Turbulent Wave-Current Boundary Layers Revisited

by

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Abstract

A theoretical model for turbulent boundary layers in wave-current flows is developed using a time-invariant eddy viscosity formulation. The model allows for the no-slip condition to be applied either at \( z = z_0 \) or \( z = 0 \). The eddy viscosity definition provides for a continuous transition between the region where the turbulence is dominated by wave motion, to the region where turbulence is dominated by the current. The start of the transition is not chosen as a constant factor of the boundary layer length scale, as in previous eddy viscosity models. Instead, it is chosen as a pre-set fraction of the boundary layer height as determined from boundary layer experiments. Thus, the model presented here is does not have fitting parameters, i.e. it is an entirely predictive model. The resulting height of the transition depends on the relative bottom roughness and the magnitude of the current shear stress relative to the maximum combined wave and current shear stress. Because the model incorporates this, the results show good agreement with data from pure waves and waves and currents over a large range of bottom roughnesses; from rippled bottoms, to flat sand bottoms, and even smooth beds. For the application of the model to practical problems, simple analytical formulas are derived, the solution procedures outlined and two examples shown.

Thesis Supervisor: Ole S. Madsen
Title: Professor, Department of Civil and Environmental Engineering
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Chapter 1

Introduction

1.1 Background

Coasts are the setting of many commercial and recreational activities. They are also the habitat of innumerable species of organisms. As with all natural environments, it is crucial to protect them while we make intelligent use of their resources. To do this effectively, it is necessary to understand the physical processes involved in the continually transforming coast. Quantitative modeling of these processes is essential to assist society in solving the problems occurring in the coastal zone.

One of the processes in question is sediment transport. Coastal sediment transport is of concern because it is a factor entering navigational channel maintenance, shore protection, beach integrity and water quality. The modeling of the velocity field throughout the water depth is essential to the prediction of sediment transport. This velocity field is affected by the combined action of waves and currents. Waves are typically the result of wind forcing on the water surface. Currents may be the result of tides, wind, density variations, river outflows and wave action. Both waves and currents can transfer energy to the sediment in the bottom.

It has been observed that the superposition of waves and a current increases the turbulence intensity at the bottom. This transmits more energy to the sediment grains
and makes them more likely to go into suspension. While the sediments are in suspension, the current can move them to another location. In the general case, sediment suspension is due to wave action, while bulk sediment transport is due to the current. To comprehend how this mechanism works, the flow dynamics at the interface between the water column and the sea floor bottom have to be understood. This interface is called the wave-current bottom boundary layer. A boundary layer is the region where velocity goes from zero at the bottom to the height where viscous effects are no longer significant. The characteristics of the wave-current bottom boundary layer are an integral part of sediment transport models for the coastal zone.

The wave-current bottom boundary layer can be studied as two separate components: one is a very thin unsteady oscillatory layer, while the other is a steady thick layer. The steady component is related to the current and is called the current bottom boundary layer. The oscillatory component is related to the waves and is called the wave bottom boundary layer. The difference in height between these two components is related to their different time scales. The velocity gradients within the wave boundary layer are relatively large compared to the velocity gradient of the current boundary layer. Since shear stress is proportional to velocity gradients, the shear stress due to the unsteady wave action is large compared to the shear stress due to the current. This is why waves dominate the turbulence at the bottom. The increased turbulence at the bottom makes the sediments go into suspension more easily.

Besides causing sediment suspension, the back and forth movement of waves also promotes changes in the bottom shape. The presence of bedforms increases turbulence intensities even more. The increased turbulence and shear due to the wave action results in an increased flow resistance for the current.

Theoretical models for turbulent boundary layers use wave, current, and bottom characteristics to predict shear stresses, boundary layer thickness, velocity profiles, energy dissipation, etc. The waves are usually considered monochromatic and described by the wave bottom orbital velocity, $U_b$, and the wave period, $T$. The current is usually
considered a quasi steady flow and described by the current shear stress, $\tau_c$, the mean current velocity, or a velocity defined at a certain elevation above the bottom. A bottom roughness length scale, $k_a$, is often used to describe the properties of the bed.

Viscosity is what relates the stresses between fluid particles to the kinematics of motion. For laminar flows, the viscosity is only dependent on the nature of the fluid. The eddy viscosity concept for turbulent flow, developed by Boussinesq, is similar to the laminar flow viscosity. It relates the stresses to the kinematics, but instead of being a constant value like the laminar viscosity, the eddy viscosity will be a function of the flow intensity itself, i.e. a function of space and time. Of the existing turbulent boundary layer models, eddy viscosity models are the simplest and most common. These models achieve closure by defining an eddy viscosity usually chosen to be a function of height above bottom and time-invariant. Certain aspects of the results of these models are very sensitive to the formulation of their eddy viscosity.

1.2 Previous Studies

Different types of theoretical models have been developed for oscillatory boundary layers. Mixing length models, advanced turbulence models and eddy viscosity models are among them. Eddy viscosity models are the most commonly used for their applicability and simplicity. These models usually scale the eddy viscosity with a time-invariant shear velocity and the distance from the bed. Shear velocities are defined as: $u_* = \sqrt{\tau/\rho}$, where $\tau$ is the bottom shear stress and $\rho$ is the fluid density. Examples of time-invariant eddy viscosity models scaled by shear velocities and distance from the bottom include: Kajiura (1968), Smith (1977), Tanaka and Shuto (1981), Christoffersen and Jonsson (1985), Grant and Madsen (1979, 1986), Madsen and Wikramanayake (1991), and Madsen and Salles (1998). Trowbridge and Madsen (1984) incorporated time variation into the eddy viscosity formulation. Davies et al. (1988) used a numerical turbulence model that is equivalent to a sophisticated time-varying eddy viscosity.
The present study is based on the work of Grant and Madsen (1986), Madsen and Wikramanayake (1991), and Madsen and Salles (1998). These three models will be discussed in some detail in the following sub-sections.

### 1.2.1 Original Grant-Madsen Model

Grant and Madsen (1986) defines a time-invariant discontinuous eddy viscosity. For the region inside the wave boundary layer, the eddy viscosity is linearly proportional to the shear velocity based on the maximum combined wave and current shear stress. For the region outside the wave boundary layer, the eddy viscosity is linearly proportional to a shear velocity based on the current shear stress.

To obtain the current velocity profile, the following eddy viscosity formulation is used

\[
\nu_t = \begin{cases} 
\kappa u_{sm} z & z \leq A_l \\
\kappa u_{sc} z & A_l \leq z 
\end{cases}
\]  

(1.1)

where \( u_{sm} \) is the shear velocity based on the maximum combined shear stress, \( u_{sc} \) is the shear velocity based on the current bottom shear stress, \( z \) is the distance from the bottom and \( \kappa \) is von Karman’s constant, which is a non-dimensional number that has a value of approximately 0.4. The model’s definition of the wave boundary layer height is \( z = A_l \). The term \( A \) is the fitting parameter of this model. The wave boundary layer length scale, \( l \), is defined as

\[
l = \frac{\kappa u_{sm}}{\omega}
\]

(1.2)

where \( \omega \) is the radian frequency of the periodic wave.

For the wave problem, (1.1) is simplified to \( \nu_t = \kappa u_{sm} z \). This eddy viscosity formulation is assumed valid for all \( z \) above the bottom. The difference in the definition of the eddy viscosity between the wave problem and the current problem is an inconsistency of the model. In addition, it is known that the wave-associated turbulence should approach
zero at the upper edge of the wave boundary layer. Therefore the constantly increasing eddy viscosity for all heights above bottom used for the solution of the wave problem is physically unrealistic.

The bottom boundary condition for the Grant-Madsen model, and most eddy viscosity models, is the no-slip condition, $u = 0$ at $z = z_o = k_n/30$, used in rough turbulent flow. This is not a problem for small roughnesses, but as seen in Madsen and Salles (1998) it is not applicable to large roughnesses because it leaves a considerable gap in the velocity profile for $z < z_o$.

Experimental data were used to select the best value for the fitting parameter $A$. Grant and Madsen (1986) suggests the best value for $A$ to be between 1.0 and 2.0, but no value of $A$ worked well for all data sets. The selection of a reasonable value for $A$ is a major issue when using this model. It defines the height where the eddy viscosity changes from a large value to a small value, and thus, it has a significant effect on the results.

The use of the Grant-Madsen model results in a logarithmic current velocity profile. It works well for distances very close to the bottom, i.e. for the region that is mainly influenced by the wave action. It also gives good results for heights within the current boundary layer that are not influenced by the wave boundary layer. It does not predict well the current velocity for the region in between. Also, the current velocity profile is not smooth because the eddy viscosity is discontinuous. This is of concern because the velocity gradient is important for sediment transport. This suggests that the formulation of the eddy viscosity in the Grant-Madsen model is too simple.

### 1.2.2 Improved Grant-Madsen Model

Based on the original Grant-Madsen model, Madsen and Wikramanayake (1991) developed an improved time-invariant continuous eddy viscosity model. In the region closest to the bottom, the eddy viscosity increases linearly proportional to the maximum combined shear velocity and the distance above the bottom. In the intermediate region, the
eddy viscosity is constant. The eddy viscosity in the upper-most region increases linearly with height and is proportional to the current shear velocity.

\[
\nu_t = \begin{cases} 
\kappa u_{sm} z & z \leq \alpha l \\
\kappa u_{sm} \alpha l & \alpha l \leq z \leq \alpha l / \epsilon \\
\kappa u_{sc} z & \alpha l / \epsilon \leq z
\end{cases}
\]  \hspace{1cm} (1.3)

A new parameter \( \epsilon \) is introduced in this improved model. It gives the relative magnitude of the current shear stress to the maximum combined shear stress,

\[
\epsilon = \frac{u_{sc}}{u_{sm}}
\]  \hspace{1cm} (1.4)

The parameter \( \alpha \) gives the fraction of the length \( l \) where the eddy viscosity varies linearly. It is the only fitting parameter of this model and thus replaces the fitting parameter \( A \) in the original Grant-Madsen model (1.1). The intermediate region is intended to be a transition between the zone where the wave turbulence is dominant to the zone where the current turbulence is dominant. The value of the eddy viscosity in this region depends on the chosen value for \( \alpha \). The parameter \( \epsilon \) scales the height of the transition zone.

By comparison with experimental data, a single value of \( \alpha = 0.5 \) was chosen by Madsen and Wikramanayake (1991). This value was a compromise in that prediction of wave orbital velocities for moderate to small values of the bottom roughness suggested a value of \( \alpha \approx 0.15 \), whereas comparison with data on currents in the presence of waves suggested \( \alpha \approx 0.5 \) or larger.

The improved Grant-Madsen model has a more complicated solution than the original Grant-Madsen model, but better physical sense. The benefits of the improved model include the use of the same definition of the eddy viscosity for the wave and current problems, and the use of a continuous eddy viscosity which results in a smooth current velocity profile. Furthermore, the results agree better with the data and there is still only one fitting parameter. The disadvantages of this model include that no single value
of $\alpha$ works well for all data sets and that it still has not solved the problem of large roughnesses, since it imposes the no-slip condition at $z = z_0$.

### 1.2.3 Madsen-Salles Hybrid Model

Madsen and Salles (1998) developed an eddy viscosity model to predict the details of the velocity profile for pure waves over very rough bottoms. They found that the orbital velocity profiles were well predicted for a constant eddy viscosity. However, this leads to a phase lag of shear stress relative to the free stream velocity of 45° which is known to be in error for smaller roughness values. To cover the entire range of roughnesses they propose a hybrid model similar to that of Madsen and Wikramanayake (1.3),

$$\nu_t = \begin{cases} 
\kappa u_{*m} (z + z_o) & z \leq \alpha l \\
\kappa u_{*m} (\alpha l + z_o) & z \geq \alpha l 
\end{cases}$$

(1.5)

Since this model was developed for waves alone, $u_{*m}$ corresponds to the shear velocity based on the maximum wave bottom shear stress. This eddy viscosity is linearly varying in the lower portion of the boundary layer and constant from a set height and above. The addition of $z_o$ to the length scale makes $u = 0$ at $z = 0$ instead of $z = z_o$. For small roughnesses this difference is insignificant, but it improved considerably the comparison with measured orbital velocity profiles for the large roughnesses of Mathisen and Madsen (1996). The reason is that this model avoids the prediction of the negative velocities for $z < z_o$ obtained when (1.3) is adopted. Partially guided by Madsen and Wikramanayake (1991) and also to approximately match their values of the constant eddy viscosity model obtained for large roughnesses, Madsen and Salles (1998) chose $\alpha = 0.5$.

Madsen and Salles (1998) also investigated the boundary layer thickness predicted by their hybrid model (1.3). They found that the wave boundary layer thickness, $\delta$, is not simply proportional to the boundary layer scaling length $l$ given by (1.2). Their results
suggest that $\delta = A l$ with

$$A = \exp \left[ 2.96 \left( \frac{A_b}{k n} \right)^{-0.071} - 1.45 \right]$$

(1.6)

where $A_b = U_b/\omega$ is the excursion amplitude of the bottom orbital velocity. Incorporating this finding into the original Grant-Madsen model, i.e. choosing the transition height $A l$ in (1.1) with $A$ given by (1.6), produced acceptable agreement between the predictions and the observations of Mathisen and Madsen (1996) of currents in the presence of waves over artificially rippled beds.

Despite the success of this extension of the Madsen-Salles' pure wave model to combined wave-current boundary layer flows, it should be emphasized that this extension is fundamentally inconsistent. The inconsistency is that the value of $A$, obtained from (1.6) and therefore based on the Madsen-Salles eddy viscosity model (1.5), is used in a different eddy viscosity model, namely the original Grant-Madsen model (1.1).

The main improvement of this model is that the bottom boundary condition is set to $u = 0$ at $z = 0$. This prevents the prediction of negative velocities for small $z$. Also, the idea that the boundary layer thickness is a function of the relative roughness makes a lot of physical sense. Drawbacks of this model are that it was developed for pure waves only and that $\alpha$ was chosen as a constant and not as a function of the wave boundary layer height $A l$.

### 1.3 Objectives

The objective of the present study is to develop an analytical model to describe the wave-current boundary layer interaction that incorporates the best features of the previously discussed models and improves their deficiencies. Since a trade-off exists between model simplicity and general applicability, this study attempts to achieve a balance between representation of physical reality and computational effort. The ultimate objective is a
model that is useful and easily incorporated in numerical circulation models for sediment transport problems.

Only the case of turbulent flow near a rough bed will be considered. The simplifying assumptions are: (1) constant water depth, (2) plane progressive monochromatic waves, (3) a steady current and (4) a non-movable bed. Also, outside the wave boundary layer the wave motion is governed by potential theory. Additionally, linear wave theory is assumed to be valid.

The features of previous models that we wish to maintain are various. We seek to maintain internal consistency by using the same definition of the eddy viscosity for both wave and current problems. We will keep using a time-invariant eddy viscosity to make computations simpler. Also, we wish to have a physically sound continuous eddy viscosity to prevent discontinuity in the velocity gradients. Another feature that we wish to include is the bottom boundary condition set at $z = 0$ for bottoms with very large roughnesses to prevent negative velocity predictions.

Following Madsen and Salles (1998) we will further explore the relationship of boundary layer thickness to relative roughness. Based on the results, we will try to develop a model that is applicable to a large range of bottom roughnesses. The main disadvantage of all the previous models is that the transition height has been set as a constant fraction of the length scale $l$. This study seeks to improve this by choosing the transition height as a fraction of the boundary layer thickness, and having the boundary layer thickness be determined by the model, i.e. the model we seek will have no free parameter to be determined by fitting the model to experimental data.

1.4 Thesis Outline

In Chapter 2, the hydrodynamic model for turbulent wave-current boundary layer flows is developed. The chapter starts with the linearized governing equation for the bottom boundary layer. The concept of an eddy viscosity is used to relate the shear stress
to the rate of strain. Assuming a time-invariant eddy viscosity, the time-varying and
time-invariant components of the governing equation are separated into two independent
equations. A new eddy viscosity is formulated to solve these equations analytically.
Then, using the new eddy viscosity definition and the boundary and matching conditions,
the equation governing the waves and the equation governing the current are solved in
terms of the basic parameters. A closure hypothesis is developed towards the end of the
chapter, and the friction factor concept is introduced to facilitate the solution of practical
problems. Finally, the procedure for obtaining the friction factor diagram is outlined in
the last section.

The results of the model are shown in the Chapter 3. Friction factor diagrams and
diagrams of other pertinent quantities of the model are presented. Also, approximate for-
mulas to represent these results analytically are developed. The procedure for obtaining
a prediction of the current velocity profile using the approximate equations is outlined.
The parameters needed to solve the problem are specified and the corresponding equa-
tions are shown. Finally, example calculations are given for the two possible methods of
specifying the current.

In Chapter 4, the predictions of the model are compared with experimental data.
The data sets come from laboratory experiments and numerical models. The results for
pure waves are presented in terms of the velocity amplitude and its phase relative to the
free stream vs. distance above the bottom. The current velocity profiles are presented
for data sets for waves and currents. Predictions of the current and maximum combined
shear velocities are also shown.

In Chapter 5, the results are summarized, the conclusions presented and future de-
velopments are suggested.
Chapter 2

Model Development

In this chapter we will discuss the development of a hydrodynamic model for turbulent wave-current boundary layer flows. The chapter starts with the linearized governing equation for the bottom boundary layer. The concept of an eddy viscosity is used to relate the shear stress to the rate of strain. Assuming a time-invariant eddy viscosity, the time-varying and time-invariant components of the governing equation are separated into two independent equations. A new eddy viscosity is formulated to solve these equations analytically. Then, using the new eddy viscosity definition and the boundary and matching conditions, the equation governing the waves and the equation governing the current are solved in terms of the basic parameters. A closure hypothesis is developed towards the end of the chapter, and the friction factor concept is introduced to facilitate the solution of practical problems. Finally, the procedure for obtaining the friction factor diagram is outlined at the end of the chapter.

2.1 Governing Equations

The linearized governing equation for the wave-current boundary layer can be written as

\[ \rho \frac{\partial \vec{u}}{\partial t} = -\nabla p + \frac{\partial \tau}{\partial z} \]  

(2.1)
where
\[ \vec{u} = \text{horizontal velocity vector} = \{u, v\} \]
\[ \rho = \text{fluid density} \]
\[ \nabla = \text{gradient operator} = \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\} \]
\[ p = \text{pressure} \]
\[ \vec{\tau} = \text{shear stress vector on horizontal planes} \]
\[ z = \text{height above bottom} \]

The linearized equation for boundary layer flow assumes uniform flow and thus neglects convective accelerations. Based on the stress-strain relationships for Newtonian fluids, a relationship between the viscous shear stress and the rate of strain can be defined for turbulent flows as

\[ \frac{\vec{\tau}}{\rho} = \nu_t \frac{\partial \vec{u}}{\partial z} \quad (2.2) \]

where \( \nu_t \) is a "virtual" or eddy viscosity. This term can be modeled as a function of space, time or both. Using (2.2) in the linearized boundary layer equation (2.1) we obtain

\[ \frac{\partial \vec{u}}{\partial t} = -\frac{1}{\rho} \nabla p + \frac{\partial}{\partial z} \left[ \nu_t \frac{\partial \vec{u}}{\partial z} \right] \quad (2.3) \]

which is an expression analogous to the Navier-Stokes equation for laminar flows, but it does not assume a constant viscosity.

For a time-invariant eddy viscosity, each term in (2.3) can be considered as the sum of an oscillatory component and a time-invariant component. The oscillatory component is related to the waves, while the time-invariant component is related to the current,

\[ \frac{\partial}{\partial t} \vec{u}_w = -\frac{1}{\rho} \nabla p_c - \frac{1}{\rho} \nabla p_w + \frac{\partial}{\partial z} \left[ \nu_t \frac{\partial \vec{u}_c}{\partial z} \right] + \frac{\partial}{\partial z} \left[ \nu_t \frac{\partial \vec{u}_w}{\partial z} \right] \quad (2.4) \]

Provided the eddy viscosity, \( \nu_t \), is independent of time, the time-varying components in (2.4) are independent of the time-invariant components and can be separated to form
two independent equations:

\[
\frac{\partial \bar{u}_w}{\partial t} = -\frac{1}{\rho} \nabla p_w + \frac{\partial}{\partial z} \left[ \nu_t \frac{\partial \bar{u}_w}{\partial z} \right]
\]  

(2.5)

and

\[
0 = -\frac{1}{\rho} \nabla p_c + \frac{\partial}{\partial z} \left[ \nu_t \frac{\partial u_c}{\partial z} \right]
\]  

(2.6)

2.1.1 Wave Problem

To solve the time-varying problem, we start with the governing equation for the waves (2.5). To make the analysis simpler, we choose the x-direction to be the same as the propagating wave, so \( \bar{u}_w = (u_w)_x = u_w \).

\[
\frac{\partial u_w}{\partial t} = -\frac{1}{\rho} \frac{\partial p_w}{\partial x} + \frac{\partial}{\partial z} \left[ \nu_t \frac{\partial u_w}{\partial z} \right]
\]  

(2.7)

At the outer edge of the wave boundary layer viscous shear dies out, so the second term on the right hand side of (2.7) becomes zero. This occurs at a small distance above the bottom, \( z = \delta \). By continuity, the governing equation at this location therefore becomes

\[
\frac{\partial u_w}{\partial t} = \frac{\partial U}{\partial t} = -\frac{1}{\rho} \frac{\partial p_w}{\partial x}
\]  

(2.8)

where \( U \) is the horizontal velocity predicted by linear potential theory at a location just outside of the boundary layer. Here we assume the pressure gradient inside the boundary layer is independent of depth and is equal to the pressure gradient just outside the boundary layer. Subtraction of (2.8) from (2.7) gives

\[
\frac{\partial u_w}{\partial t} - \frac{\partial U}{\partial t} = \frac{\partial}{\partial z} \left[ \nu_t \frac{\partial u_w}{\partial z} \right]
\]  

(2.9)
Since the velocity predicted by potential theory at the edge of the boundary layer, \( U \), is not a function of distance above the bottom, \( \frac{\partial U}{\partial z} = 0 \), (2.9) can also be written as

\[
\frac{\partial}{\partial t} (u_w - U) = \frac{\partial}{\partial z} \left[ \nu_t \frac{\partial}{\partial z} (u_w - U) \right] \quad (2.10)
\]

Because the wave-associated turbulence is restricted to a small height above the bottom, the wave boundary layer height is expected to be much smaller than the water depth, \( \delta/h \ll 1 \). So, \( U \) at the edge of the wave boundary layer, \( z = \delta \), can be regarded as being the same as \( U \) at the bottom, \( z = 0 \). From linear wave theory, the predicted orbital velocity at the bottom is

\[
U = U_b \cos \omega t \quad (2.11)
\]

where \( U_b \) is the magnitude of the bottom orbital velocity and \( \omega = 2\pi/T \). Expecting the solution to be simple harmonic, we can write it in the following form

\[
\frac{u_w - U}{U_b} = \Re \{u_d \exp(i\omega t)\} \quad (2.12)
\]

where \( u_d \) is a normalized deficit velocity, \( i = \sqrt{-1} \), and the operator \( \Re \) means the real part of the argument. Inserting (2.12) into (2.10), the equation governing the time-varying component of the velocity field within the wave boundary layer becomes

\[
\frac{d}{dz} \left[ \nu_t \frac{du_d}{dz} \right] - i\omega u_d = 0 \quad (2.13)
\]

The first boundary condition for (2.13) is that the velocity must satisfy the no-slip condition at the bottom, \( u_w = 0 \) or \( u_d = -1 \). The second boundary condition is that the velocity at the edge of the wave boundary layer should match the velocity of the free stream just outside of the boundary layer, \( u_w = U \) or \( u_d \to 0 \) as \( z \to \infty \).
2.1.2 Current Problem

Since we have assumed a time-invariant eddy viscosity, the equation governing the current will only depend on the distance above the bottom. Looking at it in the direction of the current, the governing equation for the time-invariant component of the velocity (2.6) becomes

$$0 = -\frac{1}{\rho} \nabla p_c + \frac{d}{dz} \left[ \nu_t \frac{du_c}{dz} \right]$$  \hspace{1cm} (2.14)

Integrating both sides of the equation (2.14) from the bottom, $z \equiv 0$, to an arbitrary distance $z = Z$ above the bottom

$$\int_0^Z \frac{d}{dz} \left[ \nu_t \frac{du_c}{dz} \right] dz = \int_0^Z \frac{1}{\rho} \nabla p_c dz$$  \hspace{1cm} (2.15)

and noting that the definition of the current shear stress at the bottom is

$$\frac{\tau_c}{\rho} = \lim_{z \to 0} \left[ \nu_t \frac{du_c}{dz} \right]$$  \hspace{1cm} (2.16)

we obtain a general equation for the current valid for all distances above the bottom

$$\nu_t \frac{du_c}{dz} - \frac{\tau_c}{\rho} = \frac{1}{\rho} \nabla p_c Z$$  \hspace{1cm} (2.17)

Using a “law of the wall” argument to neglect the pressure gradient term very close to the bottom, (2.17) reduces to

$$\nu_t \frac{du_c}{dz} = \frac{\tau_c}{\rho}$$  \hspace{1cm} (2.18)

Although the current boundary layer is expected to extend over the whole water depth, the previous approximation is only valid for the inner region of the current boundary layer, i.e. for small distances $Z$, above the bottom. The boundary condition for the current equation (2.18) is the no-slip condition at the bottom, $u_c = 0$ at the bottom.
2.2 **Eddy Viscosity Model**

The next step in this analysis is to define an eddy viscosity based on the objectives stated in Chapter 1. The new chosen formulation for the eddy viscosity is

\[
\nu_t = \begin{cases} 
\kappa u_{*m} (z + z_b) & z + z_b \leq \alpha l \\
\kappa u_{*m} \alpha l & \alpha l \leq z + z_b \leq \alpha l / \epsilon \\
\kappa u_{*c} (z + z_b) & \alpha l / \epsilon \leq z + z_b
\end{cases}
\]  

(2.19)

where:

\( \kappa \) = von Karman’s constant (\( \kappa \approx 0.4 \))

\( z \) = height above bottom

\( z_b \) = either \( z_o \) or 0

\( u_{*m} \) = shear velocity based on maximum combined bottom shear stress

\( u_{*c} \) = shear velocity based on current bottom shear stress

\( l = \frac{\kappa u_{*m}}{\omega} \) = length scale of wave boundary layer

\( \alpha \) = fraction of length \( l \) over which the eddy viscosity varies linearly

\( \epsilon = \frac{u_{*c}}{u_{*m}} \) = ratio between the current and the maximum shear velocities

This is the time-invariant continuous eddy viscosity formulation that will be used in both the wave problem and the current problem. It consists of three different regions: a region close to the bottom, an intermediate or transition region, and an upper region. It is known that the turbulence close to the bottom is dominated by the combined action of the waves and the current. For this reason the eddy viscosity for the lower region is scaled with the shear velocity based on the maximum combined shear stress. At a certain location above the bottom, the turbulence associated with the wave starts decreasing and the turbulence associated with the current takes over. This is modeled as a transition region with a constant eddy viscosity. In the upper region, the turbulence is independent of the waves and is due exclusively to the current. The eddy viscosity in this region is scaled with the shear velocity based on the current shear stress. Figure 2-1 shows a
Figure 2-1: Spatial representation of the eddy viscosity for the model with $z_b = z_0$, $z_r$ marks the start of the transition region ($z_r + z_b = \alpha l$) and $z_e$ marks the end of the transition region ($z_e + z_b = \alpha l / \epsilon$).

Diagram of the new eddy viscosity formulation. With this new eddy viscosity definition, the no-slip condition at the bottom will be applied at $z + z_b = z_0$. The term $z_b$ allows us to select the location of the zero-velocity either at $z = 0$ or $z = z_0$. The length $z_o$ depends on the bottom roughness and reflects the resistance that the flow experiences from the bed. For rough turbulent flow it is commonly defined as

$$z_o = \frac{k_n}{30} \quad (2.20)$$

where $k_n$ is the equivalent Nikuradse sand roughness. For smooth turbulent flow, the roughness elements are smaller than the height of the viscous sub-layer. In this case the resistance experienced by the flow depends on the viscous terms and

$$z_o = \frac{\nu}{9 u_{*m}} \quad (2.21)$$
where $\nu$ is the kinematic viscosity of the fluid.

The shear velocities based on the current and maximum combined shear stress are

$$u_c = \sqrt{\tau_c/\rho}$$ and $$u_{sm} = \sqrt{\tau_m/\rho},$$ respectively. The maximum combined shear stress is the vector sum of the maximum wave shear stress and the current shear stress

$$\bar{\tau}_m = \bar{\tau}_{wm} + \bar{\tau}_c = (\tau_{wm} + \tau_c \cos \phi, \tau_c \sin \phi)$$ (2.22)

where the maximum wave shear stress is defined as

$$\frac{\tau_{wm}}{\rho} = \lim_{z \to z_b} \left\{ \nu U_b \frac{\partial u_d}{\partial z} \right\}$$ (2.23)

and $\phi$ is the angle between the direction of the waves and the current. The magnitude of the maximum combined shear stress vector is

$$\tau_m = |\bar{\tau}_m| = \sqrt{\tau_{wm}^2 + 2\tau_{wm} \tau_c \cos \phi + \tau_c^2}$$ (2.24)

### 2.2.1 Wave Solution

Now that the vertical variation of the eddy viscosity has been defined, we can use it in the governing equation for the waves and solve it for the three regions.

In the lower region, $z + z_b < a_l$, the governing equation for the waves (2.13) becomes

$$\frac{d}{dz} \left[ \kappa u_{sm} (z + z_b) \frac{du_d}{dz} \right] - i \omega u_d = 0$$ (2.25)

in the transition region, $a_l \leq z + z_b \leq a_l /\epsilon$, (2.13) becomes

$$\frac{d}{dz} \left[ \kappa u_{sm} a_l \frac{du_d}{dz} \right] - i \omega u_d = 0$$ (2.26)
and in the upper region, \( z + z_b > \alpha l / \epsilon \), (2.13) becomes

\[
\frac{d}{dz} \left[ \kappa u_{zc}(z + z_b) \frac{du_d}{dz} \right] - i \omega u_d = 0
\]  

(2.27)

These equations can be simplified by defining a new non-dimensional variable of the form

\[
\zeta = \frac{z + z_b}{l}
\]

(2.28)

where \( l \) is given by (1.2). Using (2.28) in (2.25) we get for the lower region, \( \zeta < \alpha \),

\[
\frac{d}{d\zeta} \left[ \frac{du_d}{d\zeta} \right] - i u_d = 0
\]

(2.29)

From Hildebrand (1976) the solution of (2.29) is

\[
u_d = A \left[ \text{ker} \left( 2\sqrt{\zeta} \right) + i \text{kei} \left( 2\sqrt{\zeta} \right) \right] + B \left[ \text{ber} \left( 2\sqrt{\zeta} \right) + i \text{bei} \left( 2\sqrt{\zeta} \right) \right]
\]

(2.30)

where ker, kei, ber and bei are Kelvin functions of zeroth order. For the intermediate region, \( \alpha \leq \zeta \leq \alpha / \epsilon \), (2.26) becomes

\[
\frac{d^2 u_d}{d\zeta^2} - \frac{i}{\alpha} u_d = 0
\]

(2.31)

for which the solution is

\[
u_d = C \exp \left( \sqrt{i/\alpha} \zeta \right) + D \exp \left( -\sqrt{i/\alpha} \zeta \right)
\]

(2.32)

For the upper region, \( \zeta > \alpha / \epsilon \), (2.27) becomes

\[
\frac{d}{d\zeta} \left[ \frac{\zeta du_d}{\epsilon d\zeta} \right] - i u_d = 0
\]

(2.33)
for which the solution is

\[ u_d = E \left[ \text{ker} \left( \frac{2\sqrt{\zeta}}{\epsilon} \right) + i \text{kei} \left( \frac{2\sqrt{\zeta}}{\epsilon} \right) \right] + F \left[ \text{ber} \left( \frac{2\sqrt{\zeta}}{\epsilon} \right) + i \text{bei} \left( \frac{2\sqrt{\zeta}}{\epsilon} \right) \right] \]  

(2.34)

\( A, B, C, D, E, F \) are complex constants that will be determined applying the boundary and matching conditions. The boundary and matching conditions for the preceding equations governing the waves are

\[
\begin{align*}
   u_d &= -1 \quad \text{at} \quad z + z_b = z_o \quad \text{or} \quad \zeta = \zeta_o \\
   u_{d-} &= u_{d+} \quad \text{at} \quad z + z_b = \alpha l \quad \text{or} \quad \zeta = \alpha \\
   \frac{\partial}{\partial \zeta} [u_{d-}] &= \frac{\partial}{\partial \zeta} [u_{d+}] \quad \text{at} \quad z + z_b = \alpha l \quad \text{or} \quad \zeta = \alpha \\
   u_{d-} &= u_{d+} \quad \text{at} \quad z + z_b = \alpha l/\epsilon \quad \text{or} \quad \zeta = \alpha/\epsilon \\
   \frac{\partial}{\partial \zeta} [u_{d-}] &= \frac{\partial}{\partial \zeta} [u_{d+}] \quad \text{at} \quad z + z_b = \alpha l/\epsilon \quad \text{or} \quad \zeta = \alpha/\epsilon \\
   u_d &\rightarrow 0 \quad \text{at} \quad z + z_b \rightarrow \infty \quad \text{or} \quad \zeta \rightarrow \infty
\end{align*}
\]

where the non-dimensional form of \( z_o \) is

\[ \zeta_o = \frac{z_o}{l} \]  

(2.35)

From applying the no-slip boundary condition at the bottom, \( u_d = -1 \) at \( \zeta = \zeta_o \), to (2.30) we get

\[-1 = A \left[ \text{ker} \left( 2\sqrt{\zeta_o} \right) + i \text{kei} \left( 2\sqrt{\zeta_o} \right) \right] + B \left[ \text{ber} \left( 2\sqrt{\zeta_o} \right) + i \text{bei} \left( 2\sqrt{\zeta_o} \right) \right] \]  

(2.36)

From matching the velocity and the velocity gradients at \( \zeta = \alpha \) from above and below using (2.32) and (2.30), we have

\[ A \left[ \text{ker} \left( 2\sqrt{\alpha} \right) + i \text{kei} \left( 2\sqrt{\alpha} \right) \right] + B \left[ \text{ber} \left( 2\sqrt{\alpha} \right) + i \text{bei} \left( 2\sqrt{\alpha} \right) \right] = C \exp \left( \sqrt{i\alpha} \right) + D \exp \left( -\sqrt{i\alpha} \right) \]  

(2.37)
\[ A \text{ker}' (2\sqrt{\alpha}) + i \text{kei}' (2\sqrt{\alpha}) + B \text{ber}' (2\sqrt{\alpha}) + i \text{bei}' (2\sqrt{\alpha}) = \]
\[ \sqrt{iC} \exp(\sqrt{i\alpha}) - \sqrt{iD} \exp(-\sqrt{i\alpha}) \]  

From matching conditions at \( \zeta = \alpha/\epsilon \), velocity and velocity gradients matched from above and below using (2.34) and (2.32), we have

\[ C \exp(\sqrt{i\alpha/\epsilon}) + D \exp(-\sqrt{i\alpha/\epsilon}) = \]
\[ E \text{ker}(2\sqrt{\alpha/\epsilon}) + i \text{kei}(2\sqrt{\alpha/\epsilon}) + F \text{ber}(2\sqrt{\alpha/\epsilon}) + i \text{bei}(2\sqrt{\alpha/\epsilon}) \]

\[ \sqrt{iC} \exp(\sqrt{i\alpha/\epsilon}) - \sqrt{iD} \exp(-\sqrt{i\alpha/\epsilon}) = \]
\[ E \text{ker}'(2\sqrt{\alpha/\epsilon}) + i \text{kei}'(2\sqrt{\alpha/\epsilon}) + F \text{ber}'(2\sqrt{\alpha/\epsilon}) + i \text{bei}'(2\sqrt{\alpha/\epsilon}) \]

In the previous equations, the notation ' implies differentiation with respect to the argument of the function. From the boundary condition far away from the bottom, \( u_d \to 0 \) as \( \zeta \to \infty \), we obtain

\[ F = 0 \]  

since \( \text{ber} \) and \( \text{bei} \) become exponentially large when their arguments go to infinity (Abramowitz and Stegun, 1972). At this point we have five equations, (2.36, 2.37, 2.38, 2.39, 2.40), and five unknown constants, \( A, B, C, D, E \). These can be solved analytically in terms of \( \alpha, \zeta_0 \) and \( \epsilon \). The solution for these equations is shown Appendix A in the MATLAB file Constants.m.

### 2.2.2 Current Solution

In this section we will use the new eddy viscosity formulation (2.19) in the governing equation for the current (2.18) and then apply the boundary and matching conditions. The boundary condition for the current is the no-slip at the bottom, \( z + z_b = z_0 \). The
matching conditions are

\[ u_{c-} = u_{c+} \text{ at } z + z_b = \alpha l \]
\[ u_{c-} = u_{c+} \text{ at } z + z_b = \alpha l / \epsilon \]  \hspace{1cm} (2.42)

For the lower region, \( z + z_b < \alpha l \), we get

\[ \kappa u_{sm} (z + z_b) \frac{du_c}{dz} = u_{*c}^2 \]  \hspace{1cm} (2.43)

For the intermediate region, \( \alpha l \leq z + z_b \leq \alpha l / \epsilon \), we get

\[ \kappa u_{sm} \alpha l \frac{du_c}{dz} = u_{*c}^2 \]  \hspace{1cm} (2.44)

for the upper region, \( z + z_b > \alpha l / \epsilon \), we get

\[ \kappa u_{*c} (z + z_b) \frac{du_c}{dz} = u_{*c}^2 \]  \hspace{1cm} (2.45)

The solution for the current profile, applying the boundary and matching conditions to the previous equations is:

for \( z + z_b < \alpha l \), or \( \zeta < \alpha \),

\[ u_c = \frac{u_{*c}^2}{\kappa u_{sm}} \ln \left( \frac{z + z_b}{z_o} \right) \]
\[ = \frac{u_{*c}}{\kappa} \epsilon \ln \left( \frac{\zeta}{z_o} \right) \]  \hspace{1cm} (2.46)

for \( \alpha l \leq z + z_b \leq \alpha l / \epsilon \), or \( \alpha \leq \zeta \leq \alpha / \epsilon \),

\[ u_c = \frac{u_{*c}^2}{\kappa u_{sm}} \left[ \frac{z + z_b}{\alpha l} - 1 + \ln \left( \frac{\alpha l}{z_o} \right) \right] \]
\[ = \frac{u_{*c}}{\kappa} \left[ \frac{\zeta}{\alpha} - 1 + \ln \left( \frac{\alpha}{z_o} \right) \right] \]  \hspace{1cm} (2.47)
and for \( z + z_b > \alpha l / \epsilon \), or \( \zeta > \alpha / \epsilon \),

\[
\begin{align*}
uc &= \frac{u_{sc}}{\kappa} \left\{ \ln \left( \frac{u_{sc} (z + z_b)}{u_{sm} \alpha l} \right) + 1 + \frac{u_{sc}}{u_{sm}} \ln \left( \frac{\alpha l}{z_o} - 1 \right) \right\} \\
&= \frac{u_{sc}}{\kappa} \left\{ \ln \left( \frac{\zeta}{\alpha} \right) + 1 + \epsilon \left[ \ln \left( \frac{\alpha}{\zeta_o} - 1 \right) \right] \right\}
\end{align*}
\] (2.48)

2.3 Closure

In order to evaluate the equations, \( \alpha \), \( \epsilon \), and \( \zeta_o \) need to be specified. In addition a value for \( z_b \) has to be chosen. To close the problem we will use a friction factor concept in the definition of the bottom shear stress.

2.3.1 Wave Friction Factor

The shear velocity, \( u_{sm} = \sqrt{\frac{T_m}{\rho}} \), corresponds to the maximum combined shear stress as stated before. The maximum combined shear stress (2.24) can be written as

\[
\tau_m = \tau_{wm} \sqrt{1 + 2 \left( \frac{\tau_c}{\tau_{wm}} \right) |\cos \phi| + \left( \frac{\tau_c}{\tau_{wm}} \right)^2}
\] (2.49)

It is convenient here to define a parameter, \( \mu \), that expresses the relative magnitude of the current shear stress to the maximum wave shear stress,

\[
\mu^2 = \frac{\tau_c}{\tau_{wm}} = \left( \frac{u_{sc}}{u_{sw}} \right)^2
\] (2.50)

using the previous definition of \( \mu \), (2.49) now reads

\[
\tau_m = \tau_{wm} \sqrt{1 + 2 \mu^2 |\cos \phi| + \mu^4}
\] (2.51)
This expression can be further simplified by defining another parameter as follows

\[ C_\mu = \sqrt{1 + 2\mu^2 |\cos \phi| + \mu^4} \]  

(2.52)

Then, using this definition in (2.51) we obtain a direct relationship between the maximum combined shear velocity and the wave shear velocity

\[ u_{*m} = u_{*w} \sqrt{C_\mu} \]  

(2.53)

The minimum possible value of \( C_\mu \) is unity, this occurs in the absence of currents. The relationship of \( C_\mu \) to the previously defined parameters \( \mu \) and \( \epsilon \) is

\[ \sqrt{C_\mu} = \frac{u_{*m}}{u_{*w}} = \frac{(u_{*c}/u_{*w})}{(u_{*c}/u_{*m})} = \frac{\mu}{\epsilon} \]  

(2.54)

At this stage, it is useful to define a wave friction factor of the form originally suggested by Jonsson (1966)

\[ \tau_{wm} = \frac{1}{2} \rho f_w U_b^2 \]  

(2.55)

Rearranging the previous equation we obtain

\[ \sqrt{\frac{f_w}{2}} = \sqrt{\frac{\tau_w}{\rho U_b^2}} = \frac{u_{*w}}{U_b} = \frac{u_{*w} \omega}{A_b} \]  

(2.56)

where the wave excursion amplitude, \( A_b \), is given by

\[ A_b = \frac{U_b}{\omega} \]  

(2.57)
Using (2.53) and (2.56) in the definition of $\zeta_o$, (2.35), we obtain

$$
\zeta_o = \frac{z_o}{l} = \frac{k_n/30}{\kappa u_{wm}/\omega} = \frac{k_n/30}{\kappa u_{aw} \sqrt{C_\mu/\omega}} = \frac{\sqrt{2}}{30 \kappa C_\mu A_b} \sqrt{\frac{C_\mu}{f_w}}
$$

(2.58)

From the definition of the wave shear stress,

$$
\frac{T_{wm}}{\rho} = \lim_{z \rightarrow z_o} \left\{ \nu_t U_b \frac{\partial u_d}{\partial z} \right\}
$$

(2.59)

we get by use of (2.30)

$$
u^2_{sw} = \lim_{\zeta \rightarrow \zeta_o} \left\{ \kappa u_{wm} \zeta U_b \frac{\partial u_d}{\partial \zeta} \right\}
$$

$$
= \kappa u_{sw} \sqrt{C_\mu} U_b \lim_{\zeta \rightarrow \zeta_o} \sqrt{\zeta} \left\{ \sqrt{2 \zeta} \frac{\partial u_d}{\partial (2 \sqrt{\zeta})} \right\}
$$

$$
= \kappa u_{sw} \sqrt{C_\mu} U_b \lim_{\zeta \rightarrow \zeta_o} \sqrt{\zeta} \left\{ A \left[ \text{ker}' \left(2 \sqrt{\zeta} \right) + i \text{kei}' \left(2 \sqrt{\zeta} \right) \right] + B \left[ \text{ber}' \left(2 \sqrt{\zeta} \right) + i \text{bei}' \left(2 \sqrt{\zeta} \right) \right] \right\}
$$

(2.60)

Including (2.56) in (2.60) we obtain a relationship between the friction factor and $\zeta_o$,

$$
\sqrt{\frac{f_w}{C_\mu}} = \kappa \sqrt{2 \zeta_o} \left[ A \left[ \text{ker}' \left(2 \sqrt{\zeta_o} \right) + i \text{kei}' \left(2 \sqrt{\zeta_o} \right) \right] + B \left[ \text{ber}' \left(2 \sqrt{\zeta_o} \right) + i \text{bei}' \left(2 \sqrt{\zeta_o} \right) \right] \right]
$$

(2.61)

Also, from (2.58) we get a relationship between the friction factor and the relative roughness parameter $A_b/k_n$,

$$
C_\mu \frac{A_b}{k_n} = \frac{\sqrt{2}}{30 \kappa \zeta_o} \sqrt{\frac{C_\mu}{f_w}}
$$

(2.62)

These two equations, (2.61) and (2.62), can be used to develop a diagram that shows the dependency of the friction factor to the relative roughness. Finally, the phase angle
between the shear stress and the near bottom velocity is given by

\[ \theta_t = \arctan \left( \frac{\Im \left| A \left[ \text{ker}' \left( 2 \sqrt{\zeta_o} \right) + i \text{kei}' \left( 2 \sqrt{\zeta_o} \right) \right] + B \left[ \text{ber}' \left( 2 \sqrt{\zeta_o} \right) + i \text{bei}' \left( 2 \sqrt{\zeta_o} \right) \right] \right)}{\Re \left| \left[ \text{ker}' \left( 2 \sqrt{\zeta_o} \right) + i \text{kei}' \left( 2 \sqrt{\zeta_o} \right) \right] + B \left[ \text{ber}' \left( 2 \sqrt{\zeta_o} \right) + i \text{bei}' \left( 2 \sqrt{\zeta_o} \right) \right] \right|} \right) \]

(2.63)

### 2.3.2 Boundary Layer Thickness and Transition Height Definitions

Up to this point, the parameter \( \alpha \) has no pre-set value. This is usually "the fitting parameter," but in this section we explain how it is determined without recourse to experimental data on wave-current flows. From experiments of zero pressure-gradient turbulent flows over flat plates, two distinct regions within the boundary layer are observed. These two regions overlap at a distance above the bottom of around 15% of the boundary layer height, \( \delta \). The velocity profile for the lower 15% of the boundary layer has a logarithmic shape. When the shear stress close to the bottom is assumed to be a constant in the governing equation for a current, i.e. corresponding to the "law of the wall," a logarithmic velocity profile that agrees well with experimental data is obtained for the region where \( z/\delta < 0.15 \). The logarithmic profile suggests a linear variation of the eddy viscosity up to this height (Daily and Harleman, 1966).

We will adopt the definition of the transition height for the eddy viscosity based on the former explanation. The location where the maximum velocity is predicted at the moment when the flow is experiencing no pressure gradient, \( z_m \), will be our working definition of the wave boundary layer height, see Figure 2-2. This location is in good physical agreement with the steady boundary layer definition. We will take the location where the eddy viscosity goes from linearly increasing to constant, \( z_r \), as 15% of this height

\[ z_r = 0.15 \, z_m \]  

(2.64)
The subscript \( r \) is used to specify that it represents the real physical distance above the bottom. In non-dimensional terms,

\[
\zeta_m = \frac{z_m + z_b}{l} \quad (2.65)
\]

and

\[
\alpha_r = \frac{z_r}{l} = 0.15 \frac{z_m}{l} = 0.15 (\zeta_m - \zeta_b) \quad (2.66)
\]

where \( \alpha_r \) is the normalized distance above the bottom where the transition takes place. In terms of the parameter \( \alpha \) we have that

\[
\alpha = \frac{z_r + z_b}{l} = \alpha_r + \zeta_b \quad (2.67)
\]
2.3.3 Solution Procedure

Now that the problem is closed, the following iterative procedure can be used to develop the friction factor diagrams:

1. Specify $\zeta_b$, either $\zeta_b = 0$ or $\zeta_b = \zeta_o$.

2. Specify $\epsilon = \frac{u_{*c}}{u_{*m}}$, possible range: 0 to 1.

3. Pick $\zeta_o$, approximate range: $10^0$ to $10^{-5}$ for $C_\mu A_b/k_n$ between $10^{-1}$ and $10^{16}$

4. Select an initial estimate of $\alpha$. A good first estimate is $\alpha = \zeta_o$, given that it is the smallest value it can have.

5. Determine constants $A$, $B$, $C$, $D$ and $E$ from solving (2.36, 2.37, 2.38, 2.39, 2.40) for given $\zeta_o$, $\epsilon$ and $\alpha$.

6. Solve $u_d$ as a function of $\zeta$ for the three regions (2.30, 2.32, 2.34).

7. Solve $\frac{u_w}{U_b}$ as a function of $\zeta$ for $\cos \omega t = 1$. From (2.12),

$$\frac{u_w}{U_b} = \text{Re} \{u_d \exp (i\omega t)\} + \cos \omega t$$  \hspace{1cm} (2.68)

8. Find $\zeta_m$, i.e. $\zeta$ where $u_w/U_b$ is maximum at the time when $\cos \omega t = 1$.

9. Determine $\alpha_r = 0.15 (\zeta_m - \zeta_b)$, and $\alpha = \alpha_r + \zeta_b$.

10. With this new $\alpha$, repeat steps from 5 to 9 until convergence.

11. Determine $f_u/C_\mu$ from (2.61) and $C_\mu A_b/k_n$ from (2.62).

12. Repeat steps 4 to 12 for new values of $\zeta_o$ to get a range of $C_\mu A_b/k_n$.

13. Return to step 2 and specify a new value of $\epsilon$ until range 0 to 1 is covered.

A MATLAB code is provided in Appendix A to do these calculations. The file name is Friction_factors.m.
Chapter 3

Model Results and Approximations for Applications

In the present chapter, the results of the model are shown in the form of friction factor diagrams and $\alpha_r$ diagrams. Also, approximate formulas to represent the results analytically are developed for the model with $z_b = z_o$. The procedure for obtaining a prediction of the current velocity profile using the approximate equations is outlined. Finally, example calculations are given for two possible methods of specifying the current.

3.1 Model Results

Following the procedure outlined in Chapter 2, Section 2.3.2, a series of modified friction factor diagrams and $\alpha_r$ diagrams were developed. Figure 3-1 presents the friction factor diagram for waves without currents for both $z_b = z_o$ and $z_b = 0$. Both diagrams give the same results for $A_b/k_n > 100$. This was expected because as $A_b/k_n$ gets larger, $\zeta_o$ gets smaller, and for a small enough $\zeta_o$ there will be no difference between $\zeta_b = \zeta_o$ or $\zeta_b = 0$. For $A_b/k_n < 100$, the predicted friction factors for the model with $z_b = 0$ are smaller than the predicted friction factors for the model with $z_b = z_o$. The reason for this difference is that the eddy viscosity, and consequently the shear stress, is larger for the
model with \( z_b = z_o \). The results of the model with \( z_b = z_o \) cover the whole range of relative roughnesses, but the results of the model with \( z_b = 0 \) do not go beyond \( A_b/k_n < 1 \). The reason is that for very small \( A_b/k_n \), \( \alpha \) becomes smaller than \( \zeta_o \) and the way the model is set up does not allow for this case to be solved.

Figure 3-2 shows the dependency of the non-dimensional transition height, \( \alpha_r \), on \( A_b/k_n \) for pure waves for both \( z_b = z_o \) and \( z_b = 0 \). Here we can see that the value of \( \alpha_r \) is not a constant, but depends greatly on the relative roughness. For example, for the results of the model with \( z_b = z_o \), \( \alpha_r = 0.7 \) when \( A_b/k_n = 10^{-1} \) and becomes \( \alpha_r = 0.16 \) when \( A_b/k_n = 10^3 \). Both models predict the same results for \( A_b/k_n \) larger than \( 10^3 \). As with the friction factors, the model with \( z_b = 0 \) predicts smaller values for \( \alpha_r \) when \( A_b/k_n \) becomes small. Here too, the results of the model with \( z_b = 0 \) do not go beyond \( A_b/k_n < 1 \). This diagram is also useful to predict the wave boundary layer height. As defined before, \( \alpha_r \) is \( 15\% \) of the normalized boundary layer height \( (\alpha_r = 0.15 \frac{z_m}{l}) \), so based on this diagram we can also say that the wave boundary layer height is a non-linear function of the relative roughness. Figure 3-3 shows the phase angle between the shear stress and the near-bottom velocity. For \( A_b/k_n < 10^3 \) the model with \( z_b = 0 \) predicts larger values than the model with \( z_b = z_o \). The phase angle decreases as \( A_b/k_n \) increases for both cases.

Figures 3-4 and 3-5 show modified friction factor diagrams for the case of waves and currents. As can be observed in Figure 3-5, when the results are displayed as \( f_u/C_\mu \) vs \( C_\mu A_b/k_n \), all the lines are the same regardless of \( \epsilon \). For the model with \( z_b = 0 \), Figure 3-4, there is a small dependency on \( \epsilon \), for \( \epsilon > 0.2 \) and small \( C_\mu A_b/k_n \). Figure 3-6 shows the \( \alpha_r \) values for waves and currents for the model with \( z_b = z_o \) for different \( \epsilon \) values. The \( \alpha_r \) predictions for a fixed \( C_\mu A_b/k_n \) get larger as \( \epsilon \) increases. There is no difference in \( \alpha_r \) for \( \epsilon \leq 0.2 \). This diagram shows that the transition height and the wave boundary layer thickness are functions of \( \epsilon \) for \( \epsilon > 0.2 \). Figure 3-7 shows the phase shift between the maximum shear stress and the maximum near bottom velocity for waves and currents for the model with \( z_b = z_o \) and different \( \epsilon \) values.
Figure 3-1: Friction factor diagram for pure waves $(\epsilon = 0)$ for the model with $z_b = z_o$ (solid line), and for the model with $z_b = 0$ (dashed line).

Figure 3-2: $\alpha_r$ diagram for pure waves $(\epsilon = 0)$ for the model with $z_b = z_o$ (solid line), and for the model with $z_b = 0$ (dashed line).
Figure 3-3: Phase shift diagram for pure waves ($\epsilon = 0$) for the model with $z_b = z_o$ (solid line), and for the model with $z_b = 0$ (dashed line).

Figure 3-4: Modified wave friction factor diagram for waves and currents for the model with $z_b = 0$. 
Figure 3-5: Modified wave friction factor diagram for waves and currents for the model with $z_b = z_o$.

Figure 3-6: $\alpha_r$ diagram for waves and currents for the model with $z_b = z_o$ for different $\epsilon$ values.
Figure 3-7: Phase shift diagram for waves and currents for the model with $z_b = z_o$ for different $\epsilon$ values.
3.2 Approximate Equations

The results of the model with $z_b = z_o$ may be approximated by explicit formulas for determination of friction factors and $\alpha_r$. The expressions for determining the friction factor are the following

$$\frac{f_w}{C_\mu} = \exp \left\{ 7.02 \left( \frac{C_\mu A_b}{k_n} \right)^{-0.078} - 8.82 \right\} \quad \text{for} \quad 10^{-1} < \frac{C_\mu A_b}{k_n} < 10^2 \quad (3.1)$$

or

$$\frac{f_w}{C_\mu} = \exp \left\{ 5.61 \left( \frac{C_\mu A_b}{k_n} \right)^{-0.109} - 7.30 \right\} \quad \text{for} \quad 10^2 < \frac{C_\mu A_b}{k_n} < 10^6 \quad (3.2)$$

These formulas were taken from Madsen (1994) because the predicted friction factors are the same. Given that the friction factor is primarily dependent on the variation of the eddy viscosity very close to the bottom and the eddy viscosity presented in this analysis has the same variation at the bottom as in Madsen (1994), the prediction of the friction factor is the same. This will not be the case with velocity profiles, wave boundary layer thickness, and other results that depend on the variation of the eddy viscosity through the entire depth. Figure 3-8 shows the comparison between the results of the model and the approximation from (3.1) and (3.2).

An approximate analytical formula for determination of $\alpha_r$ was also developed. The formula is valid for all $C_\mu A_b/k_n$ between $10^{-1}$ and $10^6$. This formula is not as accurate as the expressions for friction factors, but the prediction of a value for $\alpha_r$ is less critical than the prediction of the friction factor, and having just one formula for the whole range $C_\mu A_b/k_n$ that is of interest simplifies the computations. The expression is

$$\alpha_r = Y \left\{ \exp \left[ 1.2 \left( \frac{C_\mu A_b}{k_n} \right)^{-0.2} - 2.12 \right] - 0.02 \left( \frac{C_\mu A_b}{k_n} \right)^{-0.3} \right\} \quad (3.3)$$
where $Y$ is a factor, greater or equal to 1, that will depend on the value of $\epsilon$ as follows

$$Y = \begin{cases} S\epsilon + I & \text{if } S\epsilon + I > 1 \\ 1 & \text{if } S\epsilon + I \leq 1 \end{cases} \quad (3.4)$$

where

$$S = -0.026 \left[ \log_{10} \left( \frac{C_{\mu}A_b}{k_n} \right) \right]^2 + 0.284 \left[ \log_{10} \left( \frac{C_{\mu}A_b}{k_n} \right) \right] + 0.942 \quad (3.5)$$

and

$$I = -0.013 \left[ \log_{10} \left( \frac{C_{\mu}A_b}{k_n} \right) \right] + 0.712 \quad (3.6)$$

For $\epsilon < 0.2$, $Y$ can safely be assumed equal to 1. Figure 3-9 shows the comparison of the model results for $\epsilon = 0$ and the approximation from (3.3). Figure 3-10 shows the comparison of the model results for different $\epsilon$ values and the approximation from (3.3).
Figure 3-9: Comparison of the $\alpha_r$ diagram for pure waves for the model with $z_b = z_o$ and the approximate formula.

Figure 3-10: Comparison of the $\alpha_r$ diagrams for waves and currents for the model with $z_b = z_o$ and the approximate formulas.
3.3 Solution Procedure for Practical Problems

In this section we will explain how to solve practical problems using the approximate formulas presented in Section 3.2.

3.3.1 Specifications

We first need to specify the bottom roughness as an equivalent Nikuradse sand grain roughness, $k_n$. Then, the wave motion is specified in terms of its period, $T$, and the near-bottom orbital velocity predicted by potential theory, $U_b$. The current may be specified by a current shear stress, $\tau_c$, and its angle relative to the wave motion, $\phi$. Or, the current may be specified by its magnitude at a certain height above the bottom, $u_c$ at $z_{ref}$, and its direction relative to the waves, $\phi$.

Regardless of the current specification, the relative roughness parameter $A_b/k_n$ is determined using the wave and roughness characteristics, i.e.

\[
    k_n = 30 z_o, \quad \omega = \frac{2\pi}{T} \quad \text{and} \quad A_b = \frac{U_b}{\omega}
\]

3.3.2 Current Specified by Current Shear Stress

If the current is specified by the current shear stress, $\tau_c$, and the angle relative to the waves, $\phi$, the solution procedure is as follows:

1. With the current shear stress and the fluid density, determine the current shear velocity

\[
    u_{*c} = \sqrt{\frac{\tau_c}{\rho}}
\] \hspace{1cm} (3.7)

2. Then, for the first iteration, assume $\mu \approx 0$ and $C_\mu = 1$.

3. Determine the friction factor, $f_w$, from (3.1) or (3.2) depending on $C_\mu A_b/k_n$. 

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4. Obtain the wave shear velocity, \( u_{sw} \), from

\[
  u_{sw} = \sqrt{\frac{1}{2} f_w U_b}
\]  

(3.8)

5. Find new value of \( \mu \), using

\[
  \mu = \frac{u_{sc}}{u_{sw}}
\]  

(3.9)

6. Then, determine new value of \( C_\mu \), from

\[
  C_\mu = \sqrt{1 + 2\mu^2 |\cos \phi| + \mu^4}
\]  

(3.10)

7. Repeat Steps 3 through 5 until \( C_\mu \) converges. Three significant digits after the decimal point is enough precision.

8. Get \( u_{sm}, \epsilon \) and \( l \) from

\[
  u_{sm} = u_{sw} \sqrt{C_\mu}, \quad \epsilon = \frac{u_{sc}}{u_{sm}} \quad \text{and} \quad l = \frac{\kappa u_{sm}}{\omega}
\]  

(3.11)

9. Obtain \( \alpha_r \) from (3.3).

10. This procedure is only valid for the model with \( z_b = z_o \). So,

\[
  \zeta_b = \zeta_o = \frac{z_o}{l} \quad \text{and} \quad \alpha = \alpha_r + \zeta_b
\]  

(3.12)

11. Now the current profile is, as obtained in Chapter 2, Section 2.2.2,

for \( z + z_b < \alpha l \),

\[
  u_c = \frac{u_{sc}}{\kappa} \epsilon \ln \left( \frac{z + z_b}{z_o} \right)
\]  

(3.13)
for \( \alpha l \leq z + z_b \leq \alpha l / \epsilon \),

\[
u_c = \frac{u_{sc} \epsilon}{\kappa} \left[ \frac{z + z_b}{\alpha l} - 1 + \ln \left( \frac{\alpha l}{z_o} \right) \right] \quad (3.14)\]

and for \( z + z_b > \alpha l / \epsilon \),

\[
u_c = \frac{u_{sc} \epsilon}{\kappa} \left\{ \ln \left( \frac{z + z_b}{\alpha l} \right) + 1 + \epsilon \left[ \ln \left( \frac{\alpha l}{z_o} \right) - 1 \right] \right\} \quad (3.15)\]

### 3.3.3 Current Specified by Velocity at Reference Height

The current may also be specified by its magnitude, \( u_c \), and its angle relative to the waves, \( \phi \), at a reference height, \( z_{ref} \). In the following procedure we assume that this height falls in the outside layer, i.e. \( z_{ref} + z_o > \alpha l / \epsilon \). In most cases it is safe to assume this because most of the water depth is within this layer. The procedure in this case is as follows:

1. Solve wave-current interaction (Steps 3 to 10 of Section 3.3.2) assuming \( u_{sc} = 0 \) for this first iteration. Get \( u_{sm} \), \( l \) and \( \alpha \).

2. For the first approximation of \( u_{sc} \), since no estimate of \( \epsilon \) is available, assume that \( \ln \left( \epsilon \frac{z + z_b}{\alpha l} \right) \) is small. Then, (3.15) simplifies to

\[
0 = \frac{u_{sc}^2}{\kappa u_{sm}} \left[ \ln \left( \frac{\alpha l}{z_o} \right) - 1 \right] + \frac{u_{sc}}{\kappa} - u_c \quad (3.16)
\]

3. Solve (3.16) for \( u_{sc} \) using the values of \( u_{sm} \), \( l \) and \( \alpha \) obtained in Step 1.

4. Solve wave-current interaction (Steps 3 to 10 of Section 3.3.2) with this \( u_{sc} \) to get new values for \( u_{sm} \), \( l \) and \( \alpha \), and an initial estimate of \( \epsilon \).

5. Use this estimate of \( \epsilon \) in the \( \ln \left( \epsilon \frac{z + z_b}{\alpha l} \right) \) term of (3.15), which rearranging now
becomes

\[ 0 = \frac{u_{*c}^2}{\kappa u_{*m}} \left[ \ln \left( \frac{\alpha l}{z_o} \right) - 1 \right] + \frac{u_{*c}}{\kappa} \ln \left( \frac{z + z_b}{\alpha l} \right) + 1 - u_c \]  \hspace{1cm} (3.17)

6. Solve (3.17) for \( u_{*c} \) using the values of \( u_{*m}, l, \alpha \) and \( \epsilon \) obtained in Step 4.

7. With this new estimate of \( u_{*c} \), repeat Steps 4 and 6 until value of \( u_{*c} \) converges.

8. Verify that the assumption of \( z_{ref} + z_b > \alpha l / \epsilon \) holds.

3.4 Examples

The conditions specified in the following examples correspond to the data set BVD20 by Bakker and Van Doorn (1978). The results of the full model for this data set are shown in Chapter 4, Section 4.3.

3.4.1 Example of Current Specified by Current Shear Stress

The chosen wave and bottom roughness parameters are

\[ U_b = 24.3 \text{ cm/s} \], \( T = 2 \text{ sec} \), \( k_n = 2.1 \text{ cm} \]  \hspace{1cm} (3.18)

and the current is specified by

\[ \tau_c = 0.71 \text{ Pa} \quad \text{and} \quad \phi = 0^\circ \]  \hspace{1cm} (3.19)

The angular frequency, wave orbital amplitude and \( z_o \) are

\[ \omega = \frac{2\pi}{T} = 3.142 \text{ sec}^{-1} , \quad A_b = \frac{U_b}{\omega} = 7.73 \text{ cm} \quad \text{and} \quad z_o = \frac{k_n}{30} = 0.07 \text{ cm} \]  \hspace{1cm} (3.20)
and the relative roughness parameter

\[
\frac{A_b}{k_n} = 3.7 \tag{3.21}
\]

The current shear velocity is

\[
u_* = \sqrt{\frac{\tau_c}{\rho}} = 2.66 \text{ cm/s} \tag{3.22}
\]

Given that \(A_b/k_n < 100\), assuming \(C_\mu = 1\), we get \(f_w = 0.0837\) from (3.1). Then, using this in (3.8), \(u_{sw} = 4.97 \text{ cm/s}\). And from (3.9), \(\mu = 0.535\). From (3.10), \(C_\mu = 1.286\). For the second iteration we use this new value of \(C_\mu\) to get \(f_w = 0.0951\), \(u_{sw} = 5.30 \text{ cm/s}\), \(\mu = 0.502\) and \(C_\mu = 1.252\). This procedure is repeated two more times until \(C_\mu\) converges. The end results are \(\mu = 0.505\), \(C_\mu = 1.255\), and \(u_{sm} = 5.27 \text{ cm/s}\).

Now, from (3.11), \(u_{sm} = 5.90 \text{ cm/s}\), \(\epsilon = 0.451\) and \(l = 0.751 \text{ cm}\). From (3.4), \(Y = 1.21\) and from (3.3), \(\alpha_r = 0.335\). From (3.12), \(\zeta_b = 0.093\) and \(\alpha = 0.428\). Using (3.15) to obtain the velocity at \(z = 5.9\) cm, since \(\alpha l / \epsilon - z_o = 0.643 \text{ cm}\), the magnitude of the predicted velocity turns out to be \(u_c = 22.4 \text{ cm/s}\).

### 3.4.2 Example of Current Specified by Velocity at Reference Height

The chosen wave bottom and roughness parameters are the same as the previous example

\[
U_b = 24.3 \text{ cm/s} , \quad T = 2 \text{ sec} , \quad k_n = 2.1 \text{ cm} \tag{3.23}
\]

but the current is now specified by

\[
u_c = 22.4 \text{ cm/s at } z = 5.90 \text{ cm and } \phi = 0^\circ \tag{3.24}
\]
As before, $A_b/k_n = 3.7$. The first step in this somewhat more complicated problem is to solve the wave-current interaction assuming $C_\mu = 1$. From this first iteration, $u_{*m} = 4.97$ cm/s, $l = 0.633$ cm and $\alpha = 0.399$. Solving the quadratic equation (3.16) for $u_{*c}$ using these values of $\alpha$, $l$ and $u_{*m}$, we get $u_{*c} = 6.53$ cm/s. With this estimate of $u_{*c}$ we solve the wave-current interaction again and get $u_{*m} = 8.91$ cm/s, $\epsilon = 0.732$, $l = 1.135$ cm and $\alpha = 0.459$. Now we have a first estimate of $\epsilon$. Using this estimate of $\epsilon$ inside the ln() term of (3.17) and solving for $u_{*c}$, the new estimate is $u_{*c} = 2.62$ cm/s. Repeating these steps three more times until $u_{*c}$ converges, the result is $u_{*c} = 2.66$ cm/s. We know this is correct, first because it confirms the results of the model (see Chapter 4, Section 4.3), and because it is the same value of $u_{*c}$ that was used in the previous example.
Chapter 4

Comparison with Experimental Data

In this chapter the results of the model are compared with experimental data. The data sets come from laboratory experiments and numerical models. The results for pure waves are presented in terms of the velocity amplitude and its phase relative to the free stream vs the distance above the bottom. The current velocity profiles are presented for data sets for waves and currents. Predictions of the current and maximum combined shear velocities are also shown.

4.1 Description of Data

The data sets chosen to compare the results for pure waves are: Tests 1 and 2 from Jonsson and Carlsen (1976), the data set from Van Doorn (1981), Tests 10 and 13 from Jensen (1989), and Experiment “a” from Mathisen and Madsen (1996). The general parameters for these data sets are presented in Table 4.1.

Jonsson and Carlsen (1976) obtained velocity measurements in turbulent flow near a fixed, rough bed in an oscillating water tunnel with two-dimensional roughness elements. The velocity was measured by a small propeller that determined the velocity magnitude, but not its direction. Two experiments were performed. In both experiments, the velocity was measured at various heights above the trough of the roughness elements. The mea-
Table 4.1: Experimental parameters for the data sets from a pure wave motion.

<table>
<thead>
<tr>
<th>Data Set</th>
<th>$U_b$ cm/s</th>
<th>$\omega$ s$^{-1}$</th>
<th>$k_n$ cm</th>
<th>$A_b/k_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MMa</td>
<td>17.1</td>
<td>2.80</td>
<td>28.0</td>
<td>$2.18 \times 10^{-1}$</td>
</tr>
<tr>
<td>VDW</td>
<td>26.5</td>
<td>3.14</td>
<td>2.10</td>
<td>$4.01 \times 10^{6}$</td>
</tr>
<tr>
<td>JC2</td>
<td>153</td>
<td>0.873</td>
<td>7.50</td>
<td>$2.37 \times 10^{1}$</td>
</tr>
<tr>
<td>JC1</td>
<td>211</td>
<td>0.749</td>
<td>1.59</td>
<td>$1.77 \times 10^{2}$</td>
</tr>
<tr>
<td>BJ13</td>
<td>200</td>
<td>0.646</td>
<td>0.084</td>
<td>$3.69 \times 10^{3}$</td>
</tr>
<tr>
<td>BJ10</td>
<td>200</td>
<td>0.646</td>
<td>0.0041</td>
<td>$7.55 \times 10^{4}$</td>
</tr>
</tbody>
</table>

Measurements were phase-averaged over many cycles. Grant (1977) analyzed the measured data and obtained values for $k_n$ in a more methodical way than Jonsson and Carlsen (1976). This was done by fitting a logarithmic profile to the velocity measurements very close to the bed. The results of Grant’s analysis are chosen here. Jonsson and Carlsen’s Test 1 and Test 2 are named here JC1 and JC2, respectively.

Van Doorn (1981) took velocity measurements in oscillatory flow in a wave flume 30 meters long. The waves in this flume were generated by a paddle. Two dimensional fixed artificial roughness elements were placed in a portion of the flume 15 meters long. The flow in this section of the flume was turbulent. Velocities were measured with a single-axis laser Doppler velocimeter. Velocities were measured at different heights above the crest and above the trough of the roughness elements. An equivalent Nikuradse roughness for the rough part of the flume was determined by fitting a logarithmic profile to the steady flow velocity measurements and finding the intercept on the vertical axis. The theoretical bed was taken as the bottom of the roughness elements. The measurements were averaged over 15 periods. The results presented in this analysis are the ones above the trough of the ripples. This data set is referred to here as VDW.

Bjorn Lykke Jensen (1989) carried out experiments in a U shaped oscillating water tunnel with a 10 meters long test section. The bottom was made of PVC plates, which were left as is for tests with a smooth wall. For the tests with rough bottoms, sand paper or sand grains were glued on the bottom. Flow velocities were measured with a Laser Doppler Anemometer (LDA) in the direction stream-wise and perpendicular to the flume.
The Nikuradse equivalent sand roughness was determined by fitting a logarithmic profile to the velocity measurements of the bottom region of the water column. The maximum combined friction velocity was obtained from direct measurements of the shear stress at the bottom (only for smooth wall tests) and by fitting straight lines to the logarithmic section of the profile. The smooth wall shear stress was measured directly with a hot film probe. Test 13 had sand paper on the bottom and represented rough turbulent flow. Test 10 had PVC plates on the bottom and represented smooth turbulent flow. The maximum shear velocity obtained from direct measurement for Test 10 was 7.7 cm/s, the value obtained from fitting the measurements of the velocities was 8.0 cm/s.

The value of $k_n$ used in this study for Test 10 is

$$k_n = \frac{3.3 \nu}{u_{*m}} = \frac{3.3 \times (10^{-2})}{8.0} = 0.00413 \text{ cm}$$

Jensen's Test 10 and Test 13 are referred to here as BJ10 and BJ13, respectively.

Mathisen and Madsen (1996) carried out experiments in rough turbulent oscillatory flow in a 28 m long wave flume with large bottom roughnesses. The roughness elements were fixed triangular iron bars placed on the glass bottom of the flume. The bars were 1.5 cm high spaced at 10 cm intervals perpendicular to the flume longitudinal axis. The determination of the wave roughness was done by measuring energy dissipation due to bottom friction, obtaining an energy dissipation factor, correlating this with a wave friction factor and using a modified version of the Grant-Madsen model to relate the friction factor to $A_b/k_n$. The value of $k_n$ used in this analysis is the one obtained for pure waves evaluated at $\zeta_o$ with $\varphi \neq 0$. The use of this $k_n$ is not strictly valid here because it was determined using a different model, but we will use it expecting it to be an approximate value. Mathisen and Madsen Experiment “a” is named here MMa.

The data sets chosen to compare the model results for waves and currents are: two data sets from Bakker and Van Doorn (1978), the results of three runs of the numerical model of Davies, Soulsby and King (1988), and Experiment B from Mathisen and Madsen (1996). The general parameters for these data sets are presented in Table 4.2. Table 4.3
Data Set | $U_b$ (cm/s) | $\omega$ (s$^{-1}$) | $k_n$ (cm) | $A_b/k_n$
---|---|---|---|---
MMB (1) | 18.0 | 2.39 | 15.7 | 0.48
MMB (2) | " | " | 28.0 | 0.27
BVD20 | 24.3 | 3.14 | 2.1 | 3.7
BVD10 | 25.7 | 3.14 | 2.1 | 3.9
DV05 | 50 | 0.785 | 15.0 | 4.2
DV10 | 100 | 0.785 | 15.0 | 8.5
DV15 | 150 | 0.785 | 15.0 | 12.7

Table 4.2: Experimental parameters for the data sets from combined wave and current flows.

shows the different ways that the current was specified for each data set.

Bakker and Van Doorn (1978) used the same flume as Van Doorn (1981) described above. The current was added by means of recirculating pipe system with a pump. The roughness elements for these experiments were 2 mm high and 15 mm apart. The pump flow rates corresponded to average flow rates of 10 cm/s and 20 cm/s. These data sets are referred to here as BVD10 and BVD20, for the 10 cm/s and 20 cm/s average flow rate, respectively.

Davies, Soulsby and King (1988) developed a higher order numerical closure model for wave-current boundary layer flows. This model allows for a complex time-varying eddy viscosity. The results of this model are used here as a basis for comparison of the performance of our model with the more detailed model. In the model by Davies et al. the current is specified by a mean shear stress, $\tau_c = 3.5$ Pa in all the model runs presented here. The waves are specified by the maximum orbital velocity, $U_b$, and the period, $T$. The roughness is specified as an equivalent Nikuradse roughness, $k_n$. The results of the three runs of the Davies et al. model are denoted by DV05, DV10 and DV15.

Mathisen and Madsen (1996) Experiment B was performed in the same wave flume described for Mathisen and Madsen Experiment “a.” The current was generated with a 1200 gpm pump and associated recirculation piping. This experiment is named here MMB. The results of Mathisen and Madsen (1996) show that the equivalent roughness experienced by pure waves, pure currents, and wave-current flows are the same for a
Table 4.3: Current specifications for the data sets from combined wave and current flows.

given bottom configuration. The average equivalent roughness obtained from all the experiments performed in this flume with the same bottom configuration was around 20 cm. Two different \( k_n \) values were used in the analysis presented here. For MMB (1) the lower-bound value of 15.7 cm obtained by Mathisen and Madsen for this experiment was used. The relatively large value of 28 cm obtained from Experiment “a” was used for MMB (2).

### 4.2 Comparison of Model Results with Data from Pure Waves

The comparison of the predicted wave velocity amplitude and phase with the data sets MMA, VDW, JC2 and JC1 are shown in Figures 4-1, 4-2, 4-3 and 4-4. The comparison of the predicted velocity at phase \( \cos \omega t = 1 \) with the data sets BJ13 and BJ10 are shown in Figures 4-5 and 4-6. Table 4.4 shows the predicted maximum shear velocity, boundary
Table 4.4: Calculated maximum shear velocity, boundary layer thickness and normalized transition height for the data sets from a pure wave motion.

<table>
<thead>
<tr>
<th>Data Set</th>
<th>$u_{sw}$ (cm/s)</th>
<th>$z_m$ (cm)</th>
<th>$\alpha_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MMa</td>
<td>6.61</td>
<td>3.68</td>
<td>0.584</td>
</tr>
<tr>
<td>VDW</td>
<td>5.25 4.94</td>
<td>1.29 1.11</td>
<td>0.290 0.264</td>
</tr>
<tr>
<td>JC2</td>
<td>20.20 19.90</td>
<td>13.37 12.65</td>
<td>0.217 0.208</td>
</tr>
<tr>
<td>JC1</td>
<td>19.05 19.00</td>
<td>11.95 11.77</td>
<td>0.176 0.174</td>
</tr>
<tr>
<td>BJ13</td>
<td>11.68 11.68</td>
<td>7.19 7.19</td>
<td>0.149 0.149</td>
</tr>
<tr>
<td>BJ10</td>
<td>8.46 8.46</td>
<td>4.77 4.77</td>
<td>0.136 0.136</td>
</tr>
</tbody>
</table>

The data sets presented for comparison with the model results for a pure wave motion come from a large range of relative roughness. The values of $A_b/k_n$ span five orders of magnitude, from $O(10^{-1})$ to $O(10^4)$. The smaller $A_b/k_n$ correspond to large roughness elements like ripples, $A_b/k_n$ values in the middle of the range correspond to flat sand bottoms, and the very large $A_b/k_n$ values correspond to smooth surfaces. As can be observed in the figures, both versions of the model, i.e. $z_b = z_o$ or $z_b = 0$, do a reasonable job predicting the velocity amplitude and phase profiles, given the large range of relative roughnesses shown. The underlying reason for this is that the model does not fix $\alpha_r$ to a single value, but determines it as a function of the boundary layer thickness, which depends on the relative roughness. This can be seen in Table 4.4.

The agreement between the prediction and the location of the maximum amplitude of the velocity is generally good. For all the cases, this height is somewhat over-predicted, but the ratio between the location of the maximum velocity seen in the data and that predicted by the model is the same for all the data sets. This means that the over-prediction of the boundary layer thickness is relatively the same amount in all cases. This over-prediction can be explained in two ways. First, the assumption of a constant eddy viscosity close to the edge of the boundary layer is not physically realistic. The turbulence is expected to decrease as $z$ approaches the edge of the wave boundary layer. So, the model is over-predicting the viscosity far away from the bottom. This results
Figure 4-1: Profiles of the velocity amplitude and velocity phase obtained from 1) the model with $z_b = z_o$ (solid line), 2) data set MMA: measurements from Mathisen and Madsen (1996) Experiment “a” (pluses). $A_b/k_n = 0.218$
Figure 4-2: Profiles of the velocity amplitude and velocity phase obtained from 1) the model with $z_b = z_0$ (solid line), 2) the model with $z_b = 0$ (dashed line), 3) data set VDW: measurements from Van Doorn (1981) (pluses). $A_b/k_n = 4.01$
Figure 4-3: Profiles of the velocity amplitude and velocity phase obtained from 1) the model with $z_b = z_o$ (solid line), 2) the model with $z_b = 0$ (dashed line), 3) data set JC2: measurements from Jonsson and Carlsen (1976) Test 2 (pluses). $A_b/k_n = 23.7$
Figure 4-4: Profiles of the velocity amplitude and velocity phase obtained from 1) the model with $z_b = z_o$ (solid line), 2) the model with $z_b = 0$ (dashed line), 3) data set JC1: measurements from Jonsson and Carlsen (1976) Test 1 (pluses). $A_b/k_n = 177$
Figure 4-5: Profiles of the velocity at phase $\cos \omega t = 1$ obtained from 1) the model with $z_b = z_o$ (solid line), 2) the model with $z_b = 0$ (dashed line), 3) data set BJ13: measurements from Jensen (1989) Test 13 (pluses). $A_b/k_n = 3690$

Figure 4-6: Profiles of the velocity at phase $\cos \omega t = 1$ obtained from 1) the model with $z_b = z_o$ (solid line), 2) the model with $z_b = 0$ (dashed line), 3) data set BJ10: measurements from Jensen (1989) Test 10 (pluses). $A_b/k_n = 75500$
in a larger boundary layer thickness than if this decrease had been incorporated in the model. Another explanation is that the transition height, set as 15% of the boundary layer height, may be too large. If this percentage was smaller, it would result in a smaller $\alpha_r$, and thus a smaller eddy viscosity value in the constant region. Selecting this number based on the data gives a best fit value of $\frac{z_r}{z_m} = 0.125$, instead of the 0.15 used in the model.

Four of the six results show predictions of the overshoot magnitude that are smaller than the ones seen in the data, with the exception of data sets MMA and JC1 that are just right. This is a result of the over-prediction of the eddy viscosity as explained before. If the eddy viscosity was smaller, then the magnitude of the predicted overshoot would be bigger. Given the simplicity of the model, this over-prediction of the eddy viscosity, and resulting over-prediction of boundary layer height and under-prediction of overshoot magnitude are within acceptable limits.

As for the differences between the results with $z_b = z_o$ vs $z_b = 0$, first and as expected, the model with $z_b = z_o$ sets the no-slip at $z = 0$, while the model with $z_b = 0$ predicts the zero velocity at $z = z_o$. Also, the model with $z_b = z_o$ predicts larger boundary layer thicknesses and larger maximum shear velocities. This can be explained by recalling that the value of the eddy viscosity in the constant region was directly proportional to $\alpha = \alpha_r + \zeta_b$. For a given $A_b/k_n$, the value of $\alpha$ for the case where $z_b = z_o$ will always be larger than for the case where $z_b = 0$ for two inter-connected reasons. The obvious reason being that $\zeta_b = \zeta_o$. The other reason is that $\alpha_r$ for the model with $z_b = z_o$ will be larger than $\alpha_r$ for the model with $z_b = 0$ because the boundary layer thickness is larger. And the boundary layer is larger because the eddy viscosity is larger. This is the iterative procedure that is performed within the model. The difference in the results become less noticeable as $A_b/k_n$ increases.

Another thing to notice is the lack of a prediction for $z_b = 0$ in Figure 4-1. This is due to the small value of $A_b/k_n$ of the MMA data set. For this value of $A_b/k_n$, the predicted $\alpha$ is smaller than $\zeta_o$ and the model does not provide for this situation. Figure
4-3 and 4-4 show how the difference between the two model variations start to become less significant as $A_b/k_n$ increases.

The predicted maximum shear velocity for the BJ13 data set is $u_{*m} = 11.68$ cm/s, while Jensen (1989) obtained a value of $u_{*m} = 11.00$ cm/s from fitting a logarithmic curve to the data points. This corresponds to an over-prediction of the shear stress of 13%. The predicted maximum shear velocity for the BJ10 data set is $u_{*m} = 8.46$ cm/s, while the value obtained by Jensen (1989) was 8.0 cm/s from fitting the data and 7.7 cm/s from measurements with the probe. This corresponds to an over-prediction of the shear stress of about 12%.

In general, the model gives good predictions given the large range of $A_b/k_n$. It is somewhat over-predicting boundary layer height and maximum shear velocities. The model with $z_b = 0$ can not be used for very large roughnesses. As $A_b/k_n$ gets larger, the smaller the difference between the results of the model for the two options of $z_b$.

**4.3 Comparison of Model Results with Data from Waves and Currents**

The comparison of the predicted current velocity profiles with the data sets for waves and currents are shown in Figures 4-7 through 4-17. For the data sets from Davies et al. (1988) and Mathisen and Madsen (1996) the results are presented for the two ways of specifying the current, fixing the current shear velocity and also by fixing the velocity at a specified height. The results for data sets from Bakker and Van Doorn are shown only for specification of the current velocity at a reference height.

The predicted maximum combined shear velocity and current shear velocity for the data sets from Bakker and Van Doorn, BVD10 and BVD20, are shown in Table 4.5. For both data sets, the model with $z_b = z_o$ predicts a larger maximum combined shear velocity and a larger current shear velocity than the model with $z_b = 0$. The predicted current profiles are shown in Figures 4-7 and 4-8. The model with $z_b = z_o$ predicts the
The predicted maximum shear velocity for the data sets from Davies et al. are shown in Table 4.6. These results are for the case where the current is specified by fixing its current shear velocity. The current velocity profiles are shown in Figures 4-9, 4-11 and 4-13. The prediction of the maximum combined shear velocity somewhat closer to the results of Davies et al. for the model with $z_b = 0$, but the prediction of the velocity profile is better when $z_b = z_o$. The model of Davies et al. is set to approach the no-slip velocity at $z_o$. This is why the results of model with $z_b = 0$ are closer to the data for very small distances above the bottom.

The results for the case where the current velocity at 70 cm above the bottom was specified for data sets DV05, DV10 and DV15 are shown in Table 4.7. The predicted current profiles are shown in Figures 4-10, 4-12 and 4-14. Here, the current shear velocity was allowed to change in order to give the desired velocity at the specified height. The maximum combined shear velocity predicted for these runs are smaller than those predicted when the current shear velocity was specified (see Table 4.6) and the values are closer to the predictions by Davies. On the other hand, the predicted current shear velocities are smaller than those specified by Davies et al.

For the MMB data set, three runs were performed for MMB(1) and the same three runs for MMB(2). One run specifying the same current shear velocity obtained by Mathisen and Madsen (1996) for this data set, and two runs specifying the velocities at different heights. The resulting current profiles are presented in Figures 4-15, 4-16 and 4-17. The predicted current shear velocities and maximum combined shear velocities

<table>
<thead>
<tr>
<th>Data Set</th>
<th>$u_{sc}$ (cm/s)</th>
<th>$u_{sm}$ (cm/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BVD10</td>
<td>1.28</td>
<td>5.35</td>
</tr>
<tr>
<td></td>
<td>1.16</td>
<td>5.06</td>
</tr>
<tr>
<td>BVD20</td>
<td>2.66</td>
<td>5.90</td>
</tr>
<tr>
<td></td>
<td>2.57</td>
<td>5.68</td>
</tr>
</tbody>
</table>

Table 4.5: Calculated maximum combined shear velocity and current shear velocity for the data sets from Bakker and Van Doorn (1978).
Table 4.6: Calculated maximum combined shear velocity for the runs where the current shear velocity was specified to 5.92 cm/s compared with the results from Davies et al. (1988).

<table>
<thead>
<tr>
<th>Data Set</th>
<th>$u_{*m}$ (cm/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Davies et al.</td>
</tr>
<tr>
<td>DV05</td>
<td>11.40</td>
</tr>
<tr>
<td>DV10</td>
<td>16.43</td>
</tr>
<tr>
<td>DV15</td>
<td>22.11</td>
</tr>
</tbody>
</table>

Table 4.7: Calculated maximum combined shear velocity and current shear velocity for the runs where the magnitude of the current was specified at 70 cm compared with the results of Davies et al. (1988) are shown in Tables 4.8 and 4.9.

As can be observed from Figure 4-15, when the current shear velocity is fixed, the slope of the curves are the same as the data points, but the predicted velocities increase too much close to the bottom and this results in an over-prediction of the velocity far from the bottom. We would have expected the data points to be between the predicted profile for MMB(1) and MMB(2). It is seen that even for the large roughness of $k_n = 28$ cm, the results are over-predicted by about 4 cm/s. From Figures 4-16 and 4-17 we can observe that when the magnitude of the velocity is specified, the resulting slope, and thus, the shear stress, is lower than that seen in the data. Also, from Tables 4.8 and 4.9, it is seen that when the velocity at reference heights are specified, the predicted current shear velocity will vary depending on the selected data point.

It is worth noticing that for data set BVD20, with $A_b/k_n = 3.7$, the model does a very good job in predicting the velocity profile. But for data set MMB, which has $A_b/k_n$ assumed between 0.48 and 0.27, the predicted velocities are larger than the measurements. To find out why, we first see if it may be due to the non-linearity of the waves. For this
we compute the Ursell number, given by

\[ U = \frac{HL^2}{h^3} \quad (4.2) \]

where \( H \) is the wave height, \( L \) is the wave length and \( h \) is the water depth. The larger the Ursell number, the more non-linear the waves. Data set MMB had \( H = 0.10 \, \text{m}, \) \( L = 6.04 \, \text{m} \) and \( h = 0.60 \, \text{m} \). The Ursell number for MMB is 17. Data set BVD20 had \( H = 0.095 \, \text{m}, \) \( L = 3.26 \, \text{m} \) and \( h = 0.30 \, \text{m} \). The Ursell number for BVD20 is 37. This shows that the waves from BVD20 were more non-linear than the waves from MMB. So, this does not explain why the model is over-predicting the velocities for the MMB data set.

Trowbridge and Madsen (1984) found that for long waves, \( \log(kh) < -0.2 \), the wave-induced streaming reversed and became more negative as \( A_b/k_a \) decreased. If this were the case, then it would mean that the velocity shown in the data is the result of the current velocity minus the wave streaming. Since our model does not account for the wave
streaming, the predicted velocities are larger than the ones in the data. The magnitude of the reversed wave streaming would be larger for the MMB data set than for the BVD20 due to the difference in relative roughness. Mathisen and Madsen found that, indeed, there was a reverse streaming of $\bar{u}_w = -1.2$ cm/s for the MMB data set.

The magnitude of the wave-streaming is not enough to account for the difference between the predicted velocities and the data. The other possible reason for the difference is that the values of equivalent roughness used for MMB(1), $k_n = 15.7$ cm and MMB(2), $k_n = 28$ cm, are too small. This is a real possibility, given that these values were obtained using a modified version of the Grant-Madsen model, and not using the model presented here. By trial and error we find that a value of $k_n = 45$ cm gives results that are in excellent agreement with the data, if the wave-induced streaming is not accounted for. A value of $k_n = 38$ cm is found after accounting for the wave-induced mass transport, $\bar{u}_w = -1.2$ cm/s. The comparison of these results with the data are shown in Figure 4-18.

In general, when the model with $z_b = z_o$ is used, the magnitude of the eddy viscosity in the transition region is determined as a function of $z_r$ and $z_o$. This means that the predictions will become more and more dependent on $k_n$, as $A_k/k_n$ decreases. This is the reason why there is such variability in the predicted velocity profiles for different $k_n$ values. For the very large roughness elements of MMB, the prediction of the correct value for $k_n$ is crucial.

Finally, we can say that the model results are in reasonable agreement with the data, but there is room for improvement. In the case of rough bottoms, the results are very sensitive to the value of the equivalent roughness. Based on the comparisons with all the data sets, if the current shear velocity is specified the current shear stress will be correct, but the current velocities will tend to be over-predicted. If the current velocity at a reference height is specified the predicted velocities will be closer to the measurements, but the current shear velocity will tend to be under-predicted. Also, the prediction of the magnitude of the current shear stress will depend on the reference
velocity selected. Between the two possible specifications for the model, \( z_b = z_o \) gives overall better predictions than \( z_b = 0 \).
Figure 4-7: Profiles of the current velocity obtained from 1) the model with $z_b = z_o$ (solid line), 2) the model with $z_b = 0$ (dashed line), 3) data set BVD10: measurements from Bakker and Van Doorn (1978) (pluses). Current specified at reference height.

Figure 4-8: Profiles of the current velocity obtained from 1) the model with $z_b = z_o$ (solid line), 2) the model with $z_b = 0$ (dashed line), 3) data set BVD20: measurements from Bakker and Van Doorn (1978) (pluses). Current specified at reference height.
Figure 4-9: Profiles of the current velocity obtained from 1) the model with $z_b = z_o$ (solid line), 2) the model with $z_b = 0$ (dashed line), 3) data set DV05: results of the model of Davies et al. (1988) (dotted line). Current specified by current shear stress.

Figure 4-10: Profiles of the current velocity obtained from 1) the model with $z_b = z_o$ (solid line), 2) data set DV05: results of the model of Davies et al. (1988) (dotted line). Current specified by velocity at reference height.
Figure 4-11: Profiles of the current velocity obtained from 1) the model with \( z_b = z_o \) (solid line), 2) the model with \( z_b = 0 \) (dashed line), 3) data set DV10: results of the model of Davies et al. (1988) (dotted line). Current specified by current shear stress.

Figure 4-12: Profiles of the current velocity obtained from 1) the model with \( z_b = z_o \) (solid line), 2) data set DV10: results of the model of Davies et al. (1988) (dotted line). Current specified by velocity at reference height.
Figure 4-13: Profiles of the current velocity obtained from 1) the model with $z_b = z_o$ (solid line), 2) the model with $z_b = 0$ (dashed line), 3) data set DV15: results of the model of Davies et al. (1988) (dotted line). Current specified by current shear stress.

Figure 4-14: Profiles of the current velocity obtained from 1) the model with $z_b = z_o$ (solid line), 2) data set DV15: results of the model of Davies et al. (1988) (dotted line). Current specified by velocity at reference height.
Figure 4-15: Profiles of the current velocity obtained from 1) the model with $z_b = z_o$ and $k_n = 15.7$ cm, current specified by current shear stress (solid line), 2) the model with $z_b = z_o$ and $k_n = 28$ cm, current specified by current shear stress (dashed-dotted line), 3) data set MMB, Experiment B from Mathisen and Madsen (1996): measurements above the crest (triangles) and above the through (squares) of the roughness elements.
Figure 4-16: Profiles of the current velocity obtained from 1) the model with $z_b = z_o$ and $k_n = 15.7$ cm, current specified by current shear stress, 2) the model with $z_b = z_o$ and $k_n = 15.7$ cm, current specified by velocity at height marked with circle (dashed line and dashed-dotted line), 3) data set MMB, Experiment B from Mathisen and Madsen (1996): measurements above the crest (triangles) and through (squares) of the roughness elements.
Figure 4-17: Profiles of the current velocity obtained from 1) the model with $z_b = z_o$ and $k_n = 28$ cm, current specified by current shear stress, 2) the model with $z_b = z_o$ and $k_n = 28$ cm, current specified by velocity at height marked with circle (dashed line and dashed-dotted line), 3) data set MMB, Experiment B from Mathisen and Madsen (1996): measurements above the crest (triangles) and through (squares) of the roughness elements.
Figure 4-18: Profiles of the current velocity obtained from 1) the model with $z_b = z_o$ and $k_n = 45$ cm, current specified by current shear stress, 2) the model with $z_b = z_o$ and $k_n = 38$ cm, current specified by current shear stress, 3) data set MMB, Experiment B from Mathisen and Madsen (1996): measurements above the crest (triangles) and through (squares) of the roughness elements.
Chapter 5

Conclusions

The objectives of this study were to develop an analytical model to describe the wave-current boundary layer interaction. The model needed to be simple enough to be easily incorporated into coastal sediment transport models. At the same time, the model had to be sufficiently complex to reasonably predict the characteristics of the flow for the large range of bottom roughnesses seen in the field.

The distinguishing features of the developed model are discussed in Chapter 2. They include a time-invariant continuous eddy viscosity with a vertical structure composed of three different regions. For the region immediately above the bottom, the eddy viscosity is linearly increasing and scaled by the maximum combined wave-current shear stress. For the region far away from the bottom, the eddy viscosity is linearly increasing and scaled by the current shear velocity. The region in between is a transition region where the eddy viscosity is a constant that depends on the location of the transition. This transition region makes the eddy viscosity a continuous function of \( z \), and thus, the current velocity profiles have a smooth transition between the wave and the current dominated regions. Also, the developed model allows for the no-slip condition to be applied either at \( z = z_o \) or \( z = 0 \). Furthermore, the developed model is totally predictive, which means that it does not have a free parameter to be fitted with data.

Perhaps the most innovative features of the model presented here are the definition of
the wave boundary layer thickness and determination of the transition height. The wave boundary layer thickness is defined as the distance from the bottom to the location where the flow exhibits the maximum wave velocity at a time when it is experiencing no pressure gradient. The transition height is determined as 15% of the boundary layer height, in analogy to boundary layer theory for steady flows under no pressure gradients. Since the magnitude of the eddy viscosity in the transition region depends on a determined transition height that is a function of the boundary layer thickness, and the boundary layer thickness depends on the eddy viscosity, the model has to solve this problem iteratively. This results in a transition height and boundary layer thickness that are not simple multiples of the boundary layer scale, \( l \). As shown in Chapter 3, the predicted boundary layer thickness and transition height for a wave-current problem will be a function of the relative roughness parameter \( A_b/k_n \), the relative magnitude of the current shear velocity to the maximum combined shear velocity, \( \epsilon \), and the boundary layer length scale, \( l \). It was also seen that for values of \( A_b/k_n \) smaller than 1 the version of the model that sets the no-slip condition at \( z = 0 \) is the only one that gives a prediction.

Because the transition height is chosen as a fraction of the boundary layer thickness and the boundary layer thickness is determined within the model, the results from the model are reasonable for a large range of bottom roughnesses. This is shown in Chapter 4, Section 4.2, where the model predictions are compared with data from experiments with pure waves. The data sets for pure waves include experiments with relative roughnesses characteristic of rippled beds, flat sand bottoms, and even smooth plates. For all these experiments, the results of the model were in good agreement with the data, although there was indication that the eddy viscosity was somewhat over-predicted away from the bottom.

Chapter 4 also shows the comparison of the model's current velocity profile prediction for data sets from laboratory experiments and from a numerical model. In general, the results are reasonable, but it was seen there is a dependency on the way the current is specified. For very rough bottoms, the comparisons show that the model results are very
sensitive to the chosen value of the equivalent roughness.

Finally, for the solution of practical problems, the results of the model are simplified in Chapter 3 with the introduction of approximate formulas for friction factors and for the non-dimensional transition height, $\alpha_r$. The procedures outlined can be performed manually, although the ideal method to do the iterations would be to write a simple computer code that incorporates the approximate equations.

For future research it is recommended to model the decrease in the wave-associated turbulence for heights far from the bottom. Also, a time-variation of the eddy viscosity can be studied to account for non-linear effects.
Appendix A

Matlab Programs

This appendix presents the computer codes used to obtain the results presented throughout this study.
% Friction_factors.m
% Modified Friction_factor, alpha_r and phase shift diagrams

clear all

% eps = epsilon
eps = 0.2; % from 0 to 1, (for zero: use eps = 0.01)

% N = fraction for the transition from linear to constant
N = 1 / 0.15; % chosen to give 15% of boundary layer height

for count_Zo = 1 : 100 % 100 points to draw diagrams

% Zo = zeta_not , Zb = zeta_b,
Zo = 2^(-(count_Zo*2.5-19)/8); % arbitrary, gives good numbers
Zb = Zo; % 0 or Zo

% alpha = alpha
% Alpha_iteration % subroutine to determine alpha and constants

% k = Von Karman's Constant
kappa = 0.4;

% mod friction factor: fw_mod = fwc / Cu
fw_mod = ( kappa * sqrt(2*Zo) * abs
    ( A * ( dker(2*sqrt(Zo)) + i * dkei(2*sqrt(Zo)) ) +
     B * ( dber(2*sqrt(Zo)) + i * dbei(2*sqrt(Zo)) ) ) ) ^ 2;

% CuAb/kn
CuAbkn = sqrt(2/fw_mod)/(30*kappa*Zo);

% Phase shift with shear stress
Fp = A * (dker(2*sqrt(Zo)) + i * dkei(2*sqrt(Zo)) ) +
    B * (dber(2*sqrt(Zo)) + i * dbei(2*sqrt(Zo)) )
Phase = atan2( imag(Fp), real(Fp) );

% M = matrix of results
if Zo < alpha
    M(count_Zo,1) = fw_mod;
    M(count_Zo,2) = CuAbkn;
    M(count_Zo,3) = alpha - Zb; % alpha_real
    M(count_Zo,4) = Zo;
    M(count_Zo,5) = Phase;
else % Zo > alpha, do nothing.
end

end
% Waves.m
% This program determines the velocity profiles for a periodic wave under the influence of a current

clear all

% WAVE AND CURRENT PARAMETERS

% Ub: Maximum Amplitude of the velocity predicted by potential flow theory at the bottom, (cm/s)
Ub = 100;

% w: Wave frequency = 2 * PI / Wave Period, (1/s)
w = 3.142;

% Zo: Non-Dimensionalized measure of the boundary roughness, ( )
Zo = 0.0934;

% angle: Angle between waves and current, (rad)
angle = 0;

% Uxc: Current friction velocity, (cm/s)
Uxc = 0;

% u: Ratio of the current friction velocity to the wave friction velocity = Uxc / Uxm, ( )
u = 0.01;  % for zero: use 0.01

% eps: Ratio of the current friction velocity to the combined wave and current friction velocity = Uxc / Uum, ( )
eps = u * ( 1 + 2*cos(angle)*u^2 + u^4 ) ^ (-1/4);

% Cu:
Cu = sqrt ( 1 + 2*cos(angle)*u^2 + u^4 );

% L: Boundary layer length scale = kappa * Uxm / w

% delta_bl: (non-Dimensional length of boundary layer + Zb = Y where velocity is max)

% K: Von Karman's Constant
kappa= 0.4;

% ph: Phase of wave motion
ph = 3.14159 * 0;

% MODEL PARAMETERS

% N: Bottom fraction of the wave boundary layer where the eddy viscosity varies linearly
N = 1 / 0.15;

% alpha: Non-dimensional length where the viscosity varies linearly

% Zb: Shift in Z axis
Zb = Zo;  % either Zo or 0
% Y: Non-Dimensional Vertical Axis = Z+Zb = zeta

% Ud: Non-Dimensional Deficit Velocity

% RESULTS

% This iteration finds the corresponding alpha value for the given wave-current parameters:
Alpha_iteration

% fw_mod: Modified Friction factor = fw / Cu
fw_mod = ( kappa * sqrt(2*Zo) * abs
         ( A * (dker(2*sqrt(Zo)) + i * dkei (2*sqrt(Zo))) ) +
         B * (dber(2*sqrt(Zo)) + i * dbei (2*sqrt(Zo))) ) )^2 ;

% CuAbkn: Modified Relative roughness = Cu * Ab / kn
CuAbkn = sqrt (2/fw_mod)/(30*kappa*Zo) ;

% fw: Wave Friction Factor
fw = fw_mod * Cu ;

% Abkn: Relative roughness
Abkn = CuAbkn / Cu ;

% Friction Velocities

% Uxw: Wave friction velocity, (cm/s)
Uxw = Ub * sqrt(0.5*fw) ;

% Uxm: Combined Wave-Current Friction Velocity, (cm/s)
Uxm = Uxw * sqrt(Cu) ;

% L: boundary layer length scale, (cm)
L = kappa* Uxm / w ;

% u: Recalculated u (compare with assumed u at start )
u = Uxc / Uxw ;

% VELOCITY and PHASE PROFILE

for num = 1:220
    Y = 11^((2.15*num-390)/70) + Zo ;    % arbitrary
    if Y >= Zo & Y <= alpha
        Ud = A * ker_kei(2*sqrt(Y)) + B * ber_be(2*sqrt(Y));
    elseif Y > alpha & Y <= alpha/eps
        Ud = C * exp(sqrt(i/alpha)*(Y)) + D * exp(-sqrt(i/alpha)*(Y));
    elseif Y > alpha/eps
        Ud = E * ker_kei(2*sqrt(Y/eps)) + F * ber_be(2*sqrt(Y/eps));
    end
% Phase shift of velocity
Phase_shift = atan2( imag(Ud+1), real(Ud+1) );

% Amplitude matrix
Matrix_Udl(num, 1) = ( Y - Zb ) * L ;  % Height above bottom
Matrix_Udl(num, 2) = abs(1+Ud) * Ub ;  % Velocity Amplitude
Matrix_Udl(num, 3) = Phase_shift ;  % Velocity Phase Shift

% Velocity matrix
Matrix_Ud2(num,1) = ( Y - Zb ) * L ;  % height above bottom
Matrix_Ud2(num,2) = (cos(ph)+real(Ud*exp(i*ph)))*Ub ;  % Velocity

end

% Current.m
% Current profiles
for num = 1:400
    
    Y = 11^((2.15*num-390)/200)+ Zo;  % arbitrary, gives well-spaced Y values in log scale

    % Uc = current velocity (non-dimensionalized by Uxc/kappa)
    if Y >= Zo & Y <= alpha
        Uc = eps * log( Y/Zo);
    elseif Y > alpha & Y <= alpha/eps
        Uc = eps * ( Y/alpha - 1 + log(alpha/Zo) );
    elseif Y > alpha/eps
        Uc = log( Y/(alpha/eps)) + 1 + eps * (log(alpha/Zo)-1);
    end

    Matrix_Uc(num, 1) = L * (Y - Zb) ;  % height above bottom
    Matrix_Uc(num, 2) = (Uxc/kappa) * Uc;  % current velocity

end

num = num + 1;

end
% Alpha_iteration.m

% alpha = alpha
alpha = 0;
alpha_new = Zo; % initial estimate of alpha

difference_alphas = 1; % initializing variable

while ( difference_alphas > 0.0001 )

alpha = alpha_new;

Constants % calls subroutine that computes A, B, C, D, E, F

delta_bl = 0; % initializing delta_bl, = (Zm+Zb)/ L
num = 1; % counter

while delta_bl == 0 % loop finds delta_bl for the specified alpha.

Y = Zo + (num-1)/200; % Y = zeta
if Y >= Zo & Y <= alpha
    Ud = A * ker_kei(2*sqrt(Y)) + B * ber_bei(2*sqrt(Y));
elseif Y > alpha & Y <= alpha/eps
    Ud = C * exp(sqrt(i/alpha)*(Y)) + D * exp(-sqrt(i/alpha)*(Y));
elseif Y > alpha/eps
    Ud = E * ker_kei(2*sqrt(Y/eps)) + F * ber_bei(2*sqrt(Y/eps));
end

Matrix_Ud(num, 1) = Y; % zeta
Matrix_Ud(num, 2) = real(Ud)+1; % Uw/Ub at cos(wt)= 1

% selection of delta_bl

if num > 2 & ( Matrix_Ud(num-2,2) > Matrix_Ud(num-1,2) )
    delta_bl = Matrix_Ud(num-2,1); % zeta where Uw/Ub is max
end

num = num + 1;
end

alpha_real = ( delta_bl - Zb ) / N; % N = 1 / 0.15
alpha_new = alpha_real + Zb; % alpha = alpha_r + Zb

difference_alphas = abs( (alpha_new - alpha) / alpha_new );
end

alpha_real = alpha - Zb;
% Constants.m
% File that solves the 5 equations for the constants A, B, C, D, E

a = ker_kei(2*sqrt(Zo));
b = ber_bei(2*sqrt(Zo));
c = ker_kei(2*sqrt(alpha));
d = ber_bei(2*sqrt(alpha));
g = dker_dkei(2*sqrt(alpha));
h = dber_dbei(2*sqrt(alpha));
n = ker_kei(2*sqrt(alpha)/eps);
o = dker_dkei(2*sqrt(alpha)/eps);
e = exp( alpha*sqrt(i/alpha));
f = exp(-alpha*sqrt(i/alpha));
j = 1/sqrt(alpha);
k = 1/sqrt(alpha);
l = exp((alpha/eps)*sqrt(i/alpha));
m = exp(-(alpha/eps)*sqrt(i/alpha));
p = sqrt(i);

x = f*p*k*l/e + p*k*m - f*l*o*j/(e*n) + o*j*m/n;

A = (-h*j/b + d*p*k/b - 2*d*k^2*l*f*p^2/(e*b*x) + 2*c*l*o*j*f*p*k/(e*n*x) - 2*a*d*l*o*j*f*p*k/(e*b*n*x) ) / ( -g*j + a*h*j/b + c*p*k - a*d*p*k/b - 2*c*k^2*p^2*l*f/(e*x) + 2*a*d*k^2*p^2*l*f/(e*b*x) + 2*c*l*o*j*f*p*k/(e*n*x) - 2*a*d*l*o*j*f*p*k/(e*b*n*x) ) ;

D = (A * (c*k*p*l)/e - (d*k*p*l)/(e*b) - A * (a*d*k*p*l)/(e*b) - A * (c*l*o*j)/(e*n) + (d*l*o*j)/(b*e*n) + A * (a*d*l*o*j)/(b*e*n) ) / (x) ;

E = A * (c*l)/(e*n) - (d*l)/(b*e*n) - A * (a*d*l)/(b*e*n) - D * (f*l)/(n*e) + D * m/n ;

C = A * c/e - d/(e*b) - A * (a*d)/(e*b) - D * f/e ;

B = -1/b - A * a/b ;

F = 0 ;
% Functions: Polynomial Approximations from Abramowitz and Stegun:

function y = ker(x)
y = -log(0.5*x) * ber(x) + 0.25*pi * bei(x) - 0.57721566 -
   59.05819744*(x/8)^4 + 171.36272133*(x/8)^8 - 60.6097451*(x/8)^12 +
   5.6539121*(x/8)^16 - 0.1963647*(x/8)^20 + 0.00309699*(x/8)^24 -
   0.00002458*(x/8)^28;

function y = kei(x)
y = -log(0.5*x) * bei(x) - 0.25*pi*ber(x) + 0.25*pi*bei(x) +
   6.76454936*(x/8)^2 - 142.91827687*(x/8)^6 + 124.23569650*(x/8)^10 - 21.30060904*(x/8)^14 +
   1.17509064*(x/8)^18 - 0.02695875*(x/8)^22 + 0.00029532*(x/8)^26;

function y = ber(x)
y = 1 - 64*(x/8)^4 + 113.77777774*(x/8)^8 - 32.36345652*(x/8)^12 +
   2.64191397*(x/8)^16 - 0.08349609*(x/8)^20 + 0.00122552*(x/8)^24 -
   0.0000901*(x/8)^28;

function y = bei(x)
y = 16*(x/8)^2 - 113.77777774*(x/8)^6 + 72.81777742*(x/8)^10 -
   10.56765779*(x/8)^14 + 0.52185615*(x/8)^18 - 0.01103667*(x/8)^22 +
   0.00011346*(x/8)^26;

function y = dker(x)
y = -log(0.5*x)*dber(x) - 1/x*ber(x) + 0.25*pi*dbei(x) +
   x*(-3.69113734*(x/8)^2 + 21.42034017*(x/8)^6 - 11.36433272*(x/8)^10 +
   1.41384780*(x/8)^14 - 0.06136358*(x/8)^18 + 0.00116137*(x/8)^22 -
   0.00001075*(x/8)^26);

function y = dkei(x)
y = -log(0.5*x)*dbei(x) - 1/x*dbei(x) - 0.25*pi*dber(x) + x*(0.21139217 -
   13.39858846*(x/8)^4 + 19.41182758*(x/8)^8 - 4.65950823*(x/8)^12 +
   0.33049424*(x/8)^16 - 0.00926707*(x/8)^20 + 0.00011997*(x/8)^24);

function y = dber(x)
y = x*(-4*(x/8)^2 + 14.22222222*(x/8)^6 - 6.06814810*(x/8)^10 +
   0.66047849*(x/8)^14 - 0.02609253*(x/8)^18 + 0.00045957*(x/8)^22 -
   0.0000394*(x/8)^26);

function y = dbei(x)
y = x*(0.5 - 10.66666666*(x/8)^4 + 11.37777772*(x/8)^8 -
   2.31167514*(x/8)^12 + 0.14677204*(x/8)^16 - 0.00379386*(x/8)^20 +
   0.00004609*(x/8)^24);

function y = ker_kei(x)
y = ker(x) + i * kei(x);

function y = ber_bei(x)
y = ber(x) + i * bei(x);

function y = dker_dkei(x)
y = dker(x) + i * dkei(x);

function y = dber_dbei(x)
y = dber(x) + i * dbei(x);
Bibliography


