b=\big(\Sigma,[N],I;x,(j,y),u\big)\in\mathcal{H}_\mathcal{T}$, the map $u_{\hat{0}}$ must factor through a degree-one map $\tilde{u}_{\hat{0}}\colon S^2\longrightarrow\mathbb{P}^n$; see [ACGH, p116]. Thus, it is enough to show that the space $\mathcal{H}_{\mathcal{T}'}(\mu)=\emptyset$, where $\mathcal{T}'=\big(S^2,[N],I;j,\underline{d}'\big)$, $d'_h=d_h$ if $h\in\hat{I}$ and $d'_{\hat{0}}=0$. By Corollaries C.3 and C.5 , the space $\mathcal{H}_{\mathcal{T}'}$ is a smooth manifold of dimension

$$\dim_\mathbb{C}\mathcal{H}_{\mathcal{T}'}=(d-1)(n+1)+n+|\hat{I}|+N.$$

Similarly to (1) above, it follows that $\mathcal{H}_{\mathcal{T}'}(\mu)$ is a smooth manifold of dimension $n-1+|\hat{I}|$ on which the $(2M+3)$-dimensional group $PSL_2\times\mathcal{G}_\mathcal{T}$ acts with only finite stabilizers. It follows that $\mathcal{H}_{\mathcal{T}'}(\mu)=\emptyset$ if $n<M+4$. Note that the case $\hat{I}=\emptyset$ can occur only if $d=d_{\hat{0}}=2$. Finally, if $d_{\hat{0}}=1$, the entire space $\mathcal{H}_\mathcal{T}$ is empty, since there are no holomorphic degree-one maps from Σ into $\mathbb{P}^n$.

(3) Suppose $b=\big(\Sigma,[N],\{\hat{0}\};,(\hat{0},y),u\big)\in\mathcal{H}_{\mathcal{T}^*}(\mu)$ and $u\colon\Sigma\longrightarrow\mathbb{P}^n$ factors through a k-fold cover of S^2, where $k\geq2$ and k divides d. Then b arises from the space $\mathcal{H}_{\mathcal{T}'}(\mu)$, where

$$\mathcal{T}'=\big(S^2,[N],\{\hat{0}\};\hat{0},d/k\big).$$

Similarly to the above, this space is a smooth manifold of dimension

$$\big((d/k)(n+1)+n+N\big)-\big(d(n+1)-n+N\big)=-\frac{k-1}{k}d(n+1)+2n.$$

Thus, $\mathcal{H}_{\mathcal{T}'}(\mu)=\emptyset$, provided $d\geq3$. In fact, since $\mathcal{H}_{\mathcal{T}'}(\mu)$ has a three-dimensional group of symmetry, $\mathcal{H}_{\mathcal{T}'}(\mu)=\emptyset$ unless $d=2$ and $n\geq4$.

(4) Suppose b is as in (3) and u factors through a k-fold cover of a torus T, where $k\geq2$ and k divides d. Then b arises from the space

$$\begin{aligned}\mathcal{H}_{1,d/k}(\mu)\equiv\big\{(x,y_{[N]},u)\colon x\in\mathbb{C}-\mathbb{R};\ u\colon\mathbb{C}\longrightarrow\mathbb{P}^n,\ \bar{\partial}u=0,\ u_*[T]=\frac{d}{k}\lambda;\\ u(z+a+bx)=u(z)\ \forall x\in\mathbb{C},a,b\in\mathbb{Z};\ y_l\in\mathbb{C},\ u(y_l)\in\mu_l\ \forall l\in[N]\big\},\end{aligned}$$

where $\lambda\in H_2(\mathbb{P}^n;\mathbb{Z})$ denotes the homology class of a line in $\mathbb{P}^n$. Similarly to the above, Corollary C.5 implies that $\mathcal{H}_{1,d/k}(\mu)$ is a smooth space of dimension

$$\big((d/k)(n+1)+1+N\big)-\big(d(n+1)-n+N\big)=-\Big(\frac{k-1}{k}d-1\Big)(n+1)<1.$$

Since $\mathcal{H}_{1,d/k}(\mu)$ has a one-dimensional group of symmetries of $\mathbb{C}$-translations, $\mathcal{H}_{1,d/k}(\mu)=\emptyset$.

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