INVARIENTS OF OPTIMAL MINIMAL-ORDER OBSERVER-BASED COMPENSATORS*

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Abstract

The relations characterizing the optimal minimal-order observer-based compensators for a linear time-invariant multivariable system in a certain canonical form, with a random initial state (or equivalently, known initial state with white driving noise) have been reported by Miller [1] and independently by others [2], [3]. In this note, we establish in general that the optimal compensator transfer function uniquely determines the optimal observer transfer function, and that this characterizes precisely the degrees of freedom in the compensator design. Any two realizations of the optimal compensator dynamics yield the same performance.

1. Introduction

The problem of optimally controlling a time-invariant linear multivariable system

\[ \dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \quad \mathbf{x}(0) = \mathbf{N}(0, \mathbf{L}) \quad (1) \]

\[ \mathbf{y} = \mathbf{C}\mathbf{x} \quad (2) \]

with state \( \mathbf{x} \in \mathbb{R}^n \), control \( \mathbf{u} \in \mathbb{R}^r \) and output \( \mathbf{y} \in \mathbb{R}^m \), is subject to explicit constraints, since \( \mathbf{z} \) must be an observer of \( \mathbf{x} \) (in the sense of positive definiteness). Let \( \mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2] \) be parameters of a system with this canonical form, partitioned in accordance with \( \mathbf{C} \), as

\[ \mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} \]

\[ (3) \]

The input \( \mathbf{u} \) is constrained to be generated by a minimal-order state observer [4] with state \( \mathbf{z} \in \mathbb{R}^m \), which may generally be expressed in the form

\[ \dot{\mathbf{z}} = \mathbf{Fz} + \mathbf{Gy} \quad \mathbf{z}(0) = \mathbf{0} \quad (4) \]

so as to optimize the standard performance index,*

\[ J(\mathbf{F}, \mathbf{G}, \mathbf{H}, \mathbf{M}) = \mathbb{E}\{\int_0^\infty (\mathbf{z}'\mathbf{Q}\mathbf{z} + \mathbf{u}'\mathbf{R}\mathbf{u})dt\} \quad (6) \]

where the expectation is over the distribution induced by \( \mathbf{x}_0 \), and \( \mathbb{E} = \mathbb{E}^* > 0, \mathbf{R} = \mathbf{R}^* > 0 \) (in the sense of positive definiteness).

Let \( \mathbf{P} = \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ \mathbf{P}_2 & \mathbf{P}_3 \end{bmatrix} \) denote the parameters of the optimization problem characterized by (1)-(6). Then under the additional assumptions that \( \mathbf{P}_1 \) is minimal and \( \mathbf{P}_1 > 0 \) the optimal compensator of (1)-(3) exists, is unique, and may be characterized as follows: Let

\[ \mathbf{F} = \mathbf{X}^{-1}\mathbf{P} \quad (7) \]

where \( \mathbf{F} \) is the unique** positive definite solution of the algebraic Riccati equation

\[ \mathbf{0} = \mathbf{F}\mathbf{A} + \mathbf{A}^T\mathbf{F} + \mathbf{Q} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{F} \quad (8) \]

and let

\[ * \text{Note that the parameters of (4), (5) are not independent, but are subject to explicit constraints, since } \mathbf{z} \text{ must be an observer of } \mathbf{x}^2. \]

\[ ** \text{We assume } \mathbf{A}, \mathbf{Q} \text{ completely observable.} \]

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where $R(n,m)$ is the solution of the Lyapunov equation

$$0 = (R_{22} - F_{21} F_{11}^{-1} R_{12}) W + W (R_{22} - F_{21} F_{11}^{-1} R_{12})$$

$$-\frac{W_{12}}{1} F_{12} - F_{22} \frac{F_{12}}{1}$$

(10)

Then the optimal gains are given by

$$\bar{P} = \frac{R_{22} - F_{21} F_{11}^{-1} R_{12}}{1} + \frac{R_{22} - F_{21} F_{11}^{-1} R_{12}}{1}$$

$$\bar{Q} = (R_{22} - F_{21} F_{11}^{-1} R_{12}) E + R_{11} - F_{21} F_{11}^{-1} R_{12} (E_1 + E_2)$$

(12)

$$\bar{R} = E_2$$

(13)

$$\bar{R} = E_2 + E_2$$

(14)

And the optimal cost is [3]:

$$J^* = \text{tr}(E_2) + \text{tr}(E_2 + (E_2 - E_2 - E_2)^{(n,m)})$$

(15)

where $E_2 \in R(n,m)$ is the solution of the Lyapunov equation

$$\bar{R} (R_{22} - F_{21} F_{11}^{-1} R_{12}) + (R_{22} - F_{21} F_{11}^{-1} R_{12}) E + E_2 E_2 = 0$$

(16)

The first term of (15), which will be denoted $J_0$, is the optimal performance assuming full-state feedback through the Kalman control gains $E_2$ and the second term [2] is the additional cost incurred by the observer, denoted $J^*$.

In order to apply these results to an arbitrary (non-canonical) problem $P = (A, B, C, E, \bar{Q}, \bar{R})$, it is necessary to:

(a) transform the plant to canonical form
(b) transform the performance index in a compatible manner
(c) solve for the gains of the transformed problem using (7)-(14)
(d) retransform plant and compensator to the original coordinates in an appropriate manner.

One might infer from the prior works that this procedure is either trivial or uninteresting. We shall demonstrate that it is neither. Partial results pertaining to (a)-(d) are reported independently in the theses of Blanvillain [3] and Rothschild (see [4]), and perhaps in other documents not known to the authors.

Our procedure will be to expose in Section II the appropriate transformations (not unique!) for accomplishing (a) and (b). We show that two equivalent realizations of the plant may be reduced to the same transformed problem (c). Because the transformation of a problem $P$ to a problem $\bar{P}$ is not unique, "the" optimal compensator is not unique. In Section III we show that although the compensator gains in step (c) depend on this transformation, that any two transformations yield compensators with the same performance (after the re-transformation (d)) and that such optimal compensators are in fact always related by a similarity transformation (in $R(n,m)$). The conclusion, then, is that for a given performance index, the Kronecker invariants of the plant uniquely determine the Kronecker invariants of the optimal compensator, and that is all: the engineer is free to construct whatever realization of the optimal compensator suits him best, e.g. from the standpoint of reliability, component cost, etc. Further implications are discussed in Section IV.

II. Equivalence Relations for Compensation Problems

A well-set compensator problem, $P$, consists of a sextuple* of matrices $(A, B, C, E, \bar{Q}, \bar{R})$ (dimensioned as above) such that $(A, B, C)$ is minimal, $\bar{Q}$, $\bar{R}$ are symmetric and $\bar{Q} > 0$, $\bar{Q} C \bar{C} > 0$, and $\bar{R} > 0$ in the sense of positive definite matrices. Evidently, this corresponds to the problem (1), (2), (6), subject to the constraint that $u$ is generated by an observer (4), (5) of the unmeasured states.

Two well-set problems $P_1$, $P_2$ having equal dimensions $(n,m,r)$ are said to be equivalent if there exists a nonsingular matrix $S \in R_{n \times n}$ such that

$$P_2 = (A_2, B_2, C_2, E_2, Q_2, R_2) = (S A_1 S^{-1}, S B_1 S^{-1}, S C_1 S^{-1}, S E_2 S^{-1}, S Q_1 S^{-1}, S R_2 S^{-1})$$

where subscript (1) denotes problem $P_1$. Two minimal realizations of $A_1, B_1, C_1$ are said to be in state-output canonical form if $C = [I_m, 0]$, i.e., the first $m$ states are measured exactly and independently.

We proceed to develop some simple consequences of these definitions.

Proposition 2.1

For any well-set compensation problem, $P$ there exists an equivalent problem, $\bar{P}$, in state-output canonical form $(m < n)$.

Proof (Yuksel and Bongiorno [5])

Let $C$ be the output matrix of $P$ and choose

$$S = \begin{bmatrix} C \\ T \end{bmatrix}$$

where $T \in R_{(n-m) \times n}$ is any matrix which renders $S$ nonsingular (i.e., rows of $T$ linearly independent of rows of $C$). Since $SS^T = I_n - C C^T = I_n - [I_m, 0]$ as desired. Define the other parameters of

*We have assumed zero-mean initial state for conciseness of exposition. See [3] for the case of nonzero mean initial state.
Proposition 2.2

Assume that $P_1$ and $P_2$ are equivalent well-set compensation problems with $m = n$. Then the output feedback problems consisting of (1), (2) and (4)-(6) with $F$, $G$ and $H$ set of zero have the same optimal gains, same closed-loop dynamics, and numerically equal optimum performances.

Proof

The optimal gains for this problem are well-known to be $M_i = B_i P_i C_i$, $i = 1, 2$, where $P_i$ denotes the solution of (6) with overbars replaced by subscripts (i). Let $P_1$ and $P_2$ be related by $P_2 = (S^*)^{-1} P_1 S^{-1}$ (see [3, p. 17]), and by equivalence we see immediately that $M_1 = M_2$, and that the closed-loop roots

$$\lambda[A_2 - B_2 P_2^{-1} S_2] = \lambda[A_1 - B_1 P_1^{-1} S_1] = \lambda[S[(A_1 - B_1 P_1^{-1} B_1)] S^{-1}] = \lambda[A - B P^{-1} B_1]$$

of $P_1$, we may construct an equivalent compensation problem $P_2$, which may then be reduced to any equivalent problem $\tilde{P}$ of $P_1$. What remains, is merely to show that the solution of any problem $\tilde{P}$ derived from $P_1$ is (in an appropriate sense) independent of the choice of $T_1$.

III. Equivalence of Observer-Based Compensators

The main result of this section is the following:

Theorem 3.1

Let $P$ be a well-set compensation problem $(m < n)$ and let $T_1$, $T_2 \in R^{(n-m)xn}$ define (as in Proposition 2.1) equivalent state-output problems $P^1$, $P^2$. The optimal compensators for these problems are defined by (7)-(14) (with overbars replaced by subscripts 1, 2 respectively). Then these optimal compensators are equivalent realizations of (4)-(5), and lead to the same closed-loop performance, i.e., there exists a nonsingular $U \in R^{(n-m)^2}$ such that $[P^1, G^1, H^1] = [U F^1 U^{-1}, U G^1 U^{-1}, H^1]$.

The proof of this theorem requires some preliminary results, which are stated in the following sequence of lemmas (see Appendix for proofs).

Lemma 3.1

Let $C \in R^{m \times n}$, of full rank $n < m$, be given. Any $T \in R^{(n-m)xn}$ such that $T = \begin{bmatrix} C \\ T \end{bmatrix}$ is nonsingular has a unique decomposition.

$$T = V C + H \qquad NC_T = 0 \quad (19)$$

with $N$ of full rank, $n - m$. Furthermore, if the inverse of $S$ is partitioned as $S^{-1} = \begin{bmatrix} C & T \end{bmatrix}$, then $C$, $T$ have the unique decompositions

$$C = C' (C')^{-1} - N' (NN')^{-1} V \quad (20)$$

$$T = N' (NN')^{-1} \quad (21)$$

(hence $CT = 0$ and $CC_T = T$, in particular).

Lemma 3.2

Let $C \in R^{m \times n}$, $m < n$, be given. Let $T_1$, $T_2$ be as in Lemma 3.1 with unique decompositions of form (19). For any such matrices, there exists a unique nonsingular $U \in R^{(n-m)^2}$ such that $N^2 = U N^1$.

Lemma 3.3

Let $C_T$ be as in Lemma 3.1, and let $\tilde{P}$ denote the equivalent state-output problem resulting from
the transformation $S$. Then the dependence of the parameters in $\tilde{P}$ upon $T = V C + H$ is given by

$$\tilde{A} = \begin{bmatrix} CA^T & CA^T \\ TA^T & TA^T \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} CB \\ TB \end{bmatrix},$$

$$\tilde{C} = [I - 0 \ 0] \begin{bmatrix} CC^T & CC^T \\ TC^T & TC^T \end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} \tilde{C} \tilde{B}^T \\ \tilde{T} \tilde{C} \tilde{B}^T \end{bmatrix},$$

$$\tilde{K} = \tilde{R}$$

with the consequence that in (7)-(10),

$$\tilde{K} = [\tilde{K}_1 \ \tilde{K}_2] = \tilde{K}(\tilde{C} \tilde{B})$$

(22)

where $\tilde{K}$ denotes the Kalman gains for the original plant (solution of (17), (18) sans overbars), and $\tilde{W}$ in (10) is the solution (independent of $V$) of

$$\begin{align*}
N(I - EC'(CC')^{-1}C)AN'(NN')^{-1}2 & + \\
\tilde{W}(NN')^{-1}NA'(N - C'(CC')^{-1}C) & - \\
\tilde{W}(NN')^{-1}NA'C'(CC')^{-1}C'A'(NN')^{-1}2 & + \\
N(I - EC'(CC')^{-1}C)N' & = 0
\end{align*}$$

(24)

Proof of Theorem 3.1

Evidently, we intend to make use of Lemma 3.2, but this requires us to show that the optimal compensator gains are independent of $\tilde{V}$, as well as having the desired dependence on $\tilde{W}_1$ or $\tilde{W}_2$. Let $\tilde{W}_1$, $\tilde{W}_2$ denote the solutions of (24) corresponding to $\tilde{W}_1$ and $\tilde{W}_2$. Then it is readily verified that $\tilde{W}_2 = \tilde{W}_1 U'$. Applying (22) and the decomposition (19)-(21) to (9), some algebra yields

$$\begin{align*}
\tilde{P}_1 & = [\tilde{W}_1 (N - N')^{-1} A + N_1^{-1} C(C'C)^{-1}] + \tilde{V}_1^2
\end{align*}$$

(25)

Then applying Lemma 3.2, one finds from a similar calculation that

$$\begin{align*}
\tilde{P}_2 & = U(\tilde{V}_1^{-2}) + \tilde{V}_2
\end{align*}$$

(26)

We illustrate the proof that $\tilde{P}_2 = \tilde{W}_1^{-1} U^{-1}$. First,

$$\begin{align*}
\tilde{P}_1 & = \tilde{P}_2 - \tilde{P}_1 \tilde{P}_2^{-1} \tilde{B}_2 + \tilde{B}_2 \tilde{P}_2^{-1} \tilde{P}_1 \tilde{B}_2 \\
& = \begin{bmatrix} \tilde{P}_1 \tilde{A}^{-1} - \tilde{P}_1 \tilde{C} \tilde{A}^{-1} + (\tilde{P}_1^{-1} \tilde{B} \tilde{B}^{-1} \tilde{C}) \tilde{K}^{-1} 
\end{bmatrix}
\end{align*}$$

(22),

(23)

(24)

(25)

(26)

Similar algebra demonstrates that $\tilde{P}_2 = \tilde{W}_1^{-1} U^{-1}$ as desired. The equality of the costs is established in a similar manner; for instance,

$$\begin{align*}
J_0^1 & = \text{tr} [\tilde{P}_1^{-1} \tilde{Q}] \\
& = \text{tr} \begin{bmatrix} \tilde{P}_1^{-1} \tilde{C}_{11} \tilde{C}_{11}^{-1} \tilde{C}^T \tilde{F}_1 \tilde{F}_1^{-1} \tilde{C}^T \\
\tilde{P}_1^{-1} \tilde{C}_{12} \tilde{C}_{12}^{-1} \tilde{C}^T \tilde{F}_1 \tilde{F}_1^{-1} \tilde{C}^T \
\end{bmatrix} \\
& + \text{tr} \begin{bmatrix} \tilde{P}_1^{-1} \tilde{C}_{11} \tilde{C}_{11}^{-1} \tilde{C}^T \tilde{F}_1 \tilde{F}_1^{-1} \tilde{C}^T \\
\tilde{P}_1^{-1} \tilde{C}_{12} \tilde{C}_{12}^{-1} \tilde{C}^T \tilde{F}_1 \tilde{F}_1^{-1} \tilde{C}^T \
\end{bmatrix}
\end{align*}$$

and using the fact that $\tilde{C}_{11} + \tilde{C}_{12} = \tilde{I}_{n}$, twice along with a trace identity gives

$$\begin{align*}
J_0^1 & = \text{tr} [\tilde{P}(\tilde{P}^{-1} \tilde{P})^2] + \text{tr} [\tilde{P}(\tilde{P}^{-1} \tilde{P})^2] \\
& = \text{tr} [\tilde{P} \tilde{P}] (= J_0^1, \text{ obviously})
\end{align*}$$

independent of $\tilde{V}_1, \tilde{V}_2$ (see (25)). Then

$$\tilde{P}_2 = [\tilde{W}_1^{-1} \tilde{C} \tilde{A}^{-1} + \tilde{W}_1^{-1} \tilde{B} \tilde{B}^{-1} \tilde{C}]$$

$$+ \text{tr} [\tilde{P}(\tilde{P}^{-1} \tilde{P})^2] = \text{tr} [\tilde{P}(\tilde{P}^{-1} \tilde{P})^2]$$

Also, $\tilde{H}_2 = \tilde{H}_1^{-1}$, since

$$\begin{align*}
\tilde{H}_2 & = \tilde{W}_1^{-1} \tilde{C} \tilde{A}^{-1} + \tilde{W}_1^{-1} \tilde{B} \tilde{B}^{-1} \tilde{C} \\
& = \text{tr} [\tilde{P}(\tilde{P}^{-1} \tilde{P})^2] + \text{tr} [\tilde{P}(\tilde{P}^{-1} \tilde{P})^2] \\
& = \text{tr} [\tilde{P}(\tilde{P}^{-1} \tilde{P})^2] + \text{tr} [\tilde{P}(\tilde{P}^{-1} \tilde{P})^2]
\end{align*}$$

(26)

Corollary 3.1

If $P_1$ and $P_2$ are equivalent compensation problems, then they have equivalent optimal compensators, i.e., any optimal compensator realization of one problem is also optimal for the other.

Proof

Suppose that $P_1$ and $P_2$ are related by $S_i$, and that application of $S_i$ to $P_1$ yields a state-output problem $P_i$ which results in optimal compensator parameters $[\tilde{F}_i, \tilde{G}_i, \tilde{H}_i, \tilde{H}_2]$, $i = 1, 2$. These two

$$\text{tr} [\tilde{P}(\tilde{P}^{-1} \tilde{P})^2] = \text{tr} (\tilde{P} \tilde{P})$$

giving a shorter proof.
compensators must be related by a similarity transformation, since Proposition 2.3 the state-output problems \( P_i \) are then equivalent, and the compensator of one merely corresponds to a different choice of \( T \) for the other, and Theorem 3.1 applies.

IV. Discussion and Conclusions

The degree of non-uniqueness of the solution of the optimal minimal-order observer-based compensator has been precisely displayed, and is in fact intuitively pleasing, because the engineer retains the freedom to choose a realization of the optimal compensator. Possibly this property is so pleasing that it has been presumed to hold by several authors, although in fact it represents a rather unusual special feature for a constrained optimization problem.

We could in fact have based our proofs directly on the statement of the constrained optimization problem [3, p. 80-83] rather than on the (unique) solution of the necessary conditions for the canonical problem. This approach allows us to extract a problem formulation with a unique solution prior to the application of direct methods of computation and hence remove potential convergence problems for the problem stated in the original coordinates. In fact, an attempt to express the necessary conditions in the original coordinates leads to an underdetermined Riccati equation [3, p. 90] of order \( n^2 \) rather than \((n-m)^2\) as in (10):

\[
\dot{X} + \Sigma - (\bar{X}^{-2} + \Sigma)C'(CC')^{-1}C(\bar{X}^{-2} + \Sigma)' = 0
\]

(27)

where \( \bar{X} \in R^{n^2} \). The fact that equation (10), or more explicitly, equation (24), does have a unique solution (under the stated conditions) may be useful in the study of degenerate Riccati equations; (24) appears to be a type of projection of (27).

While the observer-based compensator is known to satisfy the necessary conditions derived by Newmann [2] and by Levine, Johnson and Athans [6] for the case \( \text{dim } z = n-m \), the "separation principle" for lower-order compensators has only been established by Jameson and Rothschild [4] and others under the assumption that the compensator is an observer of the optimal gains. The direct approach proposed above may be extended to examine uniqueness of observer-based compensators of any order. Our results suggest that a major feature of the observer-based compensator formulation is to eliminate unwanted degrees of freedom (viz. equivalent realizations of the plant and of the compensators) from the design procedure in order to obtain uniqueness.

References

1. Miller, R.A., "Specific Optimal Control of the Linear Regulator Using a Minimal Order Observer", Int. J. Control, Vol. 10, pp. 139-


Appendix

Proof of Lemma 3.1

If \( S \) is nonsingular its rows must be linearly independent; thus each row of \( T \) must be linearly independent of all rows of \( C \) and all other rows of \( T \). From linear algebra, the span of the rows of \( C \) forms an \( m \)-dimensional subspace of \( R^n \); (19) is essentially a statement of the projection theorem, stating that each row of \( T \) may be uniquely decomposed into a projection on the span of the rows of \( C \) (rows of \( V \)) and a vector orthogonal to it (row of \( N \)). We have explicitly

\[ V = CT(CC')^{-1} \]

(27)

\[ N = T(I-C(CC')^{-1}C) \]

(28)

Since \( S^{-1} \) is both a left and right inverse of \( S \), \( C, T, \bar{C} \) and \( \bar{T} \) must satisfy

\[ \bar{C} = I_m \]

(29)

\[ \bar{T} = I_{n-m} \]

(30)

\[ \bar{C} = 0_{m,n-m} \]

(31)

\[ \bar{T} = 0_{n-m,m} \]

(32)

\[ \bar{C} + \bar{T} = I \]

(33)

\[ \bar{C}, \bar{T} \] admit unique decompositions

\[ \bar{C} = C^{*}V + N \quad ; \quad CN = 0 \]

(34)
Using (A9) in (A3) gives \( \tilde{T} = (CC')^{-1} \). Using (19) and (A9) in (A6) gives \( \tilde{N} + \tilde{V} = 0 \). Using (A8) in (A5) gives \( \tilde{V} = 0 \). From (19) and (A8) in (A7) imply \( \tilde{N} + \tilde{N} \tilde{V} = 0 \). Verify that these three relations among \( N, \tilde{T}, \tilde{N} \) and \( \tilde{V} \) are satisfied uniquely (in view of (A1), (A2)) by (20) and (21). We have in fact rather complicated expressions

\[
\tilde{T} = \left( I - C'(CC')^{-1}C \right) T' \left[ I - C'(CC')^{-1}C \right]^{-1} \tilde{T}
\]

(A10)

\[
\tilde{C} = \left( I - T'C'(CC')^{-1}C \right) \tilde{C}
\]

(A11)

relating \( C \) and \( T \) to \( \tilde{C} \) and \( \tilde{T} \).

Proof of Lemma 3.2

Let \( N_i \in R^{(n-m)xn} \), \( i = 1, 2 \) correspond to the decomposition (19) of \( T_i \). The matrices \( N_i \) are not completely arbitrary, in fact, but must both be of full rank and satisfy the \( (n-m) \) constraints

\[
N_i' C' = 0.
\]

Thus there are exactly \( (n-m)^2 \) degrees of freedom in the choice of \( N_i \) so that the conjecture \( N_i = UN_i \) for some nonsingular \( U \in R^{(n-m)} \) makes sense. In fact using (A2) one finds that \( U \) is the unique solution of the \( (n-m)^2 \) independent linear relations

\[
(T_i' - UC_i^{-1}C_i') = 0\quad (A12)
\]

Proof of Lemma 3.3

Equations (22) are a straightforward consequence of the definition of equivalent compensation problems and the partitions of \( S, S^{-1} \) introduced above. Now also

\[
\tilde{K} = - R^{-1} B' P = - R^{-1} B' P (C')^{-1} P R^{-1} = - R^{-1} B' P (C')^{-1} P R^{-1} \tilde{K} = R^{-1} B' P (C')^{-1} P R^{-1} \tilde{K} = \tilde{K} (C \tilde{T}) \quad (A13)
\]

as desired, where the definition of equivalence and the expression for \( \tilde{F} \) (see proof of Proposition 2.2) have been used. To prove (24), let \( \tilde{N} \) be as in (10) and apply the relations (22) with the decompositions (19)-(21). The first step gives

\[
(T\tilde{N} - TEC' (CC')^{-1} CAT) \tilde{N} + \tilde{W}^T A' T^{-1} P A' C' (CC')^{-1} CT T' - \tilde{W}^T A' C' (CC')^{-1} CAT = \tilde{W}^T A' T^{-1} P A' C' (CC')^{-1} CT T' = 0 \quad (A14)
\]

We illustrate the procedure for simplifying the first term of this expression: