

10 Introduction to Strange Attractors

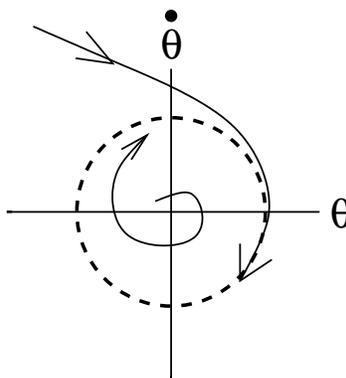
Thus far, we have studied only classical attractors such as fixed points and limit cycles. In this lecture we begin our study of *strange attractors*. We emphasize their generic features.

10.1 Dissipation and attraction

Our studies of oscillators have revealed explicitly how forced systems can reach a stationary (yet dynamic) state characterized by an energy balance:

$$\text{average energy supplied} = \text{average energy dissipated}$$

An example is a limit cycle:



Initial conditions inside or outside the limit cycle always evolve to the limit cycle.

Limit cycles are a specific way in which

$$\text{dissipation} \Rightarrow \text{attraction.}$$

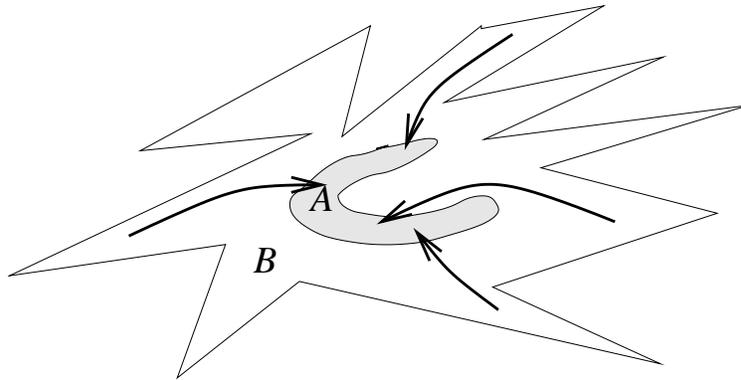
More generally, we have an n -dimensional flow

$$\frac{d}{dt}\vec{x}(t) = \vec{F}[\vec{x}(t)], \quad \vec{x} \in \mathbb{R}^n \quad (23)$$

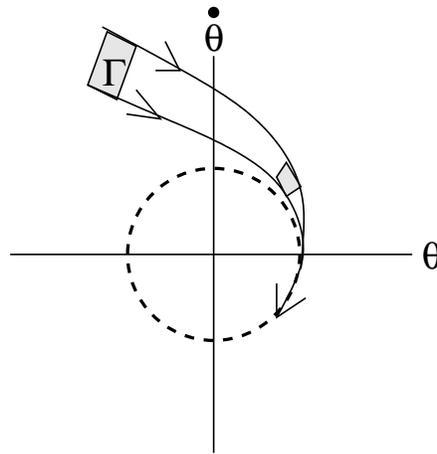
Assume that the flow $\vec{x}(t)$ is dissipative, with attractor A .

Properties of the attractor A :

- A is invariant with flow (i.e., it does not change with time).
- A is contained within B , the *basin of attraction*. B is that part of phase space from which all initial conditions lead to A as $t \rightarrow \infty$:



- A has dimension $d < n$.
Consider, for example, the case of a limit cycle:



The surface Γ is reduced by the flow to a line segment on the limit cycle (the attractor). Here

$$d = \text{attractor dimension} = 1$$

$$n = \text{phase-space dimension} = 2.$$

This phenomenon is called **reduction of dimensionality**.

Consequence: loss of information on initial conditions.

We have already quantified volume contraction. Given an initial volume V evolving according to the flow (23), the Lie derivative tells us that V changes as

$$\frac{1}{V} \frac{dV}{dt} = \nabla \cdot \dot{\vec{x}} = \sum_i^n \frac{\partial \dot{x}_i}{\partial x_i}$$

As we showed earlier, dissipation yields volume *contraction*; i.e.,

$$\frac{dV}{dt} < 0.$$

Consequently, the attractor cannot have n -dimensional volumes, so $d < n$.

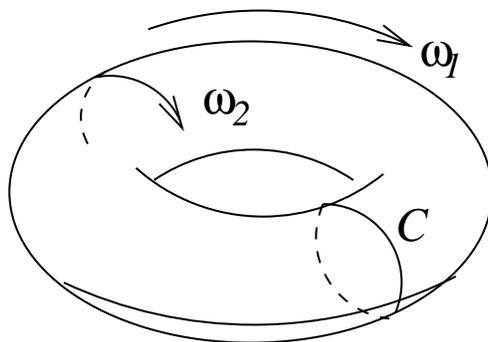
What, then, is the dimension of the attractor?

We proceed by example, by considering the case $d = 2$.

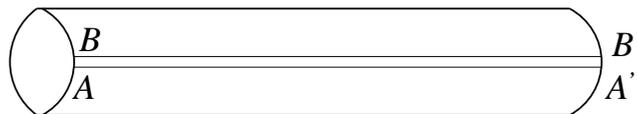
10.2 Attractors with $d = 2$

What happens when d (the dimension of the attractor) is 2?

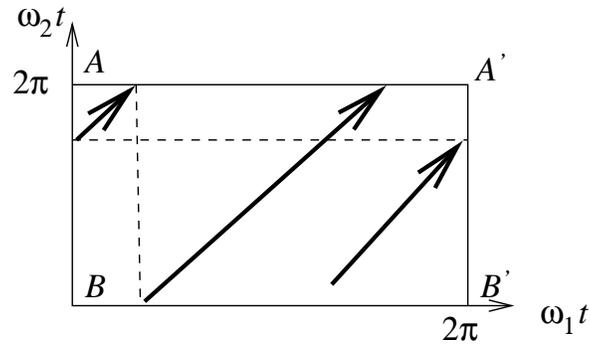
Assume a quasiperiodic attractor on a torus T^2 :



Cut the torus on a small circle C and open it:



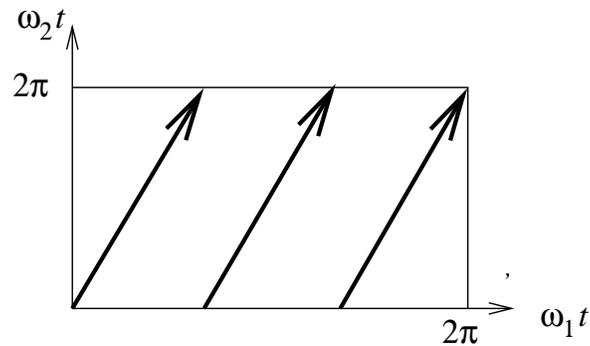
Finally, cut the long way, from A to A' , and open it again:



Note the parallel trajectories.

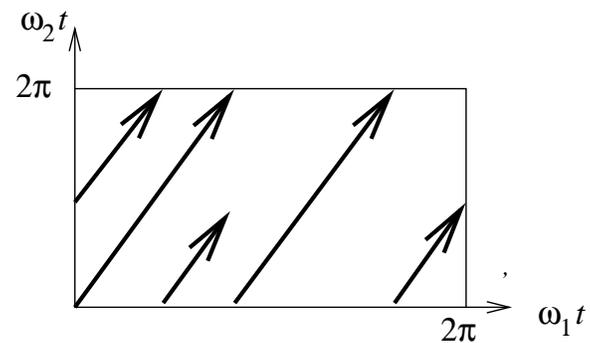
As usual, the quasiperiodic flows are characterized by two cases:
 ω_1/ω_2 rational or irrational.

- Rational. Consider, e.g., $\omega_1/\omega_2 = 1/3$:



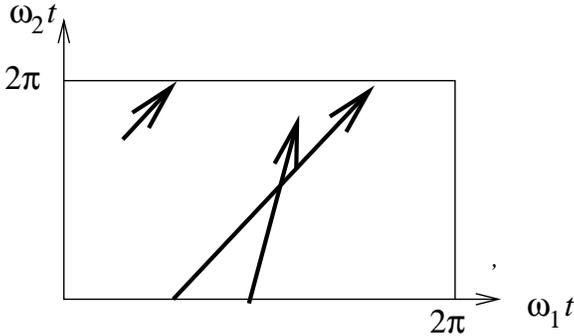
The trajectory repeats itself exactly every three times around the 2-axis,
 or each time around the 1-axis.

- Irrational.



The trajectories densely fill the plane.

Determinism forbids non-parallel trajectories, because they would cross:

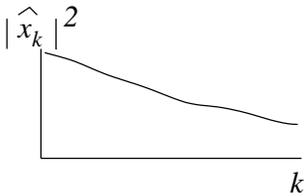


Thus a torus T^2 can only be a periodic or quasiperiodic attractor.

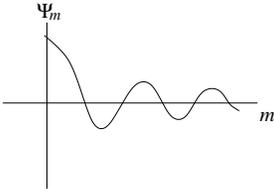
The attractor cannot be aperiodic if $d = 2$.

10.3 Aperiodic attractors

We have already shown that the power spectrum of an aperiodic signal $x(t)$ is continuous:



And the autocorrelation $\Psi_m = \langle x_j x_{j+m} \rangle$ has finite width:

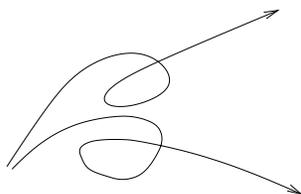


The finite width of Ψ_m implies that knowledge of no finite interval of $x(t)$ allows prediction of all future $x(t)$.

This “unpredictability” is associated with what we call “chaos.” We seek, however, a more precise definition of chaos.

On an aperiodic attractor, small differences in initial conditions *on the attractor* lead at later times to large differences, *still on the attractor*.

In phase space, trajectories on an aperiodic attractor can diverge, e.g.,



We shall see that the divergence of trajectories is exponential in time.

This phenomenon is called **sensitivity to initial conditions** (SIC). It definitively identifies *chaos*, i.e., a chaotic attractor.

Note that, despite the precision of this definition, we are left with an apparent conundrum: simultaneously we have

- attraction, such that trajectories *converge*.
- sensitivity to initial conditions, such that trajectories *diverge*.

The conundrum is solved by noting that trajectories converge *to* the attractor, but diverge *on* the attractor.

Note further that divergence on the attractor implies that the attractor dimension

$$d > 2,$$

since phase trajectories cannot diverge in two dimensions.

Thus we conclude that an aperiodic (chaotic) attractor must have phase space dimension

$$n \geq 3.$$

Assume $n = 3$. How may trajectories converge, but still remain bounded on an attractor?

The trajectories are successively **stretched** (by SIC) and **folded** (thus remaining bounded).

To illustrate these ideas, see

Figures 4.1.9–10, Abraham and Shaw

- Trajectories diverge in plane by spiralling out (stretching).
- Trajectories leave plane.
- Trajectories return to plane (folding), back to center of spiral.

At the same time, we must have volume contraction. One dimension can expand while another contracts, e.g.

Figures 4.3.1, Abraham and Shaw

10.4 Example: Rössler attractor

We proceed to consider stretching and folding in more detail, using the Rössler attractor:

$$\begin{aligned}\dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c)\end{aligned}$$

where we assume

$$a > 0.$$

Assume z and \dot{z} are small. Then in the x, y plane the system is approximated by

$$\begin{aligned}\dot{x} &= -y \\ \dot{y} &= x + ay.\end{aligned}$$

Then

$$\ddot{x} = -\dot{y} = -x + a\dot{x}$$

yielding the *negatively damped* oscillator

$$\ddot{x} - a\dot{x} + x = 0.$$

Consequently the trajectories spiral out of the origin.

How is the spreading confined? From the equation for \dot{z} , we see that, for small b ,

$$x < c \Rightarrow \dot{z} < 0$$

$$x > c \Rightarrow \dot{z} > 0$$

Thus we expect trajectories to behave as follows:

- Divergence from the origin creates $x > c$.
- $x > c \Rightarrow z$ increases $\Rightarrow x$ decreases.
- Eventually x decreases such that $x < c$.
- Then $x < c \Rightarrow z$ decreases \Rightarrow back in the x, y plane.
- The process repeats.

Thus we have

- *stretching*, from the outward spiral; and
- *folding*, from the feedback of z into x .

A sequence of figures shows how endless divergence can occur in a bounded region:

Figures 4.3.2–4, Abraham and Shaw

Trajectories never close exactly as a surface, but more like filo dough:

Figures 4.4.1–4, Abraham and Shaw

10.5 Conclusion

We arrive at the following conclusions:

- Aperiodic attractors must have

$$d > 2.$$

- Since dissipation contracts volumes,

$$d < n,$$

where n is the dimension of the phase space.

- Suppose $n = 3$. Then a chaotic attractor must have

$$2 < d < 3.$$

How can $2 < d < 3$? The attractor has a *fractal* dimension.

Chaotic attractors have three properties:

- Attraction
- SIC
- Non-integer fractal dimension.

The combination of these three properties defines a **strange attractor**. The “strangeness” arises not so much from each individual property but their combined presence.

Next we study the most celebrated strange attractor—the *Lorenz attractor*.