MASSACHUSETTS INSTITUTE OF TECHNOLOGY

15.053 – Optimization Methods in Management Science (Spring 2007)

Problem Set 5

Due March 22nd, 2007 at 4:30 pm.
You will need 119 points out of 140 to receive a grade of 5.

Problem 1: Weak and Strong Duality (36 Points; 4 Each)

The idea behind this problem is to explore what types of relationships the optimal solutions of the primal and dual can have.

Look at the following grid, for each empty box determine if the situation could occur. If it could occur, give an example. If it could not occur, explain why using weak and strong duality.

PRIMAL/ DUAL	Finite Optimal Solution	Unbounded Solution	Infeasible Solution
Finite Optimal Solution	**Part A:**	**Part B:**	**Part C:**
Unbounded Solution	**Part D:**	**Part E:**	**Part F:**
Infeasible Solution	**Part G:**	**Part H:**	**Part I:**

For example in Part F you need to determine if it is possible for the primal to be unbounded and the dual to be infeasible.

Problem 2: Dual Simplex (28 Points; 4 Points Each)

The reason we wrote this problem is to have you practice the basics of dual simplex. Also we wanted to point out the dangers of drinking and driving.

Things aren't so simple to Paris and Nicole these days, both having DUI's. They are too out of it and forgot how to use the simplex method. Since they are in a duel, they decide to use the dual simplex method to solve their problem. However they are too drunk on sugar to do so and ask you, a super talented 053 student for help.

Consider the following primal problem

$$\max ... z = -x_1 - x_2$$
$$st: x_1 + 2x_2 - x_3 = 2$$
$$x_1 - x_4 = 1$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Part A:

Take the dual.

Part B:

Graph the feasible region of the dual

Part C:

Assume x_3 and x_4 are slack variables, write the original problem and graph its feasible region. Label the corner points of the feasible set.

Part D:

Letting x3 and x4 be the initial basic variables write out the initial tableau for the simplex method. Is this current solution feasible?

Part E:

What corner point on your graph does this solution correspond to in the primal problem?

Part F:

Using your graphs show geometrically what happened on the iteration.

Part G:

What is the optimal solution to the primal and dual?

Problem 3: Justin and Kevin's Trailer Company (36 Points; 4 Each)

This problem questions you on what happens when we forget a primal constraint, and shows an application where the dual simplex is more useful than the primal simplex.

Having both been with Britney, a little bit of trailer feeling has rubbed off onto Justin and Kevin. They have decided to go into business together and open up a firm that produces trailers. Since they are so rich they sell them for next to nothing. Currently they produce three types of trailers (sorry I do not know much about trailers!!!)

- They make the white trailer, sells for $6
  - To produce each requires 1 hours of metalworking
  - To produce each requires .5 hour of woodworking

- The blue trailer, sells for $14
  - To produce each requires 2 hours of metalworking
  - To produce each requires 1 hours of woodworking

- The pink trailer (For Britney), sells for $13
  - To produce each requires 4 hours of metalworking
  - To produce each requires 4 hours of woodworking

The pair has at most 24 hours of woodworking time and 60 hours of metalworking time.

Part A:

Formulate a linear program to maximize Justin and Kevin's profits. See the next part for your decision variables.

Part B:

After converting the problem to standard form and solving it using the simplex method, the pair come up with the following optimal tableau. Write out the optimal solution that corresponds to this tableau. Note: x_4 and x_5 are the slack variables for the first and second constraints respectively.

z	x_W	x_B	x_P	x_4	x_5	RHS
1	0	9	0	11	0.5	294
0	1	6	0	4	-1	36
0	0	-1	1	-1	0.5	6

Part C:

Take the dual of the original LP. Write in words what each dual variable stands for.

Part D:

Since they were drunk, the pair forgot about that the total number of trailers that can be produced must be at most “y” where y is some positive integer. Modify your original LP in part A to take this into account.

Part E:

What is the range of y for which the trailer constraint is satisfied?

Part F:

Suppose y= 50.

Convert the constraint you added in part E to standard form by adding a slack variable x_6. Add a row to your tableau in part B to express the constraint (e.g, the slack variable you add will be the basic variable corresponding to the total produced constraint). Get the tableau in canonical form by performing row operations, and with the basic variables X_w, X_p, and X_6. Is the new tableau primal optimal? Is it primal feasible? Explain you answer.

Part G:

Solve for the new optimal solution using the dual simplex method.

Part H:

What is the optimal primal solution? Is the optimal dual objective solution?

Part I:

Can this technique be used in general when we forget an inequality constraint or do certain conditions under which the technique can be used?

Problem 4: Bjork (15 points; 5 Each)

The idea behind this question is to teach you how to interpret a dual tableau
Icelandic Singer Bjork has decided to open up her own design studio for dresses that look like dead animals. She is producing muskrat dresses, python dresses, ostrich dresses, as well as her own unique swan dresses.

In order to make the dresses, she will require cotton, synthetic ruffles, and labor. (It turns out that dead animals are not a constraint). She decides to use a linear program in order to model her problem. Her decision variables are as follows:

x_1 = number of muskrat dresses
x_2 = number of python dresses
x_3 = number of ostrich dresses
x_4 = number of swan dresses

She is assuming (unfortunately incorrectly) that there is unlimited demand for her dresses.

Here linear program is as follows:

$$
\begin{aligned}
\text{max} \quad & z = 8x_1 + 14x_2 + 30x_3 + 50x_4 \\
s.t. \quad & 1x_1 + 2x_2 + 10x_3 + 16x_4 \le 80 \quad \text{(pounds of cotton)} \\
& 1.5x_1 + 2x_2 + 4x_3 + 5x_4 \le 100 \quad \text{(quarts of ruffles)} \\
& .5x_1 + .6x_2 + 1x_3 + 2x_4 \le 340 \quad \text{(days of labor)} \\
& x_1, x_2, x_3, x_4 \ge 0
\end{aligned}
$$

Part A:

Write the initial tableau for the simplex algorithm. Use s_1, s_2, and s_3 for the slack variables.

Part B:

Write the dual of the original linear programming problem.

The z-row of the optimal tableau is as follows:

x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS
0	0	w	40	5	2	0	600

Part C:

What are the optimal basic variables, assuming that w > 0? What is the optimal objective value?

Problem 5: Duality Short Answer (20 Points; 4 Each)

This problem pinpoints some key ideas behind duality which will be useful when we learn about game theory next week

For each of the following statements say whether they are true or false. Please explain your answers. We are not assuming that the primal is a maximization problem

Part A:

Each maximization LP has a minimization dual.

Part B:

Assuming the Primal is a max problem, it is possible that there exists a dual feasible solution that has a lower objective value then some primal feasible solution.

Part C:

If the primal and dual are both feasible then they both have the same feasible region.

Part D:

The optimal solutions of the dual are to the reduced costs of the primal.

Problem 6: Duality and Abstract Formulations (5 Points)

Consider the following problem from the last problem set, in this problem you will look closer at the dual variables:

John Doe, a graduate student at MIT, recently went to see his family practitioner at the MIT Medical Center. John Doe has been complaining of the following symptoms:

- Frequent headaches and indigestion
- Sore wrists and back
- Restless sleep

The MIT family practitioner has seen these symptoms many times before. John Doe's biggest problem is stress:

Thus, John's doctor recommends John start a regular exercise routine to help manage his symptoms. The doctor recommends John do at least 2 hours of aerobic exercise and 1 hour of anaerobic exercise each week. John realizes that if he just sits in the gym for 3 hours a week he'll get bored, so he decides that he needs to mix in at least 2 hours of fun. John is considering each of the following activities and wants to determine how to optimally spend his time to satisfy his objectives.

Activity (per hour)	Aerobic (hours)	Anaerobic (hours)	Fun (hours)
Kickboxing	.54	.85	.25
Basketball	.24	.55	.25
Squash	.45	.10	.3
Softball	.25	.05	.70

Bonus: (2 Points)

Write down the primal and dual from last week

Question: (5 Points)

Interpret the dual variables and constraints. That is write what each dual variable means and what each dual constraint is modeling. Try and be very Precise in your definitions and detailed so that a person who is not an LP star like you guys could understand it.

Challenge Problem E (15 Points; 5 Each)

Part A:

Come up with a Primal Dual pair where each has multiple optimal solutions.

Part B:

Come up with a Primal Dual pair where the primal has a degenerate optimal solution while the dual has unique optimal solution.

Part C:

Come up with a Primal Dual Pair where both have degenerate optimal solutions, or explain why a pair can not exist.