15.053 February 22, 2007

- Introduction to the Simplex Algorithm

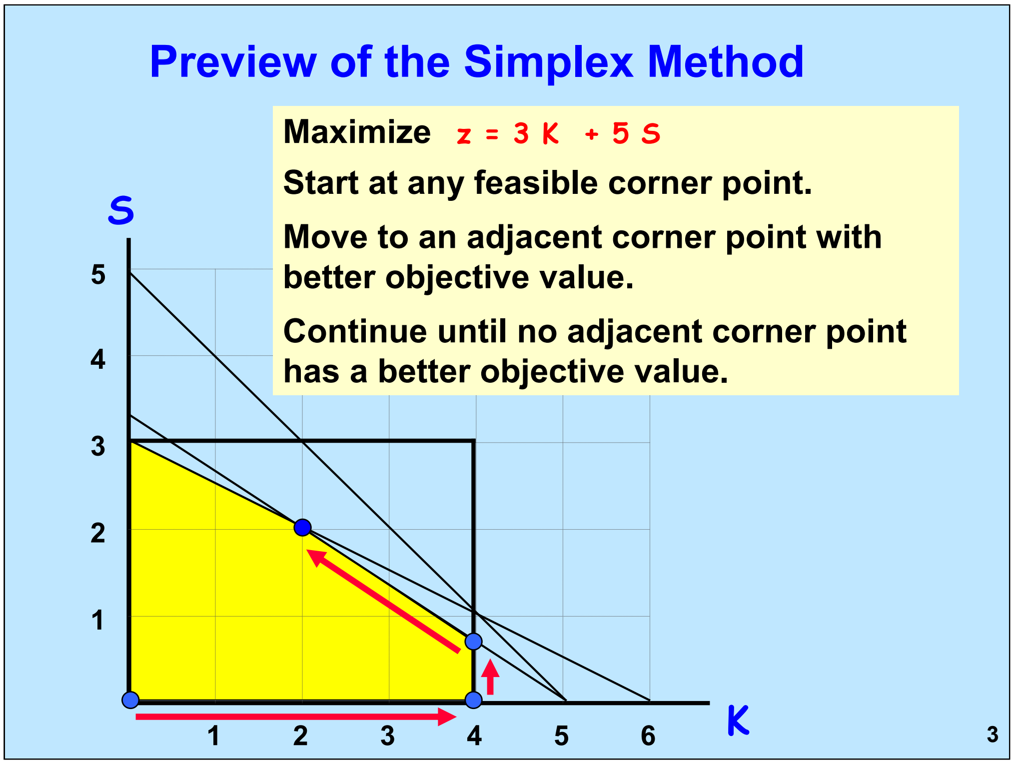
Quotes for today

Give a man a fish and you feed him for a day. Teach him how to fish and you feed him for a lifetime.

-- Lao Tzu

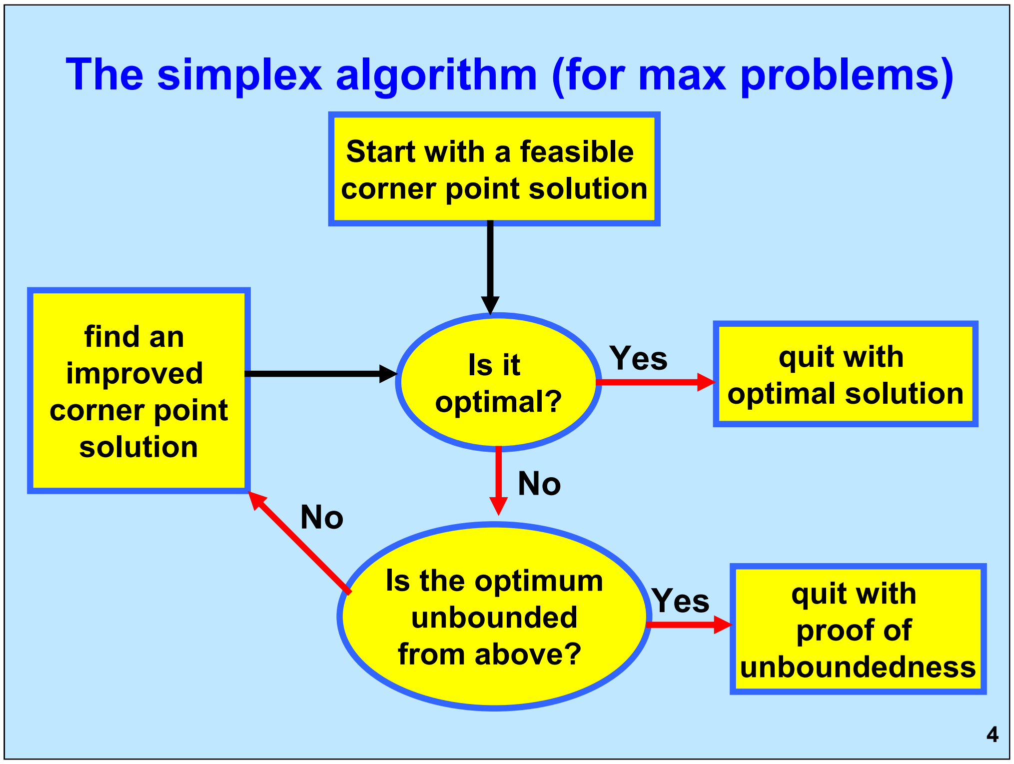
Give a man a fish dinner, and he will forget it by next week. Let a person catch the fish for himself, and he'll remember it for a lifetime.

-- Jim Orlin



This is a picture of the simplex algorithm in inequality form. In this form, the simplex algorithm moves from corner point to corner point. And each corner point is the intersection of two constraints.

When we move to equality form, the simplex algorithm still moves from corner point to corner point. And the corner points are still found by solving a system of equations. So, there are many similarities.



As you can see, this is a fairly simple structure. At the same time, it may be difficult to keep everything in one's head at the same time. That is where the two dimensional example can help out.

We will assume that we start with a feasible corner point solution. That immediately raises two questions. What does a corner point solution look like? And how do you find a corner point solution to start with? Both of these issues will be addressed shortly.

The next slides deal with something even more preliminary. We will be assuming that we start with a linear program with equality constraints and non-negativity constraints, and nothing else. So we need to get each linear program into the correct starting form. We will show how to do that on the next few slides.

Goals for this lecture

Major Issues of the Simplex Algorithm

1. **How does one get the LP into the correct starting form?**
2. **How does one recognize optimality and unboundedness?**
3. **How does one move to the next corner point solution?**

Note: we will derive the simplex algorithm in class!

Linear Programs in Standard Form

We say that a linear program is ***in standard form*** if the following are all true:

1. Non-negativity constraints for all variables.
2. All remaining constraints are expressed as equality constraints.
3. The right hand side vector, *b*, is non-negative.

An LP not in Standard Form

maximize $z = 3x_1 + 2x_2 - x_3 + x_4$

$x_1 + 2x_2 + x_3 - x_4 \leq 5$; not equality

$-2x_1 - 4x_2 + x_3 + x_4 \leq -1$; not equality

$x_1 \geq 0, x_2 \geq 0$ — x_3 and x_4 may be negative

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Excel Solver does not require that you write an LP in standard form because it will immediately transform it to standard form via software. We show next what linear programming solvers do with an LP that does not start in standard form.

Converting Inequalities into Equalities Plus Non-negatives

Before

$x_1 + 2x_2 + x_3 - x_4 \leq 5$

After

$x_1 + 2x_2 + x_3 - x_4 + s_1 = 5$

$s_1 \geq 0$

s_1 is called a *slack variable*, which measures the amount of "unused resource."

Note that $s_1 = 5 - x_1 - 2x_2 - x_3 + x_4$.

To convert a "≤" constraint to an equality, add a slack variable.

So, we transform a "≤ constraint" by

1. adding a slack variable
2. requiring that the slack variable is non-negative.

Converting RHS and "≥" constraints

- Consider the inequality $-2x_1 - 4x_2 + x_3 + x_4 \leq -1$;
- Step 1. Eliminate the negative RHS. Multiply by -1.

$$2x_1 + 4x_2 - x_3 - x_4 \geq 1$$

- Step 2. Convert to an equality

$$2x_1 + 4x_2 - x_3 - x_4 - s_2 = 1$$

$$s_2 \geq 0$$

- The variable added will be called a "*surplus variable*."

To convert a "≥" constraint to an equality, subtract a surplus variable.

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We get rid of negative right hand sides by multiplying through by -1.

We transform a "≥ constraint" by

1. adding a surplus variable
2. requiring that the slack variable is non-negative.

To be honest, I sometimes confuse the names "slack" and "surplus" because they are serving the exact same function, converting an inequality constraint to an equality constraint. They have different names because of their interpretations in practice. Often a "≤ constraint" will model a case in which we have limited resources, and the "slack" represents the amount left over. Often a "≥ constraint" will model a case in which we have to produce at least a specified amount. If we produce more than we need, we are said to have produced a surplus.

Converting Max to Min and Min to Max

Converting Max to Min: multiply objective by -1

Example: Minimize $z = 3x_1 + 2x_2$

subject to "constraints"

Has the same optimum solution(s) as

Maximize $v = -3x_1 - 2x_2$

subject to "constraints"

Minimizing z is equivalent mathematically to maximizing –z. Interestingly, practitioners often have a very strong preference. If you tell a practitioner that you are maximizing the negative of the cost, it will sound very confusing, unless you convert it somehow to maximizing profit. But mathematically, there is no important distinction.

Other transformations

See tutorial on transformations.

Why standard form?

The simplex method is designed for problems with equality constraints and non-negativity constraints.

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The tutorial covers situations in which a variable x does not start with the constraint $x \geq 0$. It is possible that in a model, some variables are constrained to be non-positive, and possibly other variables have no constraint on sign at all. In all of these cases, the LP solver will first create an equivalent program in which all variables are constrained to be non-negative.

Review: solving a system of Equations

$$2x_1 + 2x_2 + x_3 = 9$$
$$2x_1 - x_2 + 2x_3 = 6$$
$$x_1 - x_2 + 2x_3 = 5$$

	x_1	x_2	x_3		RHS
Equation 1	2	2	1	=	9
Equation 2	2	-1	2	=	6
Equation 3	1	-1	2	=	5

The set of equations with the x's written in the top row is called a tableau. We will use tableaus to illustrate the simplex algorithm.

	x_1	x_2	x_3		RHS
Equation 1	1	1	1/2	=	9/2
Equation 2	0	-3	1	=	-3
Equation 3	0	-2	3/2	=	1/2

We want column 1 to be $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Divide through equation 1 by 2.

Subtract two times equation 1 from equation 2.

Subtract equation 1 from equation 3.

For more information on solving systems of equations, see the tutorial on the website.

	x_1	x_2	x_3		RHS
Equation 1	1	0	5/6	=	7/2
Equation 2	0	1	-1/3	=	1
Equation 3	0	0	5/6	=	5/2

We want column 2 to be

0
1
0

Divide through equation 2 by -3.

Subtract equation 2 from equation 1.

Add two times equation 2 to equation 3.

	x_1	x_2	x_3		RHS
Equation 1	1	0	0	=	1
Equation 2	0	1	0	=	2
Equation 3	0	0	1	=	3

We want column 3 to be

0
0
1

Divide through equation 3 by 5/6.

Subtract equation 3 from equation 1.

Add 1/3 times equation 3 to equation 2.

	x_1	x_2	x_3		RHS
Equation 1	1	0	0	=	1
Equation 2	0	1	0	=	2
Equation 3	0	0	1	=	3

Resulting equations $x_1 = 1$, $x_2 = 2$, $x_3 = 3$.

The solution is now obvious.

The system of equations is in a very special form.

At the end, each column for a variable has a single 1 and two 0s.

The equations themselves are the same as the solution.

1. Start with a feasible corner point solution

- **Start with a tableau in "*canonical form*"**
  - **LP has equality constraints and non-negativity constraints.**
  - **There is one "*basic*" variable for each equality constraint.**
  - **The column for the basic variable for constraint j has a 1 in constraint j and 0's elsewhere.**
  - **The remaining variables are called *non-basic*.**

Standard form does not necessarily give a corner point solution. But standard form is a good place to get started.

For a corner point solution,

A “Tableau” in canonical form.

z	x_1	x_2	x_3	x_4	x_5		
1	0	2	0	0	1	=	2
0	0	2	1	0	-1	=	4
0	0	-1	0	1	2	=	1
0	1	6	0	0	3	=	3

The non-basic variables are x_2 and x_5.
z is considered to be a basic variable.

If we got rid of the non-basic variables (as in erasing the columns for x2 and x5), then the resulting equations would be the same as the solution. That is, the equations would be $x_3 = 4$, $x_4 = 1$, $x_1 = 3$. In reality, we don’t erase the columns. We just set the non-basic variables to 0, which is mathematically equivalent.

The “*basic feasible solution*” or bfs

The basic variables are x_1, x_3, x_4, and z

The non-basic variables are x_2, x_5

Set the non-basic variables to 0

z	x_1	x_2	x_3	x_4	x_5		
1	0	2	0	0	1	=	2
0	0	2	1	0	-1		4
0	0	-1	0	1	2	=	1
0	1	6	0	0	3	=	3

The basic feasible solution (bfs) is:
$x_2 = x_5 = 0$; $x_1 = 3$, $x_3 = 4$, $x_4 = 1$, $z = 2$

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We will use the term “basic feasible solution” or “bfs” throughout the rest of the semester. Every bfs is also a corner point solution, in that it is not the midpoint of a line segment joining two other solutions.

The simplex method will move from corner point to corner point along edges.

When is a basic feasible solution (bfs) optimal?

Together we will derive the optimality conditions

An example

maximize	$z = -2x_2 - x_5 + 2$
subject to	$x_1 = 3,\ x_3 = 4,\ x_4 = 1$
	$x_1, x_2, x_3, x_4, x_5 \geq 0$

What is an optimal solution for this problem?

The first example is an LP in which

1. The objective function only has terms for the nonbasic variables.

2. The coefficients of the variables in the objective function are nonpositive and only involve the nonbasic variables.

3. The only constraints on the non-basic variables are nonnegativity constraints.

So, all one needs to do is to set x_2 and x_5 optimally, which in this case sets them both to 0.

A second example

maximize	$z = -2x_2 - x_5 + 2$
subject to	$x_1 = 3 - 6x_2 - 3x_5$
	$x_3 = 4 - 2x_2 + x_5$
	$x_4 = 1 + x_2 - 2x_5$
	$x_1, x_2, x_3, x_4, x_5 \geq 0$

What is an optimal solution for this problem?

The second example is an LP in which

1. The objective function only has terms for the nonbasic variables.
2. The coefficients of the variables in the objective function are nonpositive and only involve the nonbasic variables.
3. Setting the nonbasic variables to 0 gives a feasible solution.

In this case, setting the nonbasic variables to 0 gives a feasible solution with $z = 2$. And any other solution has $x_2 \geq 0$ and $x_5 \geq 0$, and thus $z \leq 2$. So, the solution with the nonbasic variables set to 0 must be optimal.

So, all one needs to do is to set x_2 and x_5 optimally, which in this case sets them both to 0.

When are sufficient conditions for a solution to be optimal?

maximize	$z = -2x_2 - x_5 + 2$
subject to	$x_1 = 3 - 6x_2 - 3x_5$
	$x_3 = 4 - 2x_2 + x_5$
	$x_4 = 1 + x_2 - 2x_5$
	$x_1, x_2, x_3, x_4, x_5 \geq 0$

A solution x_1, x_2, x_3, x_4, x_5 is guaranteed to be optimal for an LP with non-negativity constraints whenever …..

The objective function has the following properties:

1. The coefficients of the nonbasic variables are nonpositive
2. The coefficients of the basic variables are 0.

And the feasible solution x is obtained by setting the nonbasic variables to 0.

Recognizing an Optimal bfs: Tableau Version

The basic feasible solution (bfs) is:
$x_2 = x_5 = 0$; $x_1 = 3, x_3 = 4, x_4 = 1, z = 2$

It is optimal!

z	x_1	x_2	x_3	x_4	x_5		
1	0	2	0	0	1	=	2
0	0	2	1	0	-1	=	4
0	0	-1	0	1	2	=	1
0	1	6	0	0	3	=	3

maximize $z = -2x_2 - x_5 + 2$ s.t. $x \geq 0$

The opt solution is z = 2.

In the tableau form, the objective is written as

$z + 2x_2 + x_5 = 2.$

Optimality conditions for a bfs in tableau form: the coefficients in the z-row nonnegative for the nonbasic variables.

Note that tableaus that correspond to bfs's already have the following properties:

1. The coefficients of the basic variables in the objective function are 0
2. There is a feasible solution obtained by setting the nonbasic variables to 0.

Thus the optimality condition stated above for a bfs in tableau form are the same as from the previous slides.

Optimality Conditions

z	x_1	x_2	x_3	x_4	x_5		
1	0	2	0	0	1	=	2
0	0	2	1	0	-1	=	4
0	0	-1	0	1	2	=	1
0	1	6	0	0	3	=	3

Important Fact.
If there is no negative coefficient in the z row, the basic feasible solution is optimal!

maximize $z = -2x_2 - x_5 + 2$

Is the optimum unbounded from above?

Together we will derive the conditions for unboundedness.

An example

maximize	$z = -2x_2 + x_5 + 2$
subject to	$x_1 = 3,\ x_3 = 4,\ x_4 = 1$
	$x_1, x_2, x_3, x_4, x_5 \geq 0$

What is an optimal solution for this problem?

The objective function (for a max problem) in this example satisfies the following conditions:

1. The coefficients of the basic variables in the objective are 0
2. There is a positive coefficient in the objective for a nonbasic variable
3. The only constraints on the nonbasic variables are nonnegativity constraints.

In this case, we can get a sequence of increasingly better solutions by making x_5 increasingly larger.

A second example

maximize $z = -2x_2 + x_5 + 2$

subject to

$$x_1 = 3 - 6x_2 + 3x_5$$
$$x_3 = 4 - 2x_2 + x_5$$
$$x_4 = 1 + x_2 + 2x_5$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

What is an optimal solution for this problem?

The objective function (for a max problem) in this example satisfies the following conditions:

1. The coefficients of the basic variables in the objective are 0
2. There is a positive coefficient in the objective for the nonbasic variable x_5.
3. For any fixed choice of $x_5 > 0$, there is a feasible solution in which the only positive variables are x_5 and the current basic variables.

In this case, we can get a sequence of increasingly better solutions by making x_5 increasingly larger.

Directions of Unboundedness

maximize $z = -2x_2 + x_5 + 2$

subject to

$x_1 = 3 - 6x_2 + 3x_5$

$x_3 = 4 - 2x_2 + x_5$

$x_4 = 1 + x_2 + 2x_5$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

Let $x_5 = \Delta$. **Let** $x_2 = 0$.

Assume that $\Delta \geq 0$

Then $x_1 = 3 + 3\Delta$
$x_3 = 4 + \Delta$
$x_4 = 1 + 2\Delta$
$z = \Delta$

Direction of unboundedness

$$x = \begin{pmatrix} 3 \\ 0 \\ 4 \\ 1 \\ 0 \end{pmatrix} + \Delta \begin{pmatrix} 3 \\ 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

When the solution is unbounded from above, we often keep track of the sequence of solutions whose objective is unbounded from above. This can be done very efficiently by storing a feasible x' solution and a direction of unboundedness y'. Then for every value of Δ, the solution

x' + Δy' is feasible. As Δ gets increasingly larger, the objective for x' + Δy' gets increasingly larger, and approaches infinity in the limit.

More on Directions of Unboundedness

A vector y is called a *direction of unboundedness* for a maximization problem if

1. **For all feasible solutions x and all positive numbers Δ, the vector x + Δy is feasible.**
2. **The objective value for y is positive.**

Fact: an LP is unbounded from above if and only if there is a feasible solution and there is also a direction of unboundedness.

The property of direction of unboundedness is true for linear programs, but is not true for non-linear programs. For example, one could imagine a feasible region in two dimensions that is a spiral, and that the objective goes to infinity as one moves along the spiral. But there is no direction of unboundedness as defined on the slide.

Unboundedness: Tableau Version

z	x_1	x_2	x_3	x_4	x_5		
1	0	2	0	0	-1	=	2
0	0	2	1	0	-1	=	4
0	0	-1	0	1	-2	=	1
0	1	6	0	0	-3	=	3

A maximization LP is unbounded from above if there is a bfs and a non-basic variable x_s such that

1. **The coefficient for x_s is the z-row is negative, and**
2. **All coefficients in the column for x_s are ≤ 0.**

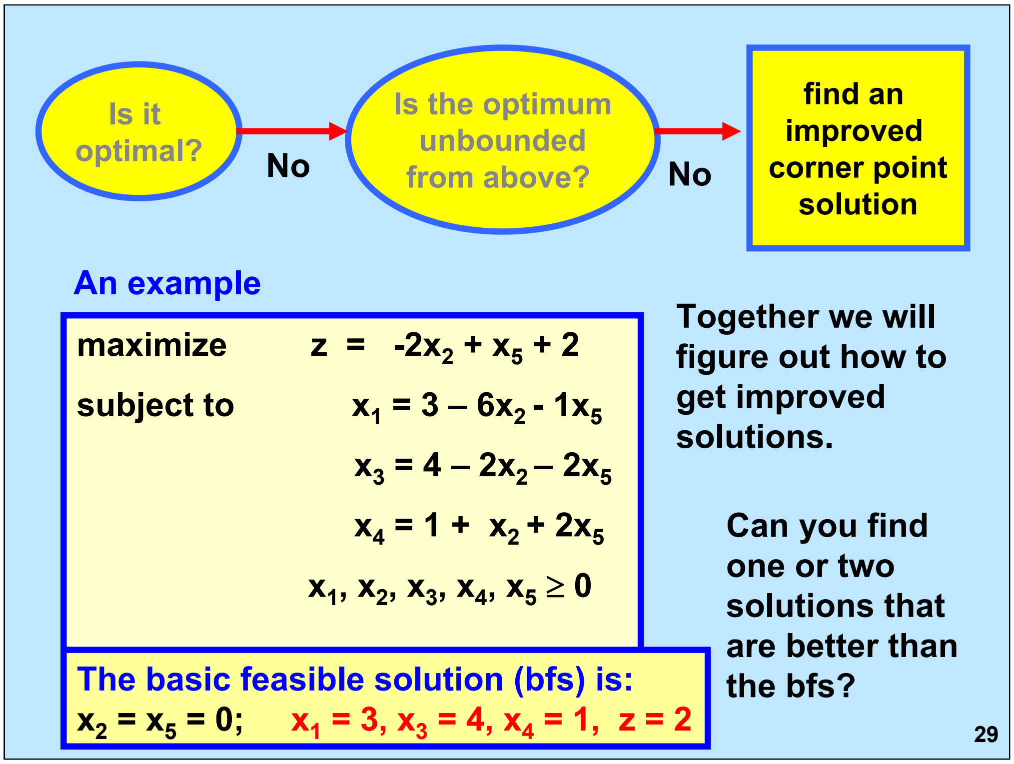
In the tableau form, the objective is written as

$z + 2x_2 - x_5 = 2$.

Unboundedness conditions when given a bfs in tableau form for a max problem: there is a negative coefficient in the z-row for some nonbasic variable x_s.

The column in the tableau for x_s is nonpositive.

For any specified value of x_s, one can adjust the values of the current basic variables to provide a feasible solution. One shows that the objective value is unbounded from above by letting x_s approach infinity.



In this example, one of the basic variables x_5 has a positive coefficient in the objective function. But the unboundedness conditions are not satisfied.

If we make x_5 a little larger than 0, we can adjust the current basic variables to give a feasible solution and this feasible solution will have a larger objective value than the current bfs.

The larger that x_5 is, the larger will be the objective value. So, we want to make x_5 as large as possible so long as the other basic variables remain non-negative.

Finding improved solutions

$$\max \quad z = -2x_2 + x_5 + 2$$

$$\begin{aligned} \text{st} \quad x_1 &= 3 - 6x_2 - 1x_5 \\ x_3 &= 4 - 2x_2 - 2x_5 \\ x_4 &= 1 + x_2 + 2x_5 \\ x_1, x_2, x_3, &x_4, x_5 \geq 0 \end{aligned}$$

We copied the equations so that there would be space to write the improved solutions.

Improving Solutions: Tableau Version

z	x_1	x_2	x_3	x_4	x_5		RHS
1	0	2	0	0	-1	=	2
0	0	2	1	0	2	=	4
0	0	-1	0	1	-2	=	1
0	1	6	0	0	1	=	3

$z = \Delta + 2$
$x_1 = 3 - 1\Delta$
$x_2 = 0$
$x_3 = 4 – 2\Delta$
$x_4 = 1 + 2\Delta$
$x_5 = \Delta$

Find a non-basic variable with a negative coefficient in the z-row. Set that variable to Δ, and keep all other non-basic variables at 0.

Choose Δ maximum

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We could look for improved solutions by just guessing the value of x5. But to do it systematically, we set it to Δ. As you can see, I am fond of using Δ as a parameter.

Once we set it to Δ, we can see how the current basic variables vary as a linear function of Δ. We then choose Δ as high as possible so that all of the current basic variables are nonnegative. In this case, we can let Δ be as large as 2. If it were any larger, than x_3 would be negative.

Mira and Marnie's M&M Adventure

Mira and Marnie, two MIT undergraduates known as the M&M sisters, recently received a gift from their parents of 2000 pounds of gray M&Ms and 6000 pounds of red M&Ms, the MIT colors. So, they decided to go into business selling large bags of "MIT M&Ms" for frat parties. They can sell a bag with 3 pounds of red M&Ms and 2 pounds of gray M&Ms for $20. They can purchase bags of 3 pounds of red M&Ms and 4 pounds of gray M&Ms for $30. How many bags should Mira & Marnie buy and sell to maximize their profit.

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M&Ms really can be bought in very large packages with quantity discounts, and you can choose the colors. You can even have custom printing (e.g., I love 15.053). See
http://www.mymms.com
for more details.

Formulation as a linear program

- Let x_1 be the number of 7 pound bags purchased (in thousands)
- Let x_2 be the number of 5 pound bags sold (in thousands)
- Measure the profit in $10,000s.

A 2-variable LP

maximize $z = -3x_1 + 2x_2$

subject to

$-3x_1 + 3x_2 \leq 6$

$-4x_1 + 2x_2 \leq 2$

$x_1 \geq 0, \; x_2 \geq 0$

maximize $z = -3x_1 + 2x_2$

subject to

$-3x_1 + 3x_2 + x_3 = 6$

$-4x_1 + 2x_2 + x_4 = 2$

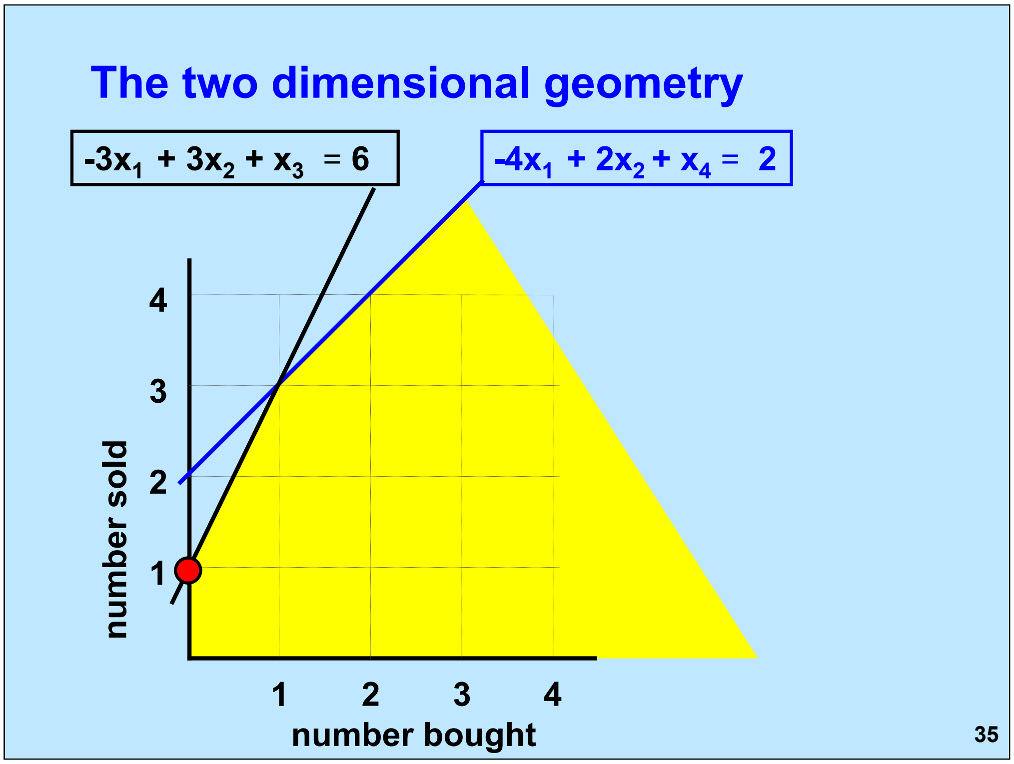
$x_1, \; x_2, \; x_3, \; x_4 \geq 0$

z	x_1	x_2	x_3	x_4		RHS
1	3	-2	0	0	=	0
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

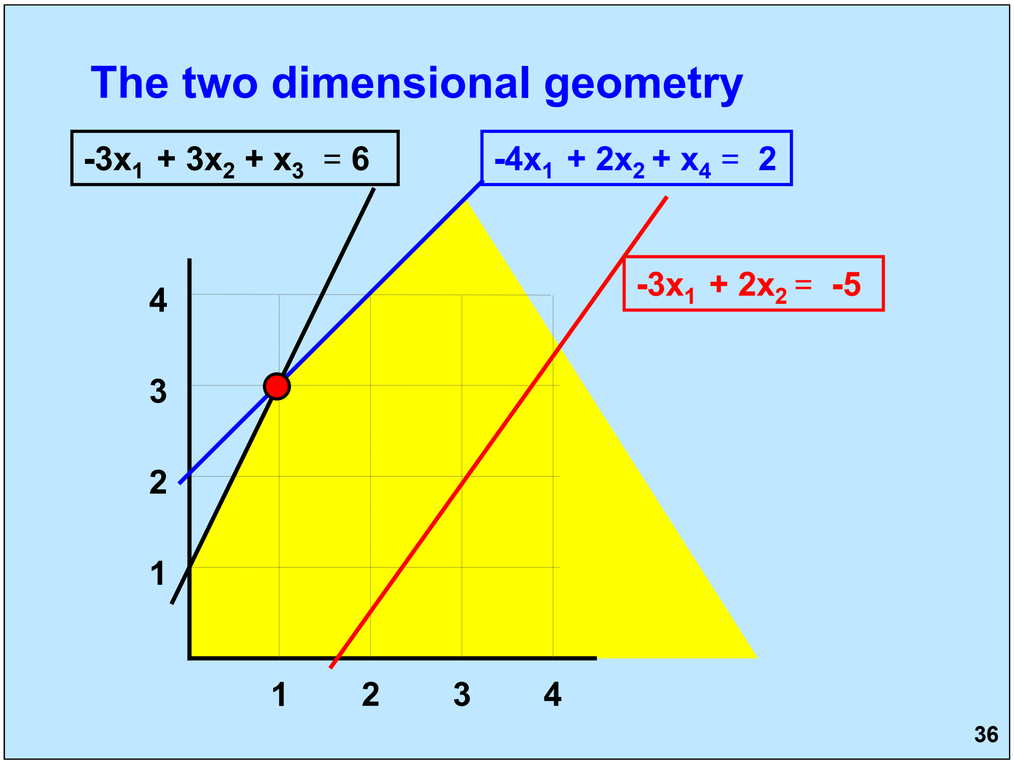
We first add slack variables x_3 and x_4.

We then express the equations in tableau form.

Note that the initial tableau is in canonical form, and there is a corresponding bfs.



For this particular LP, the feasible region is unbounded, but there will be an optimal solution.



The optimal solution will be $x_1 = 1$ and $x_2 = 3$. The slack variables will both be 0.

LP "canonical form"

The initial tableau is already in canonical form.

z	x_1	x_2	x_3	x_4		
1	3	-2	0	0	=	0
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

The basic variables are z, x_3 and x_4.

The non-basic variables are x_1 and x_2.

The basic feasible solution (bfs) for this basis is
$z = 0$, $x_1 = 0$, $x_2 = 0$, $x_3 = 6$, $x_4 = 2$

LP Canonical Form and the bfs.

z	x_1	x_2	x_3	x_4		
1	3	-2	0	0	=	0
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

The text treats z as a basic variable.

The simplex method starts with a *tableau* in *LP canonical form* (or it creates canonical form at a preprocess step.)

The first solution is the bfs for that tableau.

We will discuss next lecture what to do if there is no obvious way of getting a tableau in canonical form.

For each constraint there is a basic variable

z	x_1	x_2	x_3	x_4			
1	3	-2	0	0	=	0	Objective function.
0	-3	3	1	0	=	6	Constraint 1
0	-4	2	0	1	=	2	Constraint 2

One basic variable is z

Constraint 1: basic variable is x_3

Constraint 2: basic variable is x_4

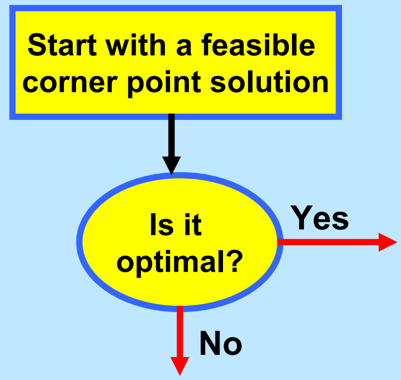
bfs

$x_1 = 0$; $x_2 = 0$;

$x_3 = 6$; $x_4 = 2$;

$z = 0$

The simplex algorithm (for max problems)



We were lucky to be able to start with a feasible bfs. We now move on to the rest of the algorithm.

Next lecture: how to find a starting bfs

On the Optimality Conditions

z	x_1	x_2	x_3	x_4		
1	3	-2	0	0	=	0
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

The cost-coefficient of x_2 is -2.

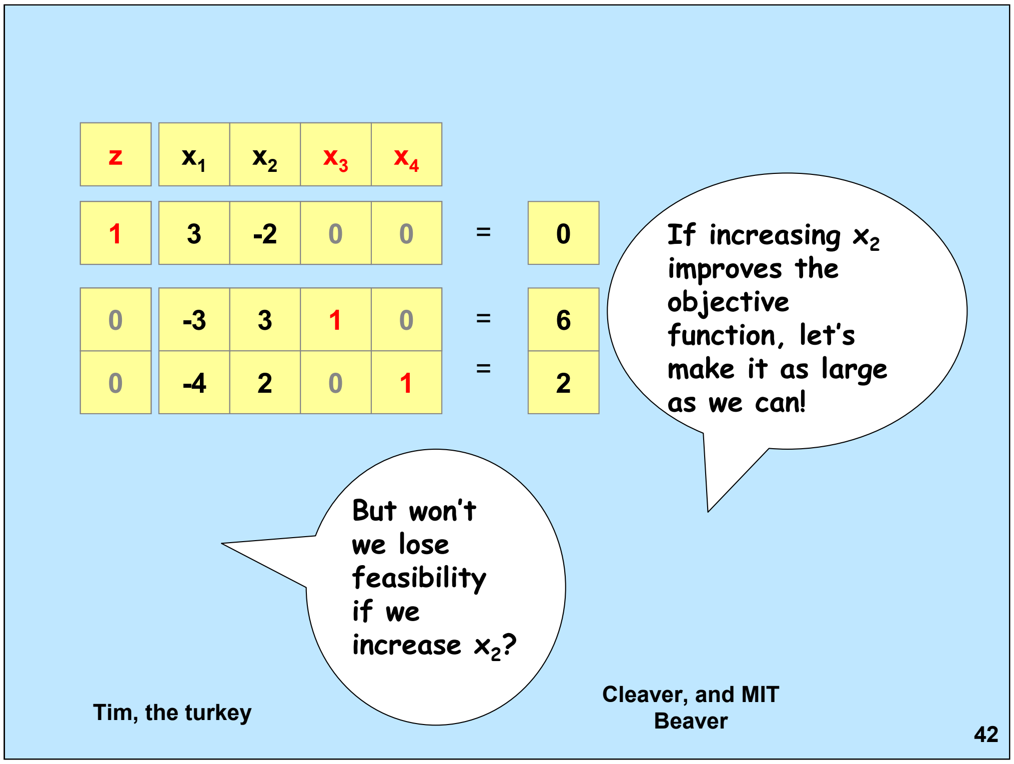
$z + 3x_1 - 2x_2 = 0$

If $x_1 = 0$, and $x_2 = 1$, then $z = 2$.

The current bfs can be improved if we can increase x_2 and hold x_1 at 0.

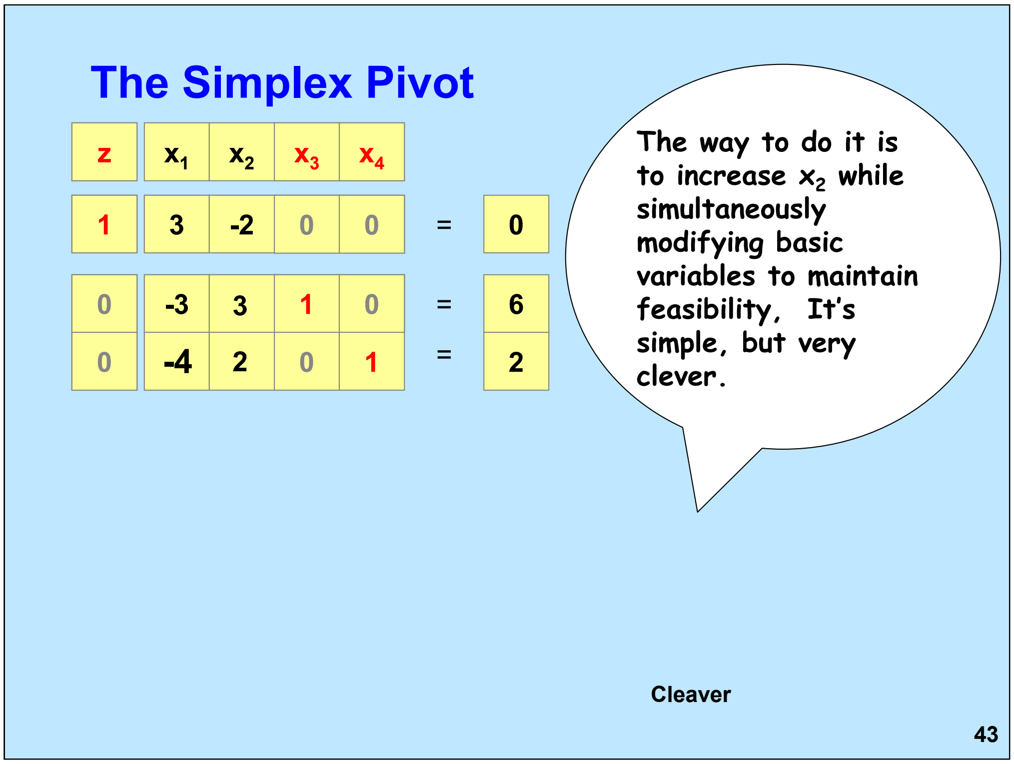
$z + 3x_1 - 2x_2 = 0$.

We can find a better solution by increasing x_2 above 0 and adjusting the current basic variables to get a feasible solution.

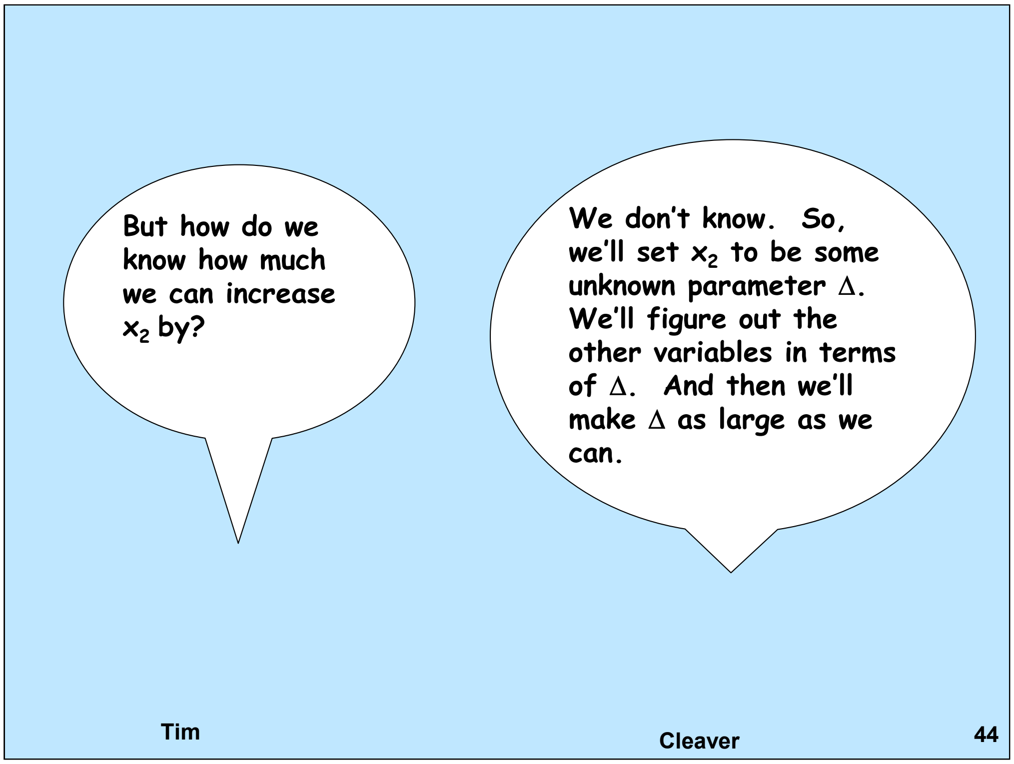


z	x_1	x_2	x_3	x_4		
1	3	-2	0	0	=	0
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

Cleaver and Tim come right to the key issues.



I like Cleaver's enthusiasm for this material.



Tim is always asking good questions, even if he doesn't know many of the answers.

The current basic feasible solution (bfs) is not optimal!

z	x_1	x_2	x_3	x_4		
1	3	-2	0	0	=	0
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

$x_2 = \Delta$

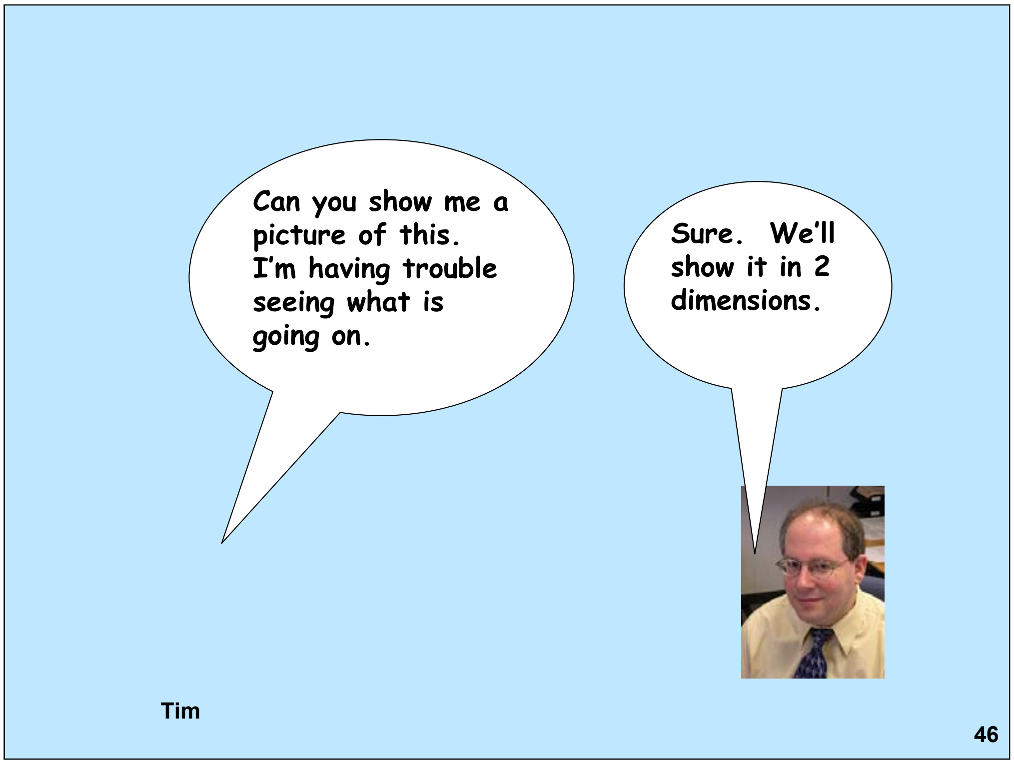
$x_1 = 0$, because we don't change any other non-basic variable.

$z = 2\Delta$.
$x_3 = 6 - 3\Delta$.
$x_4 = 2 - 2\Delta$.

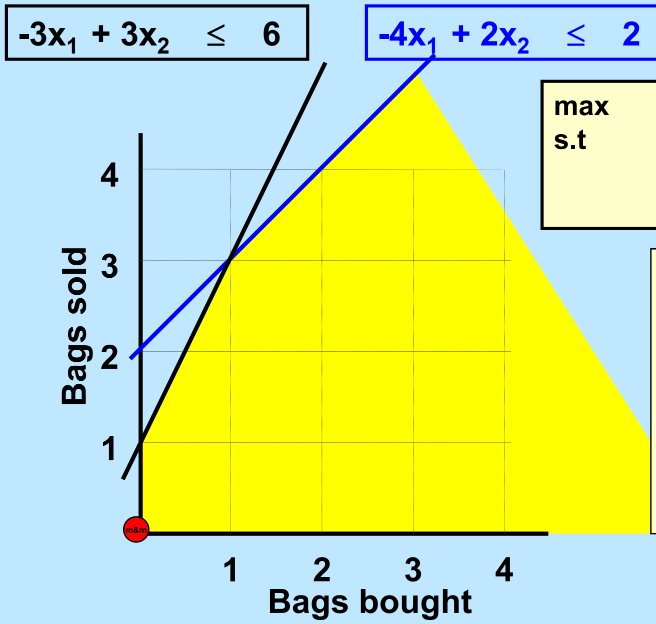
Choose Δ as large as it can be so that all variables remain non-negative. That is, the solution stays feasible.

$\Delta = 1$

$z = 2, x_1 = 0, x_2 = 1, x_3 = 3, x_4 = 0.$

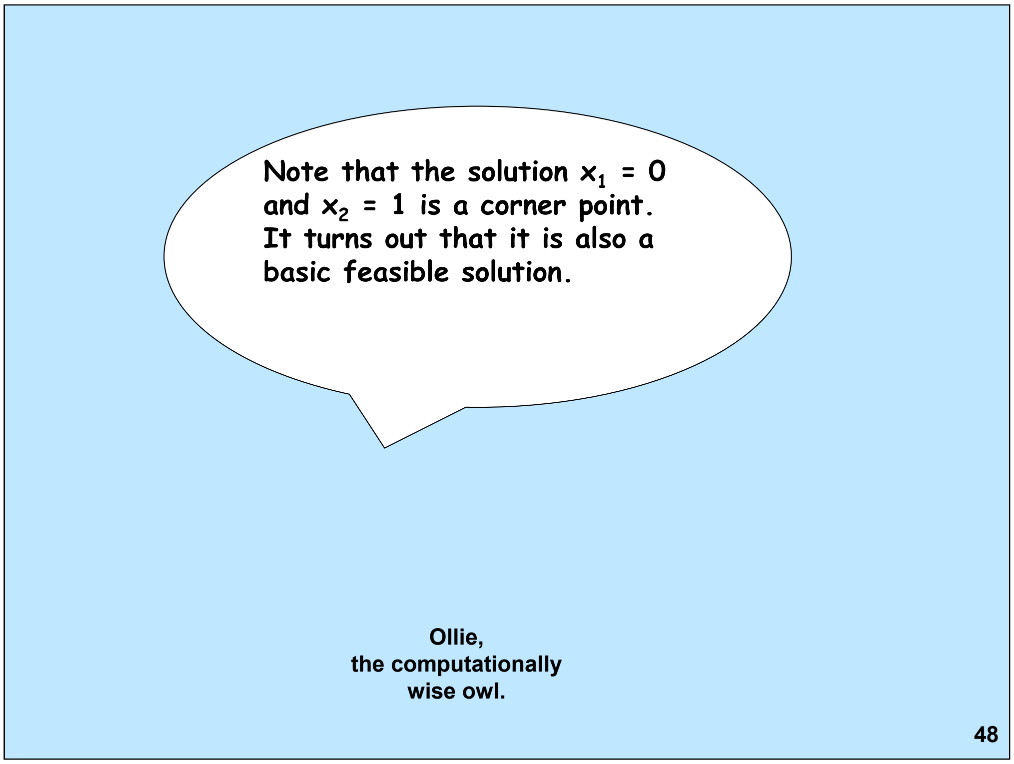


Occasionally, I put myself into the lectures as well.



$$
\begin{aligned}
\text{max} \quad & z = -3x_1 + 2x_2 \\
\text{s.t} \quad & -3x_1 + 3x_2 \leq 6 \\
& -4x_1 + 2x_2 \leq 2 \\
& x_1 \geq 0, \; x_2 \geq 0
\end{aligned}
$$

Our initial solution was to do nothing. the simplex method then realized it could do better by selling bags. But it could sell at most 1.



All bfs's correspond to corner point solutions. Ollie knew that, but decided to only tell you about a specific solution.

Pivoting to obtain the bfs

z	x_1	x_2	x_3	x_4		
1	3	-2	0	0	=	0
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

Non-basic variable x_2 becomes basic.

Choose column 2.

Basic variable x_4 becomes non-basic.

$z = 2, x_1 = 0, x_2 = 1, x_3 = 3, x_4 = 0.$

Next iteration, we want the column of x_2 to be

0
0
1

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Since x_2 replaces x_4, the column for x_2 after the iteration (pivot) will be the same as the column for x_4 before the iteration (pivot). In that way, we will still have canonical form after the pivot.

Pivoting to obtain a better solution

New Solution: basic variables z, x_2 and x_3. Nonbasics: x_1 and x_4.

z	x_1	x_2	x_3	x_4		
1	-1	0	0	1	=	2
0	3	0	1	-1.5	=	3
0	-2	1	0	.5	=	1

z = 2
x_1 = 0
x_2 = 1
x_3 = 3
x_4 = 0

Note that the bfs after the pivot is exactly what we wanted. By letting $x_2 = \Delta$ and increasing Δ from 0 to 1, we were moving along an edge of the feasible region. At the end of the edge is another corner point.

Summary of Simplex Algorithm

- Start in canonical form with a basic feasible solution

1. Check for optimality conditions
2. If not optimal, determine a non-basic variable that should be made positive
3. Increase that non-basic variable, and perform a pivot, obtaining a new bfs
4. Continue until optimal (or unbounded).

z	x_1	x_2	x_3	x_4		
1	a	0	0	0	=	3
0	b	0	1	0	=	6
0	c	1	0	0	=	3
0	d	0	0	1	=	5

To do with your partner (2 minutes)

The values a, b, c, and d are unknown

1. What are the basic variables? What is the current bfs?

2. Under what condition is the current bfs optimal?

z	x_1	x_2	x_3	x_4		
1	a	0	0	0	=	3
0	b	0	1	0	=	6
0	c	1	0	0	=	3
0	d	0	0	1	=	5

To do with your partner (3 minutes)

1. If we set x_1 to Δ, what are x_2, x_3, and x_4, all expressed in terms of Δ.
2. Assume that $b > 0$ and $d < 0$. Under what condition we will set $\Delta = 3/c$?
3. If $\Delta = 3/c$, what coefficient do we pivot on next?

Recognizing Unboundedness

If the non-cost coefficients in the entering column are ≤ 0, then the solution is *unbounded*

$$z - x_1 + x_4 = 2$$

z	x_1	x_2	x_3	x_4		RHS
1	-1	0	0	1	=	2
0	-3	0	1	-1.5	=	3
0	-2	1	0	.5	=	1

$z = 2 + \Delta$
$x_1 = \Delta$
$x_2 = 1 + 2\Delta$
$x_3 = 3 + 3\Delta.$
$x_4 = 0$

Δ can grow to ∞, and then z goes to ∞.

Next: two more iterations.

z	x_1	x_2	x_3	x_4		
1	-1	0	0	1	=	2
0	3	0	1	-1.5	=	3
0	-2	1	0	.5	=	1

$z - x_1 + x_4 = 2$

$z = 2 + \Delta$
$x_1 = \Delta$
$x_2 = 1 + 2\Delta$
$x_3 = 3 - 3\Delta.$
$x_4 = 0$

The cost coefficient of x_1 in the z-row is negative.
Set $x_1 = \Delta$ and $x_4 = 0$.

Then $\Delta = 3/3$.

Another pivot

z	x_1	x_2	x_3	x_4		
1	0	0	+1/3	+1/2	=	3
0	1	0	1/3	-1/2	=	1
0	0	1	2/3	-1/2	=	3

$z = 3$
$x_1 = 1$
$x_2 = 3$
$x_3 = 0$
$x_4 = 0$

The largest value of Δ is 3/3.

Variable x_1 becomes basic, x_3 becomes nonbasic.

So, x_1 becomes the basic variable for constraint 1.

Pivot on the coefficient with a 3.

Check for optimality

$$z + x_3/3 + x_4/2 = 3$$

z	x_1	x_2	x_3	x_4		
1	0	0	+1/3	+1/2	=	3
0	1	0	1/3	-1/2	=	1
0	0	1	2/3	1/2	=	3

$z = 3$
$x_1 = 1$
$x_2 = 3$
$x_3 = 0$
$x_4 = 0$

There is no negative coefficient in the z-row.

The current basic feasible solution is optimal!

Two views of the simplex method

- Improvement by "moving along an edge."
  - Increase Δ, and increase z.
  - An approach used in other algorithms, and that shows what is going on.

- Improvement by "moving to an adjacent corner point"
  - Move to an adjacent corner point and increase z
  - It can be viewed as a "shortcut"

Summary of Simplex Algorithm Again

- Start in canonical form with a basic feasible solution

1. Check for optimality conditions
   - Is there a negative coefficient in the cost row?
2. If not optimal, determine a non-basic variable that should be made positive
   - Choose a variable with a negative coef. in the cost row.
3. Increase that non-basic variable, and perform a pivot, obtaining a new bfs (or unboundedness)
   - We will review this step, and show a shortcut
4. Continue until optimal (or unbounded).

The Minimum Ratio Rule for determining the leaving variable.

$z - 3x_1 = 3$

z	x_1	x_2	x_3	x_4		RHS
1	-3	0	0	0	=	3
0	3	1	0	0	=	6
0	-2	0	1	0	=	1
0	2	0	0	1	=	5

$z = 3 + 2\Delta$
$x_1 = \Delta$
$x_2 = 6 - 3\Delta$
$x_3 = 1 + 2\Delta$
$x_4 = 5 - 2\Delta$

Δ = min (6/3, 5/2). At next iteration, pivot on the 3.

ratio: RHS coefficient/ entering column coefficient

s.t. entering column coefficient is positive

More on performing a pivot

- To determine the column to pivot on, select a variable with a negative cost coefficient
- To determine a row to pivot on, select a coefficient according to a minimum ratio rule
- Carry out a pivot as one does in solving a system of equations.

Next Lecture: More on the Simplex Algorithm