Experimental Investigation of
Lamb Waves in Transversely Isotropic Composite Plates

by

Mark E. Orwat

Submitted to the Department of Civil and Environmental Engineering
in partial fulfillment of the requirements for the degree of
Master of Science in Civil and Environmental Engineering

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2001

©2001 Massachusetts Institute of Technology
All rights reserved
DISCLAIMER OF QUALITY

Due to the condition of the original material, there are unavoidable flaws in this reproduction. We have made every effort possible to provide you with the best copy available. If you are dissatisfied with this product and find it unusable, please contact Document Services as soon as possible.

Thank you.

The images contained in this document are of the best quality available.
Experimental Investigation of
Lamb Waves in Transversely Isotropic Composite Plates

by

Mark E. Orwat

Submitted to the Department of Civil and Environmental Engineering
on May 21, 2001, in partial fulfillment of the
requirements for the degree of
Master of Science in Civil and Environmental Engineering

Abstract

The general complexity of Lamb waves is investigated in two materials of varying levels of isotropy, and the theory is extended to experimental applications. After an intensive review of the propagation in isotropic, homogeneous materials, the study is extended to orthotropic composite materials. The dispersive, multi-modal nature of the Lamb wave propagation is studied through the dispersion equations, and the procedure for depicting the dispersion curves is explained plainly. To study the dispersion behavior experimentally, Lamb waves are generated in aluminum and composite plates with a Q-switched Nd:YAG excitation laser and detected by a continuous Photo-EMF detection laser interferometer. The experimental dispersion curves are processed using a two dimensional fast Fourier transform (2D FFT) scheme for both materials, and compared to the theoretical dispersion curves. Finally, the methodology behind Synthetic Phase Tuning is described and the feasibility of extending the theory to composite materials is tested.

Thesis Supervisor: Shi-Chang Wooh
Title: Associate Professor of Civil and Environmental Engineering
Acknowledgments

My years at the Massachusetts Institute of Technology have been challenging. No one faces challenges alone.

Celeste has provided stability, understanding, and deep, committed love.

Edwin and Christine Orwat have given support, encouragement, and love.

Professor Shi-Chang Wooh has given me sound mentorship in all areas of academics, including the conduct of theoretical and experimental research, teaching, and publication. He always enthusiastically received me into his office and provided a mature, challenging learning environment. I value the time that he has spent with me.

My good friends in the Non-Destructive Evaluation Laboratory: Yijun Shi, Ji-yong Wang, and Jung-wuk Hong, were all quick to put aside their own studies to assist me whenever needed. They all provided a lot of laughs, advanced my cultural understanding, and furthered my research and engineering expertise.

Yijun shared with me his deep understanding of the extremely complex topic that is explored in this thesis. He spent countless hours with me testing the theory through experimentation.

Ji-yong is a true expert in experimentation, NDE methods, signal generation and analysis, and computer processing. He has often sacrificed his time to assist me, teach me, and mentor me.
## Contents

1 Introduction 10

1.1 Thesis Significance ................................................. 10

1.2 Background Knowledge of the Engineering Problem ............... 11

1.2.1 Structural Integrity Evaluation ................................. 11

1.2.2 Non-Destructive Evaluation Methods ......................... 12

1.2.3 The Inspection of Composites ................................. 13

1.2.4 Lamb Waves .................................................. 13

1.3 Thesis Organization ............................................... 15

2 Lamb Waves in Isotropic Materials 17

2.1 The Nature of Lamb Waves ......................................... 18

2.2 Wave Motion in Homogeneous, Isotropic Medium ................ 20

2.3 Wave Propagation in an Infinite Plate ............................ 23

2.3.1 Development of the Dispersion Relation ..................... 23

2.3.2 Analysis of the Dispersion Relation .......................... 28

2.3.3 Analysis of the Dispersion Curves ............................ 32

3 Lamb Waves in Transversely Isotropic Composite Materials 35

3.1 Wave Motion in Non-Homogeneous, Transversely Isotropic Medium 36

3.1.1 Derivation of Dispersion Equation ............................ 36

3.2 Wave Propagation in an Infinite Plate ............................ 40
3.2.1 Development of the Dispersion Relation ........................................ 40
3.2.2 Analysis of the Dispersion Relation ............................................. 42
3.2.3 Analysis of the Dispersion Curve .................................................. 49

4 Theoretical Dispersion Curves .............................................................. 50
4.1 Dispersion Curves for an Isotropic Aluminum Plate .......................... 50
4.1.1 Aluminum Specimen Physical Characteristics .............................. 50
4.1.2 Theoretical Aluminum Dispersion Curves ..................................... 52
4.2 Dispersion Curves for a Transversely Isotropic Composite Plate ......... 53
4.2.1 Composite Specimen Physical Characteristics .............................. 53
4.2.2 Composite Specimen Material Stiffness Constants ......................... 56
4.2.3 Theoretical Composite Dispersion Curves ..................................... 58

5 Obtaining Dispersion Curves Experimentally ........................................ 61
5.1 Experimental Procedure for Obtaining Dispersion Curves ................ 61
5.1.1 The Experimental Setup to Collect Data ...................................... 62
5.1.2 Processing the Data Into a Dispersion Curve ............................... 66
5.2 Experimental Results for Specimens .................................................. 67
5.2.1 Isotropic Aluminum Specimen .................................................... 67
5.2.2 Transversely Isotropic Composite Specimen .................................. 68

6 Synthetic Phase Tuning ........................................................................ 73
6.1 Synthetic Phase Tuning Theory ......................................................... 73
6.1.1 The objective of SPT .................................................................... 73
6.1.2 Implementing SPT to Create a Physically Tuned Mode .................. 74
6.2 Tuning Results for a Transversely Isotropic Material ......................... 79

7 Conclusions ......................................................................................... 81
A  Analysis of the Dispersion Relation-Isotropic Case  85
   A.0.1  Symmetric / Antisymmetric Modes  . . . . . . . . . . . . . . . . . . . . . . . . . . 85
   A.0.2  Cutoff Frequencies . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 89

B  Analysis of the Dispersion Relation-Orthotropic Case  91
   B.0.3  Symmetric / Antisymmetric Modes  . . . . . . . . . . . . . . . . . . . . . . . . . . 91
   B.0.4  Cutoff Frequencies . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 99

C  Formulating the Stress-Strain Relationship  105
List of Figures

1-1 A generic frequency spectrum for an isotropic material. $\omega = \text{frequency}$, $h =$ the half thickness of the material, $c =$ velocity, and $k =$ wave number. The dashed curves are asymmetric modes, the solid lines symmetric modes. 15

1-2 A generic dispersion curve for an isotropic material. $f =$ center frequency of wave, $h =$ the half thickness of the material, $c_p =$ the phase velocity of individual modes. The dashed curves are asymmetric modes, the solid lines symmetric modes. 16

2-1 A simple representation of the mode conversion that occurs within a thin material with two parallel edges. 18

2-2 The two modes, $A_0$, $S_0$, that combine to produce the multi-modal behavior of Lamb waves. 19

2-3 Propagation of a plane harmonic Lamb wave in a plate. The thickness of the plate is $2h$. 23

2-4 Flowchart explaining the iterative computer algorithm. 33

2-5 Dispersion curve (phase velocity) for an aluminum plate. 34

3-1 Dispersion curve (phase velocity) for a composite plate. 49

4-1 $kh$ vs. $\frac{\omega h}{c_T}$ curves for an aluminum plate. 53

4-2 Dispersion curve (2fh vs $c_p$) for the aluminum plate. 54

4-3 $kh$ vs. $\frac{\omega h}{c_T}$ for a T300/934 composite plate. 59
4-4 Dispersion curve, $2fh$ vs $c_p$, for a T300/934 composite plate. 60

5-1 The schematic diagram of the set-up utilizing both a source laser and a receiving laser. 63

5-2 The frequency response spectrum of the laser receiver. 65

5-3 Sample waveform received by the laser for the aluminum specimen. 66

5-4 The schematic diagram showing the dimensions of the specimen and the distance step that the laser translates. 66

5-5 The schematic diagram of the set-up utilizing a source laser and a PVDF transducer as a receiver. 67

5-6 A comparison of the three experimentally obtained waveforms for the composite plate, taken at different distances between the source and the receiving laser. 69

5-7 A comparison of the experimentally obtained dispersion curve to the theoretical dispersion curve for aluminum. 70

5-8 A comparison of the three experimentally obtained waveforms for the composite plate, taken at different distances between the source and the receiving laser. 71

5-9 A comparison of the experimentally obtained dispersion curve to the theoretical dispersion curve for the unidirectional composite specimen. 72

6-1 A comparison of the experimentally obtained dispersion curve to the theoretical dispersion curve for aluminum. 76

6-2 Experimentally created signal using a PVDF array transducer on a transversely isotropic composite plate. 80
List of Tables

4.1 Values for the wave speeds in aluminum. .................................. 51
4.2 Values for the engineering properties in the T300/934 composite specimen. 54
6.1 Characteristics of PVDF array transducer. ................................. 79
Chapter 1

Introduction

1.1 Thesis Significance

Lamb wave inspection is the preferred method used in the interrogation of thin solids bounded by parallel planes. Rayleigh and Lamb first predicted and proved the existence of the two-dimensional Lamb wave in 1889 (Graff 1975). Since then, Lamb waves have proven their usefulness as an effective Non-Destructive Evaluation (NDE) tool. The remarkable advantages of using Lamb waves as an NDE tool have attracted countless researchers, and the theoretical proof and prediction of Lamb waves in various materials are solidly developed. However, the significant complexities in the propagating of an actual Lamb wave through a material have influenced many researchers to stop their work short of the application of their theories into physical reality.

This thesis outlines the well known theory behind the propagation of a Lamb wave through both an isotropic plate and a specially orthotropic plate. This work then attempts to bridge the huge gap between the theoretical and the practical application of Lamb waves. It probes one of the more promising experimental solutions to the complexities that must be overcome in a multi modal, dispersive Lamb wave.
Readers will take from this work a solid understanding of Lamb wave theory, a methodology of expressing the wave behavior in graphical form from both tabular data and experiments, and an innovative experimental application of Lamb waves called Synthetic Phase Tuning, first developed for aluminum in 2001 (Wooh and Shi 2001).

1.2 Background Knowledge of the Engineering Problem

To understand the importance of the expansion of Lamb wave theory to experimental applications, a look at the engineering problem is needed. This section progresses from a general to a specific overview of the role of this powerful NDE method.

1.2.1 Structural Integrity Evaluation

The unexpected failure of materials and structures can be very costly in terms of the destruction of property, the suspension of operating revenue, the erosion of public confidence, and the loss of human life. From the day that a structure is built, it begins to age, weather, and lose its capacity to function as intended. Because of the usual factors of safety built into codes and design specifications, the natural degradation of material and structural strength is normally acceptable. But as the material reaches a limit in terms of serviceability or strength, failure may occur. This failure may occur in months, years, decades, or centuries.

A pro-active, responsible approach to preventing global failure of a structure is to regularly monitor the integrity of the materials for damage or other mechanisms that may erode its strength. Destructive evaluation of a structure is usually not realistic. This general category of inspection may cause an increased degradation in material strength, incur higher costs due to post-inspection repair, and impact the aesthetics of the structure. The obvious solution is to inspect structures and materials in a non-destructive way.
1.2.2 Non-Destructive Evaluation Methods

Non-destructive evaluation (NDE) is concisely defined by Robert Crane as “that class of physical and chemical tests that permit the detection and/or measurement of the significant properties of a material or structure without impairing its usefulness” (Crane 1986).

There are a wide variety of NDE techniques being utilized throughout industry and research. The type of NDE method employed depends on both the internal material complexity and the structural application in which the material is used. Material complexity ranges from homogeneous, isotropic (similar composition, many planes of symmetry) to inhomogeneous, anisotropic (varied composition, few planes of symmetry). In terms of structural application, the most effective NDE method may vary due to the combination of materials used, the method of adjoining materials, and the geometry of the structure. Currently, there is no single method that is versatile enough to inspect everything that is found in nature.

NDE methods range from simple visual inspection, to microwave inspection, to ultrasound. The most recent advances have arisen from interrogation of materials by waveforms of various frequencies. The general choice of NDE method often depends upon material properties, structural geometry, and purpose of application.

All NDE techniques may be evaluated against a comprehensive set of criteria proposed by Crane (Crane 1986). He writes that there are four qualities of NDE techniques:

1. **Accuracy** - the technique must precisely measure a property of the material which can be used to infer the presence of a flaw.

2. **Reliability** - the technique must consistently measure that same material property or consistently find and quantify flaws.

3. **Simplicity** - the technique should be able to be implemented and easily understood by repair level technicians, not just highly skilled researchers.
4. *Low cost* - the technique should be relatively inexpensive when compared to the value of the component being tested or the cost of failure or aborted mission.

### 1.2.3 The Inspection of Composites

A *structural composite* is a “material system consisting of two or more phases on a macroscopic scale, whose mechanical performance and properties are designed to be superior to those of the constituent materials acting independently” (Daniel and Ishai 1994). Most modern composite manufacturers use two phases in material design: fibers and matrix. There is an inherent complexity in the internal make up of a composite, often called anisotropy. Material properties are dependent on the plane of observation, which makes the composite a very difficult material to interrogate. When sending waveforms through a composite, attenuation of the wave is high, and the directional nature of the fibers causes skew. Normally, lower frequency wavelengths are required to allow deep penetration into the composite, with a slight loss of resolution.

Because of the improved strength of composite materials, structural elements may be constructed in longer and thinner geometries. The thin geometries cause many of these composite building materials to behave like thin plates, introducing even more complexity in the signal due to dispersion and mode conversion, as first discovered by Lamb in 1889, and further described in the next section.

Due to the large size of the structural composites, point by point methods of inspection are not realistic, as the cost of inspection would increase dramatically. A more efficient method of inspecting large areas of thin geometry is the use of *Lamb waves*, also known as guided waves.

### 1.2.4 Lamb Waves

Lamb waves are often used to inspect large areas of a thin material. If a thin, plate like material is excited with a stress wave, a Lamb wave begins to propagate along the length of
the plate. The presence of the Lamb wave depends on the existence of a boundary condition that causes multiple reflections of the stress wave, usually between two parallel surfaces. As the wave is reflected off of the boundary surface, multi-mode conversion occurs, as both symmetrical and anti-symmetrical modes begin to form. The overall wave continues to propagate along the length of the plate. Dispersion occurs, as the different frequencies propagate at different wave speeds, the higher frequencies traveling faster. When a lamb wave is received at a distance from the source, the signal is very complex, as the presence of dispersion and many modes causes distortion in the original signal. It becomes very difficult to analyze this interrogation method in the time domain.

There is little accuracy in attempting to observe the time of flight of the entire Lamb wave signal in the time domain. To effectively analyze a signal, a mode and frequency that stands out must be identified, and its respective time of arrival must be determined. It is possible to use differences between the theoretical time of arrival of a selected frequency and its actual time of arrival in order to detect the presence of a flaw or a defect.

There are two ways to graphically depict the dispersion phenomena: the frequency spectrum \( kh \) versus \( \frac{\omega_h}{c} \), and the dispersion curve \( 2f h \) versus \( c_p \). These plots are graphical representations of the the multi-modal, dispersive nature of lamb waves. An example of a frequency spectrum is shown in Fig. (1-1), while an example of a dispersion curve is shown in Fig. (1-2). The set of graphs are presented here simply to allow the reader to gain a little familiarity. The curves will be explained more fully in subsequent sections.

Attenuation is often far too high within a highly anisotropic material like a composite to effectively use Lamb waves to interrogate materials. Often, the resulting signal is composed of small amplitudes, resulting in a difficulty in discerning modes. The Synthetic Phased Tuning approach (SPT) is a novel method that tunes a particular mode of interest in a signal in order to allow it to be easily readable.
Figure 1-1: A generic frequency spectrum for an isotropic material. $\omega = \text{frequency}, \, h = \text{the half thickness of the material}, \, c = \text{velocity}, \, \text{and} \, k = \text{wave number}. \text{The dashed curves are asymmetric modes, the solid lines symmetric modes.}$

### 1.3 Thesis Organization

To allow the reader to make the step from mere theoretical knowledge of the nature of Lamb waves to actual practical application, this thesis progresses through both theoretical and experimental topics, utilizing a progressive approach. This thesis provides a natural progression from a simplified, isotropic case to a less predictable, specially isotropic one.

Chapters 2 and 3 offer a progressive look at the Lamb wave theory in isotropic and specially orthotropic materials. Both chapters evolve the dispersion relations and explain the graphical depiction of the nature of dispersion and Lamb waves.

Chapter 4 describes the experimental specimens utilized to investigate Lamb waves. This chapter also describes simple ultrasonic experiments that are helpful in calculating material properties or simply verifying known properties. The theoretical dispersion curves for both the isotropic and specially orthotropic specimens are depicted in this chapter, for later reference in Lamb wave experimentation.
Figure 1-2: A generic dispersion curve for an isotropic material. $f =$ center frequency of wave, $h =$ the half thickness of the material, $c_p =$ the phase velocity of individual modes. The dashed curves are asymmetric modes, the solid lines symmetric modes.

Chapter 5 outlines a method of creating experimental dispersion curves in the absence of known material properties or for the verification of existing properties. The curve generation is based around a broadband (wide frequency spectrum), laser generated Lamb wave. The wave event is received by both a laser receiver and a laboratory manufactured PVDF transducer.

Chapter 6 details the innovative Synthetic Phase Tuning method, and shows its application to a specially orthotropic composite material.

Chapter 7 concludes the theoretical and the experimental investigation of the complex Lamb wave phenomena.
Chapter 2

Lamb Waves in Isotropic Materials

As discussed in the introduction, the presence of Lamb waves within materials gives non-destructive technicians a very useful tool to interrogate specimens. This chapter briefly re-examines the nature of Lamb waves; describes the wave motion in a homogeneous, isotropic material; and extends the wave motion equations to infinitely long, thin plates. The derivations closely follow the notation preferred by Viktorov (Viktorov 1967). The dispersion relations are developed and dispersion curves for an isotropic material are constructed. The dispersion curve shows the relationship between frequency and the phase velocity of a particular mode. The dispersion phenomenon, where wave speed is dependent on frequency, is easily recognized.

It is important to fully understand the theory behind the most simple case of isotropy before an anisotropic composite material is examined. In a homogeneous, isotropic material such as aluminum, the material properties and the wave behavior are the same for every plane through a point.
2.1 The Nature of Lamb Waves

Lamb waves refer to elastic perturbations moving through a solid plate or layer with free boundaries, for which displacements occur both in the direction of wave propagation and perpendicular to the plane of the plate. Thus, the conditions for the presence of the Lamb wave phenomenon include:

1. Thin, plate like geometries
2. Waveform with a wavelength comparable to the thickness of the medium.

As the wave propagates, it reflects off of the boundaries back into the thin material, due to the acoustic impedance mismatch between the propagating medium and the surrounding medium. The reflected wave splits into two different waves, with shear and a longitudinal components, as shown in Fig. (2-1). This dispersive/multi-modal nature of Lamb waves causes a huge loss in the readability of signals collected during experimentation. To best understand and manage the dispersive and the multi-modal nature of Lamb waves, a graphical representation (a frequency spectrum or a dispersion curve) is needed.

![Figure 2-1: A simple representation of the mode conversion that occurs within a thin material with two parallel edges.](image)
The dispersive nature of Lamb waves occurs because of this continuous interaction with the boundary surfaces. The various frequencies which comprise the waveform travel through the material at different phase velocities, according to the relation:

\[ c_p = \frac{\omega}{k} \]  

(2.1)

For a given plate with characteristic thickness and transducer frequency, there are many propagation modes which may be grouped into two different fundamental families: symmetric modes and antisymmetric modes. The multi-modal behavior is depicted in Fig. (2-2). To best understand and manage the dispersive and the multi-modal nature of Lamb waves, a graphical representation (a frequency spectrum or a dispersion curve) is needed.

The subsequent sections in this chapter will lay out the theory behind the most simple Lamb wave case, propagation through an isotropic material.

Figure 2-2: The two modes, \( A_0, S_0 \), that combine to produce the multi-modal behavior of Lamb waves.
2.2 Wave Motion in Homogeneous, Isotropic Medium

The general wave motion of the propagating wave in an elastic medium must first be modeled before applying boundary conditions. Again, the notation is that favored by Viktorov.

The theory of elasticity for a homogeneous, isotropic elastic solid may be summarized in Cartesian tensor notation as

\[ \sigma_{ij,j} + \rho f_i = \rho \ddot{u}_i \]  

(2.2)

\[ \sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} \]  

(2.3)

\[ \epsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \]  

(2.4)

where \( \sigma_{ij} \) and \( \epsilon_{ij} \) are the stress and strain tensors at a point, \( u_i \) is the displacement vector of a material point, \( \rho \) is the mass density per unit volume of the material, and \( f_i \) is the body force per unit mass. \( \lambda \) and \( \mu \) are the Lamé constants.

The propagating wave is best modeled in terms of displacements. The governing equations in terms of displacements are obtained by substituting the expression for strain into the stress-strain relation, and the result into the stress equations of motion. The result is Navier’s equation for the media:

\[ (\lambda + \mu)u_{j,j,i} + \mu u_{i,j,j} + \rho f_i = \rho \ddot{u}_i \]  

(2.5)

In the absence of body forces, and expressing Eq. (2.5) with the vector equivalent:

\[ (\lambda + \mu)\nabla \nabla \cdot \mathbf{u} + \mu \nabla^2 \mathbf{u} = \rho \ddot{\mathbf{u}} \]  

(2.6)

By utilizing the vector identity \( \nabla^2 \mathbf{u} = \nabla \nabla \cdot \mathbf{u} - \nabla \times \nabla \times \mathbf{u} \), the equation of motion can be alternatively expressed as

\[ (\lambda + 2\mu)\nabla \nabla \cdot \mathbf{u} - \mu \nabla \times \nabla \times \mathbf{u} = \rho \ddot{\mathbf{u}} \]  

(2.7)
The equation of motion may be simplified further. The vector displacement $u$ can be expressed via Helmholtz decomposition as the gradient of a scalar and the curl of the zero divergence vector:

$$u = \nabla \phi + \nabla \times \Psi, \quad \nabla \cdot \Psi = 0 \quad (2.8)$$

where $\phi$ and $\Psi$ are the scalar and vector potentials, respectively. By substituting Eq. (2.8) into Eq. (2.6), Navier’s equation of motion becomes:

$$(\lambda + \mu)\nabla \nabla \cdot (\nabla \phi + \nabla \times \Psi) + \mu \nabla^2(\nabla \phi + \nabla \times \Psi) = \rho \left( \nabla \frac{\partial^2 \phi}{\partial t^2} + \nabla \times \frac{\partial^2 \Psi}{\partial t^2} \right) \quad (2.9)$$

By using the following identities:

$$\nabla \cdot \nabla \phi = \nabla^2 \phi; \quad \nabla \times \nabla \times \nabla \phi = 0; \quad \nabla \cdot \nabla \times \Psi = 0 \quad (2.10)$$

the equation is further refined to:

$$\nabla \left[ (\lambda + 2\mu)\nabla^2 \phi - \rho \frac{\partial^2 \phi}{\partial t^2} \right] + \nabla \times \left[ \mu \nabla^2 \Psi - \rho \frac{\partial^2 \Psi}{\partial t^2} \right] = 0 \quad (2.11)$$

This equation is satisfied if both terms vanish. By setting the bracketed terms equal to zero, we get two equations that must hold true:

$$\nabla^2 \phi = \frac{1}{c_L^2} \frac{\partial^2 \phi}{\partial t^2} \quad (2.12)$$

$$\nabla^2 \Psi = \frac{1}{c_T^2} \frac{\partial^2 \Psi}{\partial t^2} \quad (2.13)$$
$c_L$ and $c_T$ are the longitudinal (dilational) and shear (distortional) wave velocities:

\[ c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}} \]  
(2.14)

\[ c_T = \sqrt{\frac{\mu}{\rho}} \]  
(2.15)

It can be easily seen that the components $\psi_x, \psi_y, \psi_z$ satisfy the equations

\[ \nabla^2 \psi_x = \frac{1}{c_T^2} \frac{\partial^2 \psi_x}{\partial t^2} \]  
\[ \nabla^2 \psi_y = \frac{1}{c_T^2} \frac{\partial^2 \psi_y}{\partial t^2} \]  
\[ \nabla^2 \psi_z = \frac{1}{c_T^2} \frac{\partial^2 \psi_z}{\partial t^2} \]  
(2.16)

Thus, the wave equations can be written in terms of the potentials $(\phi, \psi_x, \psi_y, \psi_z)$ as:

\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{c_L^2} \frac{\partial^2 \phi}{\partial t^2} \]  
(2.17)

\[ \frac{\partial^2 \psi_x}{\partial x^2} + \frac{\partial^2 \psi_x}{\partial y^2} + \frac{\partial^2 \psi_x}{\partial z^2} = \frac{1}{c_T^2} \frac{\partial^2 \psi_x}{\partial t^2} \]  
(2.18)

\[ \frac{\partial^2 \psi_y}{\partial x^2} + \frac{\partial^2 \psi_y}{\partial y^2} + \frac{\partial^2 \psi_y}{\partial z^2} = \frac{1}{c_T^2} \frac{\partial^2 \psi_y}{\partial t^2} \]  
(2.19)

\[ \frac{\partial^2 \psi_z}{\partial x^2} + \frac{\partial^2 \psi_z}{\partial y^2} + \frac{\partial^2 \psi_z}{\partial z^2} = \frac{1}{c_T^2} \frac{\partial^2 \psi_z}{\partial t^2} \]  
(2.20)

Lastly, from Eq. (2.8), the displacement components $u_x, u_y$ and $u_z$ can be related to the potentials $\phi, \psi_x, \psi_y, \psi_z$. The result is the governing equations for the general wave motion of a propagating wave.

\[ u_x = \frac{\partial \phi}{\partial x} + \frac{\partial \psi_z}{\partial z} - \frac{\partial \psi_y}{\partial y} \]  
(2.21)

\[ u_y = \frac{\partial \phi}{\partial y} - \frac{\partial \psi_z}{\partial z} + \frac{\partial \psi_x}{\partial x} \]  
(2.22)

\[ u_z = \frac{\partial \phi}{\partial z} + \frac{\partial \psi_y}{\partial y} - \frac{\partial \psi_x}{\partial x} \]  
(2.23)
2.3 Wave Propagation in an Infinite Plate

As noted earlier, there is a thin, plate like geometrical condition which must be satisfied. Now that the governing equations for the general wave motion are fully developed, boundary conditions must be applied to ensure the presence of Lamb waves.

Consider a plane harmonic wave propagating in a plate of thickness $2h$ in the positive $x$ direction, as shown in Fig. (2-3). At the beginning, ultrasonic excitation occurs at some point in the plate. The excitation exposure is considered to be of infinitesimal duration, so the plate undergoes free vibration. As ultrasonic energy from the excitation region encounters the upper and lower bounding surfaces of the plate, mode conversions occur: longitudinal waves convert into both shear and longitudinal waves, propagating at different angles. After traveling some length within the plate, the superposition of modes causes the formation of wave packets, or Lamb waves in the plate.

![Figure 2-3: Propagation of a plane harmonic Lamb wave in a plate. The thickness of the plate is $2h$.](image)

2.3.1 Development of the Dispersion Relation

Because the plate is undergoing free vibration, the plate is assumed to be traction free. The boundary conditions are:

$$
\sigma_{zz} = \sigma_{xx} = 0
$$

(2.24)
at the top and bottom surfaces \( z = \pm h \). In the case of plane strain in the \((x, z)\) plane, we have

\[
\begin{align*}
\frac{\partial u_y}{\partial y} &= 0, \\
\frac{\partial u_y}{\partial y} &= 0
\end{align*}
\] (2.25)

The Helmholtz decomposition of the displacement vector as shown in Eq. (2.8) reduce to two equations

\[
\begin{align*}
u_x &= \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z} \\
u_z &= \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x}
\end{align*}
\] (2.26) (2.27)

where the subscript \( z \) has been omitted from \( \psi \) above for simplicity of notation. Since the wave motion does not depend on the coordinate \( y \), the vector potential \( \Psi \) has a nonzero magnitude in the direction of the \( y \) axis.

Also, from Hooke’s law the stress components \( \sigma_{xx}, \sigma_{zz} \) and \( \sigma_{xz} \) are expressed in terms of \( \phi \) and \( \psi \) as

\[
\begin{align*}
\sigma_{xx} &= \lambda \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right) + 2\mu \left( \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \psi}{\partial x \partial z} \right) \\
\sigma_{zz} &= \lambda \left( \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \phi}{\partial x^2} \right) + 2\mu \left( \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \psi}{\partial x \partial z} \right) \\
\sigma_{xz} &= \mu \left( 2 \frac{\partial^2 \phi}{\partial x \partial z} + \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2} \right)
\end{align*}
\] (2.28) (2.29) (2.30)

The potentials \( \phi \) and \( \psi \) satisfy the wave equations, refined to a two-dimensional problem due to plane strain:

\[
\begin{align*}
\frac{\partial^2 \phi}{\partial t^2} &= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{c_L^2} \frac{\partial^2 \phi}{\partial t^2} \\
\frac{\partial^2 \psi}{\partial t^2} &= \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{c_T^2} \frac{\partial^2 \psi}{\partial t^2}
\end{align*}
\] (2.31) (2.32)
To find the solutions to Eqs. (2.31) and (2.32), assume $\phi$ and $\psi$ are of the form:

$$\phi = F(z)e^{i(kx-\omega t)}$$  \hspace{1cm} (2.33)

$$\psi = G(z)e^{i(kx-\omega t)}$$  \hspace{1cm} (2.34)

which represent a standing wave in the $y$-direction and a traveling wave in the $x$-direction. Substituting Eqs. (2.33) and Eqs. (2.34) into Eqs. (2.31) and (2.32), respectively, the resulting solutions are:

$$\phi = A_s \cosh(\alpha z) + B_a \sinh(\alpha z)$$  \hspace{1cm} (2.35)

$$\psi = C_a \cosh(\beta z) + D_s \sinh(\beta z)$$  \hspace{1cm} (2.36)

The factor $e^{i(kx-\omega t)}$ is dropped for brevity and $A_s, B_a, C_a, D_s$ are arbitrary constants, $k$ is the Lamb wave number and:

$$\alpha^2 = k^2 - \frac{\omega^2}{c_L^2} = k^2 - k_L^2$$  \hspace{1cm} (2.37)

$$\beta^2 = k^2 - \frac{\omega^2}{c_T^2} = k^2 - k_T^2$$  \hspace{1cm} (2.38)

Here, $k_L$ and $k_T$ are the wave numbers of longitudinal and transverse waves respectively, and

$$k_L = \omega \sqrt{\frac{\rho}{\lambda + 2\mu}}$$  \hspace{1cm} (2.39)

$$k_T = \omega \sqrt{\frac{\rho}{\mu}}$$  \hspace{1cm} (2.40)

where $\omega$ is the angular frequency, $\lambda$ and $\mu$ are the elastic Lamé constants, $\rho$ is the density of the medium.
From Eqs. (2.35) and (2.36), the displacement components and stress components are rewritten as:

\[ u_x = jk \phi - \frac{d \psi}{dz} \quad (2.41) \]
\[ u_z = \frac{d \phi}{dz} + jk \psi \quad (2.42) \]

and

\[ \sigma_{zz} = \lambda \left( -k^2 \phi + \frac{d^2 \phi}{dz^2} \right) + 2\mu \left( \frac{d^2 \phi}{dz^2} + jk \frac{d \psi}{dz} \right) \quad (2.43) \]
\[ \sigma_{xz} = \mu \left( 2jk \frac{d \phi}{dz} - k^2 \psi - \frac{d^2 \psi}{dz^2} \right) \quad (2.44) \]

From Eqs. (2.41) and (2.42) show that for the displacement in the \( x \)-direction the motion is symmetric (antisymmetric) with regard to \( z = 0 \), if \( u_x \) only contains hyperbolic cosines (sines). The displacement in the \( z \)-direction is symmetric (antisymmetric) if \( u_z \) only contains hyperbolic sines (cosines). Thus, the modes of wave propagation in the plate is divided into two systems:

1. Symmetric modes:

\[ \phi = A_s \cosh(\alpha z) \]
\[ \psi = D_s \sinh(\beta z) \]
\[ u_x = jkA_s \cosh(\alpha z) - \beta D_s \cosh(\beta z) \]
\[ u_z = \alpha A_s \sinh(\alpha z) + jkD_s \sinh(\beta z) \]
\[ \sigma_{zz} = \lambda[(-k^2 + \alpha^2)A_s \cosh(\alpha z)] + 2\mu[\alpha^2 A_s \cosh(\alpha z) + jk\beta D_s \cosh(\beta z)] \]
\[ \sigma_{xz} = \mu[2jk\alpha A_s \sinh(\alpha z) - (k^2 + \beta^2)D_s \sinh(\beta z)] \]
2. Antisymmetric modes:

\[
\phi = B_a \sinh(\alpha z) \\
\psi = C_a \cosh(\beta z) \\
u_x = jkB_a \sinh(\alpha z) - \beta C_a \sinh(\beta z) \\
u_z = \alpha B_a \cosh(\alpha z) + jkC_a \cosh(\beta z) \\
s_{zz} = \lambda [(-k^2 + \alpha^2) B_a \sinh(\alpha z)] + 2\mu [\alpha^2 B_a \sinh(\alpha z) + jk\beta C_a \sinh(\beta z)] \\
s_{xz} = \mu [2jk\alpha B_a \cosh(\alpha z) - (k^2 + \beta^2) C_a \cosh(\beta z)]
\]

The expression relating the frequency \(\omega\) to the wave number \(k\), also called the “frequency equation”, is now obtained from the boundary conditions, Eq. (2.24). For the symmetric modes, this yields a system of two homogeneous equations for the constants \(A_s\), \(B_s\), and \(D_s\):

\[
(k^2 + \beta^2) \cosh(\alpha h) A_s + 2jk\beta \cosh(\beta h) D_s = 0 \tag{2.45}
\]
\[
2jk\alpha \sinh(\alpha h) A_s - (k^2 + \beta^2) \sinh(\beta h) D_s = 0 \tag{2.46}
\]

Since the systems are homogeneous, the determinant of the coefficients must vanish, which yields the frequency equation:

\[
\frac{(k^2 + \beta^2) \cosh(\alpha h)}{2jk\beta \cosh(\beta h)} = \frac{2jk\alpha \sinh(\alpha h)}{-(k^2 + \beta^2) \sinh(\beta h)} \tag{2.47}
\]

rewritten as

\[
\frac{\tanh(\beta h)}{\tanh(\alpha h)} = \frac{4k^2\alpha\beta}{(k^2 + \beta^2)^2} \tag{2.48}
\]

This is the Rayleigh-Lamb frequency equation for the propagation of symmetric modes in a plate.
Similarly, for the antisymmetric modes, the boundary conditions yield a system of two homogeneous equations for the constants $B_a$ and $C_a$,

\[
\begin{align*}
(k^2 + \beta^2) \sinh(\alpha h)B_a + 2jk\beta \sinh(\beta h)C_a &= 0 \quad (2.49) \\
2jk\alpha \cosh(\alpha h)B_a - (k^2 + \beta^2) \cosh(\beta h)C_a &= 0 \quad (2.50)
\end{align*}
\]

The determinant of the coefficients must vanish, which yields the frequency equation:

\[
\frac{(k^2 + \beta^2) \sinh(\alpha h)}{2jk\beta \sinh(\beta h)} = \frac{2jk\alpha \cosh(\alpha h)}{- (k^2 + \beta^2) \cosh(\beta h)}
\]

rewritten as:

\[
\frac{\tanh(\beta h)}{\tanh(\alpha h)} = \frac{(k^2 + \beta^2)^2}{4k^2\alpha\beta}
\]

This is the Rayleigh-Lamb frequency equation for the propagation of antisymmetric modes in a plate.

### 2.3.2 Analysis of the Dispersion Relation

For the Rayleigh-Lamb case, both longitudinal (L) and shear (T) waves exist for any given mode because of the mode conversion at the traction-free surfaces. By looking at the frequency equations, it is evident that simple analytical solutions do not exist. Manipulation of the dispersion equations leads to the ability to graphically depict the dispersion. The determination of the cut-off frequencies yields a starting point for the implementation of an iterative method such as the Newton-Raphson method.

#### Symmetric / Antisymmetric Modes

The displacement for symmetric and antisymmetric modes is expressed above in terms of the four constants: $A_s$, $D_s$, $B_a$ and $C_a$. They may be rewritten:
Chapter 2  Lamb Waves in Isotropic Materials

1. Symmetric modes:

From Eq. (2.46), the amplitude ratio becomes:

\[
\frac{D_s}{A_s} = \frac{2 jk \alpha \sinh(\alpha h)}{(k^2 + \beta^2) \sinh(\beta h)}
\]  \hspace{1cm} (2.53)

and the displacement can be expressed as

\[
u_x = jk A \left( \frac{\cosh(\alpha z)}{\sinh(\alpha h)} - \frac{2 \alpha \beta}{k^2 + \beta^2} \cdot \frac{\sinh(\beta z)}{\sinh(\beta h)} \right) e^{j(kx-\omega t)}
\]

\[
u_z = A \alpha \left( \frac{\sinh(\alpha z)}{\sinh(\alpha h)} - \frac{2 k^2}{k^2 + \beta^2} \cdot \frac{\sinh(\beta z)}{\sinh(\beta h)} \right) e^{j(kx-\omega t)}
\]

where \(A\) is a new constant.

2. Antisymmetric modes:

Also, from Eq. (2.50) the amplitude ratio becomes:

\[
\frac{C_a}{B_a} = \frac{2 jk \alpha \cosh(\alpha h)}{(k^2 + \beta^2) \cosh(\beta h)}
\]  \hspace{1cm} (2.56)

and the displacement can be expressed as:

\[
u_x = jk B \left( \frac{\sinh(\alpha z)}{\cosh(\alpha h)} - \frac{2 \alpha \beta}{k^2 + \beta^2} \cdot \frac{\sinh(\beta z)}{\cosh(\beta h)} \right) e^{j(kx-\omega t)}
\]

\[
u_z = B \alpha \left( \frac{\cosh(\alpha z)}{\cosh(\alpha h)} - \frac{2 k^2}{k^2 + \beta^2} \cdot \frac{\cosh(\beta z)}{\cosh(\beta h)} \right) e^{j(kx-\omega t)}
\]

where \(B\) is a new constant.
To analyze the various regions of the Rayleigh-Lamb equation, recall the definition of \( \alpha \) and \( \beta \):

\[
\alpha^2 = k^2 - \frac{\omega^2}{c_L^2} = \frac{\omega^2}{c^2} - \frac{\omega^2}{c_L^2} \quad (2.59)
\]

\[
\beta^2 = k^2 - \frac{\omega^2}{c_T^2} = \frac{\omega^2}{c^2} - \frac{\omega^2}{c_T^2} \quad (2.60)
\]

\( \alpha \) and \( \beta \) could be real or imaginary, depending on the value of the phase velocity \( c \) relative to \( c_L \) and \( c_T \). Three cases exist:

1. **Case 1:** Real \( \alpha \) and Real \( \beta \) \((0 < c < c_T < c_L)\)

2. **Case 2:** Real \( \alpha \) and Imaginary \( \beta \) \((0 < c_T < c < c_L)\)

3. **Case 3:** Imaginary \( \alpha \) and Imaginary \( \beta \) \((0 < c_T < c_L < c)\)

The first case is explained in this chapter. A complete discussion of all of the possible cases is offered in Appendix A.

**Case 1: Real \( \alpha \) and Real \( \beta \)**

\[
\alpha = \sqrt{k^2 - \frac{\omega^2}{c_L^2}} \rightarrow \text{real} \quad (2.61)
\]

\[
\beta = \sqrt{k^2 - \frac{\omega^2}{c_T^2}} \rightarrow \text{real} \quad (2.62)
\]

Thus, the frequency equation is of the same form as before:

\[
\begin{align*}
\tanh(\beta h) &= \frac{4k^2 \alpha \beta}{(k^2 + \beta^2)^2} \quad \text{(symmetric)} \\
\tanh(\alpha h) &= \frac{4k^2 \alpha \beta}{(k^2 + \beta^2)^2} \quad \text{(antisymmetric)}
\end{align*}
\quad (2.63)
\]
Cutoff Frequencies

The cutoff frequencies for the various plate modes will be obtained by considering the wavenumber $k \rightarrow 0$. For this limiting value, the Rayleigh-Lamb equation reduce to:

$$\cos(\alpha_c h) \sin(\beta_c h) = 0 \quad \text{symmetric} \quad (2.65)$$
$$\sin(\alpha_c h) \cos(\beta_c h) = 0 \quad \text{antisymmetric} \quad (2.66)$$

1. Symmetric modes:

$$\cos(\alpha_c h) = 0, \quad \alpha_c h = \frac{p\pi}{2} \quad (p = 1, 3, 5, \ldots) \quad (2.67)$$
$$\sin(\beta_c h) = 0, \quad \beta_c h = \frac{q\pi}{2} \quad (q = 0, 2, 4, \ldots) \quad (2.68)$$

Because $\alpha = \omega_c / c_L$ and $\beta = \omega_c / c_T$, we have the condition for the cutoff frequencies:

$$\frac{\omega_c h}{c_T} = \begin{cases} \frac{np}{2} \frac{c_L}{c_T} & (p = 1, 3, 5, \ldots) \\ \frac{q}{2} & (q = 0, 2, 4, \ldots) \end{cases} \quad (2.69)$$

or, in terms of the frequency and thickness product:

$$2f_c h = \begin{cases} \frac{p c_L}{2} & (p = 1, 3, 5, \ldots) \\ \frac{q c_T}{2} & (q = 0, 2, 4, \ldots) \end{cases} \quad (2.70)$$

2. Antisymmetric modes:

$$\sin(\alpha_c h) = 0, \quad \alpha_c h = \frac{p\pi}{2} \quad (p = 0, 2, 4, \ldots) \quad (2.71)$$
$$\cos(\beta_c h) = 0, \quad \beta_c h = \frac{q\pi}{2} \quad (q = 1, 3, 5, \ldots) \quad (2.72)$$
As in the case of symmetric modes, because $\bar{\alpha} = \omega_c/c_L$ and $\bar{\beta} = \omega_c/c_T$ we have the condition for the cutoff frequencies:

$$\frac{\omega_c h}{c_T} = \begin{cases} \frac{\pi p c_L}{2 c_T} & (p = 0, 2, 4, \ldots) \\ \frac{\pi q}{2} & (q = 1, 3, 5, \ldots) \end{cases}$$  \hspace{1cm} (2.73)

or, in terms of the frequency and thickness product,

$$2f_c h = \begin{cases} \frac{pc_L}{2c_T} & (p = 0, 2, 4, \ldots) \\ \frac{qc_T}{2} & (q = 1, 3, 5, \ldots) \end{cases}$$  \hspace{1cm} (2.74)

### 2.3.3 Analysis of the Dispersion Curves

By knowing the appropriate dispersion relation based upon a comparison of the wave speed and phase velocities (Cases 1 to 3), the graphical depiction of dispersion may be generated. The cutoff frequencies are first calculated, and then a simple iterative process is applied in a computer algorithm that satisfies the dispersion relation through a wide range of frequencies. Fig. (2-4) lays out the algorithm in the form of a flow chart. The dispersion curve for an isotropic aluminum plate is shown in Fig. (2-5).
Figure 2-4: Flowchart explaining the iterative computer algorithm.
Figure 2-5: Dispersion curve (phase velocity) for an aluminum plate
Chapter 3

Lamb Waves in Transversely Isotropic Composite Materials

The general nature of Lamb waves, as discussed in previous sections, does not change with the complexity of the material. Anisotropic materials, however, will cause attenuation of a waveform’s amplitude and will lead to a significantly degraded signal. Unmodified, the result is a decrease in usefulness of the Lamb wave inspection method.

This chapter initially refers to the description of the simplifying steps utilized in stating the stress-strain relation for a composite material, located in Appendix C. The chapter then describes the wave motion in a non-homogeneous, transversely isotropic material in the longitudinal direction and further extends the wave motion equations to infinitely long, thin composite plates. It then lays out the development of the dispersion relation and subsequently plots the dispersion curves for a transversely isotropic material. A good reference for the following derivations is located in a paper titled “Ultrasonic Plate Waves in Paper” (Habeger, Mann, and Baum 1979).

The theory is more complex than the case of isotropic materials discussed previously. Three important assumptions are made early in the derivation: the material is specially orthotropic, is transversely isotropic, and undergoes plane strain.
3.1 Wave Motion in Non-Homogeneous, Transversely Isotropic Medium

This section utilizes the stress-strain relation to derive dispersion equations for both symmetric and antisymmetric modes. The equations are a function of wave number, stiffness, and thickness. The general wave motion of the propagating wave must first be modeled before applying boundary conditions.

3.1.1 Derivation of Dispersion Equation

The starting point for the derivation of the dispersion equation is the stress-strain relationship for a transversely isotropic material, developed in Appendix C. After making two simplifying assumptions: special orthotropy and transverse isotropy, only five stiffness constants are needed to fully characterize a unidirectional composite. The stress strain relationship becomes:

\[
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12}
\end{pmatrix} =
\begin{pmatrix}
C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{(C_{22} - C_{23})}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{55}
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\varepsilon_{23} \\
\varepsilon_{13} \\
\varepsilon_{12}
\end{pmatrix}
\]

(3.1)

The strain-displacement relation can be written as:

\[
\varepsilon_{ij} = \frac{u_{i,j} + u_{j,i}}{2}, \quad \text{for} \quad i, j = 1, 2, 3
\]

(3.2)
The strain vector on the right side of Eq. (3.1) can be rewritten using the strain-displacement relation:

\[
\begin{pmatrix}
    u_{1,1} \\
    u_{2,2} \\
    u_{3,3} \\
    \frac{(u_{2,3} + u_{3,2})}{2} \\
    \frac{(u_{1,3} + u_{3,1})}{2} \\
    \frac{(u_{1,2} + u_{2,1})}{2}
\end{pmatrix}
\]  

(3.3)

Inertial forces (density $\times$ acceleration) in one principle direction yield stresses in all three. The equation of motion in an elastic medium is

\[
\sum_{j=1}^{3} \sigma_{ij,j} = \rho \ddot{u}_i, \quad i = 1, 2, 3. 
\]  

(3.4)

By considering an infinite plate ($y = \text{“}2\text{”} \rightarrow \infty$), a plane strain condition is assumed. The displacement $u_2$ and all associated derivatives with respect to $y$ vanish. Substituting Eqs. (3.1) and (3.3) into Eq. (3.4) yields:

\[
\rho \ddot{u}_1 = \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3}
\]  

(3.5)

and, with $C = \text{constant}$, complete the first derivatives of the stress terms:

\[
\rho \ddot{u}_1 = (C_{11} u_{1,1})_1 + (C_{12} u_{2,2})_1 + (C_{13} u_{3,3})_1 + (2C_{66} \frac{(u_{1,2} + u_{2,1})}{2})_2 + (2C_{55} \frac{(u_{1,3} + u_{3,1})}{2})_3
\]  

(3.6)

The plane strain assumption is applied, the “2” terms fall out of the equation.

\[
\rho \ddot{u}_1 = C_{11} u_{1,11} + C_{13} u_{3,31} + C_{55} (u_{1,33} + u_{3,13})
\]  

(3.7)
similarly,

$$\rho \ddot{u}_3 = C_{33} u_{3,33} + C_{13} u_{1,13} + C_{55}(u_{1,13} + u_{3,11})$$  \hspace{1cm} (3.8)

For a plane wave with displacements in the \(x\) and \(z\) directions only, the displacement components can be written as

$$u_1 = U_{10} e^{j k_z z} e^{j (k_x x - \omega t)}$$  \hspace{1cm} (3.9)

The first two terms represent the amplitude while the final term represents the propagation in the \(x\)-direction.

Similarly,

$$u_3 = U_{30} e^{j k_z z} e^{j (k_x x - \omega t)}$$  \hspace{1cm} (3.10)

Substituting Eqs. (3.9) and (3.10) into Eqs. (3.7) and (3.8), and canceling out the negative of the exponential term \((-e^{j k_z z} e^{j (k_x x - \omega t)})\), one obtains

$$\rho U_{10} \omega^2 = C_{11} U_{10} k_x^2 + (C_{13} + C_{55}) U_{30} k_x k_z + C_{55} U_{10} k_z^2$$  \hspace{1cm} (3.11)

and:

$$\rho U_{30} \omega^2 = C_{55} U_{30} k_z^2 + (C_{13} + C_{55}) U_{10} k_x k_z + C_{33} U_{30} k_x^2$$  \hspace{1cm} (3.12)

For a given frequency \(\omega\) and wavenumber \(k_x\), Eqs. (3.11) and (3.12) can be used to find the wavenumber \(k_z\) and the corresponding value of wave amplitude ratio, \(R = \frac{U_{30}}{U_{10}}\). Given a value of \(k_z\), \(R\) can be obtained as:

$$R = \frac{U_{30}}{U_{10}} = \frac{(\rho \omega^2 - C_{11} k_x^2 - C_{55} k_z^2)}{(C_{55} + C_{13}) k_x k_z}$$  \hspace{1cm} (3.13)
Solving for $U_{30}$, and substituting into Eq. (3.11), the following quadratic equation results:

\[(C_{55} + C_{13}) k_z^2 = (\rho \omega^2 - C_{11} k_x^2 - C_{55} k_z^2)(\rho \omega^2 - C_{55} k_x^2 - C_{33} k_z^2)\]  

(3.14)

$k_z^2$ must satisfy this quadratic equation.

$k_z^2$ can be represented in terms of $k_x$:

\[k_z^2 = \frac{k_x^2 [B \pm \sqrt{B^2 - 4D}]}{2}\]  

(3.15)

where

\[B = \frac{-\rho [C_{33}(C_{11}/\rho - \omega^2/k_x^2) - C_{13}(2C_{55} + C_{13})/\rho - C_{55}\omega^2/k_x^2]}{C_{33}C_{55}}\]  

(3.16)

and

\[D = \frac{\rho^2(\omega^2/k_x^2 - C_{55}/\rho)(\omega^2/k_x^2 - C_{11}/\rho)}{C_{33}C_{55}}\]  

(3.17)

The solution of Eq. (3.15) may result in two very different values, depending upon the sign of the bracketed term. Define the solutions of $k_z$ as $k_{zp} = \pm (jk_{zp})$ and $k_z = \pm (jk_{zm})$. Here, $k_{zp}^2$ and $k_{zm}^2$ are defined as the two opposite values of $k_z^2$ obtained from Eq. (3.15) with + and - signs in the bracket:

\[k_{zp}^2 = \frac{k_x^2 [B + \sqrt{B^2 - 4D}]}{2}\]  

(3.18)

\[k_{zm}^2 = \frac{k_x^2 [B - \sqrt{B^2 - 4D}]}{2}\]  

(3.19)
Also, \( R_p \) and \( R_m \) are the values of \( R \), respectively, when \( k_{zp} \) and \( k_{zm} \) are substituted into Eq. (3.13), excluding \(-j\).

\[
R_p = \frac{(\rho \omega^2 - C_{11} k_{x}^2 + C_{55} k_{zp}^2)}{(C_{55} + C_{13}) k_{x} k_{zp}} \\
R_m = \frac{(\rho \omega^2 - C_{11} k_{x}^2 + C_{55} k_{zm}^2)}{(C_{55} + C_{13}) k_{x} k_{zm}}
\]

### 3.2 Wave Propagation in an Infinite Plate

Now that the equations for the general wave motion are fully developed, boundary conditions must be applied to ensure the presence of Lamb waves.

#### 3.2.1 Development of the Dispersion Relation

The equations in the previous section represent the bulk waves traveling in an unbounded medium or half space. The plate wave solution is obtained if these bulk waves add up such that the free boundary conditions are met at \( z = \pm h \). The two possible plate wave displacements have the following forms:

\[
u_1 = e^{j(k_x x - \omega t)}[Me^{-k_{zp}z} + Ne^{k_{zp}z} + Pe^{-k_{zm}z} + Qe^{k_{zm}z}] \\
\]

\[
u_3 = e^{j(k_x x - \omega t)}(-jR_p[Me^{-k_{zp}z} - Ne^{k_{zp}z}] - jR_m[Pe^{-k_{zm}z} - Qe^{k_{zm}z}])
\]

where \( M, N, P, Q \) are arbitrary constants. The terms preceded by these constants (ie: \( Me^{-k_{zp}z} \)) are modulation terms.

\[
e^{jx} = \cos(x) + j \sin(x)
\]
Thus, the modulation terms do not contribute significantly to the amplitude’s magnitude, only the phase.

The free boundary conditions to be satisfied at $z = \pm h$ gives

$$\sigma_{33} = C_{33}u_{3,3} + C_{13}u_{1,1} = 0 \quad (3.26)$$

where:

$$\sigma_{31} = C_{55}u_{1,3} + C_{55}u_{3,1} = 0 \quad (3.27)$$

Substituting $u_1$ and $u_3$ from Eqs. (3.22) and (3.23) into Eqs. (3.26) and (3.27) imposes the four following conditions on $M$, $N$, $P$ and $Q$:

$$\begin{pmatrix}
G_p e^{-k_{zp}h} & G_p e^{k_{zp}h} & G_m e^{-k_{zm}h} & G_m e^{k_{zm}h} \\
G_p e^{k_{zp}h} & G_p e^{-k_{zp}h} & G_m e^{k_{zm}h} & G_m e^{-k_{zm}h} \\
-H_p e^{-k_{zp}h} & H_p e^{k_{zp}h} & -H_m e^{-k_{zm}h} & H_m e^{k_{zm}h} \\
-H_p e^{k_{zp}h} & H_p e^{-k_{zp}h} & -H_m e^{k_{zm}h} & H_m e^{-k_{zm}h}
\end{pmatrix}
\begin{pmatrix}
M \\
N \\
P \\
Q
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix} \quad (3.28)$$

where

$$G_{p,m} = C_{33}k_{zp,m}R_{p,m} + C_{13}k_x \quad (3.29)$$

$$H_{p,m} = k_{zp,m} - k_x R_{p,m} \quad (3.30)$$

There are non-zero solutions for $M$, $N$, $P$ and $Q$ only if the determinant of the matrix in Eq. (3.28) is equal to zero, which yields

$$G_p H_m \cosh(k_{zp}h) \sinh(k_{zm}h) - G_m H_p \sinh(k_{zp}h) \cosh(k_{zm}h) = 0 \quad (3.31)$$
Lamb Waves in Transversely Isotropic Composite Materials

\[ G_p H_m \sinh(k_{zp} h) \cosh(k_{zm} h) - G_m H_p \cosh(k_{zp} h) \sinh(k_{zm} h) = 0 \] (3.32)

which further results in:

\[ \frac{\tanh(k_{zp} h)}{\tanh(k_{zm} h)} = \frac{G_p H_m}{G_m H_p} \] (3.33)

for symmetric modes and

\[ \frac{\tanh(k_{zp} h)}{\tanh(k_{zm} h)} = \frac{G_m H_p}{G_p H_m} \] (3.34)

for antisymmetric modes. These are the dispersion equations for orthotropic composite plates with transverse isotropy.

3.2.2 Analysis of the Dispersion Relation

For the Rayleigh-Lamb case, both longitudinal (L) and shear (T) waves exist for any given mode because of the mode conversion at the traction-free surfaces. By looking at the frequency equations, it is evident that simple analytical solutions do not exist. Manipulation of the dispersion equations leads to the ability to graphically depict the dispersion. The determination of the cut-off frequencies yields a starting point for the implementation of an iterative method such as the Newton-Raphson method.

Symmetric / Antisymmetric Modes

As in the derivation for isotropic materials, both the symmetric and antisymmetric modes may have a combination of imaginary and real values for \( k_{zp} \) and \( k_{zm} \) in the solution. Under some conditions, \( k_{zp} \) and \( k_{zm} \) could be either real or imaginary, according to Eq. (3.15). Therefore, there exist three different cases for the function \( f(k_x, \omega) \).

1. Case 1: Real \( k_{zp} \) and \( k_{zm} \)
2. Case 2: Real $k_{zp}$, imaginary $k_{zm}$

3. Case 3: Imaginary $k_{zp}$ and $k_{zm}$

Only the first case is explained for Symmetric modes in this chapter. A complete discussion of all of the possible cases, for both modes, is offered in Appendix B.

**Symmetric modes:**

The dispersion equation for symmetric modes is:

$$\frac{\tanh(k_{zp}h)}{\tanh(k_{zm}h)} = \frac{G_pH_m}{G_mH_p}$$  \hspace{1cm} (3.35)

The general function in terms of $k_z$ and $\omega$ for symmetric modes is

$$f(k_z, \omega) = G_pH_m \cosh(k_{zp}h) \sinh(k_{zm}h) - G_mH_p \sinh(k_{zp}h) \cosh(k_{zm}h)$$  \hspace{1cm} (3.36)

**Case 1: Real $k_{zp}$ and $k_{zm}$**

In this case, from Eq. (3.15) we have:

$$-B + \sqrt{B^2 - 4D} \geq 0$$  \hspace{1cm} (3.37)

$$-B - \sqrt{B^2 - 4D} \geq 0 .$$  \hspace{1cm} (3.38)

The dispersion equation is correspondingly:

$$\frac{\tanh(k_{zp}h)}{\tanh(k_{zm}h)} = \frac{G_pH_m}{G_mH_p}$$  \hspace{1cm} (3.39)

The function $f(k_z, \omega)$ is expressed as:

$$f(k_z, \omega) = G_pH_m \cosh(k_{zp}h) \sinh(k_{zm}h) - G_mH_p \sinh(k_{zp}h) \cosh(k_{zm}h)$$  \hspace{1cm} (3.40)
where $k_{zp}$ and $k_{zm}$ are:

\[
\begin{align*}
k_{zp}^2 &= \frac{k_x^2[-B + \sqrt{B^2 - 4D}]}{2} \\
k_{zm}^2 &= \frac{k_x^2[-B - \sqrt{B^2 - 4D}]}{2}
\end{align*}
\]

and $H_p$ and $H_m$ are also real:

\[
\begin{align*}
H_p &= \frac{(-\rho \omega^2 + C_{11}k_x^2 + C_{13}k_{zp}^2)}{(C_{55} + C_{13})k_{zp}} \\
H_m &= \frac{(-\rho \omega^2 + C_{11}k_x^2 + C_{13}k_{zm}^2)}{(C_{55} + C_{13})k_{zm}}
\end{align*}
\]

and $G_p$ and $G_m$ are also real:

\[
\begin{align*}
G_p &= \frac{C_{33}\rho \omega^2 + (C_{13}C_{55} + C_{13}^2 - C_{33}C_{11})k_x^2 - C_{33}C_{55}k_{zp}^2}{(C_{55} + C_{13})k_x} \\
G_m &= \frac{C_{33}\rho \omega^2 + (C_{13}C_{55} + C_{13}^2 - C_{33}C_{11})k_x^2 - C_{33}C_{55}k_{zm}^2}{(C_{55} + C_{13})k_x}
\end{align*}
\]

**Cutoff Frequencies**

Cut-off frequencies are utilized in the graphical representation of a dispersive system.

The cut-off frequencies refer to the frequencies ($\omega_c$) at which the phase velocity becomes infinitely large ($c_p \to \infty$) or the wavenumber approaches zero ($k_x \to 0$). Cut-off frequencies are determined using the dispersion equations, Eq. (3.33) and (3.34), when $k_x = 0$. Notice that we consider this issue in region III where $k_{zp}$ and $k_{zm}$ are both imaginary. For this, the parameters $\tilde{k}_{zp,m}, G_{p,m}$, and $H_{p,m}$ must be calculated.
From Eqs. (3.11) and (3.12), it follows that when \( k_x = 0 \), the amplitude terms cancel out, and:

\[
\rho \omega^2 = C_{33} k_{zp}^2
\]

(3.48)

and

\[
\rho \omega^2 = C_{55} k_{zm}^2
\]

(3.49)

where we assume that \( C_{55} \geq C_{33} \).

In this case (\( k_x = 0 \)), Eq. (3.30) may be multiplied by \( k_x \) to ensure the denominators are non-zero, and \( G_p \) and \( G_m \) can be obtained as

\[
k_x G_p = C_{33} k_{zp} R_p k_x + C_{13} k_x^2
\]

\[
= \frac{C_{33}(\rho \omega^2 - C_{55} k_{zp}^2)}{(C_{55} + C_{13})}
\]

\[
= \frac{C_{33}(C_{33} k_{zp}^2 - C_{55} k_{zp}^2)}{(C_{55} + C_{13})}
\]

(3.50)

(3.51)

and

\[
k_x G_m = C_{33} k_{zm} R_m k_x + C_{13} k_x^2
\]

\[
= \frac{C_{33}(\rho \omega^2 - C_{55} k_{zm}^2)}{(C_{55} + C_{13})}
\]

\[
= \frac{C_{33}(C_{55} k_{zm}^2 - C_{55} k_{zm}^2)}{(C_{55} + C_{13})}
\]

= 0

\[\text{In our analysis, we assume that both } k_{zp} \text{ and } k_{zm} \text{ are real. From Eq. (3.15) we can see that } k_{zp} \geq k_{zm}.\]
Similarly, $\tilde{H}_p$ and $\tilde{H}_m$ can be obtained as

$$
\tilde{H}_p = \frac{(\rho \omega^2 + C_{13}\bar{k}_{zp}^2)}{(C_{55} + C_{13})\bar{k}_{zp}}
\quad = \frac{(C_{33}\bar{k}_{zp}^2 + C_{13}\bar{k}_{zp}^2)}{(C_{55} + C_{13})\bar{k}_{zp}}
\quad = \frac{\bar{k}_{zp}(C_{33} + C_{13})}{(C_{55} + C_{13})}
\quad (3.52)
$$

and

$$
\tilde{H}_m = \frac{(\rho \omega^2 + C_{13}\bar{k}_{zm}^2)}{(C_{55} + C_{13})\bar{k}_{zm}}
\quad = \frac{(C_{55}\bar{k}_{zm}^2 + C_{13}\bar{k}_{zm}^2)}{(C_{55} + C_{13})\bar{k}_{zm}}
\quad = \bar{k}_{zm}
\quad (3.53)
$$

1. Symmetric modes:

The dispersion equation for symmetric modes can be rewritten as

$$
\frac{\sin(\bar{k}_{zp}h) \cos(\bar{k}_{zm}h)}{\cos(\bar{k}_{zm}h) \sin(\bar{k}_{zp}h)} = \frac{k_x G_p \bar{H}_m}{k_x G_m \bar{H}_p}
\quad (3.54)
$$

The function in terms of $k_x$ and $\omega$ can be thus obtained as

$$
f(k_x, \omega) = k_x G_p \bar{H}_m \cos(\bar{k}_{zp}h) \sin(\bar{k}_{zm}h) - k_x G_m \bar{H}_p \sin(\bar{k}_{zp}h) \cos(\bar{k}_{zm}h)
\quad (3.55)
$$

Notice that $k_x G_m = 0$ for $k_x = 0$, we have

$$
f(0, \omega_c) = -k_x G_p \bar{H}_m \cos(\bar{k}_{zp}h) \sin(\bar{k}_{zm}h)
\quad (3.56)
$$
Therefore $f(0, \omega_c) = 0$ gives the condition for cut-off frequencies $\omega_c$ (assume $C_{33} \neq C_{55}$)

$$\cos(\bar{k}_{zp} h) \sin(\bar{k}_{zm} h) = 0 \quad (3.57)$$

This can be satisfied if

$$\cos(\bar{k}_{zp} h) = 0, \quad \bar{k}_{zp} h = \frac{p\pi}{2} \quad (p = 1, 3, 5, \cdots) \quad (3.58)$$

or

$$\sin(\bar{k}_{zm} h) = 0, \quad \bar{k}_{zm} h = \frac{q\pi}{2} \quad (q = 0, 2, 4, \cdots) \quad (3.59)$$

Substituting Eqs. (B.71) and (B.72) into Eqs. (B.81) and (B.82), the cut-off frequencies for symmetric modes become:

$$\omega_c h = \frac{p\pi}{2} \sqrt{\frac{C_{33}}{\rho}}, \quad (p = 1, 3, 5, \cdots) \quad (3.60)$$

$$\omega_c h = \frac{q\pi}{2} \sqrt{\frac{C_{55}}{\rho}}, \quad (q = 0, 2, 4, \cdots) \quad (3.61)$$

2. Antisymmetric modes:

The dispersion equation for antisymmetric modes can be rewritten as

$$\frac{\sin(\bar{k}_{zp} h) \cos(\bar{k}_{zm} h)}{\cos(\bar{k}_{zm} h) \sin(\bar{k}_{zp} h)} = \frac{k_x G_m \tilde{H}_p}{k_x G_p \tilde{H}_m} \quad (3.62)$$
The function in terms of \( k_x \) and \( \omega \) can be obtained as

\[
f(k_x, \omega) = k_x G_p \bar{H}_m \sin(\bar{k}_{zp} h) \cos(\bar{k}_{zm} h) - k_x G_m \bar{H}_p \cos(\bar{k}_{zp} h) \sin(\bar{k}_{zm} h) \quad (3.63)
\]

Notice that \( k_x G_m = 0 \) for \( k_x = 0 \), we have

\[
f(0, \omega_c) = k_x G_p \bar{H}_m \sin(\bar{k}_{zp} h) \cos(\bar{k}_{zm} h) \quad (3.64)
\]

Therefore \( f(0, \omega_c) = 0 \) gives the condition for cut-off frequencies \( \omega_c \) (assume \( C_{33} \neq C_{55} \))

\[
\sin(\bar{k}_{zp} h) \cos(\bar{k}_{zm} h) = 0 
\]

(3.65)

This can be satisfied if

\[
\sin(\bar{k}_{zp} h) = 0, \quad \bar{k}_{zp} h = \frac{p\pi}{2} \quad (p = 0, 2, 4, \ldots) 
\]

(3.66)

or

\[
\cos(\bar{k}_{zm} h) = 0, \quad \bar{k}_{zm} h = \frac{q\pi}{2} \quad (q = 1, 3, 5, \ldots) 
\]

(3.67)

Substituting Eqs. (B.71) and (B.72) into Eqs. (B.89) and (B.90), the cut-off frequencies for antisymmetric modes become:

\[
\omega_{ch} = \frac{p\pi}{2} \sqrt{\frac{C_{33}}{\rho}}, \quad (p = 0, 2, 4, \ldots) 
\]

(3.68)

\[
\omega_{ch} = \frac{q\pi}{2} \sqrt{\frac{C_{55}}{\rho}}, \quad (q = 1, 3, 5, \ldots) 
\]

(3.69)
3.2.3 Analysis of the Dispersion Curve

The dispersion curves may be created from a simple computer algorithm that satisfies the dispersion relation through a wide range of frequencies. See Fig. (2-4) for a summary of the algorithm. The dispersion curve (frequency versus phase velocity) for a transversely isotropic composite plate is shown in Fig. (3-1).

![Dispersion curve (phase velocity) for a composite plate.](image)

Figure 3-1: Dispersion curve (phase velocity) for a composite plate.
This chapter provides the theoretical dispersion curves for actual experimental specimens: an isotropic aluminum plate, and a transversely isotropic (unidirectional) composite plate. The curves are developed based upon the specimen’s material properties. The properties are initially taken from tables and manufacturer specifications, and later partially verified in simple ultrasound experiments. The derived stiffness constants are entered into an iterative computer algorithm, explained previously, which applies the applicable dispersion relationship and generates the corresponding graphical representation.

4.1 Dispersion Curves for an Isotropic Aluminum Plate

4.1.1 Aluminum Specimen Physical Characteristics

The isotropic specimen utilized for experimentation is a thin aluminum plate. The plate has length and width dimensions much greater than its thickness. The dimensions are 130.76 cm \( \times \) 160.93 cm, with a thickness of .317 cm, as measured by a micrometer. The density of the specimen, \( \rho = 2.7 \text{g/cm}^3 \). Because the aluminum is isotropic, it is characterized by an infinite number of planes of material symmetry through a point. Only two properties are...
needed in the application of the dispersion relation: the longitudinal wave speed, $c_L$, and the transverse wave speed, $c_T$. They are both listed in Table (4.1):

<table>
<thead>
<tr>
<th>Property</th>
<th>Value (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_L$</td>
<td>6,320</td>
</tr>
<tr>
<td>$c_T$</td>
<td>3,130</td>
</tr>
</tbody>
</table>

Table 4.1: Values for the wave speeds in aluminum.

To verify the above wave speeds, two simple experiments are performed. To check the shear wave speed, a Panametrics Shear Wave Transducer (V155) with a frequency of 5.0 MHz is set in pulse-echo mode to receive signals originating from a Panametric signal generator. A shear wave event is generated in the specimen, reflected off of an edge, and received by the same transducer. Because of the isotropy of the material, rotation of the transducer does not vary the received waveform. The received waveform has a repeated signal, from which an accurate time of flight for the shear wave may be obtained.

As extracted from the waveform, $\Delta t = 1.99 \mu s$. The shear wave speed is calculated as:

$$c_T = \frac{2t}{\Delta t}$$

(4.1)

where,

$c_T = \text{shear wave speed}$

$t = \text{thickness of the specimen}$

$\Delta t = \text{time of flight for the shear wave (through twice the thickness)}$

$$c_T = \frac{23.17\text{mm}}{1.99\mu s}$$

(4.2)

$$c_T = 3,186 \frac{\text{m}}{\text{s}}$$
The order of magnitude of the accepted shear wave speed for Aluminum is similar to the experimentally determined wave speed. Due to variances in the manufacture of aluminum, the experimental wave speed is the value to be incorporated in the creation of the dispersion curves.

A second ultrasound experiment allows the verification of the longitudinal wave speed within the specimen. A Matec Ultrasonic Inspection System, manufactured by Matec Instruments, is used to verify the value of \( c_L \). The immersion ultrasound tank is filled with water at room temperature. A 5.0 MHz flat transducer is used in a vertical fashion, in a “pulse echo” mode. The flat transducer is unfocused, resulting in the presence of both front-face and back-face reflections when the aluminum specimen is submerged in the tank, perpendicular to the flat face of the transducer.

The difference between two distinct back-face reflections of the specimen is \( \Delta t = 1.0 \) \( \mu s \). The corresponding longitudinal wave speed for the aluminum specimen (\( c_L \)) is equal to twice the thickness of the specimen divided by \( \Delta t \).

The longitudinal wave speed is calculated as:

\[
\frac{c_L}{\Delta t} = \frac{2l}{\Delta t} = 6,340 \frac{m}{s} \tag{4.3}
\]

The order of magnitude of the accepted longitudinal wave speed for aluminum is similar to the experimentally determined wave speed.

### 4.1.2 Theoretical Aluminum Dispersion Curves

With the two wave speeds introduced in the previous subsection, the dispersion relations outlined in Chapter (2) may be evaluated over a range of frequencies. The relevant curves for an isotropic aluminum plate are shown below. The frequency spectrum (wave number versus frequency) is in Fig. (4-1) and the dispersion curve (frequency versus phase velocity) is in Fig. (4-2).
4.2 Dispersion Curves for a Transversely Isotropic Composite Plate

4.2.1 Composite Specimen Physical Characteristics

The composite plate utilized for experimentation is a thin, 7 ply, unidirectional composite manufactured by Fiberite (a division of Cytec Industries). It has T-300 fibers, with 934 resin (350 cure epoxy). The material’s nomenclature is T300/934. The specimen is also plate like, with length and width much greater than its thickness. The dimensions are 26.96 cm × 75.42 cm, with an average thickness of .092 cm, as measured by a micrometer. The density of the specimen, \( \rho = 1.45 \text{g/cm}^3 \). With the transversely isotropic assumption made earlier, the specimen’s required engineering properties are limited to those in the Table (4.2).

To verify the above engineering constants, ultrasound experiments similar to those for the aluminum specimen are performed.
Figure 4-2: Dispersion curve \((2f/h \text{ vs } c_p)\) for the aluminum plate.

<table>
<thead>
<tr>
<th>Engineering Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E_1)</td>
<td>153.1 GPa</td>
</tr>
<tr>
<td>(E_2)</td>
<td>10.90 GPa</td>
</tr>
<tr>
<td>(\nu_{12})</td>
<td>.30</td>
</tr>
<tr>
<td>(\nu_{21})</td>
<td>.02</td>
</tr>
<tr>
<td>(\nu_{23})</td>
<td>.25</td>
</tr>
<tr>
<td>(G_{12})</td>
<td>5.59 GPa</td>
</tr>
</tbody>
</table>

Table 4.2: Values for the engineering properties in the T300/934 composite specimen.

To check the \(G_{12}\) and \(E_3\) (using \(G_{23}\)) constants, a Panametrics Shear Wave Transducer (V155) with a frequency of 5.0 MHz is set in pulse-echo mode to receive signals originating from a Panametric signal generator. Because of the anisotropy of the material, rotation of the transducer will vary the received waveform. There is a clear shift in the time of flight and a change in the amplitude as the transducer is rotated from the fast axis to the slow axis. The received waveforms have repeated signals, from which accurate time of flights for the shear waves may be obtained.
As extracted from the waveform, the time of flights are: fast axis: $\Delta t = 1.18 \, \mu s$, slow axis: $\Delta t = 1.44 \, \mu s$. The shear wave speeds are calculated as:

$$c_T = \frac{2t}{\Delta t} \quad (4.4)$$

where,

$c_T$ = shear wave speed along either the fast axis and slow axis

$t$ = thickness of the specimen

$\Delta t$ = time of flight for the shear wave (through twice the thickness)

$$c_{T(fast)} = \frac{2.92 \text{mm}}{1.18 \mu \text{s}}$$
$$c_{T(fast)} = 1,559.32 \frac{\text{m}}{\text{s}} \quad (4.5)$$

$$c_{T(slow)} = \frac{2.92 \text{mm}}{1.44 \mu \text{s}}$$
$$c_{T(slow)} = 1,277.78 \frac{\text{m}}{\text{s}}$$

Based on the shear wave speeds:

$$G_{12} = \rho c_{T(fast)}^2$$
$$G_{12} = (1.45)(1.55932)^2 \quad (4.6)$$
$$G_{12} = 3.53$$

$$G_{23} = \rho c_{T(slow)}^2$$
$$G_{23} = (1.45)(1.27778)^2 \quad (4.7)$$
$$G_{23} = 2.37$$
From this value for $G_{23}$, we can check the value for $E_3 = E_2$:

\[
E_3 = 2(1 + \nu_{23})G_{23}
\]

\[
E_3 = 2(1 + .25)(2.37) \tag{4.8}
\]

\[
E_3 = E_2 = 5.93
\]

Because $E_2$ is experimentally shown to be lower, $\nu_{21}$ is affected.

\[
\nu_{21} = \frac{\nu_{12}E_2}{E_1}
\]

\[

\nu_{21} = \frac{(.30)(5.93)}{153.1} \tag{4.9}
\]

\[
\nu_{21} = .0116
\]

The order of magnitude of the material properties for the composite plate is similar to the experimentally determined values. The experimental properties will be incorporated in the creation of the dispersion curves.

### 4.2.2 Composite Specimen Material Stiffness Constants

During experimentation, researchers reference the engineering properties (elastic modulus, poisson’s ratio). However, when formulating stress-strain relations and manipulating data, it becomes more convenient to express the engineering constants in terms of elastic (stiffness) constants, $C_{ij}$. As developed in a previous chapter, the required constants for this research include: $C_{11}, C_{13}, C_{33}$, and $C_{55}$.

The relations between the constants simplify due to the transversely isotropic assumption, with $E_2 = E_3$ and $\nu_{12} = \nu_{13}$. Dayal lists the simplified relationships as (Dayal and
Kinra 1989):

\[
\begin{align*}
C_{11} &= \frac{E_1}{(1 - \nu_{12}\nu_{21})} \\
C_{13} &= \frac{\nu_{12}E_2}{(1 - \nu_{12}\nu_{21})} \\
C_{33} &= \frac{E_2}{(1 - \nu_{12}\nu_{21})} \\
C_{55} &= G_{12} = G_{13}
\end{align*}
\] (4.10)

By substituting the given material properties into the above relations, the stiffness constants are:

\[
\begin{align*}
C_{11} &= 153.64 \\
C_{13} &= 1.79 \\
C_{33} &= 5.91 \\
C_{55} &= 3.53
\end{align*}
\] (4.11)

A second ultrasound experiment allows the verification of \(C_{33}\) through the determination of the longitudinal wave speed within the specimen. As utilized for the aluminum specimen, the Matec Ultrasonic Inspection System’s immersion ultrasound tank is filled with water at room temperature. A 5.0 MHz flat transducer is used in a vertical fashion, in a “pulse echo” mode. The flat transducer is unfocused, resulting in the presence of both front-face and back-face reflections when the composite specimen is submerged in the tank, perpendicular to the flat face of the transducer. distinct back-face reflections of the specimen is \(\Delta t = .76 \mu\text{s}\). The corresponding longitudinal wave speed for the composite specimen \(c_{L}\) is equal to twice the thickness of the specimen divided by \(\Delta t\).
The longitudinal wave speed is calculated as:

\[ c_L = \frac{2l}{\Delta t} \]

\[ = 2,863 \text{ m/s} \]  

The stiffness constant, \( C_{33} \), is derived from the longitudinal wave speed using the equation:

\[ C_{33} = \rho c_L^2 \]

\[ = 1.45 \times (2,863)^2 \]

\[ = 11.89 \text{ GPa} \]  

The experimental value for this stiffness constant is incorporated in the creation of the dispersion curves.

### 4.2.3 Theoretical Composite Dispersion Curves

With the stiffness constants introduced in the previous subsection, the dispersion relations outlined in Chapter (3) may be evaluated over a range of frequencies. The relevant curves for a transversely isotropic composite plate are shown below. The frequency spectrum (wave number versus frequency) is in Fig. (4-3) and the dispersion curves (frequency versus phase velocity) are in Fig. (4-4).
Figure 4-3: $kh$ vs. $\frac{\omega h}{c_p}$ for a T300/934 composite plate
Figure 4-4: Dispersion curve, $2fh$ vs $c_p$, for a T300/934 composite plate.
Chapter 5

Obtaining Dispersion Curves Experimentally

This chapter describes the procedure for experimentally obtaining a dispersion curve utilizing broadband frequency methods. It details the experimental collection of the data from a broadband source and describes the processing of the raw data into a dispersion curve format. The procedure utilizes the two dimensional fast Fourier transform (2D FFT) to process the data into the standard dispersion curve. This experimental method of obtaining the curves enables both the verification of known specimen material properties and the determination of unknown properties. Results are shown for both aluminum and composite specimens.

5.1 Experimental Procedure for Obtaining Dispersion Curves

Experimental NDE methods that utilize Lamb waves are heavily dependent upon an accurate set of dispersion curves. As shown in the theoretical chapters, error in the initial material parameters will have a significant effect on the positioning of the curves. To effectively use Lamb waves to interrogate a material, the dispersive behavior and the phase
velocities of the various modes must be determined with a high degree of confidence. An experimentally obtained dispersion curve can help to minimize the potential source of error caused by a specimen’s assumed material properties.

Material properties are most often determined from table values or manufacture specifications for a material of interest. These two sources will normally yield values that are good approximations to the actual values. The manufacture of materials is a very imprecise process. The bigger the geometry of the material structure, the more variance will be in critical parameters; such as density, thickness, and modulus. This variance is most evident in anisotropic materials such as composites, where layup sequences may cause irregularities in geometries. Thus, a tabular value or a manufacturer specification is at best a good approximation of the average material properties for the material, not necessarily the exact values for a particular specimen.

Experimentally creating dispersion curves allows for the verification of known specimen material properties and the determination of unknown properties. The procedure outlined below utilizes a broadband frequency method. Nd:YAG laser generated waveforms are collected using a Photo-EMF detection laser interferometer as a receiver, processed with a two dimensional fast Fourier transform, and plotted with wave number and frequency to obtain the standard dispersion curve. The procedure lasts about three hours in duration and yields fairly accurate curves.

5.1.1 The Experimental Setup to Collect Data

The experimental setup to allow the acquisition of waveform data has three primary components: the source laser, the receiving laser, and the data display. A schematic diagram of the setup is shown in Fig. (5-1).

The figure lays out the general positioning of the major components. The excitation laser is mounted to a sliding table to allow easy translation of the source. The sliding table is manipulated with a hand crank divided into 10 gradations, with .1 inch translation for
Figure 5-1: The schematic diagram of the set-up utilizing both a source laser and a receiving laser.

every full turn of the crank. The laser is located at the focal length distance away from the specimen. The specimen is placed vertically, perpendicular to both the source and the receiver. Care is made to ensure that the source is at the same focal length throughout the translation of the laser. The receiving laser is mounted on a fixed table, on the same side of the specimen as the source laser. It is likewise focused for its fixed position.

The experimental procedure is based upon the use of a pulse laser as an excitation source. The laser’s thermoelastic energy is focused at a point for only a short duration of time, creating a Lamb wave that is broadband. A broadband wave contains a large window
of varying frequencies. As the waveform propagates through the plate, mode conversion occurs, and the frequencies act in a dispersive manner. The laser used is a Q-switched, Nd:YAG 266 nm / 30 mJ / 5 ns short pulse laser manufactured by New Wave Research. The laser is concentrated using a semi concave lens, and a focal length of 10 mm is maintained to the specimen throughout the experiment. The laser energy ablates the surface of the specimen, even though a water couplant is maintained at the excitation location. The laser is remotely fired from a control box made by Uniblitz.

The laser receiver is a Photo-EMF receiver made by Verdi. It is powered by a source laser Nd:YVO₄. The laser is first focused at a power of .1 watt, which is low enough in intensity to prevent surface ablation. The focusing is accomplished by comparing the laser beam to a reference beam that travels within the receiving laser. The receiver is complemented with a shutter, wired to the same control box as the source laser. The power of the receiving laser is then increased to 1 watt in order to allow it to receive the broadband, multi-modal waveform traveling through the specimen. The coupling of the source laser and the shutter enables the shutter to open upon trigger of the source. The aperture of the shutter opens and closes rapidly, minimizing exposure time and preventing ablation by the high powered receiving laser. The frequency response of the receiver is shown in Fig. (5-2). The response is clearly broadband, with the disadvantage of cutoff frequency around 250 kHz. The laser receiver cannot pick up the lower frequencies in the waveform.

The propagating waveform is acquired by a TDS-210 two channel oscilloscope, acting in “normal”, “average 16” data collection mode. An example of an obtained waveform is shown in Fig. (5-3). The data is stopped, and moved by way of a GPIB interface cable to a computer for processing.

The source laser is translated away from the receiver, using a uniform distance step of .8255 mm. The data collection is repeated and new waveform data is collected for a total of 128 waveforms. A schematic of the specimen’s excitation dimensions is in Fig. (5-4).

For the composite specimen, the point of the initial measurement and the distance step are the same as those for aluminum, described in Fig. (5-4).
Figure 5-2: The frequency response spectrum of the laser receiver.

An alternate method to acquire the waveform data may be used if a laser receiver is not readily available. The receiver may be replaced by a polyvinylidene fluoride (PVDF) transducer as the receiving transducer. The other two components: the source laser and the data display, do not change. A schematic diagram of the setup is shown in Fig. (5-5). The transducer is fabricated to simulate the laser receiver, which collects data in a circular fashion. The diameter of the transducer is 2 mm. Initial testing showed that the PVDF received waveforms were comparable to the laser received waveforms.

The figure lays out the general positioning of the major components. The receiving transducer is mounted to the specimen in a fixed location, on the same side of the specimen as the source laser. Couplant is used to ensure good contact.
Figure 5-3: Sample waveform received by the laser for the aluminum specimen

Figure 5-4: The schematic diagram showing the dimensions of the specimen and the distance step that the laser translates.

5.1.2 Processing the Data Into a Dispersion Curve

The 2D FFT scheme is an efficient way to process the raw data into a dispersion curve (Costley, Daniel, and Berthelot 1993). The 128 laser generated signals are stored in matrix form, the columns representing time, the rows representing space: \( u_z(x, t) \). A 2D FFT processing scheme converts this raw data into frequency and wavenumber information using
Figure 5-5: The schematic diagram of the set-up utilizing a source laser and a PVDF transducer as a receiver.

The relation:

\[ \dot{u}_z = \int \int u_z(x, t) \, dx \, dt \]  

(5.1)

The magnitudes of each element of the resulting matrix is the frequency spectrum information, which may be further plotted into the graphical depiction of the frequency-wavenumber domain. An analysis of the slopes of these curves will yield the standard dispersion curve plot.

5.2 Experimental Results for Specimens

5.2.1 Isotropic Aluminum Specimen

The procedure outlined above is applied to the aluminum specimen. Three waveforms from various points of excitation are captured in Fig. (5-6). A delay in the time of flights is fairly obvious. After processing the 128 waveforms with the 2D FFT scheme, an experimental curve is plotted. The results are depicted on the left hand graph in Fig. (5-7). The right graph is the theoretical comparison. There is very good agreement between the ex-
perimentally obtained plot and the theoretical plot. Improvements in resolution may occur by increasing the sampling rate during data collection. The results verify that the initial material parameters were accurate.

5.2.2 Transversely Isotropic Composite Specimen

The procedure is extended to the composite specimen. Three waveforms from various points of excitation are captured in Fig. (5-8). A delay in the time of flights is obvious. The presence of only a small number of propagating modes is reflected in the waveforms. After processing the 128 waveforms with the 2D FFT scheme, an experimental curve is plotted. The results are depicted on the left hand graph in Fig. (5-9). The right graph is the theoretical comparison. The experimentally obtained dispersion curve is not very complete. We see a faint agreement in the lower $2f h$ values, but no matching curve in the higher $2f h$ values. This result is expected since only a small number of modes were propagating in the plate as indicated by the collected signals. Attenuation is extremely high, which may effect the resolution of the experimental curve. A possible improvement is to decrease the distances between the excitation laser and the receiving laser.
Figure 5-6: A comparison of the three experimentally obtained waveforms for the composite plate, taken at different distances between the source and the receiving laser.
Figure 5-7: A comparison of the experimentally obtained dispersion curve to the theoretical dispersion curve for aluminum.
Figure 5-8: A comparison of the three experimentally obtained waveforms for the composite plate, taken at different distances between the source and the receiving laser.
Figure 5-9: A comparison of the experimentally obtained dispersion curve to the theoretical dispersion curve for the unidirectional composite specimen.
Chapter 6

Synthetic Phase Tuning

This chapter extends the Lamb wave theory to an effective experimental application: Synthetic Phase Tuning (SPT). SPT improves the robustness of Lamb waves as a non-destructive evaluation tool and facilitates the utilization of Lamb waves in more complex materials. The theory behind SPT is explained, along with the details covering the experimental implementation of SPT in a lab environment.

6.1 Synthetic Phase Tuning Theory

6.1.1 The objective of SPT

The signal obtained from a propagating Lamb wave tends to be very complex, due to the Lamb wave’s dispersive and multi-modal nature. The waveform may have many peaks, from which one propagating mode must be isolated and analyzed in order to obtain any information of value. Selection of a mode to analyze is often a judgment call, resulting in the introduction of significant human error. A Lamb wave propagating through a transversely isotropic composite will have the additional complexity of attenuation of the signal amplitudes.
To counter the signal complexity, a novel method for tuning individual Lamb wave modes, called Synthetic Phase Tuning (SPT), had been recently developed (Wooh and Shi 2001). After numerical processing a waveform using the method's algorithms, a tuned mode is easily discernible and very usable in the time domain. Experimental implementation of the method is very straightforward and efficient, as the linear phased array used to transmit and receive the signal does not need to be continually relocated along the length of the specimen.

### 6.1.2 Implementing SPT to Create a Physically Tuned Mode

In the Synthetic Phase Tuning (SPT) technique, individually recorded waveforms are processed numerically, allowing the transmitting and receiving of a synthetic signal. SPT takes advantage of the physical characteristics of an array transducer in order to influence a wave by constructive interference. A sequential firing of individual elements at various time delays allows for constructive interference of a particular mode in an evenly spaced array. One designated mode has its amplitude increased relative to the other modes in the signal, making it a discernible mode in which to more accurately calculate time of flight measurements.

The operating mode outlined in this chapter is the Pseudo Pulse-Echo (PPE). In this mode, a single multi-element array is used to both transmit and receive a signal that reflects off of a discontinuity or edge. Although each of the elements in the multi-element array act independently, their respective signals interfere with each other within the material to form a synthetic waveform. To take advantage of this observation, the SPT technique may be used to influence the interference in a constructive way.

There are three steps in the PPE procedure (Wooh and Shi 2001):

1. Signal generation and recording.

2. Synthetic reconstruction of the emitted wave.
3. Synthetic construction of the received wave.

**Signal generation and recording**

The initial step is to record a large number of signals to assist in the numerical processing. A narrow band (tone-burst) or a broadband (spike pulse) excitation signal may be used.

When collecting data using a tone-burst signal, the starting point of SPT is an accurate dispersion curve. The dispersion relation for isotropic and transversely isotropic materials were derived in Chapters (2) and (3). These chapters also explain the creation of the dispersion curve. To ensure accuracy of the phase velocities and the mode characteristics of the curves, the procedure laid out in Chapter (4) may be used.

From the dispersion curve, the dispersive and multi-modal behavior of the Lamb wave is evaluated in order to choose the best center frequency for the tuning of the modes. The x-axis of the dispersion curve contains the values for $2f_h$. The best choice of $2f_h$ is found by looking for a vertical region of the dispersion curve that has two characteristics:

1. **Mode separation** - The modes should all have distinct phase velocities ($c_p$), as indicated by the vertical spacing between the points of the dispersion curve for a particular frequency. The space between the modes makes it easier to distinguish between them.

2. **Dispersive behavior** - Good dispersion means that a small change in frequency will result in different velocities. The preferred part of the curve is steeply sloped, away from the non-dispersive areas of horizontal segments.

Fig. (6-1) shows a good choice for a $2f_h$ value. The dispersion curve is for the transversely isotropic composite specimen used in Chapter (4). The best choice is $2f_h = 2$. Note that there exists other choices having the two characteristics listed above.

Once the $2f_h$ value is known, the center frequency can easily be determined using the value for the specimen half thickness. For this example, the composite specimen has a half
Figure 6-1: A comparison of the experimentally obtained dispersion curve to the theoretical dispersion curve for aluminum.

thickness of \((1/2) \times (1.08 \text{ mm})\).

\[
2fh = 2 \quad f = \frac{2}{(2)(.54)} \quad f = 1.85 \text{MHz}
\]  

At this center frequency, all the modes present may be constructively tuned using the SPT technique.

To actually collect the data, place a PVDF array transducer of 16 elements and constant inter-element spacing on a specimen. Element index \(m\) is the transmitting element, \(n\) is the index of the receiving elements. Activate two of the array elements; one transmitter, one receiver. The best choice are the elements closest to the edge of reflection. Generate a narrow band tone-burst signal with the predetermined center frequency using a function
generator and a power amplifier. The narrow bandwidth of the signal will minimize the dispersion present in the signal. Because of the close proximity of the face of the array transducer to the edge of the specimen, the wave will reflect off of the edge, reverse itself, and travel back to the receiving element of the transducer. This returned signal is recorded using a digital oscilloscope. While continuing to use the same transmitting element, change the receiving element and repeat the procedure.

Alternatively, the experimental data may be collected using a broadband source, like that utilized to experimentally obtain dispersion curves in Chapter (5). The translating of the excitation laser is, in effect, an array, provided that the inter-element spacing is kept constant. The data collected earlier may be processed using SPT techniques to simulate a 16 element array transducer.

**Synthetic reconstruction of the emitted wave**

Once all the signals have been recorded, the data is numerically processed to construct a synthetic, virtual wave that simulates the waveform leaving the transducer and encountering the discontinuity or edge. The signals are processed using a time delay algorithm.

The algorithm begins with the determination of the time delay needed to tune a particular wave mode. The relation is based upon the inter-element spacing in the array transducer (d). \( c_p \) is taken for the desired mode, from the dispersion curve.

\[
\Delta \tau = \frac{d}{c_p} \tag{6.2}
\]

With the first element as the reference element \( t_1 = 0 \), the calculated \( \Delta \tau \) is used to find the delay in each \( m \)th element of the array using the relation:

\[
t_{m+1} = t_m + \Delta \tau \tag{6.3}
\]

for \( m = 1 \) to \( N - 1 \).
This time delay profile enables the construction of the synthetic waveform received by the \( n \)th element. The signal reconstruction consists of a summation of all of the time shifted waveforms:

\[
 s_n(t) = \sum_{m=1}^{N} S_{mn}(t - (m - 1)\Delta\tau) \\
 = \sum_{m=1}^{N} S_{mn}(t - \frac{(m - 1)d}{c_p}) 
\]

(6.4)

with \( S_{mn}(t) \) the waves transmitted by the \( m \)th element and received by the \( n \)th element.

**Synthetic construction of the received wave**

The reflected wave is actually reversed, causing the received waveform to be different than the initial, synthetically constructed signal. A second tuning of the signal is necessary.

The time delay has to be altered to reflect the reversal of the waveform:

\[
 t_{n-1} = t_n + \Delta\tau' 
\]

(6.5)

with:

\[
 \Delta\tau' = \frac{d}{c_p} 
\]

(6.6)

This second profile of time delays may be identical to the first, thus tuning the same mode in reverse. However, a different mode may be tuned (Wooh and Shi 2001).

The fully constructed signal becomes:

\[
 s_n(t) = \sum_{n=N}^{1} S_n(t - (n - 1)\Delta\tau') 
\]

(6.7)
Practical application

The synthetic construction of the waveform is easily applied to tune an actual chosen mode of the Lamb wave by utilizing an array controller. All the elements may be fired numerically with a time shift profile as determined above. The propagating wave mode may be constructively reinforced as each element of the array is activated. The reflected waveform is similarly tuned. The result will be a highly discernible, tuned mode, useful in the NDE of a composite.

6.2 Tuning Results for a Transversely Isotropic Material

To ensure the feasibility of synthetic phase tuning for a transversely isotropic composite material, experimentation is done to collect a strong signal able to be processed in the aforementioned scheme.

The composite specimen described in Chapter (4) is placed horizontally. A sixteen element transducer made of polyvinylidene fluoride (PVDF) piezo-polymer film is located 8.5 cm from the manufacturer’s edge of the composite. The elements of the array are parallel to the edge of the plate and perpendicular to the fiber direction. The transducer characteristics are listed in Table (6.1).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of elements, (N)</td>
<td>16</td>
</tr>
<tr>
<td>Inter-element spacing, (d), (mm)</td>
<td>.7</td>
</tr>
<tr>
<td>Element width, (a), (mm)</td>
<td>1</td>
</tr>
<tr>
<td>Transducer aperture, (D), (mm)</td>
<td>40.0</td>
</tr>
</tbody>
</table>

Table 6.1: Characteristics of PVDF array transducer.

Two of the array elements are active; one transmitter, one receiver. The elements chosen A five-cycle tone-burst signal with a center frequency of 1.65 MHz is generated by a
function generator and power amplifier. The array is rotated until the face of the transducer is parallel to the edge of the material, as indicated by the strength of signal. As shown in Fig. (6-2), there is clear evidence of the existence of a multi-mode behavior in the unidirectional composite.

![Graph](image.png)

Figure 6-2: Experimentally created signal using a PVDF array transducer on a transversely isotropic composite plate.

The receiver may now be changed, and the data collection started. After recording data while using all 16 elements as both transmitters and receivers, the time delay profile and the corresponding synthetic wave may be generated.
Chapter 7

Conclusions

Although the theory behind Lamb wave propagation is very complex, its application to the interrogation of thin, plate like structures has a lot of potential. A simple case of a homogeneous, isotropic material immediately satisfies many of Robert Crane's criteria described in the introduction. Already proved to be reliable and accurate in aluminum, the experimental Synthetic Phase Tuning method discussed is clearly applicable to orthotropic composite materials.

When compared against Crane's criteria, Lamb wave inspection in all materials is favorable. Although the underlying theory will always remain very complex to the average Non-destructive evaluation technician, the application aspect is becoming increasingly straightforward. SPT should dramatically improve the accuracy at which technicians can discern a signal's peak in the time domain. This simplicity in waveform analysis will improve the reliability of the technician's NDE observations, as he will be better equipped to identify changes in signals due to the presence of defects.

Experimentally, both the isotropic and the transversely isotropic specimens produced interesting results while obtaining dispersion curves experimentally. The introduction of a Photo-EMF detection laser interferometer as a waveform receiver allowed the accurate collection of a broadband frequency spectrum, with only a small cut-off frequency. Although
a very expensive piece of equipment, the laser receiver's reliability and repeatability are much better than a PVDF transducer alone. The dispersion frequency / wavenumber plot for the aluminum specimen showed excellent agreement. The composite plot, however, revealed similarity, but only at lower frequencies. The higher frequencies were either attenuated through the composite or appeared to be insignificant due to the intense energy shown by the lower frequencies. In the composite, only the longitudinal direction was studied. The transverse direction may generate different curve properties, and is worth further investigation.

Synthetic Phase Tuning appears to be very promising when extended to composite materials. The signal obtained using a narrow band, tone-burst event in a composite revealed the multi-modal nature of the waveform, and showed that the multi-element array used in the procedure was capable of receiving a Pseudo Pulse-Echo signal through the orthotropic composite.

The experimental process laid out in this thesis is a very efficient way of experimenting with Lamb waves. The same waveform data recorded to experimentally obtain the dispersion curves may be used as input into the SPT process, thus eliminating many hours of data collection. The inter-element spacing of the excitation laser hits must be kept constant in order to properly simulate an equally spaced, multi-element array transducer.

The theory is explained in this thesis in a very straightforward manner. A number of experiments are outlined to allow a more accurate application of the SPT method. The extension of SPT to orthotropic, composite materials is discussed and the feasibility of the method is proven. Future studies must complete the waveform recording for the composite, synthetically construct the waveform, and apply SPT to a multitude of various fiber layup angles.
Bibliography


Appendix A

Analysis of the Dispersion Relation-Isotropic Case

This appendix describes the complete analysis of the dispersion relation for isotropic materials. For the Rayleigh-Lamb case, both longitudinal (L) and shear (T) waves exist for any given mode because of the mode conversion at the traction-free surfaces. By looking at the frequency equations, it is evident that simple analytical solutions do not exist. Manipulation of the dispersion equations leads to the ability to graphically depict the dispersion. The determination of the cut-off frequencies yields a starting point for the implementation of an iterative method such as the Newton-Raphson method.

A.0.1 Symmetric / Antisymmetric Modes

The displacement for symmetric and antisymmetric modes is expressed above in terms of the four constants: $A_s, D_s, B_a$ and $C_a$. They may be rewritten:
Symmetric modes:

From Eq. (2.46), the amplitude ratio becomes:

\[
\frac{D_s}{A_s} = \frac{2j k \alpha \sinh(\alpha h)}{(k^2 + \beta^2) \sinh(\beta h)}
\]

and the displacement can be expressed as

\[
u_x = j k A \left( \frac{\cosh(\alpha z)}{\sinh(\alpha h)} - \frac{2 \alpha \beta}{k^2 + \beta^2} \cdot \frac{\cosh(\beta z)}{\sinh(\beta h)} \right) e^{j(kz - \omega t)}
\]

\[
u_z = A \alpha \left( \frac{\sinh(\alpha z)}{\sinh(\alpha h)} - \frac{2k^2}{k^2 + \beta^2} \cdot \frac{\sinh(\beta z)}{\sinh(\beta h)} \right) e^{j(kz - \omega t)}
\]

where \(A\) is a new constant.

Antisymmetric modes:

Also, from Eq. (2.50) the amplitude ratio becomes:

\[
\frac{C_a}{B_a} = \frac{2j k \alpha \cosh(\alpha h)}{(k^2 + \beta^2) \cosh(\beta h)}
\]

and the displacement can be expressed as:

\[
u_x = j k B \left( \frac{\sinh(\alpha z)}{\cosh(\alpha h)} - \frac{2 \alpha \beta}{k^2 + \beta^2} \cdot \frac{\sinh(\beta z)}{\cosh(\beta h)} \right) e^{j(kz - \omega t)}
\]

\[
u_z = B \alpha \left( \frac{\cosh(\alpha z)}{\cosh(\alpha h)} - \frac{2k^2}{k^2 + \beta^2} \cdot \frac{\cosh(\beta z)}{\cosh(\beta h)} \right) e^{j(kz - \omega t)}
\]

where \(B\) is a new constant.
Analysis of the Dispersion Relation-Isotropic Case

To analyze the various regions of the Rayleigh-Lamb equation, recall the definition of $\alpha$ and $\beta$:

$$\alpha^2 = k^2 - \frac{\omega^2}{c_L^2} = \frac{\omega^2}{c^2} - \frac{\omega^2}{c_L^2} \quad (A.7)$$

$$\beta^2 = k^2 - \frac{\omega^2}{c_T^2} = \frac{\omega^2}{c^2} - \frac{\omega^2}{c_T^2} \quad (A.8)$$

$\alpha$ and $\beta$ could be real or imaginary, depending on the value of the phase velocity $c$ relative to $c_L$ and $c_T$. Three cases exist:

**Case 1: Real $\alpha$ and Real $\beta$, ($0 < c < c_T < c_L$)**

$$\alpha = \sqrt{k^2 - \frac{\omega^2}{c_L^2}} \quad \text{→ real} \quad (A.9)$$

$$\beta = \sqrt{k^2 - \frac{\omega^2}{c_T^2}} \quad \text{→ real} \quad (A.10)$$

Thus, the frequency equation is of the same form as before:

$$\frac{\tanh(\beta h)}{\tanh(\alpha h)} = \frac{4k^2\alpha\beta}{(k^2 + \beta^2)^2} \quad (\text{symmetric}) \quad (A.11)$$

$$\frac{\tanh(\beta h)}{\tanh(\alpha h)} = \frac{(k^2 + \beta^2)^2}{4k^2\alpha\beta} \quad (\text{antisymmetric}) \quad (A.12)$$

**Case 2: Real $\alpha$ and Imaginary $\beta$, ($0 < c_T < c < c_L$)**

$$\alpha = \sqrt{k^2 - \frac{\omega^2}{c_L^2}} \quad \text{→ real} \quad (A.13)$$

$$\beta = \sqrt{k^2 - \frac{\omega^2}{c_T^2}} \quad \text{→ imaginary} \quad (A.14)$$
The frequency equation becomes:

\[
\frac{\tan(\beta h)}{\tanh(\alpha h)} = \frac{4\alpha\beta k^2}{(k^2 - \beta^2)^2} \quad \text{(symmetric)} \tag{A.15}
\]

\[
\frac{\tan(\bar{\beta} h)}{\tanh(\alpha h)} = -\frac{4\alpha\bar{\beta} k^2}{(k^2 - \bar{\beta}^2)^2} \quad \text{(antisymmetric)} \tag{A.16}
\]

where \(\bar{\beta}\) is the conjugate of \(\beta\)

\[
\bar{\beta} = \sqrt{\frac{\omega^2}{c_T^2} - k^2} \tag{A.17}
\]

**Case 3: Imaginary \(\alpha\) and Imaginary \(\beta\) \((0 < c_T < c_L < c)\)**

\[
\alpha = \sqrt{k^2 - \frac{\omega^2}{c_L^2}} \quad \rightarrow \text{imaginary} \quad \tag{A.18}
\]

\[
\beta = \sqrt{k^2 - \frac{\omega^2}{c_T^2}} \quad \rightarrow \text{imaginary} \quad \tag{A.19}
\]

The frequency equations become:

\[
\frac{\tan(\bar{\beta} h)}{\tanh(\alpha h)} = -\frac{4\bar{\alpha}\bar{\beta} k^2}{(k^2 - \bar{\beta}^2)^2} \quad \text{(symmetric)} \tag{A.20}
\]

\[
\frac{\tan(\bar{\beta} h)}{\tanh(\alpha h)} = -\frac{4\bar{\alpha}\bar{\beta} k^2}{(k^2 - \bar{\beta}^2)^2} \quad \text{(antisymmetric)} \tag{A.21}
\]

where \(\bar{\alpha}\) and \(\bar{\beta}\) are the conjugates of \(\alpha\) and \(\beta\) respectively,

\[
\bar{\alpha} = \sqrt{\frac{\omega^2}{c_L^2} - k^2} \quad \tag{A.22}
\]

\[
\bar{\beta} = \sqrt{\frac{\omega^2}{c_T^2} - k^2} \quad \tag{A.23}
\]
A.0.2 Cutoff Frequencies

The cutoff frequencies for the various plate modes will be obtained by considering the wavenumber $k \to 0$. For this limiting value, the Rayleigh-Lamb equation reduce to:

$$\cos(\alpha_c h) \sin(\beta_c h) = 0 \quad \text{symmetric} \quad (A.24)$$

$$\sin(\alpha_c h) \cos(\beta_c h) = 0 \quad \text{antisymmetric} \quad (A.25)$$

**Symmetric modes**

$$\cos(\alpha_c h) = 0, \quad \alpha_c h = \frac{p\pi}{2} \quad (p = 1, 3, 5, \ldots) \quad (A.26)$$

$$\sin(\beta_c h) = 0, \quad \beta_c h = \frac{q\pi}{2} \quad (q = 0, 2, 4, \ldots) \quad (A.27)$$

Because $\alpha = \omega_c/c_L$ and $\beta = \omega_c/c_T$, we have the condition for the cutoff frequencies:

$$\frac{\omega_c h}{c_T} = \begin{cases} \frac{\pi p c_L}{2 c_T} & (p = 1, 3, 5, \ldots) \\ \frac{\pi q}{2} & (q = 0, 2, 4, \ldots) \end{cases} \quad (A.28)$$

or, in terms of the frequency and thickness product:

$$2f_c h = \begin{cases} \frac{p c_L}{2} & (p = 1, 3, 5, \ldots) \\ \frac{q c_T}{2} & (q = 0, 2, 4, \ldots) \end{cases} \quad (A.29)$$
Antisymmetric modes

\[
\begin{align*}
\sin(\bar{\alpha}_c h) &= 0, \quad \bar{\alpha}_c h = \frac{p\pi}{2} \quad (p = 0, 2, 4, \ldots) \quad (A.30) \\
\cos(\bar{\beta}_c h) &= 0, \quad \bar{\beta}_c h = \frac{q\pi}{2} \quad (q = 1, 3, 5, \ldots) \quad (A.31)
\end{align*}
\]

As in the case of symmetric modes, because \( \bar{\alpha} = \omega_c/c_L \) and \( \bar{\beta} = \omega_c/c_T \) we have the condition for the cutoff frequencies:

\[
\frac{\omega_c h}{c_T} = \begin{cases} 
\frac{\pi p c_L}{2 c_T} & (p = 0, 2, 4, \ldots) \\
\frac{\pi q}{2} & (q = 1, 3, 5, \ldots)
\end{cases} \quad (A.32)
\]

or, in terms of the frequency and thickness product,

\[
2f_c h = \begin{cases} 
\frac{pc_L}{2} & (p = 0, 2, 4, \ldots) \\
\frac{qCT}{2} & (q = 1, 3, 5, \ldots)
\end{cases} \quad (A.33)
\]
Appendix B

Analysis of the Dispersion Relation-Orthotropic Case

This appendix describes the complete analysis of the dispersion relation for orthotropic materials. For the Rayleigh-Lamb case, both longitudinal (L) and shear (T) waves exist for any given mode because of the mode conversion at the traction-free surfaces. By looking at the frequency equations, it is evident that simple analytical solutions do not exist. Manipulation of the dispersion equations leads to the ability to graphically depict the dispersion. The determination of the cut-off frequencies yields a starting point for the implementation of an iterative method such as the Newton-Raphson method.

B.0.3 Symmetric / Antisymmetric Modes

Symmetric modes:

The dispersion equation for symmetric modes is:

\[
\frac{\tanh(k_{zp}h)}{\tanh(k_{zm}h)} = \frac{G_pH_m}{G_mH_p}
\]  

(B.1)
The general function in terms of $k_z$ and $\omega$ for symmetric modes is

$$ f(k_z, \omega) = G_p H_m \cosh(k_{zp} h) \sinh(k_{zm} h) - G_m H_p \sinh(k_{zp} h) \cosh(k_{zm} h) $$  \hspace{1cm} (B.2)

where it is assumed that $k_{zp}$ and $k_{zm}$ are real. However, under some conditions, $k_{zp}$ and $k_{zm}$ could be imaginary, according to Eq. (3.15). Therefore, there exist three different cases for the function $f(k_z, \omega)$.

**Case 1: Real $k_{zp}$ and $k_{zm}$**

In this case, from Eq. (3.15) we have:

$$ -B + \sqrt{B^2 - 4D} \geq 0 $$  \hspace{1cm} (B.3)

$$ -B - \sqrt{B^2 - 4D} \geq 0 . $$  \hspace{1cm} (B.4)

The dispersion equation is correspondingly:

$$ \frac{\tanh(k_{zp} h)}{\tanh(k_{zm} h)} = G_p H_m \frac{G_m}{G_p H_p} $$  \hspace{1cm} (B.5)

The function $f(k_z, \omega)$ is expressed as:

$$ f(k_z, \omega) = G_p H_m \cosh(k_{zp} h) \sinh(k_{zm} h) - G_m H_p \sinh(k_{zp} h) \cosh(k_{zm} h) $$  \hspace{1cm} (B.6)

where $k_{zp}$ and $k_{zm}$ are:

$$ k_{zp}^2 = \frac{k_x^2[-B + \sqrt{B^2 - 4D}]}{2} $$  \hspace{1cm} (B.7)

$$ k_{zm}^2 = \frac{k_x^2[-B - \sqrt{B^2 - 4D}]}{2} $$  \hspace{1cm} (B.8)
and $H_p$ and $H_m$ are also real:

$$H_p = \frac{(-\rho \omega^2 + C_{11} k_x^2 + C_{13} k_{zp}^2)}{(C_{55} + C_{13}) k_{zp}}$$  \hspace{1cm} (B.9)

$$H_m = \frac{(-\rho \omega^2 + C_{11} k_x^2 + C_{13} k_{zm}^2)}{(C_{55} + C_{13}) k_{zm}}$$  \hspace{1cm} (B.10)

and $G_p$ and $G_m$ are also real:

$$G_p = \frac{C_{33} \rho \omega^2 + (C_{13} C_{55} + C_{13}^2 - C_{33} C_{11}) k_x^2 - C_{33} C_{55} k_{zp}^2}{(C_{55} + C_{13}) k_x}$$  \hspace{1cm} (B.12)

$$G_m = \frac{C_{33} \rho \omega^2 + (C_{13} C_{55} + C_{13}^2 - C_{33} C_{11}) k_x^2 - C_{33} C_{55} k_{zm}^2}{(C_{55} + C_{13}) k_x}$$  \hspace{1cm} (B.13)

**Case 2: Real $k_{zp}$, imaginary $k_{zm}$**

In this case, from Eq. (3.15) we have:

$$-B + \sqrt{B^2 - 4D} \geq 0$$  \hspace{1cm} (B.14)

$$-B - \sqrt{B^2 - 4D} < 0$$  \hspace{1cm} (B.15)

The dispersion equation is:

$$\frac{\tanh(k_{zp} h)}{\tanh(k_{zm} h)} = -\frac{G_p H_m}{G_m H_p}$$  \hspace{1cm} (B.16)

The function $f(k_x, \omega)$ is expressed as:

$$f(k_x, \omega) = G_p H_m \cosh(k_{zp} h) \sinh(k_{zm} h) + G_m H_p \sinh(k_{zp} h) \cosh(k_{zm} h)$$  \hspace{1cm} (B.17)
where $k_{zp}$ and $k_{zm}$ are real ($k_{zm} = j \overline{k}_{zm}$ is imaginary)

\[
k_{zp}^2 = \frac{k_{zp}^2[B + \sqrt{B^2 - 4D}]}{2}
\]

\[
k_{zm}^2 = \frac{k_{zm}^2[B + \sqrt{B^2 - 4D}]}{2}
\]

and $H_p$ and $\tilde{H}_m$ are real ($H_m = j \tilde{H}_m$ is imaginary)

\[
H_p = \frac{(-\rho \omega^2 + C_{11}k_x^2 + C_{13}k_{zp}^2)}{(C_{55} + C_{13})k_{zp}}
\]

\[
\tilde{H}_m = \frac{(\rho \omega^2 - C_{11}k_x^2 + C_{13}k_{zm}^2)}{(C_{55} + C_{13})k_{zm}}
\]

and $G_p$ and $G_m$ are real:

\[
G_p = \frac{C_{33}\rho \omega^2 + (C_{13}C_{55} + C_{13}^2 - C_{33}C_{11})k_x^2 - C_{33}C_{55}k_{zp}^2}{(C_{55} + C_{13})k_x}
\]

\[
G_m = \frac{C_{33}\rho \omega^2 + (C_{13}C_{55} + C_{13}^2 - C_{33}C_{11})k_x^2 + C_{33}C_{55}k_{zm}^2}{(C_{55} + C_{13})k_x}
\]

**Case 3: Imaginary $k_{zp}$ and $k_{zm}$**

In this case, from Eq. (3.15) we have:

\[-B + \sqrt{B^2 - 4D} < 0 \quad \text{(B.25)}
\]

\[-B - \sqrt{B^2 - 4D} < 0 . \quad \text{(B.26)}
\]

The dispersion equation is:

\[
\frac{\tan(k_{zp}h)}{\tan(k_{zm}h)} = \frac{G_p\tilde{H}_m}{G_mH_p}
\]
The function $f(k_x, \omega)$ is thus expressed as:

$$f(k_x, \omega) = G_p \bar{H}_m \cos(k_{zp}h) \sin(k_{zm}h) - G_m \bar{H}_p \sin(k_{zp}h) \cos(k_{zm}h)$$  \hspace{1cm} (B.28)

where $k_{zp}$ and $k_{zm}$ are real ($k_{zp} = jk_z$ is imaginary, $k_{zm} = jk_m$ is imaginary)

$$k_{zp}^2 = \frac{k_z^2[B - \sqrt{B^2 - 4D}]}{2}$$  \hspace{1cm} (B.29)

$$k_{zm}^2 = \frac{k_z^2[B + \sqrt{B^2 - 4D}]}{2}$$  \hspace{1cm} (B.30)

and $\bar{H}_p$ and $\bar{H}_m$ are real ($H_p = j\bar{H}_p$ is imaginary, $H_m = j\bar{H}_m$ is imaginary)

$$H_p = \frac{(\rho \omega^2 - C_{11} k_z^2 + C_{13} k_{zp}^2)}{(C_{55} + C_{13}) k_{zp}}$$  \hspace{1cm} (B.31)

$$\bar{H}_m = \frac{(\rho \omega^2 - C_{11} k_z^2 + C_{13} k_{zm}^2)}{(C_{55} + C_{13}) k_{zm}}$$  \hspace{1cm} (B.32)

and $G_p$ and $G_m$ are real:

$$G_p = \frac{C_{33} \rho \omega^2 + (C_{13} C_{55} + C_{13}^2 - C_{33} C_{11}) k_z^2 + C_{33} C_{55} k_{zp}^2}{(C_{55} + C_{13}) k_z}$$  \hspace{1cm} (B.34)

$$G_m = \frac{C_{33} \rho \omega^2 + (C_{13} C_{55} + C_{13}^2 - C_{33} C_{11}) k_z^2 + C_{33} C_{55} k_{zm}^2}{(C_{55} + C_{13}) k_z}$$  \hspace{1cm} (B.35)

**Antisymmetric modes:**

The dispersion equation for antisymmetric modes is:

$$\frac{\tanh(k_{zp}h)}{\tanh(k_{zm}h)} = \frac{G_m H_p}{G_p H_m}$$  \hspace{1cm} (B.36)
The general function in terms of $k_x$ and $\omega$ for antisymmetric modes is

$$f(k_x, \omega) = G_p H_m \sinh(k_{zp} h) \cosh(k_{zm} h) - G_m H_p \cosh(k_{zp} h) \sinh(k_{zm} h) \quad (B.37)$$

where we assume that $k_{zp}$ and $k_{zm}$ are real. However, under some condition, $k_{zp}$ and $k_{zm}$ could be imaginary, according to Eq. (3.15). Therefore we have three different cases for the function $f(k_x, \omega)$.

**Case 1: Real $k_{zp}$ and $k_{zm}$**

In this case, from Eq. (3.15) we have:

$$-B + \sqrt{B^2 - 4D} \geq 0 \quad (B.38)$$

$$B - \sqrt{B^2 - 4D} \geq 0 . \quad (B.39)$$

$$\frac{\tanh(k_{zp} h)}{\tanh(k_{zm} h)} = \frac{G_m H_p}{G_p H_m} \quad (B.40)$$

The function $f(k_x, \omega)$ is thus expressed as:

$$f(k_x, \omega) = G_p H_m \sinh(k_{zp} h) \cosh(k_{zm} h) - G_m H_p \cosh(k_{zp} h) \sinh(k_{zm} h) \quad (B.41)$$

where $k_{zp}$ and $k_{zm}$ are:

$$k_{zp}^2 = \frac{k_x^2 [-B + \sqrt{B^2 - 4D}]}{2} \quad (B.42)$$

$$k_{zm}^2 = \frac{k_x^2 [-B - \sqrt{B^2 - 4D}]}{2} \quad (B.43)$$
and $H_p$ and $H_m$ are also real:

$$H_p = \frac{(-\rho \omega^2 + C_{11} k_x^2 + C_{13} k_{zp}^2)}{(C_{55} + C_{13}) k_{zp}^2}$$  \hspace{1cm} (B.44)$$

$$H_m = \frac{(-\rho \omega^2 + C_{11} k_x^2 + C_{13} k_{zm}^2)}{(C_{55} + C_{13}) k_{zm}^2}$$  \hspace{1cm} (B.45)

and $G_p$ and $G_m$ are also real:

$$G_p = \frac{C_{33} \rho \omega^2 + (C_{13} C_{55} + C_{13}^2 - C_{33} C_{11}) k_x^2 - C_{33} C_{55} k_{zp}^2}{(C_{55} + C_{13}) k_x}$$  \hspace{1cm} (B.47)$$

$$G_m = \frac{C_{33} \rho \omega^2 + (C_{13} C_{55} + C_{13}^2 - C_{33} C_{11}) k_x^2 - C_{33} C_{55} k_{zm}^2}{(C_{55} + C_{13}) k_x}$$  \hspace{1cm} (B.48)

**Case 2: Real $k_{zp}$, imaginary $k_{zm}$**

In this case, from Eq. (3.15) we have:

$$-B + \sqrt{B^2 - 4D} \geq 0$$  \hspace{1cm} (B.49)$$

$$B - \sqrt{B^2 - 4D} < 0$$  \hspace{1cm} (B.50)

The dispersion equation is correspondingly:

$$\frac{\tanh(k_{zp} h)}{\tan(k_{zm} h)} = \frac{G_m H_p}{G_p H_m}$$  \hspace{1cm} (B.51)$$

The function $f(k_x, \omega)$ is expressed as:

$$f(k_x, \omega) = G_p \bar{H}_m \sinh(k_{zp} h) \cos(k_{zm} h) - G_m H_p \cosh(k_{zp} h) \sin(k_{zm} h)$$  \hspace{1cm} (B.52)$$
where $k_{zp}$ and $\tilde{k}_{zm}$ are real ($k_{zm} = j\tilde{k}_{zm}$ is imaginary)

\[
k_{zp}^2 = \frac{k^2_x[-B + \sqrt{B^2 - 4D}]}{2}
\]

(B.53)

\[
\tilde{k}_{zm}^2 = \frac{k^2_x[B + \sqrt{B^2 - 4D}]}{2}
\]

(B.54)

and $H_p$ and $\tilde{H}_m$ are real ($H_m = j\tilde{H}_m$ is imaginary)

\[
H_p = \frac{(-\rho\omega^2 + C_{11}k_x^2 + C_{13}k_{zp}^2)}{(C_{55} + C_{13})k_{zp}}
\]

(B.55)

\[
\tilde{H}_m = \frac{(\rho\omega^2 - C_{11}k_x^2 + C_{13}\tilde{k}_{zm}^2)}{(C_{55} + C_{13})\tilde{k}_{zm}}
\]

(B.56)

and $G_p$ and $G_m$ are real:

\[
G_p = \frac{C_{33}\rho\omega^2 + (C_{13}C_{55} + C_{13}^2 - C_{33}C_{11})k_x^2 - C_{33}C_{55}k_{zp}^2}{(C_{55} + C_{13})k_x}
\]

(B.58)

\[
G_m = \frac{C_{33}\rho\omega^2 + (C_{13}C_{55} + C_{13}^2 - C_{33}C_{11})k_x^2 + C_{33}C_{55}\tilde{k}_{zm}^2}{(C_{55} + C_{13})k_x}
\]

(B.59)

**Case 3: Imaginary** $k_{zp}$ and $k_{zm}$

In this case, from Eq. (3.15) we have:

\[
-B + \sqrt{B^2 - 4D} < 0 \quad \text{(B.60)}
\]

\[
B - \sqrt{B^2 - 4D} < 0 \quad \text{(B.61)}
\]

The dispersion equation is correspondingly:

\[
\frac{\tan(\tilde{k}_{zp}h)}{\tan(\tilde{k}_{zm}h)} = \frac{G_m\tilde{H}_p}{G_pH_m}
\]

(B.62)
The function \( f(k_x, \omega) \) is expressed as:

\[
f(k_x, \omega) = G_p \bar{H}_m \sin(\bar{k}_{zp} h) \cos(\bar{k}_{zm} h) - G_m \bar{H}_p \cos(\bar{k}_{zp} h) \sin(\bar{k}_{zm} h)
\]  

(B.63)

where \( \bar{k}_{zp} \) and \( \bar{k}_{zm} \) are real (\( k_{zp} = j \bar{k}_{zp} \) is imaginary, \( k_{zm} = j \bar{k}_{zm} \) is imaginary)

\[
\bar{k}_{zp}^2 = \frac{k_z^2 [B - \sqrt{B^2 - 4D}]}{2}
\]  

(B.64)

\[
\bar{k}_{zm}^2 = \frac{k_z^2 [B + \sqrt{B^2 - 4D}]}{2}
\]  

(B.65)

and \( \bar{H}_p \) and \( \bar{H}_m \) are real (\( H_p = j \bar{H}_p \) is imaginary, \( H_m = j \bar{H}_m \) is imaginary)

\[
\bar{H}_p = \frac{\rho \omega^2 - C_{11} k_x^2 + C_{13} \bar{k}_{zp}^2}{(C_{55} + C_{13}) \bar{k}_{zp}}
\]  

(B.66)

\[
\bar{H}_m = \frac{\rho \omega^2 + C_{11} k_x^2 + C_{13} \bar{k}_{zm}^2}{(C_{55} + C_{13}) \bar{k}_{zm}}
\]  

(B.67)

and \( G_p \) and \( G_m \) are real:

\[
G_p = \frac{C_{33} \rho \omega^2 + (C_{13} C_{55} + C_{13}^2 - C_{33} C_{11}) k_x^2 + C_{33} C_{55} \bar{k}_{zp}^2}{(C_{55} + C_{13}) k_x}
\]  

(B.69)

\[
G_m = \frac{C_{33} \rho \omega^2 + (C_{13} C_{55} + C_{13}^2 - C_{33} C_{11}) k_x^2 + C_{33} C_{55} \bar{k}_{zm}^2}{(C_{55} + C_{13}) k_x}
\]  

(B.70)

### B.0.4 Cutoff Frequencies

Cut-off frequencies are utilized in the graphical representation of a dispersive system.

The cut-off frequencies refer to the frequencies \( (\omega_c) \) at which the phase velocity becomes infinitely large \( (c_p \to \infty) \) or the wavenumber approaches zero \( (k_x \to 0) \). Cut-off frequencies are determined using the dispersion equations, Eq. (3.33) and (3.34), when
\( k_x = 0 \). Notice that we consider this issue in region III where \( k_{zp} \) and \( k_{zm} \) are both imaginary. For this, the parameters \( \tilde{k}_{zp,m} \), \( G_{p,m} \), and \( H_{p,m} \) must be calculated.

From Eqs. (3.11) and (3.12), it follows that when \( k_x = 0 \), the amplitude terms cancel out, and:

\[
\rho \omega^2 = C_{33} \tilde{k}_{zp}^2 \quad \text{(B.71)}
\]

and

\[
\rho \omega^2 = C_{55} \tilde{k}_{zm}^2 \quad \text{(B.72)}
\]

where we assume that \( C_{55} \geq C_{33} \). \(^1\)

In this case \((k_x = 0)\), Eq. (3.30) may be multiplied by \( k_x \) to ensure the denominators are non-zero, and \( G_p \) and \( G_m \) can be obtained as

\[
k_x G_p = C_{33} \tilde{k}_{zp} R_p k_x + C_{13} k_x^2 \\
= \frac{C_{33}(\rho \omega^2 - C_{55} \tilde{k}_{zp}^2)}{(C_{55} + C_{13})} \\
= \frac{C_{33}(C_{33} \tilde{k}_{zp}^2 - C_{55} \tilde{k}_{zp}^2)}{(C_{55} + C_{13})} \quad \text{(B.73)}
\]

\[
= \frac{\rho \omega^2(C_{33} - C_{55})}{(C_{55} + C_{13})}
\]

\(^1\)In our analysis, we assume that both \( k_{zp} \) and \( k_{zm} \) are real. From Eq. (3.15) we can see that \( \tilde{k}_{zp} \geq \tilde{k}_{zm} \).
and

\[ k_x G_m = C_{33} \bar{k}_{zm} R_m k_x + C_{13} k_x^2 \]
\[ = \frac{C_{33}(\rho \omega^2 - C_{55} \bar{k}_{zm}^2)}{(C_{55} + C_{13})} \]
\[ = \frac{C_{33}(C_{55} \bar{k}_{zm}^2 - C_{55} \bar{k}_{zm}^2)}{(C_{55} + C_{13})} \] \hspace{1cm} (B.74)
\[ = 0 \]

Similarly, \( \bar{H}_p \) and \( \bar{H}_m \) can be obtained as

\[ \bar{H}_p = \frac{(\rho \omega^2 + C_{13} \bar{k}_{zp}^2)}{(C_{55} + C_{13})\bar{k}_{zp}} \]
\[ = \frac{(C_{33} \bar{k}_{zp}^2 + C_{13} \bar{k}_{zp}^2)}{(C_{55} + C_{13})\bar{k}_{zp}} \] \hspace{1cm} (B.75)
\[ = \frac{\bar{k}_{zp}(C_{33} + C_{13})}{(C_{55} + C_{13})} \]

and

\[ \bar{H}_m = \frac{(\rho \omega^2 + C_{13} \bar{k}_{zm}^2)}{(C_{55} + C_{13})\bar{k}_{zm}} \]
\[ = \frac{(C_{55} \bar{k}_{zm}^2 + C_{13} \bar{k}_{zm}^2)}{(C_{55} + C_{13})\bar{k}_{zm}} \] \hspace{1cm} (B.76)
\[ = \bar{k}_{zm} \]

**Symmetric modes:**

The dispersion equation for symmetric modes can be rewritten as

\[ \frac{\sin(\bar{k}_{zp}h)}{\cos(\bar{k}_{zm}h)} \frac{\cos(\bar{k}_{zm}h)}{\sin(\bar{k}_{zp}h)} = \frac{k_x G_p \bar{H}_m}{k_x G_m \bar{H}_p} \] \hspace{1cm} (B.77)
The function in terms of \( k_x \) and \( \omega \) can be thus obtained as

\[
f(k_x, \omega) = k_x G_p \bar{H}_m \cos(k_{zp} \theta) \sin(k_{zm} \theta) - k_x G_m \bar{H}_p \sin(k_{zp} \theta) \cos(k_{zm} \theta)
\]

(B.78)

Notice that \( k_x G_m = 0 \) for \( k_x = 0 \), we have

\[
f(0, \omega_c) = -k_x G_p \bar{H}_m \cos(k_{zp} \theta) \sin(k_{zm} \theta)
\]

(B.79)

Therefore \( f(0, \omega_c) = 0 \) gives the condition for cut-off frequencies \( \omega_c \) (assume \( C_{33} \neq C_{55} \))

\[
\cos(k_{zp} \theta) \sin(k_{zm} \theta) = 0
\]

(B.80)

This can be satisfied if

\[
\cos(k_{zp} \theta) = 0, \quad k_{zp} \theta = \frac{p\pi}{2} \quad (p = 1, 3, 5, \ldots) ,
\]

(B.81)

or

\[
\sin(k_{zm} \theta) = 0, \quad k_{zm} \theta = \frac{q\pi}{2} \quad (q = 0, 2, 4, \ldots).
\]

(B.82)

Substituting Eqs. (B.71) and (B.72) into Eqs. (B.81) and (B.82), the cut-off frequencies for symmetric modes become:

\[
\omega_c \theta = \frac{p\pi}{2} \sqrt{\frac{C_{33}}{\rho}}, \quad (p = 1, 3, 5, \ldots)
\]

(B.83)

\[
\omega_c \theta = \frac{q\pi}{2} \sqrt{\frac{C_{55}}{\rho}}, \quad (q = 0, 2, 4, \ldots)
\]

(B.84)
Antisymmetric modes:

The dispersion equation for antisymmetric modes can be rewritten as

\[
\frac{\sin(k_{zp}h) \cos(k_{zm}h)}{\cos(k_{zm}h) \sin(k_{zp}h)} = \frac{k_x G_m \tilde{H}_p}{k_x G_p \tilde{H}_m}
\]  

(B.85)

The function in terms of \( k_x \) and \( \omega \) can be obtained as

\[
f(k_x, \omega) = k_x G_p \tilde{H}_m \sin(k_{zp}h) \cos(k_{zm}h) - k_x G_m \tilde{H}_p \cos(k_{zp}h) \sin(k_{zm}h)
\]  

(B.86)

Notice that \( k_x G_m = 0 \) for \( k_x = 0 \), we have

\[
f(0, \omega_c) = k_x G_p \tilde{H}_m \sin(k_{zp}h) \cos(k_{zm}h)
\]  

(B.87)

Therefore \( f(0, \omega_c) = 0 \) gives the condition for cut-off frequencies \( \omega_c \) (assume \( C_{33} \neq C_{55} \))

\[
\sin(k_{zp}h) \cos(k_{zm}h) = 0
\]  

(B.88)

This can be satisfied if

\[
\sin(k_{zp}h) = 0, \quad k_{zp}h = \frac{p\pi}{2}, \quad (p = 0, 2, 4, \cdots),
\]  

(B.89)

or

\[
\cos(k_{zm}h) = 0, \quad k_{zm}h = \frac{q\pi}{2}, \quad (q = 1, 3, 5, \cdots).
\]  

(B.90)
Substituting Eqs. (B.71) and (B.72) into Eqs. (B.89) and (B.90), the cut-off frequencies for antisymmetric modes become:

\[ \omega_c h = \frac{p\pi}{2} \sqrt{\frac{C_{33}}{\rho}}, \quad (p = 0, 2, 4, \ldots) \quad \text{(B.91)} \]

\[ \omega_c h = \frac{q\pi}{2} \sqrt{\frac{C_{55}}{\rho}}, \quad (q = 1, 3, 5, \ldots) \quad \text{(B.92)} \]
Appendix C

Formulating the Stress-Strain Relationship

This appendix describes the simplifying steps utilized in stating the stress-strain relation for various types of materials, depending upon its state of isotropy. Daniel and Ishai have a thorough and detailed summary in their text (Daniel and Ishai 1994). The conclusion of this section is a reduced stiffness matrix containing 5 constants. These constants fully characterize a material. The material properties utilized in the previous chapter arise from simplifying assumptions based upon the isotropic nature of the material. A similar set of assumptions must be formulated in order to generate the dispersion relation for an anisotropic material. This section describes this process. Any point in a continuum can be characterized by nine stress components ($\sigma_{ij}$, where $i, j = 1, 2, 3$). Similarly, the state of deformation is characterized by nine strain components ($\epsilon_{ij}$). Hooke’s Law relates the stress and strain components by equating:

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}, \quad \text{for} \quad i, j, k, l = 1, 2, 3$$

(C.1)

The $C$ tensor in the above equation is populated with the stiffness components of a given material. In the case of a general, anisotropic composite, where there are no planes of
Appendix C

Formulating the Stress-Strain Relationship

symmetry, all strain components contribute to the overall stress in a material. To fully characterize the material, 81 elastic constants must be determined. The generalized Hooke’s law contains a fully populated stiffness matrix:

\[
\begin{pmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{pmatrix}
=\begin{pmatrix}
C_{1111} & C_{1122} & C_{1133} & C_{1212} & C_{1113} & C_{1132} & C_{1123} & C_{1112} & C_{1131} & C_{1121} \\
C_{2211} & C_{2222} & C_{2233} & C_{2213} & C_{2212} & C_{2232} & C_{2231} & C_{2221} \\
C_{3311} & C_{3322} & C_{3333} & C_{3313} & C_{3312} & C_{3332} & C_{3331} & C_{3321} \\
C_{2311} & C_{2322} & C_{2333} & C_{2313} & C_{2312} & C_{2332} & C_{2331} & C_{2321} \\
C_{1311} & C_{1322} & C_{1333} & C_{1313} & C_{1312} & C_{1332} & C_{1331} & C_{1321} \\
C_{1211} & C_{1222} & C_{1233} & C_{1213} & C_{1212} & C_{1232} & C_{1231} & C_{1221} \\
C_{3211} & C_{3222} & C_{3233} & C_{3213} & C_{3212} & C_{3232} & C_{3231} & C_{3221} \\
C_{3111} & C_{3122} & C_{3133} & C_{3113} & C_{3112} & C_{3132} & C_{3131} & C_{3121} \\
C_{2111} & C_{2122} & C_{2133} & C_{2113} & C_{2112} & C_{2132} & C_{2131} & C_{2121}
\end{pmatrix}
\begin{pmatrix}
\epsilon_{11} \\
\epsilon_{22} \\
\epsilon_{33} \\
\epsilon_{23} \\
\epsilon_{13} \\
\epsilon_{12} \\
\epsilon_{32} \\
\epsilon_{31} \\
\epsilon_{21}
\end{pmatrix}
\]

(C.2)

Every position in the C-tensor is filled with the appropriate stiffness constant.

With closer inspection of an elementary cube, symmetry must exist between the stress and strain tensors in order for the element to remain in equilibrium:

\[
\sigma_{ij} = \sigma_{ji} \\
\epsilon_{ij} = \epsilon_{ji}
\]

(C.3)

The lower three rows of the generalized Hooke’s relation are omitted, and the number of stiffness constants reduces to 36. The standard notation simplifies the stiffness matrix.

\[C_{1111} = C_{11}, \quad C_{1122} = C_{12}, \quad C_{1133} = C_{13}, \quad C_{1123} = 2C_{14}, \quad \cdots, \quad C_{1121} = C_{61}\]

The stress-strain
relation can be written as:

\[
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12}
\end{pmatrix} = \begin{pmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\
C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\
C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}
\end{pmatrix}
\begin{pmatrix}
\epsilon_{11} \\
\epsilon_{22} \\
\epsilon_{33} \\
\epsilon_{23} \\
\epsilon_{13} \\
\epsilon_{12}
\end{pmatrix}
\] (C.4)

Or, in tensor notation:

\[
\sigma_i = C_{ij} \epsilon_j, \quad \text{for} \quad i, j = 1, 2, 3, 4, 5, 6
\] (C.5)

Other symmetries within the stiffness matrix arise from energy considerations. The formula for work per unit volume is:

\[
W = \frac{1}{2} C_{ij} \epsilon_i \epsilon_j
\] (C.6)

The derivative of the work formula with respect to \(\epsilon_i\), then \(\epsilon_j\), yields \(C_{ij}\).

\[
\frac{\partial^2 W}{\partial \epsilon_i \partial \epsilon_j} = C_{ij}, \quad \text{for} \quad i, j = 1, 2, 3
\] (C.7)

By reversing the order of differentiation, the same answer is derived. Thus, the order of differentiation of \(W\) is immaterial, and:

\[
C_{ij} = C_{ji}
\] (C.8)

This symmetry in the stiffness matrix forces the lower triangle of constants to be equivalent to the upper triangle. The important result is a reduction to 21 independent stiffness constants. The resulting stress-strain relation for a general anisotropic composite material
The first simplifying assumption made is that the material is specially orthotropic. Condition: The condition that allows this assumption to be valid for a composite material is that they are fabricated in exact composite laminae. Thus, the material has three mutually perpendicular axis of material symmetry.

\[
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12}
\end{pmatrix}
= 
\begin{pmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & 2C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & 2C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & 2C_{66}
\end{pmatrix}
\begin{pmatrix}
\epsilon_{11} \\
\epsilon_{22} \\
\epsilon_{33} \\
\epsilon_{23} \\
\epsilon_{13} \\
\epsilon_{12}
\end{pmatrix}
\] 

\( (C.9) \)

Result: The stiffness matrix reduces considerably. Nine independent stiffness constants will fully characterize a material. The second simplifying assumption made is that the material is transversely isotropic. Condition: The first condition that allows this assumption to be valid for a composite material is that the material is specially orthotropic, as described above. Additionally, one of the material’s principal planes is a plane of isotropy. Many unidirectional composites with fibers packed in a hexagonal array, with a high fiber volume ratios, may be considered transversely isotropic. The 2-3 plane (transverse to the fibers) is the plane of isotropy.
At every point there is a plane on which the mechanical properties are the same in all directions. By assuming the 2-3 plane is the plane of isotropy, subscripts 2 and 3 become interchangeable, as well as 5 and 6. Thus,

\[ C_{12} = C_{13} = C_{22} = C_{33} = C_{55} = C_{66} \]  

(C.11)

The stiffness constant, \( C_{44} \) is not independent.

\[ C_{44} = \frac{(C_{22} - C_{23})}{2} \]  

(C.12)

The stress-strain relation reads:

\[
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12}
\end{pmatrix} =
\begin{pmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{22} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{(C_{22} - C_{23})}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{55}
\end{pmatrix}
\begin{pmatrix}
\epsilon_{11} \\
\epsilon_{22} \\
\epsilon_{33} \\
\epsilon_{23} \\
\epsilon_{13} \\
\epsilon_{12}
\end{pmatrix}
\]  

(C.13)

*Result:* The stiffness matrix reduces further. Five independent stiffness constants will fully characterize a material.